Chapter 6

Existence Results of Non-Instantaneous Impulsive Fractional Integro-differential Equation

Existence of mild solution for non-instantaneous impulses fractional order integro-differential equations with local and nonlocal conditions on Banach space is established in this chapter. Existence results with local and nonlocal conditions are obtained through operator semigroup theory using generalized Banach contraction theorem and Krasnoselkii's fixed point theorem respectively. Finally, illustration is discussed to validate derived results.

Considering non-instantaneous impulses fractional order integro-differential equations

$$
\begin{aligned}
{}^{c}D^{\beta}x(t) &= Ax(t) + f\left(t, x(t), \int_0^t a(t,s,x(s))ds\right),\ t \in [s_k, t_{k+1}),\ k = 1, 2, \cdots, p \\
x(t) &= I_k(k, x(t)),\ t \in [t_k, s_k)
\end{aligned}
$$

with local condition $x(0) = x_0$ and nonlocal condition $x(0) = x_0 + h(x)$ over the interval $[0, T_0]$ in a Banach space $\mathcal{X}$. Here $A : \mathcal{X} \to \mathcal{X}$ is linear operator, $Kx = \int_0^t a(t,s,x(s))ds$ is nonlinear Volterra integral oprator on $\mathcal{X}$, $f : [0, T_0] \times \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is nonlinear function and $I_k : [0, T_0] \times \mathcal{X}$ are set of nonlinear functions applied in the interval $[t_k, s_k)$ for all $i = 1, 2, \cdots, p$.

6.1 Introduction

Fractional order differential equations have gained lot of attensions to many researchers due to hereditory attributes and long term memory descriptions. In fact, many models in science and engineering like seepage flow in porous media, anomalous diffusion, the nonlinear oscillations of earthquake, fluid dynamics traffic model, electromagnetism and population dynamics are now revisited in terms of fractional differential equations. More details and applications are found in monographs in [89, 90] and in articles of [7, 8, 9, 10, 11, 12, 13]. Due to wide range of applications in various fields fractional order differential equations became fertile branch of Applied Mathematics. The study of existence of mild solutions of fractional differential, integro-differential and evolution equations using different fixed point theorem were found in [91, 14, 15].

The extension of classical conditions for Cauchy problem are nonlocal conditions, which gives better effect than classical conditions in many physical phenomena in the field of science and engineering [92]. Existence results for nonlocal Cauchy problem using various techniques are found in [93, 16, 17, 18, 19, 94, 95].

On the other hand, evolutionary processes undergo abrupt change in the state either at a fixed moment of time or in a small interval of time are modelled into instantaneous impulsive evolution or non-instantaneous impulsive evolution equation respectively. Applications of the instantaneous impulsive evolution equation and existence results for integer order instantaneous impulsive evolution equations are found in [20, 96, 39, 33]. Existence results for fractional instantaneous impulsive equation are found in [40, 41, 42, 21, 22, 43, 73, 74]. In some evolutionary process non-instantaneous impulses are more accurate instead of instantaneous impulses. Existence of mild solution of non-instantaneous impulsive fractional differential equation with local initial condition has been studied by Li and Xu [98]. Meraj and Pandey [99] studied existence of mild solutions of nonlocal semilinear evolution equation using Krasnoselskii's fixed point theorem.

6.2 Preliminaries

Basic definitions and theorems of fractional calculus and functional analysis are discuss in this section, which will help us to prove our main results.

Definition 6.2.1. *[5] The Riemann-Liouville fractional integral operator of* $\beta > 0$, *of function* $h \in L_1(\mathbb{R}_+)$ *is defined as*

$$J^{\beta}_{t_0+}h(t) = \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-q)^{\beta-1}h(q)dq,$$

provided the integral on right side exist. Where, $\Gamma(\cdot)$ *is gamma function.*

Definition 6.2.2. *[6] The Caputo fractional derivative of order* $\beta > 0$, $n-1 < \beta < n$, $n \in \mathbb{N}$, *is defined as*

$${}^{c}D^{\beta}_{t_0+}h(t) = \frac{1}{\Gamma(n-\beta)}\int_{t_0}^{t}(t-q)^{n-\beta-1}\frac{d^n h(q)}{dq^n}dq$$

where, the function $h(t)$ *has absolutely continuous derivatives up to order* $(n-1)$.

Theorem 6.2.1. *(Banach Fixed Point Theorem)[100] Let* E *be closed subset of a Banach Space* $(\mathcal{X}, ||\cdot||)$ *and let* $T : E \to E$ *contraction then,* T *has unique fixed point in* E.

