Chapter 5

Congruence between Mild and Classical Solutions of Generalized Fractional Impulsive Evolution Equation

In many physical phenomenon like, motion of a vehicle in traffic or body dynamics of persons having infectious disease can be modeled in terms of generalized impulsive evolution equations in which, perturbing forces are different after every impulses. This chapter, includes existence and uniqueness of mild and classical solution of generalized fractional order impulsive evolution equations

$$\begin{aligned} {}^{c}D^{\beta}x(t) &= Ax(t) + g_k(t, x(t)) \quad t \in (t_{k-1}, t_k)\ k = 1, 2, \cdots, p \\ \Delta x(t_k) &= I_k(x(t_k)), \quad t = t_k,\ k = 1, 2, \cdots, p \\ x(0) &= x_0 \end{aligned} \tag{5.0.1}$$

in which perturbing force is different after every impulse over the interval $[0, T]$ on a Banach space X. A is bounded linear operator on X and $I_k : [0, T] \times X \to X$ for $k = 1, 2, \cdots, p+1$. Also established conditions in which mild and classical solutions are congruent.

5.1 Introduction

From the past few decades, the theory of impulsive equations is extensively used to model the phenomena having abrupt changes like removal of insertion of biomass, abrupt harvesting, satellite motion and many other real world problems in medicine, biology, mechanics and control theory [59, 60, 61, 62, 106, 64, 20, 65, 66].

Many researchers studied existence and uniqueness of mild and classical solution of linear integer order impulsive evolution equation with local and non-local condition by assuming sectorial and semi group properties of evolution equation [67, 68, 69, 70, 71]. Liu, Anguraj and Arjunan [72, 39] were studied existence and uniqueness of mild and classical solution of integer order quasi-linear impulsive evolution equation with local and non-local condition and also studied conditions in which mild and classical solution of impulsive evolution equation are coincides.

Several non-linear integer order model in physical, engineering and biological sciences are remodel into simpler fractional order model [44, 7, 8, 9, 10, 12, 85, 86, 51]. Due to remodeling of integer order differential equations onto fractional order differential equation, many mathematicians are interested to study the qualitative properties viz existence, uniqueness and stability of solution the fractional order differential equations [11, 15, 5, 6].

Fractional differential operators does not have sectorial as well as semi group properties. In view of this fact, Balachandran *et. al.* [21] were studied the existence and uniqueness of the mild solution of impulsive fractional equation on the Banach space by inverting differential operator by applying fractional integral operator. This approach was also studied by Wang *et. al.* [88]. The system of fractional order differential equation taken by Balachandran was modified by Kataria and Prakashkumar by introducing new Fredholm type operator in system and studied the existence and uniqueness of mild solution of the equation [74]. Kataria and Prakashkumar [73] introduced the concept of classical solution and discussed existence and uniqueness of mild and classical solutions of the fractional order impulsive evolution with perturbing forces after every impulses.

Recently Shah et. al. [33] derived the integer order model having different perturbing forces after every impulse and discussed existence and uniqueness of classical and mild solutions of generalized impulsive evolution equation on a Banach space. To enhance generalized model here studied the existence and uniqueness of solutions of generalized fractional order evolution equation.

Theorem 5.1.1. *(Banach fixed point theorem) [97] If X is Banach space and $F : X \to X$ be a contraction mapping on Banach space X then there exist unique fixed point $x \in X$, such that $Fx = x$.*

Notations 5.1.1. *Following notations are introduced for the convenience.*

(N1) $X =$ *Banach space and* $D(A) =$ *Domain of an operator* A.

(N2) $\mathbb{R}_+ = [0, \infty)$

(N3) $C([0,T], X) = \big\{x : [0,T] \to X \mid x \text{ is continuous}\big\}$ *with norm* $||x|| = sup_t||x(t)||$

(N4) $PC([0,T], X) = Closure\Big(\big\{x : [0,T] \to X; x \in C([t_{k-1}, t_k], X),\ and\ x(t_k^-)\ and\ x(t_k^+)$ $exist, = 1, 2, \cdots, p\ with\ x(t_k^-) = x(t_k)\big\}\Big)$ *with norm* $||x||_{PC} = sup_{t\in[0,T]}||x(t)||$

