# Chapter 4

# Congruency between classical and mild solutions of Caputo fractional impulsive evolution equation on Banach Space

The sufficient conditions for existence and uniqueness of mild and classical solution of fractional order impulsive integro-differential equations of the following form is established in this chapter. And also derived conditions in which mild and classical solution are congruence.

$$\begin{aligned} {}^cD^{\beta}x(t) &= Ax(t) + f(t, x(t), Tx(t), Sx(t)) \quad t \neq t_k,\ k = 1, 2, \cdots, p \\ \Delta x(t_k) &= I_k(x(t_k)), \quad t = t_k,\ k = 1, 2, \cdots, p \\ x(0) &= x_0 \end{aligned} \tag{4.0.1}$$

over the interval $[0, T_0]$ in a Banach space $X$. Here, $A$ is bounded linear operator on $X$ and $f : [0, T_0] \times X \times X \times X \to X$,$T, S : X \to X$ are defined by $Tx(t) = \int_0^t h(t, s, x(s))ds$ and $Sx(t) = \int_0^{T_0} k(t, s, x(s))ds$; where $h : D_0 \times X \to X,\ D_0 = \{(t, s); 0 \leq s \leq t \leq T_0\}$ and $k : D_1 \times X \to X,\ D_1 = \{(t, s); 0 \leq t, s \leq T_0\}$ are the operators satisfying condition of the hypotheses.

## 4.1 Introduction

Mathematical problems having such nonlinear equations arise in many physical situations like heat flow in materials with viscoelastic behavior [44]. Many physical phenomena like seepage flow in porous media [7], anomalous diffusion, wave and transport [8, 9, 10] and other problems in fluid dynamics [12] need modern mathematical modeling and solutions. In fact fractional differential equations are considered as an alternative model for nonlinear model [13]. This is because of their non local property unlike integer order differential equations [14] which means, that the next state of the system depends not only upon its current state but also upon its entire historical states.

On the other-hand due to sudden change at certain moments in dynamics of the systems, such systems are modeled in to impulsive differential equations [20]. The existence and uniqueness of integer order impulsive system for many systems are studied by researchers like Gao, Liu, Rogovchenko, Anguraj and Arjunan [39] using various conditions.

Existence and uniqueness of mild solutions of impulsive fractional differential equations with classical conditions using transition matrix and semi-group theory are derived by Benchohra [40], Mophou [41, 87] and Ravichandran and Arjunan [42]. However, both the Riemann-Liouville and the Caputo fractional differential operators do not possess semi-group or commutative properties, which are inherent to the derivatives on integer order [21, 22, 46]. Therefore, there should be another approach to study fractional differential equation. The existence and uniqueness of the mild solution of impulsive fractional order evolution equation with initial classical and nonlocal conditions introduced by [22]. Balachandran et. al.[46], existence and uniqueness of mild solution of impulsive fractional differential equations with delay using Banach fixed point theorem was explore by Wang et. al. [56, 88], existence and uniqueness of the solutions of fractional order impulsive differential equation with order lies between one and two using fixed point theorem was derived by Kataria and

Patel [57], in this chapter we extended our work and discusses sufficient conditions for existence and uniqueness of classical solutions and conditions in which the mild solution becomes classical solution.

## 4.2 Notations

(N1) $X$ = Banach space and $D(A)$ = Domain of an operator $A$.

(N2) $\mathbb{R}_+ = [0, \infty)$

(N3) $C([0,T_0],X) = \big\{x : [0,T_0] \to X/x \, is \, continuous\big\}$ with norm $||x|| = sup_t||x(t)||$

(N4) $PC([0,T_0],X) = Closure\Big(\big\{x : [0,T_0] \to X; x \in C([t_{k-1},t_k],X), \; and \, x(t_k^-) \, and \, x(t_k^+)$ $exist, \; k = 1,2,\cdots,p \, with \, x(t_k^-) = x(t_k)\big\}\Big)$ with norm $||x||_{PC} = sup_{t\in[0,T_0]}||x(t)||$