Theorem 6.2.2. *(Krasnoselskii's Fixed Point Theorem)[100] Let* E *be closed convex nonempty subset of a Banach Space* $(\mathcal{X}, ||\cdot||)$ *and* P *and* Q *are two operators on* E *satisfying:*

(1) $Px + Qy \in E$, *whenever* $x, y \in E$,

(2) P *is contraction,*

(3) Q *is completely continuous*

then, the equation $Px + Qx = x$ *has unique solution.*

Definition 6.2.3. *(Completely Continuous Operator)[108] Let X and Y be Banach spaces. Then the operator $T : D \subset X \to Y$ is called completely continuous if it is continuous and maps any bounded subset of D to relatively compact subset of Y.*

6.3 Equation with Local Conditions

Sufficient conditions for the existence and uniqueness of the equation:

$$\begin{aligned} {}^cD^\beta x(t) &= Ax(t) + f\left(t, x(t), \int_0^t a(t,s,x(s))ds\right),\ t \in [s_k, t_{k+1}),\ i = 1, 2, \cdots, p \\ x(t) &= g_k(t, x(t)),\ t \in [t_k, s_k) \\ x(0) &= x_0 \end{aligned} \tag{6.3.1}$$

over the interval $[0, T]$ in the Banach space $\mathcal{X}$, is derived in this section.

Definition 6.3.1. *The function $x(t)$ is called mild solution of the impulsive fractional equation* (6.3.1) *over the interval if $x(t)$ satisfies the integral equation*

$$x(t) = \begin{cases} X(t)x_0 + \int_0^t (t-s)^{\beta-1} Y(t-s) f(t, x(s), Kx(s))ds, & t \in [0, t_1) \\ g_k(t, x(t)), & t \in [t_k, s_k) \\ X(t-s_k)g_k(s_k, x(s_k)) + \int_{s_k}^t (t-s)^{\beta-1} Y(t-s) f(t, x(s), Kx(s))ds, & t \in [s_k, t_{k+1}) \end{cases} \tag{6.3.2}$$

where,

$$Kx(t) = \int_0^t a(t,s,x(s))ds,\ X(t) = \int_0^\infty \zeta_\beta(\theta) S(t^\beta\theta)d\theta,\ Y(t) = \beta\int_0^\infty \theta\zeta_\beta(\theta)S(t^\beta\theta)d\theta$$

are the linear operators defined on $\mathcal{X}$. Here, $\zeta_\beta(\theta)$ is probability density function over the interval $[0, \infty)$ defined by

$$\zeta_\beta(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-\beta n-1}\frac{\Gamma(n\beta+1)}{n!}sin(n\pi\beta)$$

and the operator $S(t)$ is semi-group generated by evolution operator A.

Assumptions 6.3.1. *Assumptions for the existence and uniqueness of the mild solution of fractional evolution equation with non instantaneous impulses.*
(A1) The evolution operator A generates C_0 semigroup $S(t)$ for all $t \in [0, T]$.
(A2) The function $f : [0, T] \times \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is continuous with respect to t and there exist a positive constants f_1^* and f_2^* such that $||f(t, x_1, y_1) - f(t, x_2, y_2)|| \leq f_1^*||x_1 - y_2|| + f_2^*||x_1 - y_2||$ for $x_1, y_1, x_2, y_2 \in B_{r_0} = \{x \in \mathcal{X}; ||x|| \leq r_0\}$ for some r_0.
(A3) The operator $K : [0, T] \times \mathcal{X} \to \mathcal{X}$ is continuous and there exist a constant k^* such that $||Kx - Ky|| \leq k^*||x - y||$ for $x, y \in B_{r_0}$.
(A4) The functions $g_k : [t_k, s_k] \times \mathcal{X}$ are continuous and there exist a positive constants $0 < g_k^* < 1$ such that $||g_k(t, x(t)) - g_k(t, y(t))|| \leq g_k^*||x - y||$.

Lemma 6.3.1. *([91])If the evolution operator A generates C_0 semi group $S(t)$ then the operators $X(t)$ and $Y(t)$ are strongly continuous and bounded. This means there exist positive constant M such that $||X(t)x|| \leq M||x||$ and $||Y(t)x|| \leq \frac{M}{\Gamma(\beta)}||x||$ for all $t \in [0, T]$.*

Theorem 6.3.2. *If assumptions (A1)-(A4) are satisfied, then the semilinear fractional integro-differential equation with non-instantanious impulses* (6.3.1) *has unique mild solution.*

Proof. Define the operator $\mathcal{F}$ on $\mathcal{X}$ by

$$\mathcal{F}x(t) = \begin{cases} \mathcal{F}_1 x(t), & t \in [0, t_1) \\ \mathcal{F}_{2k} x(t), & t \in [t_k, s_k) \\ \mathcal{F}_{3k} x(t), & t \in [s_k, t_{k+1}) \end{cases}$$

where, $\mathcal{F}_1$, $\mathcal{F}_{2k}$ and $\mathcal{F}_{3k}$ are

$$\mathcal{F}_1 x(t) = X(t)x_0 + \int_0^t (t-s)^{\beta-1} Y(t-s) f(t, x(s), Kx(s)) ds, \qquad t \in [0, t_1)$$

$$\mathcal{F}_{2k} x(t) = g_k(t, x(t)), \qquad t \in [t_k, s_k)$$

$$\mathcal{F}_{3k} k x(t) = X(t-s_k) g_k(s_k, x(s_k)) + \int_{s_k}^t (t-s)^{\beta-1} Y(t-s) f(t, x(s), Kx(s)) ds, \qquad t \in [s_k, t_{k+1})$$

for all $k = 1, 2, \cdots p$.

