Chapter 1

Introduction

1.1 Historical Background

Recently fractional calculus emerged as new branch of mathematics due to wide range of applications in physics and engineering viz problem of anomalous diffusion [8, 9] modeled using fractional calculus. The fractional calculus is generalization of integral and derivatives to non- integer order having a long history of approximately 300 years [3]. Many authors on this topic cite a particular date as the birthday of so called fractional calculus. In a letter dated September 30th, 1695 L' Hopital wrote to Leibniz asking him about half derivative of $f(x) = x$ and Leibniz replied " An apparent paradox, from one day useful consequences will be drawn". In these words, fractional calculus was born. After this reply, many stalwart mathematicians like Fourier, Euler and Laplace started study of fractional calculus and its consequences. Few of them found their own notations and methodology and definition that fit into the concept of a non integer order integral and derivatives. The most popular definitions are the Riemann- Liouville and Grunwald- Letnikov definitions [3, 2, 4, 5, 6]. However differentiation and integration of non-integer order were not acceptable for geometrical and physical interpretation unlike integer order [10].

Due to non-local property of the fractional differential equations, the non-linear systems like the non-linear oscillations of earthquake [11], seepage flow in porous media [7] and traffic flow in fluid dynamics and viscoelastic problems are modeled in to fractional differential equations [2, 8, 12]. Abel was the first one to used fractional derivative to solve tautochrone problem which involves integral equation which was later termed as Riemann- Liouville integral of order β.

The existence and uniqueness of solutions of fractional differential equations with classical condition $x(0) = x_0$ using fixed point theory studied by Delbosco and Rodino [11], Cheng and Guozhu [14] and El-Borai [15]. In many physical situations, initial condition of the form $x(0) = x_0 - g(x)$ can be more descriptive than classical condition called nonlocal condition. Study of existence and uniqueness of solutions with nonlocal condition was initiated by Byszewski [16]. The study of fractional differential equations with same conditions was initiated by Guerekata [18] and Balachandran, Trujillo and Park [19, 17]. There are problems for which history is required to be modeled into delay differential equations. Existence and Uniqueness of solution of such fractional differential equations was studied by Balachandran et. al.[22].

Besides existence and uniqueness, many researchers now a days are also interested in the solution's behavior with respect to initial conditions. Study of behavior of the solution is known as stability of differential equation. Matignon [28] derived necessary and sufficient conditions for asymptotic stability of linear fractional differential equations followed by other mathematicians [29, 30, 31]. Most of nonlinear fractional differential equations do not have analytical solution, so approximations and numerical techniques have to be used to solve fractional differential equations. The decomposition method and variational iteration method are extensively studied by Odibat and Momani [24, 25]. Apart from these iterative methods, few numerical methods have been developed till now. Diethelm et. al. [26] generalized Adams-Bashforth method for fractional differential equations. Odibat and Momani [27] combined generalized trapezoidal rule with generalized Euler method to develop a new technique to solve fractional differential equations.

1.2 Preliminaries

Definition 1.2.1 ([1]). *The gamma function is defined, for $z \in \mathbb{C}$ as:*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dz, \quad Re(z) > 0.$$

This integral representation $\Gamma(z)$ is valid for right half plane. Extend it to left half plane barring negative integers using analytic continuation by:

$$\Gamma(z) = \lim_{n \to \infty} \frac{n! n^z}{z(z+1)\cdots(z+n)}.$$

Thus $\Gamma(z) : C - \{0, -1, -2, ...\} \to C$, fig. 1.1 presents the graph of the function

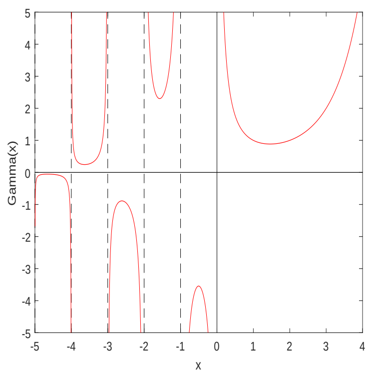

fig. 1.1

Definition 1.2.2 ([1]). *The complementary error function (erfc)is defined, as for $z \in \mathbb{C}$ as:*