(N5) $AC([0,T_0],X) = \big\{x : [0,T_0] \to X/x \; is \; absolutely \; continuous\big\}$ with norm $||x|| = sup_t||x(t)||$

(N6) $B(X) = \big\{A : X \to X/A \; is \; bouneded \; and \; linear\big\}$ with norm $||A||_{B(X)} = sup\{||A(y)||; y \in X \& ||y|| \leq 1\}$

(N7) $C^\beta([0,T_0],X) = \big\{x : [0,T_0] \to X/{}^cD^\beta x(t) \; exist \; and \; continuous \; at \; each$ $t \in [0,T_0]\big\}$

with these definitions and properties, sufficient conditions for existence and uniqueness of solutions are derived as follows:

## 4.3 Mild Solution

In this section sufficient conditions are derived for existence and uniqueness of mild solution of the equation (4.0.1).

**Definition 4.3.1.** *([88]) A function $x(t) \in PC([0,T_0],X)$ is a mild solution of the equation (4.0.1) if it satisfies*

$$\begin{aligned}x(t) = x_0 &+ \frac{1}{\Gamma(\beta)}\sum_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-s)^{\beta-1}A(s)x(s)ds + \frac{1}{\Gamma(\beta)}\int_{t_k}^{t}(t-s)^{\beta-1}A(s)x(s)ds\\ &+ \frac{1}{\Gamma(\beta)}\sum_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-s)^{\beta-1}f(s,x(s),Tx(s),Sx(s))ds\\ &+ \frac{1}{\Gamma(\beta)}\int_{t_k}^{t}(t-s)^{\beta-1}f(s,x(s),Tx(s),Sx(s))ds + \sum_{0<t_k<t}I_kx(t_k^-).\end{aligned} \tag{4.3.1}$$

### 4.3.1 Assumptions

(H1) $A(t): X \to X$ is continuous bounded linear operator and there exists a positive constant $M$, such that $||A(t)x - A(t)y||_{B(X)} \leq M||x-y||$, for all $x, y \in X$.

(H2) $f : [0,T_0] \times X \times X \times X \to X$ is continuous and there exists positive constants $L_1, L_2$ and $L_3$, such that $||f(t,x_1,x_2,x_3) - f(t,y_1,y_2,y_3)|| \leq L_1||x_1-y_1|| + L_2||x_2-y_2|| + L_3||x_3-y_3||$ for all $x_1,x_2,x_3,y_1,y_2$ and $y_3$ in $X$.

(H3) $h : D_0 \times X \to X$ and $k : D_1 \times X \to X$ are continuous and there exists positive constants $H$ and $K$, such that $||h(t,s,x) - h(t,s,y)|| \leq H||x-y||$ and $||k(t,s,x) - k(t,s,y)|| \leq K||x-y||$ for all $x$ and $y$ in $X$.

(H4) The functions $I_k : X \to X$ are continuous and there exist positive constants $I_k^*$ for all $k = 1,2,\cdots,p$, such that $||I_kx - I_ky|| \leq I_k^*||x-y||$ for all $x$ and $y$ in $X$.

Set, $\gamma = \frac{T_0^\beta}{\Gamma(\beta+1)}$ and further assume that,

(H5) $q = \left\{\gamma\big[(p+1)[M + L_1 + T_0HL_2 + T_0KL_3]\big] + \sum I_k^*\right\} < 1$

Define $F : PC([0,T_0],X) \to PC([0,T_0],X)$ by

$$\begin{aligned} Fx(t) &= x_0 + \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} A(s)x(s)ds + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} A(s)x(s)ds \\ &+ \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} f(s, x(s), Tx(s), Sx(s))ds \\ &+ \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} f(s, x(s), Tx(s), Sx(s))ds + \sum_{0<t_k<t} I_k x(t_k^-). \end{aligned} \tag{4.3.2}$$

Thus it can be said that equation (4.3.1) has unique mild solution if $F$ defined by (4.3.2) has unique fixed point. This means $F$ is well defined bounded operator on $PC([0,T_0],X)$ and $F$ is contraction [23].