In view of this operator $\mathcal{F}$, the equation (6.3.2) has unique solution if and only if the operator equation $x(t) = \mathcal{F}x(t)$ has unique solution. This is possible if and only if each of $x(t) = \mathcal{F}_1 x(t)$, $x(t) = \mathcal{F}_{2k} x(t)$ and $x(t) = \mathcal{F}_{3k} x(t)$ has unique solution over the interval $[0, t_1)$, $[t_k, s_k)$ and $[s_k, t_{k+1})$ for all $k = 1, 2, \cdots, p$ respectively. Consider $x_1(t), x_{2k}(t)$ and $x_{3k}(t)$ be the solutions of $x(t) = \mathcal{F}_1 x(t)$, $x(t) = \mathcal{F}_{2k} x(t)$ and $x(t) = \mathcal{F}_{3k} x(t)$ respectively. Defining,

$$x(t) = \begin{cases} x_1(t), & [0, t_1) \\ x_{2k}(t), & [t_k, s_k) \\ x_{3k}(t), & [s_k, t_{k+1}) \end{cases}$$

then one can easily show that $x(t)$ is unique solution of $x(t) = \mathcal{F}x(t)$.
For all $t \in [0, t_1)$ and $x, y \in B_{r_0}$,

$$\begin{aligned} &||\mathcal{F}_1^{(n)} x(t) - \mathcal{F}_1^{(n)} y(t)|| \\ &\qquad \leq \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} (t-\tau_1)^{\beta-1} (\tau_1 - \tau_2)^{\beta-1} \cdots (\tau_{n-1} - s)^{\beta-1} ||Y(t-\tau_1)|| \\ &\qquad\quad ||Y(\tau_1 - \tau_2)|| \cdots ||Y(\tau_{n-1} - s)|| ||f(s, x(s), Kx(s)) - f(s, y(s), Ky(s))|| ds d\tau_{n-1} \cdots d\tau_1 \end{aligned}$$

By applying assumption (A1), (A2) and (A3) and lemma 6.3.1 we get,

$$\begin{aligned}||\mathcal{F}_1^{(n)}x(t)-\mathcal{F}_1^{(n)}y(t)|| &\leq \int_0^{t_1}\int_0^{t_1}\cdots\int_0^{t_1} t_1^{n(\beta-1)}\frac{M^n}{(\Gamma(\beta))^n}\big[f_1^*||x-y||+f_2^*k^*||x-y||\big]dsd\tau_{n-1}\cdots d\tau_1\\ &\leq \frac{t_1^{n(\beta-1)}M^n(f_1^*+f_2^*k^*)}{(n-1)!(\Gamma(\beta))^n}\int_0^{t_1}(t_1-s)^{n-1}ds||x-y||\\ &\leq \frac{t_1^{n\beta}M^n(f_1^*+f_2^*k^*)}{n!(\Gamma(\beta))^n}||x-y||\\ &\leq c^*||x-y||.\end{aligned}$$

Considering supremum over interval $[0,t_1)$ we get $||\mathcal{F}_1^{(n)}x-\mathcal{F}_1^{(n)}y|\leq c^*||x-y||\to 0$ for fixed t_1. Therefore there exist m such that $\mathcal{F}_1^{(m)}$ is contraction on B_{r_0}. Thus by general Banach contraction theorem the operator equation $x(t)=\mathcal{F}_1x(t)$ has unique solution over the interval $[0,t_1)$.

For all $k=1,2,\cdots,p$, $t\in[t_k,s_k)$ and $u,v\in\mathcal{X}$ and assuming (A4)

$$||\mathcal{F}_{2k}x(t)-\mathcal{F}_{2k}y(t)||=||g_k(t,x(t))-g_k(t,y(t))||\leq g_k^*||x-y||.$$

Then $\mathcal{F}_{2k}$ is contraction and by Banach fixed point theorem the operator equation $x(t)=\mathcal{F}_{2k}x(t)$ has unique solution for the interval $[t_k,s_k)$ for all $k=1,2,\cdots,p$. This means for all $k=1,2,\cdots,p$, $x(t)=g_k(t,x(t))$ has unique solution for all $t\in[t_k,s_k)$. Lipschitz continuity of g_k leads to uniqueness of the solution at point s_k also.

For all $k=1,2,\cdots,p$, $t\in[s_k,t_{k+1})$ and $x,y\in B_{r-0}$,

$$\begin{aligned}&||\mathcal{F}_{3k}^{(n)}x(t)-\mathcal{F}_{3k}^{(n)}y(t)||\\ &\qquad\leq \int_{s_k}^{t}\int_{s_k}^{\tau_1}\cdots\int_{s_k}^{\tau_{n-1}}(t-\tau_1)^{\beta-1}(\tau_1-\tau_2)^{\beta-1}\cdots(\tau_{n-1}-s)^{\beta-1}||Y(t-\tau_1)||\\ &\qquad\quad ||Y(\tau_1-\tau_2)||\cdots||Y(\tau_{n-1}-s)||||f(s,x(s),Kx(s))-f(s,y(s),Ky(s))||dsd\tau_{n-1}\cdots d\tau_1\end{aligned}$$