$$erfc(z) = \frac{2}{\pi^{(1/2)}} \int_z^\infty e^{-t^2} dt$$

fig. 1.2 presents the graph of complementary error function

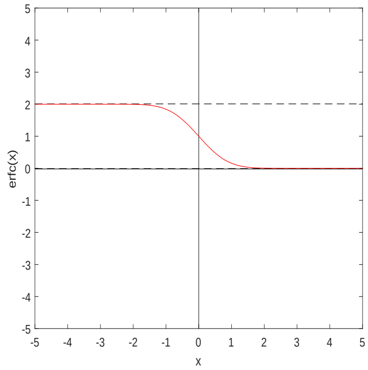

fig. 1.2

some special values of the function are $erfc(\infty) = 0$, $erfc(0) = 1$, $erfc(-\infty) = 2$ and some relations are

$$\text{erfc(-z)=2-erfc(z)}$$
$$\int_0^\infty erfc(z)dz = \frac{1}{\pi^{1/2}}$$
$$\int_z^\infty erfc^2(z)dz = \frac{2-2^{1/2}}{\pi^{1/2}}$$

Definition 1.2.3 ([2]). *The Riemann-Liouville fractional integral operator of order $\beta > 0$, of function $f \in L_1(\mathbb{R}_+)$, is defined as*

$$I_{0+}^{\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)ds, \tag{1.2.1}$$

where $\Gamma(\cdot)$ is gamma function.

Definition 1.2.4 ([2]). *The Riemann-Liouville fractional derivative of order $\beta > 0$, $n-1 < \beta < n$, $n \in \mathbb{N}$, is defined as*

$$D_{0+}^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\beta-1} f(s)ds, \tag{1.2.2}$$

where the function $f(t)$ has absolutely continuous derivatives up to order $(n-1)$.

This derivative has singularity at zero and also requires special initial condition which is lacking physical interpretation. To overcome this difficulty, Caputo [32] interchanged the role of operators and defined the fractional derivatives as follows:

Definition 1.2.5 ([2])**.** *The Caputo fractional derivative of order* $\beta > 0$, $n-1 < \beta < n$, $n \in \mathbb{N}$, *is defined as*

$$ {}^{c}D_{0+}^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} \frac{d^n f(s)}{dt^n} ds. \tag{1.2.3} $$

Where the function $f(t)$ *has absolutely continuous derivatives up to order* $(n-1)$.

Moreover if $0 < \beta < 1$, then

$$ {}^{c}D_{0+}^{\beta} f(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \frac{df(s)}{dt} ds. $$

1.3 Properties

The Riemann-Liouville integral I^{β} and Caputo derivative ${}^{c}D_{0+}^{\beta}$ satisfies following properties For $\beta, \alpha > 0$ and f having absolutely continuous derivatives up to suitable order, then

(1)$I_{0+}^{\beta} I_{0+}^{\alpha} f(t) = I_{0+}^{\beta+\alpha} f(t)$

(2)$I_{0+}^{\beta} I_{0+}^{\alpha} f(t) = I_{0+}^{\alpha} I_{0+}^{\beta} f(t)$

(3)$I_{0+}^{\beta}(f(t) + g(t)) = I_{0+}^{\beta} f(t) + I_{0+}^{\beta} g(t)$

(4)$I_{0+}^{\beta}\ {}^{c}D_{0+}^{\beta} f(t) = f(t) - f(0), \quad 0 < \beta < 1$

(5)${}^{c}D_{0+}^{\beta} I_{0+}^{\beta} f(t) = f(t)$

(6)${}^{c}D_{0+}^{\beta} f(t) = I_{0+}^{1-\beta} f'(t), \quad 0 < \beta < 1$

(7)${}^{c}D_{0+}^{\beta}\ {}^{c}D_{0+}^{\alpha} f(t) \neq {}^{c} D_{0+}^{\beta+\alpha} f(t)$

(8)${}^{c}D_{0+}^{\beta}\ {}^{c}D_{0+}^{\alpha} f(t) \neq {}^{c}D_{0+}^{\alpha}\ {}^{c}D_{0+}^{\beta} f(t)$

which are mentioned in Kilbas *et. al.* [5] and Samko *et. al.* [6].