**Lemma 4.3.1.** *If the operators $A, f, T, S$ and $I_k$ for $k = 1, 2, \cdots, p$ are continuous then $F$ is bounded operator on $PC([0,T_0],X)$.*

*Proof.* Let a sequence $\{x_n\}$ converges to $x$ in $PC([0,T_0],X)$. Therefore $||x_n - x|| \to 0$ as $n \to \infty$.

Consider,

$$\begin{aligned} ||Fx_n - Fx||_{PC} \leq & \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} ||A(s)x_n(s) - A(s)x(s)||ds \\ &+ \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} ||A(s)x_n(s) - A(s)x(s)||ds \\ &+ \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \Big\{ \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} ||f(s, x_n(s), Tx_n(s), Sx_n(s)) \\ &- f(s, x(s), Tx(s), Sx(s))||ds \Big\} \\ &+ \frac{1}{\Gamma(\beta)} \Big\{ \int_{t_k}^{t} (t-s)^{\beta-1} ||f(s, x_n(s), Tx_n(s), Sx_n(s)) \\ &- f(s, x(s), Tx(s), Sx(s))||ds \Big\} \\ &+ \sum_{0<t_k<t} ||I_k x_n(t_k^-) - I_k x(t_k^-)|| \end{aligned}$$

Assuming the continuity of $A, f, T, S$ and $I_k$ for $k = 1, 2, \cdots, p$ the right side of above expression tends to zero as $n \to \infty$. Therefore $F$ is continuous on $PC([0,T_0],X)$

and hence $F$ is bounded. □

**Theorem 4.3.2.** *If the hypotheses (H1)-(H5) are satisfied, then the fractional impulsive integro-differential equation (4.0.1) has unique mild solution in* $PC([0,T_0],X)$ *for* $0<\beta\leq 1$.

*Proof.* To show equation (4.0.1) has unique mild solution it is sufficient to show $F$ defined (4.3.2) is contraction. Let $x$ and $y$ in $PC([0,T_0],X)$ and consider,

$$\begin{aligned}||Fx-Fy||_{PC} \leq &\frac{1}{\Gamma(\beta)}\sum_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-s)^{\beta-1}||A(s)x(s)-A(s)y(s)||ds\\
&+\frac{1}{\Gamma(\beta)}\int_{t_k}^{t}(t-s)^{\beta-1}||A(s)x(s)-A(s)y(s)||ds\\
&+\frac{1}{\Gamma(\beta)}\sum_{0<t_k<t}\Big\{\int_{t_k-1}^{t_k}(t_k-s)^{\beta-1}||f(s,x(s),Tx(s),Sx(s))\\
&-f(s,y(s),Ty(s),Sy(s))||ds\Big\}\\
&+\frac{1}{\Gamma(\beta)}\Big\{\int_{t_k}^{t}(t-s)^{\beta-1}||f(s,x(s),Tx(s),Sx(s))\\
&-f(s,y(s),Ty(s),Sy(s))||ds\Big\}\\
&+\sum_{0<t_k<t}||I_kx(t_k^-)-I_ky(t_k^-)||\end{aligned}$$

Applying hypotheses (H1)-(H4), it leads to

$$
\begin{aligned}
||Fx - Fy|| \le & \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} M ||x-y|| ds \\
& + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} M ||x-y|| ds \\
& + \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} \big\{ L_1 + T_0 H L_2 + T_0 K L_3 \big\} ||x-y|| ds \\
& + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} ds \big\{ L_1 + T_0 H L_2 + T_0 K L_3 \big\} ||x-y|| ds \\
& + \sum_{0<t_k<t} I_k^* ||x-y|| \\
& \le \left\{ \frac{T_0^\beta}{\Gamma(\beta+1)} \big[ (p+1)[M + L_1 + T_0 H L_2 + T_0 K L_3] \big] + \sum I_k^* \right\} ||x-y|| \\
& = \left\{ \gamma \big[ (p+1)[M + L_1 + THL_2 + TKL_3] \big] + \sum I_k^* \right\} ||x-y||
\end{aligned}
$$

Assuming hypotheses (H5) it follows that, $||Fx - Fy||_{PC} \le q||x-y||$ with $q < 1$. Hence, by Banach fixed point theorem [97] the equation (4.0.1) has unique mild solution. □

### 4.3.2 Remarks

(1) This method suggests not only the existence and uniqueness of mild solution but it also suggests method to find approximate solution of impulsive fractional differential equation (4.0.1).