Appalling assumption (A1), (A2) and (A3) and lemma 6.3.1 and we get,

$$\begin{aligned}||\mathcal{F}_{3k}^{(n)}x(t) - \mathcal{F}_{3k}^{(n)}y(t)|| &\leq \int_{s_k}^{t_{k+1}}\int_{s_k}^{t_{k+1}}\cdots\int_{s_k}^{t_{k+1}}(t_{k+1}-s_k)^{n(\beta-1)}\frac{M^n}{(\Gamma(\beta))^n}\\ &\quad\big[f_1^*||x-y|| + f_2^*k^*||x-y||\big]dsd\tau_{n-1}\cdots d\tau_1\\ &\leq \frac{(t_{k+1}-s_k)^{n(\beta-1)}M^n(f_1^*+f_2^*k^*)}{(n-1)!(\Gamma(\beta))^n}\int_{s_k}^{t_{k+1}}(t_{k+1}-s)^{n-1}ds||x-y||\\ &\leq \frac{(t_{k+1}-s_k)^{n\beta}M^n(f_1^*+f_2^*k^*)}{n!(\Gamma(\beta))^n}||x-y||\\ &\leq c^*||x-y||.\end{aligned}$$

Considering supremum over interval $[s_k, t_{k+1})$
we get $||\mathcal{F}_{3k}^{(n)}x - \mathcal{F}_{3k}^{(n)}y| \leq c^*||x-y|| \to 0$ for fixed sub-interval $[s_k, t_{k+1})$ for all $k = 1, 2, \cdots, p$. Therefore, there exist m such that $\mathcal{F}_{3k}^{(m)}$ is contraction on B_{r_0}. Thus by Banach contraction theorem the operator equation $x(t) = \mathcal{F}_{3k}x(t)$ has unique solution over the interval $[s_k, t_{k+1})$ for all $k = 1, 2, \cdots, p$.
Hence, the operator equation $x(t) = \mathcal{F}x(t)$ has unique solution over the interval $[0, T]$ which is nonhing but mild solution of the equation (6.3.1). □

Example 6.3.2.1. *The fractional order integro-differential equation:*

$$\begin{aligned}{}^cD_t^\beta x(t,u) &= x_{uu}(t,u) + x(t,u)x_u(t,u) + \int_0^t e^{-x(s,u)}ds, && t \in [0, \frac{1}{3}) \cup [\frac{2}{3}, 1]\\ x(t,u) &= \frac{x(t,u)}{2(1+x(t,u))}, && t \in [\frac{1}{3}, \frac{2}{3})\end{aligned} \tag{6.3.3}$$

over the interval $[0, 1]$ *with initial condition* $x(0, u) = x_0(u)$ *and boundary condition* $x(t, 0) = x(t, 1) = 0$*. The equation* (6.3.3) *can be reformulated as fractional order abstract equation in* $\mathcal{X} = L^2([0, 1], \mathbb{R})$ *as:*

$$\begin{aligned}{}^cD^\beta z(t) &= Az(t) + f(t, z(t), Kz(t)), && t \in [0, \frac{1}{3}) \cup [\frac{2}{3}, 1]\\ z(t) &= g(t, z(t)) && t \in [\frac{1}{3}, \frac{2}{3})\end{aligned} \tag{6.3.4}$$

over the interval $[0, 1]$ *by defining* $z(t) = x(t, \cdot)$*, operator* $Ax = x''$ *(second order*

derivative with respect to x). The functions f and g over respected domains are defined as $f(t, z(t), Kz(t)) = \frac{(z^2(t))'}{2} + \int_0^t e^{-z(s)}ds$ *and* $g(t, z(t)) = \frac{z(t)}{2(1+z(t))}$ *respectively.*
(1) The linear operator A over the domain $D(A) = \{x \in \mathcal{X}; x''$ *exist and continuous with* $x(0) = x(1) = 0\}$ *is self-adjoint, with compact resolvent and is the infinitesimal generator of* C_0 *semigroup* $S(t)$ *over the interval* $[0, 1]$ *given by*

$$S(t)x = \sum_{n=1}^{\infty} exp(-n^2\pi^2 t) < x, \phi_n > \phi_n, \qquad (6.3.5)$$

where $\phi_n(s) = \sqrt{2}sin(n\pi s)$ *for all* $n = 1, 2, \cdots$ *is the orthogonal basis for the space* X.
(2) The function $K : [0, 1] \times [0, 1] \times \mathcal{X} \to \mathcal{X}$ *is continuous with respect to t and differentiable with respect to z for all z and hence K is Lipschitz continuous with respect to z. This means there exist positive constant* k^* *such that* $||K(t, z_1) - K(t, z_2)|| \leq k^*||z_1 - z_2||$.
(3) The function $f : [0, 1] \times \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ *is continuous with respect to t and is differential with respect to argument z and Kz. Therefore there exist positive constats* f_1^* *and* f_2^* *such that* $||f(t, z_1, Kz_1) - f(t, z_2, Kz_2)|| \leq f_1^*||z_1 - z_2|| + f_2^*||Kz_1 - Kz_2||$, $z_1, z_2 \in B_{r_0}$ *for some* r_0.
(4) The impulse g is continuous with respect to t and Lipchitz continous with respect to z with Lipschitz constant $g^* = 1/2 < 1$.
Therefore by theorem-6.3.2 the equation (6.3.4) *has unique solution over* $[0, 1]$. *Hence the equation* (6.3.3) *has unique solution over the interval* $[0, 1]$.