(N5) $AC([0,T], X) = \big\{x : [0,T] \to X \mid x \text{ is absolutely continuous}\big\}$ *with norm* $||x|| = sup_t||x(t)||$

(N6) $B(X) = \big\{A : X \to X \mid A \text{ is bouneded and linear}\big\}$ *with norm* $||A||_{B(X)} = sup\{||A(y)||; y \in X \& ||y|| \leq 1\}$

(N7) $C^\beta([0,T], X) = \big\{x : [0,T] \to X \mid {}^cD^\beta x(t) \text{ exist and continuous at each } t \in [0,T]\big\}$

5.2 Mild Solution

Existence and uniqueness condition of mild solution of equation (5.0.1) discussed in this section

Definition 5.2.1. *The function $x(t)$ is mild solution of equation* (5.0.1) *if it satisfies integral equation of the form:*

$$x(t) = x_0 + \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - q)^{\beta-1} A(q)x(q)dq + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-q)^{\beta-1} A(q)x(q)dq$$
$$+ \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - q)^{\beta-1} g_k(q, x(q))dq + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-q)^{\beta-1} g_{k+1}(q, x(q))dq$$
$$+ \sum_{0<t_k<t} I_k x(t_k^-). \tag{5.2.1}$$

for all $t \in (t_k, t_{k+1})$.

Assumptions 5.2.1. *Presently we have following presumptions for the existence and uniqueness of mild solution of equation* (5.0.1)*. (i) Let the linear operator $A(t)$ on X is continuous and bounded. Further there exist a positive constant such that $||A(t)|| \leq M$. (ii) Perturbing functions g_k's are Lipschitz continuous with Lipschitz constant L_k for all $k = 1, 2, \cdots, p{+}1$. Take $L = max\{L_1, L_2, \cdots, L_{p+1}\}$. (iii) Jumps I_k's are such that there exist positive constants K_k's satisfying $||I_k x(t_k) - I_k y(t_k)|| \leq K_k||x(t_k) - y(t_k)||$, with $K = max\{K_1, K_2, \cdots, K_{p+1}\}$.*

We define an operator F on a Banach space $PC([0, T], X)$ as follow, and to show the equation (5.0.1) has unique solution, we have to show F is continuous, bounded and contraction.

$$Fx(t) = x_0 + \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - q)^{\beta-1} A(q)x(q)dq + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-q)^{\beta-1} A(q)x(q)dq$$
$$+ \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - q)^{\beta-1} g_k(q, x(q))dq \quad .$$
$$+ \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-q)^{\beta-1} g_{k+1}(q, (q))dq + \sum_{0<t_k<t} I_k x(t_k^-) \tag{5.2.2}$$

Theorem 5.2.1. *Under the assumptions(5.2.1), then the operator F defined in (6.3.2) is continuous and bounded on PC([0,T],X)*

Proof. To show continuity of the operator F, let us consider a sequence $\{x_n\}$ converges to x in $PC([0,T],X)$. i.e. $||x_n - x|| \to 0$ as $n \to \infty$.
Consider,

$$\begin{aligned}||Fx_n - Fx||_{PC} \leq & \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - q)^{\beta-1} ||A(q)|| ||x_n(q) - x(q)|| dq \\ & + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-q)^{\beta-1} ||A(q)|| ||x_n(q) - x(q)|| dq \\ & + \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - q)^{\beta-1} ||g_k(q, x_n(q)) - g_k(q, x(q))|| dq \\ & + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-q)^{\beta-1} ||g_{k+1}(q, x_n(q)) - g_{k+1}(q, x(q))|| dq \\ & + \sum_{0<t_k<t} ||I_k x_n(t_k^-) - I_k x(t_k^-)||\end{aligned}$$

By our assumptions(5.2.1) we get, $||Fx_n - Fx|| \leq C||x_n - x||$ where $C = \frac{(M+L+K)(p+1)T^\beta}{\Gamma(\beta+1)}$.
This gives the continuity of operator F.
Now to show F is bounded on PC([0,T],X) , let us consider arbitrary $x(t) \in PC([0,T],X)$,

$$\begin{aligned}|Fx(t)| \leq & \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - q)^{\beta-1} |A(q)||x(q)| dq + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-q)^{\beta-1} |A(q)||u(q)| dq \\ & + \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - q)^{\beta-1} |g_k(q, x(q))| dq \\ & + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-q)^{\beta-1} |g_{k+1}(q, x(q))| dq + \sum_{0<t_k<t} |I_k x(t_k^-)|\end{aligned}$$

