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Existence and Uniqueness of Solution of Fractional Order Differential Equations.

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Existence and Uniqueness of Solution of Fractional Order Differential Equations.

1 Historical Background

Recently fractional calculus emerged as a new branch of mathematics due to a wide range of applications in physics and engineering vizvproblem of anomalous diffusion [8, 9] modeled using fractional calculus. The fractional calculus is a generalization of integral and derivatives to non- integer order having a long history of approximately 300 years [2]. Many Researchers on this topic cite a particular date as the birthday of so called fractional calculus. In a letter dated September 30th, 1695 L' hospital wrote to Leibniz asking him about half derivative of $f(x) = x$ and Leibniz replied " An apparent paradox, from one day useful consequences will be drawn". In these words, fractional calculus was born. After this reply, many stalwart mathematicians like Euler, Laplace and Fourier started a study of fractional calculus and its consequences. few of them found their own notations and methodology and definition that fits into the concept of a non integer order integral and derivatives. The most popular definitions are the Riemann- Liouville and Grunwald- Letnikov definition [2, 3, 4, 5, 6]. However, differentiation and integration of non-integer order were not acceptable for geometrical and physical interpretation unlike integer order [10].

Due to non-local property of the fractional differential equations, the non-linear systems like the non-linear oscillations of earthquake [11], seepage flow in porous media [7] and traffic flow in fluid dynamics and viscoelastic problems are modeled in to fractional differential equations [3, 8, 12] studied. Abel was the first one to used fractional derivative to solve tautochrone problem which involves integral equation which was later termed as the Riemann-Liouville integral of order β.

The existence and uniqueness of solutions of fractional differential equations with classical condition $x(0) = x_0$ using fixed point theory has been studied by Delbosco and Rodino

[11], Cheng and Guozhu [14] and El-Borai [15]. In many physical situations, the initial condition of the form $x(0) = x_0 - g(x)$ can be more descriptive than classical condition called nonlocal condition. Study of existence and uniqueness of solutions with nonlocal condition was initiated by Byszewski [16]. The study of fractional differential equations with same conditions was initiated by Guerekata [18] and Balachandran, Trujillo and Park [19, 17]. There are problems for which past history is required to be modeled into delay differential equations. The existence and Uniqueness of solution of such fractional differential equations were studied by Balachandran et. al.[22].

Besides existence and uniqueness, many researchers now a days are also interested in the solution's behavior with respect to initial condition. The study of the behavior of the solution is known as stability of differential equation. Matignon [28] derived necessary and sufficient condition for asymptotic stability of linear fractional differential equations followed by other mathematicians [29, 30, 31]. Most of nonlinear fractional differential equations do not have analytical solution, so approximations and numerical techniques have to be used to solve fractional differential equations. The decomposition method and variational iteration method have been extensively studied by Odibat and Momani [24, 25]. Apart from this semi-analytic iterative methods, few numerical methods have been developed till now. Diethelm et. al. [26] generalized Adams-Bashforth method for fractional differential equations. Odibat and Momani [27] combined generalized trapezoidal rule with a generalized Euler method to develop a new technique to solve fractional differential equations.

2 Preliminaries

Definition 2.1 ([1]). *The gamma function is defined, for $z \in \mathbb{C}$ as:*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dz, \quad Re(z) > 0.$$

This integral representation $\Gamma(z)$ is valid for right half plane. Extend it to left half plane barring negative integers using analytic continuation by:

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)\cdots(z+n)}.$$

Thus $\Gamma(z) : C - \{0, -1, -2, ...\} \to C$, fig. 1.1 presents the graph of the function

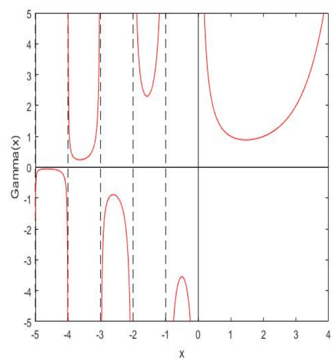

fig. 1.1

Definition 2.2 ([3]). *The Riemann-Liouville fractional integral operator of order $\beta > 0$, of function $f \in L_1(\mathbb{R}_+)$, is defined as*

$$I_{0+}^{\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s) ds, \tag{2.1}$$

where $\Gamma(\cdot)$ is gamma function.

Definition 2.3 ([3]). *The Riemann-Liouville fractional derivative of order $\beta > 0$, $n-1 < \beta < n$, $n \in \mathbb{N}$, is defined as*

$$D_{0+}^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\beta-1} f(s) ds, \tag{2.2}$$

where the function $f(t)$ has absolutely continuous derivatives up to order $(n-1)$.