6.4 Equation with Nonlocal Conditions

Sufficient conditions for the existence of the equation:

$$
\begin{aligned}
{}^cD^\beta x(t) &= Ax(t) + f\left(t, x(t), \int_0^t a(t,s,x(s))ds\right),\ t \in [s_i, t_{i+1}),\ i = 1, 2, \cdots, p \\
x(t) &= g_i(t, x(t)),\ t \in [t_i, s_i) \\
x(0) &= x_0 + h(x)
\end{aligned}
\tag{6.4.1}
$$

in the Banach space $\mathcal{X}$, is derived in this section.

Definition 6.4.1. *The function $x(t)$ is called mild solution of the impulsive fractional equation* (6.3.1) *over the interval if $x(t)$ satisfies the integral equation*

$$
x(t) = \begin{cases} X(t)(x_0 + h(x)) + \int_0^t (t-s)^{\beta-1} Y(t-s) f(t, x(s), Kx(s))ds, & t \in [0, t_1) \\ g_k(t, x(t)), & t \in [t_k, s_k) \\ X(t - s_k) g_k(s_k, x(s_k)) + \int_{s_k}^t (t-s)^{\beta-1} Y(t-s) f(t, x(s), Kx(s))ds, & t \in [s_k, t_{k+1}) \end{cases}
\tag{6.4.2}
$$

where,

$$
Kx(t) = \int_0^t a(t,s,x(s))ds,\ X(t) = \int_0^\infty \zeta_\beta(\theta) S(t^\beta \theta) d\theta,\ Y(t) = \beta \int_0^\infty \theta \zeta_\beta(\theta) S(t^\beta \theta) d\theta
$$

are the linear operators defined on $\mathcal{X}$. Here, $\zeta_\beta(\theta)$ is probability density function over the interval $[0, \infty)$ defined by

$$
\zeta_\beta(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-\beta n - 1} \frac{\Gamma(n\beta + 1)}{n!} sin(n\pi\beta)
$$

and the operator $S(t)$ is semi-group generated by evolution operator A.

Assumptions 6.4.1. *Assumptions for the existence of the mild solution of fractional evolution equation with non-instantenous impulses.*

(B1) The evolution operator A generates C_0 semigroup $S(t)$ for all $t \in [0, T]$.

(B2) The function $f(t,\cdot,\cdot)$ *is continuous and* $f(\cdot,x,y)$ *is measurable on* $[0,T]$*. Also there exist* $\beta \in (0,\beta)$ *with* $m_f \in L^{\frac{1}{\beta}}([0,T],\mathbb{R})sx$ *such that* $|f(t,x,y)| \leq m_f(t)$ *for all* $x, y \in \mathcal{X}$.

(B3) The operator $K : [0,T] \times \mathcal{X} \to \mathcal{X}$ *is continuous and there exist a constant* k^* *such that* $||Kx - Ky|| \leq k^*||x-y||$.

(B4) The operator $h : \mathcal{X} \to \mathcal{X}$ *is Lipschiz continuous with respect to u with Lipschitz constant* $0 < h^* \leq 1$.

(B5) The functions $g_k : [t_k, s_k] \times \mathcal{X}$ *are continuous and there exist a positive constants* $0 < g_k^* < 1$ *such that* $||g_k(t,x(t)) - g_k(t,y(t))|| \leq g_k^*||x-y||$.

Theorem 6.4.1. *(Existence Theorem)If assumptions (B1)– (B5) are satisfied, then the nonlocal semi-linear fractional order integro-differential equation* (6.4.2) *has mild solution provided* $Mh^* < 1$ *and* $Mg^* < 1$.

Proof. From the lemma-(6.3.1) $||X(t)|| \leq M$ for all $x \in B_k = \{x \in \mathcal{X} : ||x|| \leq k\}$ for any positive constnat k. Therefore,

$$|X(t)(x_0 + h(x))| \leq M(|x_0| + h^*||x|| + |h(0)|). \tag{6.4.3}$$

According to (B2) $f(\cdot,x,y)$ is measurable on $[0,T]$ and one can easily shows that $(t-s)^{\beta-1} \in L^{\frac{1}{1-\beta}}[0,t]$ for all $t \in [0,T]$ and $\beta \in (0,\beta)$. Let

$$b = \frac{\beta-1}{1-\beta} \in (-1,0), \qquad M_1 = ||m_f||_{L^{\frac{1}{\beta}}}.$$

By Holder's inequality and assumption (B2), for $t \in [0,T]$,

$$\begin{aligned} \int_0^t |(t-s)^{\beta-1}Y(t-s)f(s,x(s),Kx(s))|ds &\leq \frac{M}{\Gamma(\beta)}\left(\int_0^t (t-s)^{\frac{\beta-1}{1-\beta}}ds\right)^{1-\beta} M_1 \\ &\leq \frac{MM_1}{\Gamma(\beta)(1+b)^{1-\beta}}T^{(1+b)(1-\beta)}. \quad (6.4.4) \end{aligned}$$

For $t \in [0, t_1)$ and for positive r we define F_1 and F_2 on B_r as,

$$\begin{aligned} F_1x(t) &= X(t)(x_0 + h(x)) \\ F_2x(t) &= \int_0^t (t-s)^{\beta-1} Y(t-s) f(t, x(s), Kx(s)) ds \end{aligned}$$

then, $x(t)$ is mild solution of the semilinear fractional integro-differential equation if and only if the operator equation $x = F_1x + F_2x$ has solution for $x \in B_r$ for some r. Therefore the existence of a mild solution of (6.3.1) over the interval $[0, t_1)$ is equivalent to determining a positive constant r_0, such that $F_1 + F_2$ has a fixed point on B_{r_0}.