By assumptions(5.2.1) we get $|Fx(t)| \leq C||x||$. Therefore, F is bounded operator on $PC([0,T],X)$. □

Theorem 5.2.2. *(Existence and Uniqueness) Under the assumptions(5.2.1) generalize impulsive evolution equation* (5.0.1) *for* $0 < \beta < 1$ *has unique mild solution in* $PC([0,T],X)$ *if* $C = \frac{(M+L+K)(p+1)T^\beta}{\Gamma(\beta+1)} < 1$.

Proof. For each $x(t), y(t) \in PC([0,T], X)$ we have,

$$\begin{aligned}||Fx - Fy||_{PC} \leq & \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - q)^{\beta-1} ||A(q)|| ||x(q) - y(q)|| dq \\ & + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t - q)^{\beta-1} ||A(q)|| ||x(q) - y(q)|| dq \\ & + \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - q)^{\beta-1} ||g(q, x(q)) - g(q, y(q))|| dq \\ & + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t - q)^{\beta-1} ||g(q, x(q)) - g(q, y(q))|| dq \\ & + \sum_{0<t_k<t} ||I_k x(t_k^-) - I_k y(t_k^-)||\end{aligned}$$

Again by assumptions(5.2.1) we get $||Fx - Fy|| \leq C||x - y||$ and F is contraction. Hence by theorem (5.1.1), F has unique fixed point if $C < 1$. So the sequence $x_{n+1} = Fx_n$ converges uniformly to x in $PC([0,T], X)$ which is mild solution of the equation (5.0.1). □

Remark 5.2.2.1. *We have*

(i) The conditions in the hypotheses of theorem(5.2.2) is not necessary, i.e. equation (5.0.1) *may have mild solution if one of the conditions from the assumptions(5.2.1) are not satisfied.*

(ii) If all I_k's are constants then condition on C can be weaker by $\frac{(M+L)(p+1)T^\beta}{\Gamma(\beta+1)}$.

5.3 Classical Solution

Existence and uniqueness conditions of classical solution of generalized fractional evolution equation (5.0.1) been discussed in this section.

Definition 5.3.1. $x(t)$ *is a classical solution of the equation* (5.0.1) *for* $0 < \beta < 1$ *if* $x(t) \in PC([0,T], X) \cap C^\beta(K', X)$ *where,* $K' = [0,T] - \{t_1, t_2, \cdots, t_p\}$, $x(t) \in D(A)$ *for* $t \in K'$ *and satisfies* (5.0.1) *on* $[0,T]$.

Theorem 5.3.1. *If ,*

(i) $A : X \to X$ *is continuous bounded linear operator and there exists a positive constant* M*, such that* $||A(t)|| \leq M$.

(ii) $g_i \in C^{\beta}([0,T] \times X, X)$ *such that, there exists positive constants* L_i*, such that* $||g_i(t, x(t)) - g_i(t, y(t))|| \leq L_i||x - y||$.

(iii) The number C_i *defined by* $C_i = \frac{(M+L_i)(t_i - t_{i-1})^{\beta}}{\Gamma(\beta+1)} < 1$

(iv) $r_{i-1} \in D(A)$.

are satisfied then the fractional evolution equation

$$\begin{aligned} {}^{c}D^{\beta}_{t_{i-1}+}x(t) &= A(t)x(t) + g_i(t, x(t)) \\ x(t_{i-1}) &= r_{i-1} \end{aligned} \tag{5.3.1}$$

$(0 < \beta < 1)$ *and for* $i = 1, 2, \cdots, p$ *has unique classical solution over interval* $[t_{i-1}, t_i)$ *which satisfies,*

$$x(t) = r_{i-1} + \frac{1}{\Gamma(\beta)} \int_{t_{i-1}}^{t} (t-q)^{\beta-1} A(q)x(q)dq + \frac{1}{\Gamma(\beta)} \int_{t_{i-1}}^{t} (t-q)^{\beta-1} g_i(q, x(q))dq. \tag{5.3.2}$$

Moreover, we can define $x(t_i)$ *in such a way that,* $x(t_i)$ *is left continuous and* $x(t_i) \in D(A)$.