(2) This condition is not necessary condition. This means equation (4.0.1) may have mild solution if one of the (H1) to (H5) are not satisfied.

(3) If all $I_k$'s are constants then assumption (H5) can replace by
$$q = \left\{ \gamma \big[ (p+1)[M + L_1 + T_0 H L_2 + T_0 K L_3] \big] \right\} < 1$$

## 4.4 Classical Solution

**Definition 4.4.1.** *A solution $x(t)$ is a classical solution of the equation (4.0.1) for $0 < \beta < 1$ if $x(t) \in PC([0,T_0],X) \cap C^{\beta}(J',X)$ where, $J' = [0,T_0] - \{t_1,t_2,\cdots,t_p\}$, $x(t) \in D(A)$ (Domain of A) for $t \in J'$ and satisfies (4.0.1) on $[0,T_0]$.*

**Lemma 4.4.1.** *If conditions,*

*(B1)* $A : X \to X$ *is continuous bounded linear operator and there exists a positive constant $M$, such that $||A(t)x - A(t)y||_{B(X)} \leq M||x-y||$, for all $x,y \in X$.*

*(B2)* $f \in C^{\beta}([0,T_0]\times X\times X\times X, X)$ *such that, there exists positive constants $L_1, L_2$ and $L_3$, such that $||f(t,x_1,x_2,x_3) - f(t,y_1,y_2,y_3)|| \leq L_1||x_1-y_1|| + L_2||x_2-y_2|| + L_3||x_3-y_3||$.*

*(B3)* $h : D_0 \times X \to X$ *and* $k : D_1 \times X \to X$ *are continuous and there exists positive constants $H$ and $K$, such that $||h(t,s,x) - h(t,s,y)|| \leq H||x-y||$ and $||k(t,s,x) - k(t,s,y)|| \leq K||x-y||$ for all $x$ and $y$ in $X$.*

*(B4) Let,* $\gamma = \frac{T_0^{\beta}}{\Gamma(\beta+1)}$ *and* $q = \left\{\gamma\big[(p+1)[M + L_1 + T_0HL_2 + T_0KL_3]\big]\right\} < 1.$

*(B5)* $x_0 \in D(A)$.

*are satisfied then the fractional evolution equation*

$$\begin{aligned} {}^{c}D^{\beta}x(t) &= Ax(t) + f(t,x(t),Tx(t),Sx(t)) \\ x(0) &= x_0 \end{aligned} \tag{4.4.1}$$

$(0 < \beta \leq 1)$ *has unique classical solution over $[0,T_0]$ which satisfies*

$$x(t) = x_0 + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}A(s)x(s)ds + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}f(s,x(s),Tx(s),Sx(s))ds.$$

*Proof.* Applying Riemann- Liouville integral operator both side of equation (4.4.1) to obtain,

$$x(t) = x_0+\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}A(s)x(s)ds+\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}f(s,x(s),Tx(s),Sx(s))ds. \tag{4.4.2}$$

It can be easily shown that $x(t)$ defined above is in $C^\beta([0,T_0],X)$. Therefore to show, $x(t)$ is classical solution, it is sufficient to show that $x(t)\in D(A)$ for all $t\in[0,T_0]$. Define a sequence $x_n(t)$ by,

$$\begin{aligned} x_n(t) =& x_0 + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}A(s)x_{n-1}(s)ds \\ &+ \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}f(s,x_{n-1}(s),Tx_{n-1}(s),Sx_{n-1}(s))ds. \end{aligned}$$