Step:1 $||F_1x + F_2y|| \leq r_0$ for some positive r_0.

Let $x, y \in B_{r_0}$, choose

$$r_0 = M\frac{|x_0| + |h(x)|}{1 - Mh^*} + \frac{MM_1}{(1 - Mh^*)\Gamma(\beta)(1+b)^{1-\beta}} t_1^{(1+b)(1-\beta)}$$

and consider

$$\begin{aligned} |F_1x(t) + F_2y(t)| &\leq \left| X(t)(x_0 + h(x)) \right| + \left| \int_0^t (t-s)^{\beta-1} Y(t-s) f(t, y(s), Ky(s)) ds \right| \\ &\leq M(|x_0| + h^*||x|| + |h(0)|) + \frac{MM_1}{\Gamma(\beta)(1+b)^{1-\beta}} t_1^{(1+b)(1-\beta)} \\ &\qquad \text{(using inequalities (6.4.3) and (6.4.4))} \\ &\leq r_0 \quad \text{(since, } Mh^* < 1) \end{aligned}$$

Therefore, $||F_1x + F_2y|| \leq r_0$ for every pair $x, y \in B_{r_0}$.

Step: 2 F_1 is contraction on B_{r_0}.

For any $x, y \in B_{r_0}$ and $t \in [0, t_1)$, we have $|F_1x(t) - F_1y(t)| \leq Mh^*||x - y||$ Taking supremum over $[0, t_1)$, $||F_1x - F_1y|| \leq Mh^*||x - y||$. Since, $Mh^* < 1$, F_1 is contraction.

Step: 3 F_2 is completely continuous operator on B_{r_0}

Let $\{x_n\}$ be the sequence in B_{r_0} converging to $x \in B_{r_0}$ and consider,

$$\begin{aligned}|F_2x_n(t) - F_2x(t)| &\leq \int_0^t (t-s)^{\beta-1}|Y(t-s)||f(s, x_n(s), Kx_n(s)) - f(s, x(s), Kx(s))|ds \\ &\leq \frac{M}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} \sup_{s\in[0,t_1)} |f(s, x_n(s), Kx_n(s)) - f(s, x(s), Kx(s))|ds \\ &\leq \frac{Mt_1^\beta}{\Gamma(\beta+1)} \sup_{s\in[0,t_1)} |f(s, x_n(s), Kx_n(s)) - f(s, x(s), Kx(s))|\end{aligned}$$

which implies

$$||F_2u_n - F_2u|| \leq \frac{Mt_1^\beta}{\Gamma(\beta+1)} \sup_{s\in[0,t_1)} |f(s, x_n(s), Kx_n(s)) - f(s, x(s), Kx(s))|$$

Continuity of f and K leads to $||F_2x_n - F_2x|| \to 0$ as $n \to \infty$. Thus F_2 is continuous. To show $\{F_2x(t), x \in B_{r_0}\}$ is relatively compact it is sufficient to show that the family of functions $\{F_2x, x \in B_{r_0}\}$ is uniformly bounded and equicontinuous, and for any $t \in [0, t_1)$, $\{F_2x(t), x \in B_{r_0}\}$ is relatively compact in $\mathbb{X}$.

Clearly for any $u \in B_{r_0}$, $||F_2x|| \leq r_0$, which means that the family $\{F_2x(t), x \in B_{r_0}\}$ is uniformly bounded.

For any $x \in B_{r_0}$ and $0 \leq \tau_1 < \tau_2 < t_1$,

$$
\begin{aligned}
|F_2x(\tau_2) - F_2(\tau_1)| &= \bigg| \int_0^{\tau_2} (\tau_2 - s)^{\beta-1} Y(\tau_2 - s) f(s, x(s), Kx(s)) ds \\
&\quad - \int_0^{\tau_1} (\tau_1 - s)^{\beta-1} Y(\tau_1 - s) f(s, x(s), Kx(s)) ds \bigg| \\
&= \bigg| \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\beta-1} Y(\tau_2 - s) f(s, x(s), Kx(s)) ds \\
&\quad + \int_0^{\tau_1} (\tau_2 - s)^{\beta-1} Y(\tau_2 - s) f(s, x(s), Kx(s)) ds \\
&\quad - \int_0^{\tau_1} (\tau_1 - s)^{\beta-1} Y(\tau_1 - s) f(s, x(s), Kx(s)) ds \bigg| \\
&\leq \bigg| \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\beta-1} Y(\tau_2 - s) f(s, x(s), Kx(s)) ds \bigg| \\
&\quad + \bigg| \int_0^{\tau_1} \big[(\tau_2 - s)^{\beta-1} - (\tau_1 - s)^{\beta-1} \big] Y(\tau_2 - s) f(s, x(s), Kx(s)) ds \bigg| \\
&\quad + \bigg| \int_0^{\tau_1} (\tau_1 - s)^{\beta-1} \big[Y(\tau_2 - s) - Y(\tau_1 - s) \big] f(s, x(s), Kx(s)) ds \bigg| \\
&\leq I_1 + I_2 + I_3,
\end{aligned}
$$