This derivative has singularity at zero and also requires special initial condition which is lacking physical interpretation. To overcome this difficulty, Caputo [32] interchanged the role of operators and defined the fractional derivatives as follows:

Definition 2.4 ([3]). *The Caputo fractional derivative of order $\beta > 0$, $n-1 < \beta < n$, $n \in \mathbb{N}$, is defined as*

$$^{c}D^{\beta}_{0+}f(t) = \frac{1}{\Gamma(n-\beta)}\int_0^t (t-s)^{n-\beta-1}\frac{d^n f(s)}{dt^n}ds. \tag{2.3}$$

Where the function $f(t)$ has absolutely continuous derivatives up to order $(n-1)$.

Moreover if $0 < \beta < 1$, then

$$^{c}D^{\beta}_{0+}f(t) = \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\frac{df(s)}{dt}ds.$$

3 Research work

In this work, qualitative property like existence and uniqueness of the solution of fractional order differential equations studied. Fractional order differential equation with and without impulses governing physical phenomena is also considered. Thesis consist of Six chapters:

Through out the all chapters $^{c}D^{\beta}$ denotes Caputo fractional order differential operator.

3.1 Chapter one

The impulsive fractional quasiliner integro-differential equation

$$\begin{aligned} ^{c}D^{\beta}x(t) &= A(t,x)x(t) + f(t, x(t), Tx(t), Sx(t)) \quad t \neq t_k,\ k = 1,2,\cdots,p \\ \Delta x(t_k) &= I_k(x(t_k)), \quad t = t_k,\ k = 1,2,\cdots,p \end{aligned}$$

Where, $A(t,x)$ is bounded quasi-linear operator on X and $f : [0,T_0] \times X \times X \times X \to X$, $T, S : X \to X$ are defined by $Tx(t) = \int_0^t h(t,s,x(s))ds$ and $Sx(t) = \int_0^{T_0} k(t,s,x(s))ds$, $h : D_0 \times X \to X,\ D_0 = \{(t,s); 0 \le s \le t \le T_0\}$ and $k : D_1 \times X \to X,\ D_1 = \{(t,s); 0 \le t, s \le T_0\}$ are continuous, with local condition $x(0) = x_0$ and nonlocal condition $x(0) = x_0 - g(x)$ over the interval $[0,T_0]$ in a Banach space X. Existence and uniqueness of mild solution of the problem obtained. Inclusion of Fredholm integral operator S in the equation is more relavent in modelling of many physical phenomena arises in the field of viscoelasticity. An example is added to illustrate the efficacy of the method.

3.2 Chapter Two

The work discussed in 1st chapter is extended in 2nd chapter by adding a delay condition on it and derived sufficient conditions for existence and uniqueness of mild solution of the integro-differential equations:

$$\begin{aligned} {}^cD^\beta x(t) &= A(t,x)x(t) + f(t, x(\phi(t)), Tx(t), Sx(t)) \quad t \neq t_k,\ k = 1, 2, \cdots, p \\ \Delta x(t_k) &= I_k(x(t_k)), \quad t = t_k,\ k = 1, 2, \cdots, p \\ x(0) &= x_0 - g(x), \end{aligned}$$

over the interval $[0, T_0]$ in a Banach space X. Where, $A(t, x)$ is bounded quasi linear operator on X and $f : [0, T_0] \times X \times X \times X \to X$, $T, S : X \to X$ are defined by $Tx(t) = \int_0^t h(t, s, x(\psi(s)))ds$ and $Sx(t) = \int_0^{T_0} k(t, s, x(\xi(s)))ds$, where $h : D_0 \times X \to X$, $D_0 = \{(t, s); 0 \leq s \leq t \leq T_0\}$ and $k : D_1 \times X \to X$, $D_1 = \{(t, s); 0 \leq t, s \leq T_0\}$ are the operators satisfying condition of the hypotheses.

3.3 Chapter Three

The sufficient conditions for existence and uniqueness of mild and classical solution of fractional order impulsive integro-differential equations of the following form is established in this chapter. And also derived conditions in which mild and classical solution are congruence.

$$\begin{aligned} {}^cD^\beta x(t) &= Ax(t) + f(t, x(t), Tx(t), Sx(t)) \quad t \neq t_k,\ k = 1, 2, \cdots, p \\ \Delta x(t_k) &= I_k(x(t_k)), \quad t = t_k,\ k = 1, 2, \cdots, p \\ x(0) &= x_0 \end{aligned}$$

over the interval $[0, T_0]$ in a Banach space X. Here, A is bounded linear operator on X and $f : [0, T_0] \times X \times X \times X \to X$,$T, S : X \to X$ are defined by $Tx(t) = \int_0^t h(t, s, x(s))ds$ and $Sx(t) = \int_0^{T_0} k(t, s, x(s))ds$. Where $h : D_0 \times X \to X$, $D_0 = \{(t, s); 0 \leq s \leq t \leq T_0\}$ and $k : D_1 \times X \to X$, $D_1 = \{(t, s); 0 \leq t, s \leq T_0\}$ are the operators satisfying condition of the hypotheses.