Clearly for each $n$, $x_n(t)\in D(A)$ because $x_0\in D(A)$ and applying assumptions $(B1)$ to $(B4)$ the sequence $x_n$ converges uniformly to $x$. Hence, $x(t)\in D(A)$ for all $t$. Therefore, $x(t)$ is classical solution of (4.4.1) which is of the form (4.4.2) and is also mild solution of the equation (4.4.1). Moreover, uniqueness of mild solution gives uniqueness of classical solution. Hence, $x(t)$ is unique classical solution of (4.4.1) satisfies mild solution (4.4.2). □

Next lemma is generalization of the lemma-(4.4.1)

**Lemma 4.4.2.** *If conditions,*

*(C1)* $A: X\to X$ *is continuous bounded linear operator and there exists a positive constant* $M$*, such that* $||A(t)x-A(t)y||_{B(X)}\le M||x-y||$*, for all* $x,y\in X$.

*(C2)* $f\in C^\beta([0,T_0]\times X\times X\times X, X)$ *such that, there exists positive constants* $L_1, L_2$ *and* $L_3$*, such that* $||f(t,x_1,x_2,x_3)-f(t,y_1,y_2,y_3)||\le L_1||x_1-y_1|| + L_2||x_2-y_2|| + L_3||x_3-y_3||$.

*(C3)* $h: D_0\times X\to X$ *and* $k: D_1\times X\to X$ *are continuous and there exists positive constants* $H$ *and* $K$*, such that* $||h(t,s,x)-h(t,s,y)||\le H||x-y||$ *and*

$||k(t,s,x) - k(t,s,y)|| \leq K||x-y||$ *for all* $x$ *and* $y$ *in* $X$.

*(C4) Let,* $\gamma = \frac{u_0^{\beta}}{\Gamma(\beta+1)}$ *and* $q = \Big\{\gamma\big[(p+1)[M + L_1 + t_0HL_2 + t_0KL_3]\big]\Big\} < 1$ *where,* $u_0 = (t_k - t_{k-1})$

*(C5)* $q_{k-1} \in D(A)$.

*are satisfied then the fractional evolution equation*

$$\begin{aligned} {}^cD^{\beta}_{t_{k-1}+}x(t) &= Ax(t) + f(t, x(t), Tx(t), Sx(t)) \\ x(t_{k-1}) &= q_{k-1} \end{aligned} \tag{4.4.3}$$

$(0 < \beta < 1)$ *and for* $k = 1, 2, \cdots, p$ *has unique classical solution over interval* $[t_{k-1}, t_k)$ *which satisfies,*

$$\begin{aligned} x(t) =& q_{k-1} + \frac{1}{\Gamma(\beta)}\int_{t_{k-1}}^{t}(t-s)^{\beta-1}A(s)x(s)ds \\ &+ \frac{1}{\Gamma(\beta)}\int_{t_{k-1}}^{t}(t-s)^{\beta-1}f(s, x(s), Tx(s), Sx(s))ds. \end{aligned}$$

*Moreover, one can define* $x(t_k)$ *in such a way that,* $x(t_k)$ *is left continuous and* $x(t_k) \in D(A)$.

*Proof.* Replacing 0 by $t_{k-1}$ and $T_0$) by $t_k$ and applying lemma-(4.4.1) we get, equation (4.4.3) has unique solution which satisfies

$$\begin{aligned} x(t) =& q_{k-1} + \frac{1}{\Gamma(\beta)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta-1}A(s)x(s)ds \\ &+ \frac{1}{\Gamma(\beta)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta-1}f(s, x(s), Tx(s), Sx(s))ds.. \end{aligned} \tag{4.4.4}$$