where,

$$
\begin{aligned}
I_1 &= \bigg| \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\beta-1} Y(\tau_2 - s) f(s, x(s), Kx(s)) ds \bigg| \\
&\leq \int_{\tau_1}^{\tau_2} |(\tau_2 - s)^{\beta-1} Y(\tau_2 - s) f(s, x(s), Kx(s))| ds \\
&\leq \frac{MM_1}{\Gamma(\beta)(1+b)^{1-\beta}} (\tau_2 - \tau_1)^{(1+b)(1-\beta)} \quad \text{(Applying inequality (6.4.4) over interval } [\tau_1, \tau_2]),
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= \left| \int_0^{\tau_1} \big[(\tau_2 - s)^{\beta-1} - (\tau_1 - s)^{\beta-1}\big] Y(\tau_2 - s) f(s, x(s), Kx(s)) ds \right| \\
&\leq \frac{M}{\Gamma(\beta)} \left(\int_0^{\tau_1} \big[(\tau_2 - s)^{\beta-1} - (\tau_1 - s)^{\beta-1}\big] |f(s, x(s), Kx(s))| ds \right) \\
&\leq \frac{M}{\Gamma(\beta)} \left(\int_0^{\tau_1} \big[(\tau_2 - s)^{\beta-1} - (\tau_1 - s)^{\beta-1}\big]^{\frac{1}{1-\beta}} ds \right)^{1-\beta} M_1 \quad \text{(Applying Holder inequality)} \\
&\leq \frac{MM_1}{\Gamma(\beta)} \left(\int_0^{\tau_1} \big[(\tau_2 - s)^{b} - (\tau_1 - s)\big] ds \right)^{1-\beta} \\
&\leq \frac{MM_1}{\Gamma(\beta)(1+b)^{1-\beta}} \big(\tau_1^{1+b} - \tau_2^{1+b} + (\tau_2 - \tau_1)\big)^{1-\beta} \\
&\leq \frac{MM_1}{\Gamma(\beta)(1+b)^{1-\beta}} (\tau_2 - \tau_1)^{(1+b)(1-\beta)}
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &= \left| \int_0^{\tau_1} (\tau_1 - s)^{\beta-1} \big[Y(\tau_2 - s) - Y(\tau_1 - s)\big] f(s, x(s), Kx(s)) ds \right| \\
&\leq \int_0^{\tau_1} |(\tau_1 - s)^{\beta-1} Y(\tau_2 - s) - Y(\tau_1 - s) f(s, x(s), Kx(s))| ds \\
&\leq \int_0^{\tau_1} |(\tau_1 - s)^{\beta-1} f(s, x(s), Kx(s))| ds \sup_{s \in [\tau_1, \tau_2]} |Y(\tau_2 - s) - Y(\tau_1 - s)| \\
&\leq \frac{M_1}{(1+b)^{1-\beta}} t^{(1+b)(1-\beta)} \sup_{s \in [\tau_1, \tau_2]} |V(\tau_2 - s) - V(\tau_1 - s)| \quad \text{(Applying Holder's inequality).}
\end{aligned}
$$

The integrals I_1 and I_2 are vanishes if $\tau_1 \to \tau_2$ as they contain term $(\tau_2 - \tau_1)$. By assumption (B1), the integral I_3 also vanishes as $\tau_1 \to \tau_2$. Therefore $|F_2 x(\tau_2) - F_2(\tau_1)|$ tends to zero as $\tau_1 \to \tau_2$ for independent choice of $x \in B_{r_0}$. Hence, the family $\{F_2 x, x \in B_{r_0}\}$ is equicontinuous.

For to show that the family $X(t) = \{F_2 x(t), x \in B_{r_0}\}$ for all $t \in [0, t_1)$ is relatively compact. It is obvious that $X(0)$ is relatively compact.

Let $t_0 \in [0, t_1)$ be fixed and for each $\epsilon \in [0, t_1)$, define an operator F_ϵ on B_{r_0} by formula

$$
F_\epsilon x(t) = \int_0^{t-\epsilon} (t - s)^{\beta-1} Y(t - s) f(t, x(s), Kx(s)) ds.
$$

Compactness of the operator $Y(t)$ leads to relative compactness of the set

$X_\epsilon(t) = F_\epsilon x(t), x \in B_{r_0}$ in $\mathcal{X}$.