Proof. Defining G on X by

$$Gx(t) = r_{i-1} + \frac{1}{\Gamma(\beta)} \int_{t_{i-1}}^{t} (t-q)^{\beta-1} A(q)x(q)dq + \frac{1}{\Gamma(\beta)} \int_{t_{i-1}}^{t} (t-q)^{\beta-1} g_i(q, x(q))dq. \tag{5.3.3}$$

Then assuming conditions (i), Lipschitz continuity of g_i and (iii) one can easily show that G is continuous, bounded and contraction. Therefore, by Banach fixed point theorem G has unique solution in X which is mild solution of the equation (5.3.1). On the other hand, under the assumptions (i),(ii) and (iv) the function $x(t)$ defined by:

$$x(t) = r_{i-1} + \frac{1}{\Gamma(\beta)}\int_{t_{i-1}}^{t}(t-q)^{\beta-1}A(q)x(q)dq + \frac{1}{\Gamma(\beta)}\int_{t_{i-1}}^{t}(t-q)^{\beta-1}g_i(q,x(q))dq$$

in $C^\beta([t_{i-1},t_i))$ and satisfies the equation (5.3.1) with $x(t) \in D(A)$ for all $t \in [t_{i-1},t_i)$. Thus, the mild solution (5.3.2) is classical solution of the eqution (5.3.1). Assuming condition *(iii)* we get, uniqueness of mild solution and this leads to uniqueness of classical solution of equation over the interval $[t_{i-1},t_i)$.
Finally, by defining $x(t)$ at t_i^- by

$$x(t_i^-) = r_{i-1} + \frac{1}{\Gamma(\beta)}\int_{t_{i-1}}^{t_i}(t_i-q)^{\beta-1}A(q)x(q)dq + \frac{1}{\Gamma(\beta)}\int_{t_{i-1}}^{t_i}(t-q)^{\beta-1}g_i(q,x(q))dq,$$

we get left continuity of the function $x(t)$ at point $t = t_i$. □

Following theorem for existence and uniqueness of classical solution of equation (5.0.1) on the Banach space with $\Delta x(t_k) = r_k$.

Theorem 5.3.2. *If ,*

(i) $A : X \to X$ *is continuous bounded linear operator and there exists a positive constant* M*, such that* $||A(t)|| \le M$ *over the interval* $[0,T]$.

(ii) $g_i \in C^\beta([0,T] \times X, X)$ *such that, there exists positive constants* L_k*, such that* $||g_k(t,x(t)) - g_k(t,y(t))|| \le L_k||x-y||$ *for each* $k = 1,2,\cdots p, p+1$*. Here force* g_k *is applied over the interval* $[t_p,T]$.

(iii) The number C_k *defined by* $C_k = \frac{(M+L_k)(t_k-t_{k-1})^\beta}{\Gamma(\beta+1)} < 1$ *for all* $k = 1,2,\cdots,p,p+1$.

(iv) $r_{k-1} \in D(A)$ *for all* $k = 1,2,\cdots,k$.

then the impulsive fractional evolution equation

$$
\begin{aligned}
{}^cD^{\beta}_{t_{k-1}+}x(t) &= Ax(t) + g_k(t, x(t)) \\
\Delta x(t_k) &= r_k \\
x(0) &= x_0
\end{aligned}
\tag{5.3.4}
$$

$(0 < \beta < 1)$ *and for* $i = 1, 2, \cdots, p$ *has unique classical solution over interval* $[0, T]$

Proof. $I_1 = [0, t_1)$ the equation (5.3.4) becomes,

$$
\begin{aligned}
{}^cD^{\beta}_{0}x(t) &= Ax(t) + g_1(t, x(t)) \\
x(t_0) &= x_0
\end{aligned}
$$

Applying conditions of the theorem (5.3.1) and (5.3.2), the above equation has unique classical solution satisfying

$$x(t) = x_0 + \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-q)^{\beta-1}A(q)x(q)dq + \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-q)^{\beta-1}g_1(q, x(q))dq$$

Define $x_1(t) = x(t)$ for all $t \in [0, t_1)$ then $x_1(t)$ is left continuous at $t = t_1$ therefore $x_1(t_1^-) \in D(A)$.

