Chapter 7

Existence and uniqueness of solutions of fractional order hybrid differential equations with nonlocal conditions

This chapter established sufficient conditions for the mild solution of the fractional hybrid equations:

$$\begin{aligned} {}^{c}D^{\beta}\big[x(t) - f(t, x(t))\big] &= g(t, x(t)) \\ x(0) &= h(x) \end{aligned} \tag{7.0.1}$$

with nonlocal condition over the interval $[0, T_0]$ on partially ordered Banach space X. The nonlocal condition in this equation is more relevant in many physical phenomena in physics.

7.1 Introduction

From the past decates fractional differential equations is one of the renowned subject due to its various applications in all sciences [44, 7, 8, 9, 10, 12, 47, 48, 85, 86, 51, 52, 53, 54]. This is due to their nonlocal property means, the successive state is depend on all its previous states [14] and this nonlocal property helped to remodel their systems in terms of fractional order systems [13]. Existence and uniqueness of solution of fractional order differential equation with classical condition was initiated by Delbosco and Rodino [11] and subsequently extended results by several researchers [14, 15, 16, 17, 75, 76, 77, 78, 79].

The systems having both continuous and discrete behaviour are called hybrid systems. Due to its dual nature many researchers took interst to stydy these hybrid systems. Existence and uniqueness of solution of hybrid first order systems using fixed point theorem was extensively studied by Krasnoselskii, Burton and Dhage [80, 82, 83, 84].

The study of existence and uniqueness of solution of Caputo fractional hybrid differential equations was initiated by by Herzallah and Beleanu [35].Recently, existence and uniqueness of fractional order systems with classical condition was studied by Somjaiwang and Ngiamsunthorn [34].

In this article we are modifying the work of Somjaiwang and Ngiamsunthorn and study existence and uniqueness of mild solution of fractional hybrid system (7.0.1) with nonlocal conditions.

7.2 Notations

(N1) X = Banach space equiped with partial order.

(N2) $\mathbb{R}_+ = [0, \infty)$

(N3) $C([0, T_0], X) = \big\{x : [0, T_0] \to X / x \; is \; continuous\big\}$ with norm $||x|| = sup_t ||x(t)||$

Definition 7.2.1. *([80]) An operator $T : X \to X$ is called nondecresing if the order relation preserved under T, this means, for each $x, y \in X$ such that $x \leq y$ implies $Tx \leq Ty$.*

Definition 7.2.2. *([80]) The order relation $\leq$ and a metric d are compatible on nonempty set X then convergence of subsequence $\{x_{n_k}\}$ implies the convergence of $\{x_n\}$ for any monotone sequence $\{x_n\}$ in X.*

Definition 7.2.3. *([80]) An upper semi-continuous and nondecresing function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *is* $\mathcal{D}$ *-function if* $\psi(0) = 0$

Definition 7.2.4. *([81]) An operator* $T : X \to X$ *is called partially continuous at* $a \in X$ *if for any* $\epsilon > 0$, *there exist* $\delta > 0$ *such that* $||Tx - Ta|| < \epsilon$ *for all* x *comparable to* a *in* X *with* $||x - a|| < \delta$ *and* T *is continuous on* X *if* T *is partially continuous at every* $a \in X$. *In particular, if* T *is partiallly continuous on* X, *then it is continuous on every chain* C *in* X. *An operator* T *is called partially bounded if* $T(C)$ *is bounded for every chain* C *in* X. *An operator* T *is said to be uniformly partially bounded if all the chains* $T(C)$ *in* X *are bounded.*

Definition 7.2.5. *([81]) An operator* $T : X \to X$ *is called partially compact if for any chain* C *in* X, *the set* $T(C)$ *are relatively compact subset of* X. *An operator* T *is said to be partially totally bounded if for any totally ordered and bounded subset* C *of* X, *the set* $T(C)$ *is a relatively compact subset of* X. *If* T *is partially continuous and partially totally bounded then* T *is partially completely continuous operator on* X.

Definition 7.2.6. *([81]) A mapping* $T : X \to X$ *is partially nonlinear* $\mathcal{D}$*-LIpschitz if there is a* $\mathcal{D}$*-function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $||Tx - Ty|| \leq \psi||x - y||$ *for all compatible points* $x, y \in X$. *If* $\psi(r) < r$ *then* T *is called* $\mathcal{D}$*-contraction.*

Theorem 7.2.1. *([81]) Let order and norm are compatible on a partially order Banach space. Let* P *and* Q *are nondecresing operators on* X *such that:*
(a) P *is partially bounded nonlinear* $\mathcal{D}$*-contraction.*
(b) Q *is partially continuous and partially compact.*
(c) There exist an element $x_0 \in X$ *such that* $x_0 \leq Px_0 + Qx_0$. *Then there exist a solution* x^* *in* X *of operator equation* $Px + Qx = x$. *In addition, the sequence* $\{x_n\}$ *of successive iteration* $x_{n+1} = Px_n + Qx_n$ *converges monotonically to* x^*.

Definition 7.2.7. *Function* $x(t) \in X$ *is called mild solution of fractional order* *hybrid system 7.0.1 if $x(t)$ satisfies,*

$$x(t) = h(x) - f(t_0, h(x)) + f(t, x(t)) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} g(s, x(s))ds \qquad (7.2.1)$$

for all value of $t \in [0, T_0]$.

7.3 Assumptions

(A1) The functions $f : [0, T_0] \times X \to X$ and $g : [0, T_0] \times X \to X$ are nondectresing and continuous.

(A2) The function $h : X \to X$ is continuous and nondecresing in x.