1.4 Caputo Derivative of some fundamental function

Caputo derivative of some well know function discussed in this section. And for convenient write ${}^cD^\beta$ in place of ${}^cD^\beta_{0+}$

Caputo Derivative of constant function

Consider, $f(t) = c$ and then by definition (1.2.5)

$$
\begin{aligned}
{}^cD^\beta f(t) &= \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} \frac{d^n f(s)}{ds^n} ds. \\
{}^cD^\beta (c) &= \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} \frac{d^n c}{ds^n} ds = 0
\end{aligned}
$$

Caputo Derivative of power function

Consider, $f(t) = x^p$ be a power function in the interval $(0, t)$ and for $n-1 < \beta < n$, $n \in N$, $\beta \in R$, $and\ n-1 < p$

$$
\begin{aligned}
{}^cD^\beta f(t) &= \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} \frac{d^n s^p}{ds^n} ds. \\
&= \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} p(p-1)(p-2)...(p-(n-1)) s^{(p-n)} ds \\
&= \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} \frac{p!}{(p-n)!} s^{(p-n)} ds \\
&= \frac{\Gamma(p+1)}{\Gamma(n-\beta)\Gamma(p+1-n)} \int_0^t (t-s)^{n-\beta-1} s^{(p-n)} ds
\end{aligned}
$$

Assume that $s = \lambda t$ for $0 \leq \lambda \leq 1$ then

$$\begin{aligned}
{}^{c}D^{\beta}f(t) &= \frac{\Gamma(p+1)}{\Gamma(n-\beta)\Gamma(p+1-n)} \int_0^t (1-\lambda)^{n-\beta-1} t^{n-\beta-1} (\lambda t)^{(p-n)} t d\lambda \\
&= \frac{\Gamma(p+1)}{\Gamma(n-\beta)\Gamma(p+1-n)} t^{p-\beta} \int_0^1 (\lambda)^{p-n} (1-\lambda)^{n-\beta-1} d\lambda \\
&= \frac{\Gamma(p+1)}{\Gamma(n-\beta)\Gamma(p+1-n)} t^{p-\beta} \alpha(p-n-1, n-\beta) \\
&= \frac{\Gamma(p+1)}{\Gamma(n-\beta)\Gamma(p+1-n)} t^{p-\beta} \frac{\Gamma(n-\beta)\Gamma(p+1-n)}{\Gamma(p-\beta+1)} \\
&= \frac{\Gamma(p+1)}{\Gamma(p+1-\beta)} t^{p-\beta}
\end{aligned}$$

for $n \in N$, $\frac{d^n}{dt^n} t^p = \frac{\Gamma(p+1)}{\Gamma(p+1-n)} t^{p-n}$

Therefor, for $(n-1) < p$, Caputo fractional derivative of power function is generalization of integer order derivative of power function.

Caputo derivative of the exponential function

Consider, $f(t) = e^{at}$, for $n-1 < \beta \leq n$, $n \in \mathbb{N}$, $\beta \in \mathbb{R}$

$$\begin{aligned}
{}^{c}D^{\beta}f(t) &= \frac{1}{\Gamma(n-\beta)} \int_{-\infty}^t (t)(t-s)^{n-\beta-1} \frac{d^n}{ds^n} e^{as} ds \\
&= \frac{a^n}{\Gamma(n-\beta)} \int_{-\infty}^t (t-s)^{n-\beta-1} e^{as} ds
\end{aligned}$$

Consider, $(t-s) = p \Rightarrow -ds = dp$ and when $s = -\infty \Rightarrow p = \infty$, $s = t \Rightarrow p = 0$. Therefore

$$\begin{aligned}
{}^{c}D^{\beta}f(t) &= \frac{a^n}{\Gamma(n-\beta)} \int_0^{\infty} p^{n-\beta-1} e^{a(t-p)} ds \\
&= \frac{a^n e^{at}}{\Gamma(n-\beta)} \int_0^{\infty} p^{n-\beta-1} e^{-ap} ds
\end{aligned}$$

Substituting $ap = u \Rightarrow adp = du$ and when $p = 0 \Rightarrow u = 0$, $p = \infty \Rightarrow u = \infty$.

$$
\begin{aligned}
{}^{c}D^{\beta} f(t) &= \frac{a^n e^{at}}{\Gamma(n-\beta)} \int_0^\infty (\frac{u}{a})^{n-\beta-1} e^{-u} \frac{1}{a} ds \\
&= \frac{a^n e^{at}}{a^{n-\beta}\Gamma(n-\beta)} \int_0^\infty u^{n-\beta-1} e^{-u} du \\
&= \frac{a^{\beta} e^{at} \Gamma(n-\beta)}{\Gamma(n-\beta)} \\
&= a^{\beta} e^{at}
\end{aligned}
$$

1.5 Mittag-Leffler Function

In the theory of fractional differential equation, the Mittag-Leffler function plays an important role, which is generalization of an exponential function. It is proved that they are the solution of fractional differential equations with constant coefficient. Due to its attractive feature, it is referred as the Queen Function of the fractional calculus.