Define,

$$\begin{aligned} x(t_k) =& q_{k-1} + \frac{1}{\Gamma(\beta)}\int_{q_{k-1}}^{t_k}(t-s)^{\beta-1}A(s)x(s)ds \\ &+ \frac{1}{\Gamma(\beta)}\int_{t_{k-1}}^{t}(t-s)^{\beta-1}f(s, x(s), Tx(s), Sx(s))ds, \end{aligned}$$

then to show, $x(t_k)$ is left continuous consider increasing sequence $\{s_m\}$ converges to $t_k$ *i.e.,* $||s_m - t_k|| \to 0$ *as* $m \to \infty$. *One can easily show that* $||x(s_m) - x(t_k)|| \to 0$ *as* $m \to \infty$. *Hence,* $x(t)$ *is left continuous at* $t_k$ *and* $x(t_k) \in D(A)$. □

**Theorem 4.4.3.** *If assumptions* $(H1)-(H5)$ *is satisfied,* $f \in C^{\beta}([0,T_0] \times X \times X \times X, X)$ *and* $x_0 \in D(A)$ *then the equation,*

$$\begin{aligned} {}^cD^{\beta}x(t) &= Ax(t) + f(t, x(t), Tx(t), Sx(t)) \quad t \neq t_k, \; k = 1, 2, \cdots, p \\ \Delta x(t_k) &= r_k, \quad t = t_k, \; k = 1, 2, \cdots, p \\ x(0) &= x_0 \end{aligned} \tag{4.4.5}$$

$(0 < \beta < 1)$ *has unique classical solution over the interval* $[0, T_0]$ *which satisfies*

$$\begin{aligned} x(t) = x_0 &+ \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} A(s)x(s)ds + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} A(s)x(s)ds \\ &+ \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} f(s, x(s), Tx(s), Sx(s))ds \\ &+ \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} f(s, x(s), Tx(s), Sx(s))ds + \sum_{0<t_k<t} r_k \end{aligned} \tag{4.4.6}$$

*Proof.* Consider the interval $I_1 = [0, t_1)$ then, the equation (4.4.5) becomes,

$$\begin{aligned} {}^cD^{\beta}x(t) &= Ax(t) + f(t, x(t), Tx(t), Sx(t)) \\ x(0) &= x_0. \end{aligned} \tag{4.4.7}$$

Using lemma-(4.4.2), equation (4.4.7) has unique classical solution which satisfies the equation

$$\begin{aligned} x_1(t) = &x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} A(s)x(s)ds \\ &+ \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, x(s), Tx(s), Sx(s))ds \end{aligned} \tag{4.4.8}$$

and define, $x_1(t_1)$ as

$$x_1(t_1) = x_0 + \frac{1}{\Gamma(\beta)} \int_0^{t_1} (t_1 - s)^{\beta-1} A(s)x(s)ds + \frac{1}{\Gamma(\beta)} \int_0^{t_1} (t_1 - s)^{\beta-1} f(s, x(s), Tx(s), Sx(s))ds.$$

Also, $x_1(t)$ is left continuous at $t = t_1$ and $x_1(t_1) \in D(A)$.

On the interval $I_2 = [t_1, t_2)$, the equation (4.4.5) becomes

$$\begin{aligned} {}^cD^{\beta}_{t_1+}x(t) &= Ax(t) + f(t, x(t), Tx(t), Sx(t)) \\ x(t_1) &= x_1(t_1) + r_1. \end{aligned} \tag{4.4.9}$$

Since, $x_1(t) + r_k \in D(A)$ therefore applying lemma-(4.4.2) equation (4.4.9) has unique classical solution which satisfies

$$\begin{aligned} x_2(t) = &[x_1(t_1) + r_1] + \frac{1}{\Gamma(\beta)} \int_{t_1}^{t} (t - s)^{\beta-1} A(s)x(s)ds \\ &+ \frac{1}{\Gamma(\beta)} \int_{t_1}^{t} (t - s)^{\beta-1} f(s, x(s), Tx(s), Sx(s))ds \end{aligned} \tag{4.4.10}$$

and define, $x_2(t_2)$ as

$$x_2(t_2) = [x_1(t_1) + r_1] + \frac{1}{\Gamma(\beta)} \int_{t_1}^{t_2} (t_2 - s)^{\beta-1} A(s)x(s)ds + \frac{1}{\Gamma(\beta)} \int_{t_1}^{t_2} (t_2 - s)^{\beta-1} f(s, x(s), Tx(s), Sx(s))ds.$$

Also, $x_2(t)$ is left continuous at $t = t_2$ and $x_2(t_2) \in D(A)$.