(A3) Function f is $\mathcal{D}$-contraction with ϕ such that $\phi(r) < r$.

(A4) There exist a constant M_f such that $|f(t, x)| \leq M_f$ for all $t \in [0, T_0]$ and $x \in X$.

(A5) For each $x, y \in X$ with $x \geq y$, $h(x) - h(y) \geq f(t, h(x)) - g(t, h(y))$ for all $t \in [0, T_0]$.

(B1) There exist a positive constants M_g and M_h such that $|g(t, x)| \leq M_g$ and $|h(x)| \leq M_h$.

(B2) There exist a function $u \in X$ such that u is lower solution of the equation (7.0.1).

7.4 Main Results

In this section we discuss the existence and uniqueness of mild solution of fractional order hybrid system with nonlocal condition of the system (7.0.1.)

Theorem 7.4.1. *If assumptions (A1)-(A4) and (B1)-(B3) are satisfied then fractional order hybrid system (7.0.1) has unique mild solution in partially order Banach space X.*

Proof. We use theorem (7.2.1) to prove existence and uniqueness of mild solution of 7.0.1. Defining $Px(t) = f(t, x)$ and
$Q = h(x) - f(t_0, h(x)) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} g(s, x(s))ds$ the equation (7.2.1) becomes $x = Px + Qx$. By theorem (7.2.1), this operator equation has unique solution if conditions in hypotheses of the theorem (7.2.1) is satisfied.

Step-1 Using the conditions (A1),(A2) and (B1), we can easily prove that the operators P and Q are nondecresing. Also by assuming (A1)-(A4), prove that P is partially bounded nonlinear $\mathcal{D}$-contraction on X.

Step-2 This section prove that Q is partially continuous and partially compact on X. Let, $\{x_n\}$ be a sequence in a chain C in X such that $x_n \to x$. Then,

$$\begin{aligned}||Qx_n(t) - Qx(t)|| \leq & ||hx_n(t) - hx(t)|| + ||f(t_0, x_n(t)) - f(t_0, x(t))|| \\ & + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} ||g(s, x_n(s)) - g(s, x(s))|| ds.\end{aligned}$$

Assuming continuity of f, h and g and dominated convergence theorem, $||Qx_n(t) - Qx(t)|| \to 0$ as $n \to \infty$ for all $t \in [0, T_0]$.
Moreover the sequence $\{Qx_n\}$ is equicontinous on X as let, $t_1, t_2 \in [0, T_0]$ with $t_1 < t_2$ then,

$$\begin{aligned}|Qx_n(t_2) - Qx_n(t_1)| \leq & |h(x_n(t_2)) - h(x_n(t_1))| + |f(t_0, h(x_n(t_2))) - f(t_0, h(x_n(t_1)))| \\ & + \left| \frac{1}{\Gamma(\beta)} \int_{t_0}^{t_1} \left[(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1} \right] g(s, x_n(s)) ds \right| \\ & + \frac{1}{\Gamma(\beta)} \int_{t_1}^{t_2} (t_2 - s)^{\beta-1} |g(s, x_n(s))| ds.\end{aligned}$$

From (A1),(A2) and (B1), $|Qx_n(t_2) - Qx_n(t_1)| \to 0$ as $t_2 - t_1 \to 0$. Therefore, the sequence $\{Qx_n\}$ is equicontinuous. Thus $Qx_n \to Qx$ is uniformly on chain C and hence, Q is partially continuous on X.

Step -3 In this step we will show that Q is partially compact. Let $x \in C$ where C is chain in X. Then,

$$\begin{aligned}
||Qx(t)|| &\leq ||h(x)|| + ||f(t_0, h(x))|| + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} ||g(s, x(s))|| ds \\
&\leq M_h + M_f + \frac{M_g}{\beta\Gamma(\beta)} (T - t_0)^{\beta} = K
\end{aligned}$$

for all $t \in [0, T_0]$. Therefore $||Qx(t)|| \leq K$ and hence $Q(C)$ is uniformly bounded.

Since,

$$\begin{aligned}
|Qx(t_2) - Qx(t_1)| \leq\ & |h(x(t_2)) - h(x(t_1))| + |f(t_0, h(x(t_2))) - f(t_0, h(x(t_1)))| \\
&+ \left| \frac{1}{\Gamma(\beta)} \int_{t_0}^{t_1} \left[(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}\right] g(s, x(s)) ds \right| \\
&+ \frac{1}{\Gamma(\beta)} \int_{t_1}^{t_2} (t_2 - s)^{\beta-1} |g(s, x(s))| ds
\end{aligned}$$

and $|Qx(t_2) - Qx(t_1)| \to 0$ as $t_2 \to t_1$ uniformly for all $x \in C$. Therefore, $Q(C)$ is relatively compact. Hence, Q is partially compact.

Step-4 By hypothesis (B2), the fractional hybrid system has lower solution u defined on $[0, T_0]$. That is

$$\begin{aligned}
{}^cD^{\beta}\left[u(t) - f(t, u(t))\right] &= g(t, u(t)) \\
u(0) &\leq u(x)
\end{aligned}$$

Applying integral operator both side we get,

$$u(t) \leq h(u) - f(t_0, h(u)) + f(t, u(t)) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} g(s, u(s)) ds$$

for all $t \in [0, T_0]$. Therefore $u \leq Pu + Qu$.

Thus, from these steps conclude that, the operators P and Q satisfy all the conditions of the hypotheses of the theorem (7.2.1) and hence the equation (7.0.1) has unique mild solution. $\square$