3.4 Chapter Four

This chapter, includes existence and uniqueness of mild and classical solution of generalized fractional order impulsive evolution equations

$$\begin{aligned} {}^{c}D^{\beta}x(t) &= Ax(t) + g_k(t, x(t)) \quad t \in (t_{k-1}, t_k)\ k = 1, 2, \cdots, p \\ \Delta x(t_k) &= I_k(x(t_k)), \quad t = t_k,\ k = 1, 2, \cdots, p \\ x(0) &= x_0 \end{aligned}$$

in which perturbing force is different after every impulse over the interval $[0, T_0]$ on a Banach space X. A is bounded linear operator on X and $I_k : [0, T_0] \times X \to X$ for $k = 1, 2, \cdots, p+1$. Also established conditions in which mild and classical solutions are congruent.

3.5 Chapter Five

Existence of mild solution for not-instantaneous impulses fractional order integro-differential equations with local and nonlocal conditions in Banach space is established in this paper.

$$\begin{aligned} {}^{c}D^{\beta}x(t) &= Ax(t) + f\left(t, x(t), \int_0^t a(t, s, x(s))ds\right),\ t \in [s_k, t_{k+1}),\ k = 1, 2, \cdots, p \\ x(t) &= I_k(k, x(t)),\ t \in [t_k, s_k) \end{aligned}$$

with local condition $x(0) = x_0$ and nonlocal condition $x(0) = x_0 + h(x)$ over the interval $[0, T_0]$ in a Banach space X. Here $A : X \to X$ is linear operator, $Kx = \int_0^t a(t, s, x(s))ds$ is nonlinear Volterra integral oprator on X, $f : [0, T_0] \times X \times X \to X$ is nonlinear function and $I_k : [0, T_0] \times X$ are set of nonlinear functions applied in the interval $[t_k, s_k)$ for all $i = 1, 2, \cdots, p$.

3.6 Chapter Six

This chapter established sufficient conditions for the mild solution of the fractional hybrid equations:

$$\begin{aligned} {}^{c}D^{\beta}\left[x(t) - f(t, x(t))\right] &= g(t, x(t)) \\ x(0) &= h(x) \end{aligned}$$

with nonlocal condition over the interval $[0, T_0]$ on partially ordered Banach space X. The nonlocal condition in this equation is more relevant in many physical phenomena in physics.

Papers Published, Communicated and Presented

Published and communicated papers :

1 H. R. Kataria, Prakashkumar H. Patel, Existence and Uniqueness of Non-Local Cauchy Problem for Fractional Differential Equation On Banach Space, International Journal of Pure and Applied Mathematics, Volume 120, issue 7, No. 6 , 2018, page 10237-10252 ,p- ISSN : 1311-8080 e- ISSN : 1314-3395 UGC Journal List Number: 23425.Scopus indexed journal.

2 H. R. Kataria and Prakashkumar H. Patel, Nonlinear Cauchy problem for abstract impulsive fractional quasilinear evolution equation with delay, The International journal of analytical and experimental modal analysis, Volume 11, issue 7, July 2019, page 72-81,e- ISSN : 0886-9367, Scopus indexed journal.

3 H. R. Kataria and Prakashkumar H. Patel, Congruency between classical and mild solutions of Caputo fractional impulsive evolution equation on Banach Space, International Journal of Advanced Science and Technology, Volume 29, No. 3s, March 2020, page. 1923-1936, ISSN: 2005-4238, Scopus indexed journal.

4 H. R. Kataria and Prakashkumar H. Patel, Existence and Uniqueness of solutions of Fractional Order Hybrid Differential Equations with Nonlocal Conditions, International Journal of Scientific Research in Science, Engineering and Technology, Volume 4, Issue 5, March-April-2018, page 1-7, e-ISSN : 2394-4099 p-ISSN : 2395-1990, UGC Journal List Number: 47147.

5 H. R. Kataria and Prakashkumar H. Patel, Jyotindra C. Prajapati, Congruence between Mild and Classical Solutions of Generalized Fractional Impulsive Evolution Equation, Communicated.

6 H. R. Kataria and Prakashkumar H. Patel, Existence Results of Not-Instantaneous Impulsive Fractional Integro-differential Equation, Communicated.

Presented papers:

1 Paper entitled "Existence Results For Impulsive Fractional Delay Differential Equations On Banach Spaces." was presented at the International Conference On Discrete And Computational Mathematics(ICDCM-2017) organized by Department of Mathematics, The Gandhigram Rural Institute- Deemed University Gandhigram - 624 302, Dindigul District, Tamil Nadu, India. held on February 16-18, 2017.

2 Paper entitled "Existence and Uniqueness of solutions of Fractional Order Hybrid Differential Equations with Nonlocal Conditions." was presented at the 2nd International Conference on Current Research Trends in Engineering and Technology (SIEICON-2018) organized by Sigma Institute of Engineering (SIE), Vadodara, Gujarat, India, held on 09 March 2018.

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