Moreover,

$$\begin{aligned}|F_2x(t) - F_\epsilon x(t)| &= \Big| \int_0^t (t-s)^{\beta-1} Y(t-s) f(t, x(s), Kx(s)) ds \\ &\quad - \int_0^{t-\epsilon} (t-s)^{\beta-1} Y(t-s) f(t, x(s), Kx(s)) ds \Big| \\ &\leq \int_\epsilon^t |(t-s)^{\beta-1} Y(t-s) f(t, x(s), Kx(s))| ds \\ &\leq \frac{MM_1}{\Gamma(\beta)(1+b)^{1-\beta}} (t-\epsilon)^{(1+b)(1-\beta)} \quad \text{(Applying inequality (6.4.4))}.\end{aligned}$$

Therefore, $X(t)$ is relatively compact as it is very closed to relatively compact set $X_\epsilon(t)$. Thus, by Ascoli-Arzela theorem the operator F_2 is completely continous on B_{r_0}. Hence, using Krasnoselskii's fixed point theorem $F_1 + F_2$ has fixed point on B_{r_0} which is mild solution of the equation (6.4.1) over the interval $[0, t_1)$.

On the interval $[t_k, s_k)$ for all $k = 1, 2, \cdots, p$ and for positive r we define F_1 and F_2 on B_r as,

$$\begin{aligned}F_1x(t) &= g_k(t, x(t)) \\ F_2x(t) &= 0\end{aligned}$$

then, $x(t)$ is mild solution of the semi linear fractional integro-differential equation if and only if the operator equation $x = F_1x + F_2x$ has solution for $x \in B_r$ for some r. Therefore the existence of a mild solution of (6.3.1) over the interval $[t_k, s_k)$ is equivalent to determining a positive constant r_0, such that $F_1 + F_2$ has a fixed point on B_{r_0}. In fact, it is obvious due to assumption (B5). On the interval $[s_k, t_{k+1})$ for all $k = 1, 2, \cdots, p$ and for positive r we define F_1 and F_2 on B_r as,

$$\begin{aligned}F_1x(t) &= X(t-s_k) g_k(s_k, x(s_k)) \\ F_2x(t) &= \int_{s_k}^t (t-s)^{\beta-1} Y(t-s) f(t, x(s), Kx(s)) ds\end{aligned}$$

then, $x(t)$ is mild solution of the semilinear fractional integro-differential equation if and only if the operator equation $x = F_1x + F_2x$ has solution for $x \in B_r$ for some r. Therefore the existence of a mild solution of (6.3.1) over the interval $[s_k, t_{k+1})$ is equivalent to determining a positive constant r_0, such that F_1+F_2 has a fixed point on B_{r_0}.

Selecting,

$$r_0 = M\frac{|x_0| + |g(\cdot, z)|}{1 - Mg^*} + \frac{MM_1}{(1 - Mg^*)\Gamma(\beta)(1+b)^{1-\beta}}(t - s_k)^{(1+b)(1-\beta)}$$

and using similar arguments for interval $[0, t_1)$ and by Krasnoselskii's fixed point theorem F_1+F_2 has fixed point on B_{r_0} which is mild solution of the equation (6.4.1) over the interval $[s_k, t_{k+1})$.

□

Example 6.4.1.1. *Fractional partial integro-differential system with nonlocal conditions:*

$$\begin{aligned} {}^cD^{\frac{1}{2}}x(t,u) &= x_{uu}(t,u) + \frac{1}{50}\int_0^t e^{-x(s,u)}ds, && t \in [0, \tfrac{1}{3}) \cup [\tfrac{2}{3}, 1] \\ x(t,u) &= \frac{x(t,u)}{10(1+x(t,u))}, && t \in [\tfrac{1}{3}, \tfrac{2}{3}) \end{aligned} \tag{6.4.5}$$

over the interval $[0,1]$ *with initial condition* $x(0,u) = x_0(u) + \sum_{i=1}^{2}\frac{1}{3^i}x(1/i,u)$ *and boundary condition* $x(t,0) = x(t,1) = 0$.

The equation (6.4.5) *can be reformulated as fractional order abstract equation in* $\mathcal{X} = L^2([0,1],\mathbb{R})$ *as:*

$$\begin{aligned} {}^cD^{\beta}z(t) &= Az(t) + f(t, z(t), Kz(t)), && t \in [0, \tfrac{1}{3}) \cup [\tfrac{2}{3}, 1] \\ z(t) &= g(t, z(t)) && t \in [\tfrac{1}{3}, \tfrac{2}{3}) \end{aligned} \tag{6.4.6}$$

over the interval $[0,1]$ *by defining* $z(t) = x(t,\cdot)$, *operator* $Ax = x''$ *(second order derivative with respect to x). The functions f and g over respected domains are defined as* $f(t,z(t),Kz(t)) = \frac{1}{50}\int_0^t e^{-z(s)}ds$ *and* $g(t,z(t)) = \frac{z(t)}{10(1+z(t))}$ *respectively.*

The equation (6.4.6) *satisfies the conditions (B1-B5) of the hypothesis with* $Mh^* < 1$ *and* $Mg^* < 1$. *Hence the equation* (6.4.6) *has a mild solution over the interval* $[0,1]$.

6.5 Conclusion

Existence of mild solution of non-instantaneous impulses semilinear fractional evolution equation with local and nonlocal conditions over the Banach space is established in this chapter. The result of local evolution equation is obtained through general Banach contraction theorem while, for nonlocal evolution equation is obtained through Krasnoselskii's fixed point theorem.