Continuing this process on $I_k = [t_{k-1}, t_k)$, equation (4.4.5) becomes

$$\begin{aligned} {}^cD^{\beta}_{t_{k-1}}x(t) &= Ax(t) + f(t, x(t), Tx(t), Sx(t)) \\ x(t_{k-1}) &= x_{k-1}(t_{k-1}) + r_{k-1}. \end{aligned} \tag{4.4.11}$$

and applying lemma-(4.4.2) equation (4.4.11) has unique classical solution which satisfies

$$\begin{aligned} x_k(t) &= [x_{k-1}(t_{k-1}) + r_{k-1}] + \frac{1}{\Gamma(\beta)} \int_{t_{k-1}}^{t} (t-s)^{\beta-1} A(s)x(s)ds \\ &+ \frac{1}{\Gamma(\beta)} \int_{t_{k-1}}^{t} (t-s)^{\beta-1} f(s, x(s), Tx(s), Sx(s))ds. \end{aligned} \tag{4.4.12}$$

and define, $x_k(t_k)$ as

$$\begin{aligned} x_k(t_k) = [x_{k-1}(t_{k-1}) + r_{k-1}] &+ \frac{1}{\Gamma(\beta)} \int_{t_{k-1}}^{t_k} (t_k-s)^{\beta-1} A(s)x(s)ds \\ &+ \frac{1}{\Gamma(\beta)} \int_{t_{k-1}}^{t_k} (t_k-s)^{\beta-1} f(s, x(s), Tx(s), Sx(s))ds. \end{aligned}$$

Also, $x_k(t)$ is left continuous at $t = t_k$ and $x_k(t_k) \in D(A)$. Now define

$$x(t) = \begin{cases} x_1(t) & t \in [0, t_1) \\ x_k(t) & t \in [t_{k-1}, t_k) \\ x_{p+1}(t) & t \in [t_p, t_{p+1}] \end{cases} \tag{4.4.13}$$

This $x(t)$ define by equation (4.4.13) is unique classical solution of the equation (4.4.5). Now we prove equation (4.4.13) satisfies (4.4.6). If $t \in [0, T_0]$ then there exist $k$ such that $t \in I_k = [t_{k-1}, t_k)$. Therefore

$$\begin{aligned} x(t) &= x_k(t) \\ &= [x_{k-1}(t_{k-1}) + r_{k-1}] + \frac{1}{\Gamma(\beta)} \int_{t_{k-1}}^{t} (t-s)^{\beta-1} A(s)x(s)ds \\ &+ \frac{1}{\Gamma(\beta)} \int_{t_{k-1}}^{t} (t-s)^{\beta-1} f(s, x(s), Tx(s), Sx(s))ds \end{aligned}$$

Putting value of $x_{k-1}(t_{k-1}), x_{k-2}(t_{k-2}), \cdots, x_1(t_1)$ in above equation and solving to get, equation (4.4.6). Therefore, under the given assumption equation (4.4.5) has classical solution which satisfied equation (4.4.6). □

Next theorem shows the existence and uniqueness of classical solution which derives from mild solution.

**Theorem 4.4.4.** *Assume the hypotheses* $(H1)-(H5)$ *are satisfied. Let* $x(t)$ *be mild solution of equation (4.0.1) obtained in theorem-(4.3.2). Assume that* $x_0 \in D(A)$ *and* $I_k x(t_k) \in D(A)$ *for* $k = 1, 2, \cdots, p$ *and* $f \in ((0, T_0) \times X \times X \times X, X)$. *Then* $x(t)$ *give rise to unique classical solution of (4.0.1).*