Over the interval $[t_1, t_2)$ the equation (5.3.4) becomes,

$$
\begin{aligned}
{}^cD^{\beta}_{t_1}x(t) &= Ax(t) + g_2(t, x(t)) \\
x(t_1) &= x_1 = x(t_1^-) + r_1
\end{aligned}
$$

Since, $x(t_1^-)$ and r_1 is in $D(A)$ therefore $x(t_1) = x_1 \in D(A)$ and applying conditions of the theorem (5.3.1) and (5.3.2), above equation has unique classical solution satisfying

$$x(t) = x_1 + \frac{1}{\Gamma(\beta)}\int_{t_1}^{t}(t-q)^{\beta-1}A(q)x(q)dq + \frac{1}{\Gamma(\beta)}\int_{t_1}^{t}(t-q)^{\beta-1}g_1(q, x(q))dq$$

Defining $x_2(t) = x(t)$ for $t \in [t_1, t_2)$ then $x_(t)$ is left continuous and $x(t_2^-) \in D(A)$. Continuing this process we arrived at

$$
\begin{aligned}
{}^cD^{\beta}_{t_{k-1}} u(t) &= Ax(t) + g_i(t, x(t)) \\
x(t_{k-1}) = x_1 &= x(t_{k-1}^-) + r_{k-1}.
\end{aligned}
$$

over the interval $[t_{k-1}, t_k)$ for all $k = 1, 2, \cdots, p+1$ and applying conditions of the theorem (5.3.1) and (5.3.2), the equation has unique classical solution satisfying

$$
x(t) = x_{k-1} + \frac{1}{\Gamma(\beta)} \int_{t_{k-1}}^{t} (t-q)^{\beta-1} A(q) x(q) dq + \frac{1}{\Gamma(\beta)} \int_{t_{k-1}}^{t} (t-q)^{\beta-1} g_k(q, x(q)) dq
$$

and defining $x_k(t) = x(t)$ for all $t \in [t_{k-1}, t_k)$ then $x_k(t)$ is left continuous so $x(t_{k-1}^-) \in D(A)$.

Define,

$$
x(t) = \begin{cases} x_1(t), & t \in [0, t_1) \\ x_2(t), & t \in [t_1, t_2) \\ \cdots \\ x_k(t), & t \in [t_{k-1}, t_k) \\ \cdots \\ x_{p+1}(t), & t \in [t_p, T] \end{cases} \tag{5.3.5}
$$

then it can be easily prove that $x(t) \in PC([0,T], X) \cap C^{\beta}(K', X)$ and $x(t) \in D(A)$ for all $t \in [0,T]$, $x(t)$ is classical solution of (5.3.4).

Moreover, if $y(t)$ is another classical solution of (5.3.4) then one can easily show that $w(t) = x(t) - y(t)$ is solution of ${}^cD^{\beta} w(t) = 0$ with initial condition $w(0) = 0$ without impulses. This leads to $w(t) = 0$ for all $t \in [0,T]$. i.e. equation (5.3.4) has unique classical solution over the interval $[0,T]$. □

Remark 5.3.2.1. *One can easily see that the classical solution* (5.3.5) *of* (5.0.1) *defined in Theorem-(5.3.2) also satisfies the equation:*

$$\begin{aligned} x(t) = x_0 &+ \frac{1}{\Gamma(\beta)} \sum_{0<t_k\leq t_i} \int_{t_k-1}^{t_k} (t_k-q)^{\beta-1} A(q)x(q)dq + \frac{1}{\Gamma(\beta)} \int_{t_i}^{t} (t-q)^{\beta-1} A(q)x(q)dq \\ &+ \frac{1}{\Gamma(\beta)} \sum_{0<t_k\leq t_i} \int_{t_k-1}^{t_k} (t_k-q)^{\beta-1} g_k(q,x(q))dq + \frac{1}{\Gamma(\beta)} \int_{t_i}^{t} (t-q)^{\beta-1} g_{i+1}(s,x(s))ds \\ &+ \sum_{0<t_k\leq t_i} q_k \end{aligned} \tag{5.3.6}$$