One parameter Mittag-Leffler function([2]) is defined as:

$$
E_{\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)}, \quad \beta > 0, \; z \in \mathbb{C}. \tag{1.5.1}
$$

For $0 < \beta < 1$, it interpolates between the hyper geometric function $\frac{1}{1-z}$ and exponential function e^z.

The two parameter Mittag-Leffler function ([2]) is defined as:

$$
E_{\beta,\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + \alpha)}, \quad \beta, \alpha > 0, \; z \in \mathbb{C}. \tag{1.5.2}
$$

The Mittag-Leffler function for matrix A ([2])is defined as:

$$
E_{\beta,\alpha}(A) = \sum_{n=0}^{\infty} \frac{A^n}{\Gamma(\beta n + \alpha)}, \quad \beta, \alpha > 0, \; A \in \mathbb{R}^{n \times n}. \tag{1.5.3}
$$

1.6 Laplace Transform

Laplace transform is an integral transform that converts a function of a real variable (often time) to a function of a complex variable (complex frequency). The Laplace transform of $f(t)$, for all $t > 0$,is the function $F(s)$, which is unilateral transform define by follow, where s is complex frequency parameter

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t) dt$$

Laplace transform of the power function, $f(t) = t^n$ is

$$\mathcal{L}(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$$

and conovlution of two function is

$$f * g = \int_0^t f(u)g(t-u)du = \int_0^t f(t-u)g(u)du$$

1.6.1 Laplace transform of the basic fractional operators

Lemma 1.6.1 ([2]). *The Laplace transform of fractional integral operator (1.2.3) is* $\mathcal{L}(I^\beta f(t)) = s^{-\beta} F(s)$

Proof. By definition (1.2.3)

$$\begin{aligned} I^\beta f(t) &= \frac{1}{\Gamma(\beta)} (t-s)^{\beta-1} f(s) ds \\ &= \frac{1}{\Gamma(\beta)} t^{\beta-1} * f(t) \end{aligned}$$

$$\begin{aligned} \mathcal{L}(I^\beta f(t)) = \mathcal{L}({}^c D^{-\beta} f(t)) &= \frac{1}{\Gamma(\beta)} \mathcal{L}(t^{\beta-1} * f(t)) \\ &= \frac{1}{\Gamma(\beta)} \frac{\Gamma(\beta)}{s^\beta} F(s) \qquad (1.6.1) \\ &= s^{-\beta} F(s) \end{aligned}$$

$\square$

Theorem 1.6.2 ([2]). *Laplace transform of fractional differential operator (1.2.5) is*

$$\mathcal{L}({}^{c}D^{\beta}f(t)) = s^{\beta}F(s) - \sum_{k=0}^{n-1} s^{\beta-k-1}f^{k}(0), \beta \in \mathbb{R}$$

Proof. For $n \in \mathbb{N}$

$$\mathcal{L}(\frac{d^n f(t)}{dt^n}) = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1}f^{k}(0). \tag{1.6.2}$$

Let us consider $n-1 < \beta \leq n$, $n \in \mathbb{N}$, $\beta \in \mathbb{R}$

$$\begin{aligned}
\mathcal{L}({}^{c}D^{\beta}f(t)) &= \mathcal{L}(I^{n-\beta}D^{n}f(t)) \\
&= \mathcal{L}(I^{n-\beta}g(t)) \quad \text{(where } g(t) = D^{n}f(t)\text{)} \\
&= s^{-(n-\beta)}\mathcal{L}(D^{n}f(s)) \\
&= s^{-(n-\beta)}\big(s^{n}F(s) - \sum_{k=0}^{n-1} s^{n-k-1}f^{k}(0)\big) \quad \text{(from (1.6.2))} \\
&= s^{\beta}F(s) - \sum_{k=0}^{n-1} s^{\beta-k-1}f^{k}(0)
\end{aligned}$$

$\square$

The Laplace transform of Caputo fractional derivative (1.2.5) is generalization of Laplace transform of integer order derivative.