*Proof.* Let $x(t)$ be mild solution of (4.0.1). Replace, $r_k = I_k x(t_k)$ for all $k = 1, 2, \cdots, p$. Also given that, $(H1)-(H5)$ satisfied, $x_0$ and $I_k x(t_k) \in D(A)$ for all $k = 1, 2, \cdots, p$. Therefore, by theorem-(4.4.3) equation (4.0.1) has unique classical solution say $y(t)$ which satisfy equation

$$\begin{aligned}
y(t) = x_0 &+ \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} A(s) y(s) ds \\
&+ \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t - s)^{\beta-1} A(s) y(s) ds \\
&+ \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} f(s, y(s), Ty(s), Sy(s)) ds \\
&+ \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t - s)^{\beta-1} f(s, y(s), Ty(s), Sy(s)) ds + \sum_{0<t_k<t} I_k y(t_k).
\end{aligned}$$

Set, $z(t) = x(t) - y(t)$ then $z(t)$ satisfies evolution equation ${}^cD^\beta z(t) = 0$ with initial condition $z(0) = 0$ and without impulses. Hence, $z(t) \equiv 0$ is only solution of the evolution equation ${}^cD^\beta z(t) = 0$. Hence, $x(t) = y(t)$ and therefore $x(t)$ give rise to classical solution of (4.0.1). □

## 4.5 Example

Consider an evolution equation

$$
\begin{aligned}
&{}^cD^{\beta}x(t) = \frac{1}{10}\int_0^1 (t-s)x(s)ds + \frac{1}{10}\int_0^t se^{-\frac{x(s)}{4}}ds + \frac{1}{10}\int_0^1 (t-s)^2 e^{x(s)} \\
&\Delta x(\frac{1}{2}) = \frac{1}{10}x(\frac{1}{2}^-) \qquad\qquad (4.5.1)\\
&x(0) = x_0
\end{aligned}
$$

over the interval $[0,1]$. Then,

$$f \equiv \frac{1}{10}\int_0^t se^{-\frac{x(s)}{4}}ds + \frac{1}{10}\int_0^1 (t-s)^2 e^{x(s)}$$

with

$$
\begin{aligned}
Tx(s) &= \frac{1}{10}\int_0^t se^{-\frac{x(s)}{4}}ds \\
Sx(s) &= \frac{1}{10}\int_0^1 (t-s)^2 e^{x(s)}.
\end{aligned}
$$

Impulse at

$$I_1 x(\frac{1}{2}) = \frac{1}{10}x(\frac{1}{2}^-).$$

One can easily show that

$||Tx - Ty|| \leq \frac{1}{10}||x-y||,$

$||Sx - Sy|| \leq \frac{1}{10}||x-y||,$

and $||I_1x(\frac{1}{2}) - I_1y(\frac{1}{2})|| \leq \frac{1}{10}||x(\frac{1}{2}) - y(\frac{1}{2})||.$

Therefore $||f(t,x(t),Tx(t),Sx(t))) - f(t,y(t),Ty(t),Sy(t)))|| \leq \frac{1}{5}||x(t)-y(t)||.$

Hence,$q = \left\{\gamma\big[(p+1)[M + L_1 + T_0HL_2 + T_0KL_3]\big] + \sum I_k^*\right\} = \frac{1}{\Gamma(\beta)}\frac{6}{10} < 1,$

for any $0 < \beta \leq 1$.Thus, equation (5.4.1)has unique mild solution.

Moreover,$f \in C^{\beta}((0,1) \times X \times X \times X, X)$ and choosing, $x_0 \in D(A)$

one can obtained unique classical solution of equation which is arise from mild solution.

## 4.6 Discussion

The work of Aaunguraj and Arjunan [39] modified in this chapter by first order impulsive evolution equation to fractional order . However they used operator semi-group theory to prove the results. But, fractional order system does not have semi-group property so this paper having different approach to prove modified results. We can also get existence and uniqueness of mild and classical solutions by weaken Lipschitz type conditions.

## 4.7 Conclusion

Many researchers discussed existence and uniqueness of mild solutions but only mild solution is not sufficient in many of the practical situations and requires classical solution. So there has to be sufficient conditions for classical solutions and conditions in which mild solution becomes classical solution.