and replacing r_k with $I_k(x(t_k))$, we can obtain unique classical solution of (5.0.1) *satisfying*

$$\begin{aligned} x(t) = x_0 &+ \frac{1}{\Gamma(\beta)} \sum_{0<t_k\leq t_i} \int_{t_k-1}^{t_k} (t_k-q)^{\beta-1} A(q)x(q)dq + \frac{1}{\Gamma(\beta)} \int_{t_i}^{t} (t-q)^{\beta-1} A(q)x(q)dq \\ &+ \frac{1}{\Gamma(\beta)} \sum_{0<t_k\leq t_i} \int_{t_k-1}^{t_k} (t_k-q)^{\beta-1} g_k(q,x(q))dq + \frac{1}{\Gamma(\beta)} \int_{t_i}^{t} (t-q)^{\beta-1} g_{i+1}(q,x(q))dq \\ &+ \sum_{0<t_k\leq t_i} I_k x(t_k). \end{aligned} \tag{5.3.7}$$

5.4 Congruence between Classical and Mild Solutions

Theorem 5.4.1. *If*

(i) Assumptions-5.2.1,

(ii) Perturbed operators $g_k \in C^{\beta}([0,T] \times X, X)$ for all $k = 1, 2, \cdots, p+1$,

(iii) x_0 and $I_k \in D(A)$ for all $k = 1, 2, \cdots, p$

(iv) The positive integer $\frac{(M+L+K)(p+1)T^{\beta}}{\Gamma(\beta+1)} < 1$

then the classical and mild solutions of the equation (5.0.1) *are coincides.*

Proof. Since $C_k = \frac{(M+L_k)(t_k - t_{k-1})^\beta}{\Gamma(\beta+1)} \leq \frac{(M+L+K)(p+1)T^\beta}{\Gamma(\beta+1)} < 1$. Moreover the perturbed operators $g_k \in C^\beta([0,T] \times X, X)$ and x_0 and $I_k \in D(A)$ for all $k = 1, 2, \cdots, p+1$ and by Theorem (5.3.2) the equation (5.0.1) has unique classical solution over interval $[0,T]$ satisfying the equation (5.3.7).

On the other hand condition (i) and (iv) are satisfies so equation (5.0.1) has unique mild solution say $y(t)$. Apart from this conditions (ii) and (iii) are also satisfied i.e. ${}^C D^\beta x(t)$ exist for all $t \in K'$. therefore $y(t)$ is also classical solution of equation (5.0.1). But classical solution is unique under given conditions therefore $x(t) = y(t)$ for all $t \in [0,T]$, hence mild and classical solutions of the equations (5.0.1) are coincides. □

Example 5.4.1.1. *Consider an evolution equation*

$$\begin{aligned}
{}^c D^\beta x(t) &= \frac{1}{10}\int_0^1 (t-q)x(q)dq + g_k(t, x(t)), \quad k = 1, 2 \\
\Delta x(\frac{1}{2}) &= \frac{1}{5} x(\frac{1}{2}^-) \\
x(0) &= x_0
\end{aligned} \tag{5.4.1}$$

over the interval $[0,1]$*. Where* $g_1(t,x) = te^{-\frac{x(t)}{5}}$ *and* $g_2(t,x) = \frac{1}{5}sin(x)$*. Then* $M = \frac{1}{20}$, $L_1 = \frac{1}{5}$, $L_2 = \frac{1}{5}$ *and* $J_1 = \frac{1}{5}$*. Therefore* $L = \frac{1}{5}$, $K = 15$ *and* $C = \frac{1}{10\ \Gamma\beta+1} < 1$*. By theorem(5.2.2),* (5.4.1) *has unique mild solution.*

Moreover, $g \in C^\beta((0,1) \times X, X)$ *and choosing* $x_0 \in D(A)$ *we get unique classical solution of equation which is arise from mild solution.*

From theorem (5.4.1) we can conclude that

(i) Obtained conditions are sufficient but not necessary for existence and uniqueness of classical solution.

(ii) Conditions obtained for congruence of mild and classical solutions are also sufficient conditions.