1.6.2 Laplace transform of Mittag-Leffler Function

Lemma 1.6.3 ([2]). *Laplace transform of one variable Mittag-leffler function (1.5.1) is* $\mathcal{L}(E_\beta(-\lambda t^\beta)) = \frac{s^{\beta-1}}{s^\beta+\lambda}$

Proof.

$$\begin{aligned}
\mathcal{L}(E_\beta(-\lambda t^\beta)) &= \mathcal{L}\left\{\sum_{n=0}^{\infty}\frac{(-\lambda t^\beta)^n}{\Gamma(\beta n+1)}\right\} \\
&= \sum_{n=0}^{\infty}\mathcal{L}\left(\frac{(-\lambda t^\beta)^n}{\Gamma(\beta n+1)}\right) \\
&= \sum_{n=0}^{\infty}\frac{(-\lambda)^n}{\Gamma(\beta n+1)}\mathcal{L}(t^{\beta n}) \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n\lambda^n}{s^{\beta n+1}} \\
&= \frac{1}{s}\sum_{n=0}^{\infty}\frac{(-1)^n\lambda^n}{s^{\beta n}} \\
&= \frac{1}{s}\frac{s^\beta}{s^\beta+\lambda} \\
&= \frac{s^{\beta-1}}{s^\beta+\lambda}
\end{aligned}$$

□

Lemma 1.6.4 ([2]). *Laplace transform of two variable Mittag-leffler function (1.5.2) is* $\mathcal{L}(t^{\alpha-1}E_{\beta,\alpha}(-\lambda t^\beta) = \frac{s^{\beta-\alpha}}{s^\beta+\lambda}$

Proof.

$$\begin{aligned}
\mathcal{L}(t^{\alpha-1}E_{\beta,\alpha}(-\lambda t^\beta)) &= \mathcal{L}\left\{\sum_{n=0}^{\infty}\frac{t^{\alpha-1}(-\lambda t^\beta)^n}{\Gamma(\beta n+\alpha)}\right\} \\
&= \sum_{n=0}^{\infty}\mathcal{L}\left(t^{\alpha-1}\frac{(-\lambda t^\beta)^n}{\Gamma(\beta n+\alpha)}\right) \\
&= \sum_{n=0}^{\infty}\frac{(-\lambda)^n}{\Gamma(\beta n+\alpha)}\mathcal{L}(t^{\beta n+\alpha-1})
\end{aligned}$$

□

$$
\begin{aligned}
\mathcal{L}(t^{\alpha-1}E_{\beta,\alpha}(-\lambda t^{\beta})) &= \sum_{n=0}^{\infty}\frac{(-1)^n\lambda^n}{s^{\beta n+\alpha}} \\
&= \frac{1}{s^{\alpha}}\sum_{n=0}^{\infty}\frac{(-1)^n\lambda^n}{s^{\beta n}} \\
&= \frac{1}{s^{\alpha}}\frac{s^{\beta}}{s^{\beta}+\lambda} \\
&= \frac{s^{\beta-\alpha}}{s^{\beta}+\lambda}
\end{aligned}
$$

1.7 Fourier Transform and Inverse Fourier Transform

The mathematical technique that transforms a time function $f(t)$ to a frequency function $\mathcal{F}$ is known as Fourier transform and with synthesis equation for Fourier series, the inverse Fourier transform is also derived. Consider $f(t)$ such that it is define and absolutely integrate on real line then the Fourier transform is

$$
f(t) = \mathcal{F}^{-1}F(\omega) = \frac{1}{2\pi}\int_{\infty}^{\infty} e^{-i\omega t}F(\omega)d\omega
$$

convolution is given by

$$
\mathcal{F}(f(t) * g(t)) = F(\omega)G(\omega)
$$

and Fourier transform of n-order derivative of $f(t)$ is

$$
\mathcal{F}(f^{(n)}(t)) = (-i\omega)^n\mathcal{F}(f(t))
$$

Theorem 1.7.1. *Consider a function $f(t)$ such that whose fractional order Caputo derivative and Fourier transform exist then*

$$
F(^{c}D^{\beta}(f(t)) = (-i\omega)^{\beta}F(f(t))
$$

Proof. Define for $t \in \mathbb{R}$, $\beta \in \mathbb{R}$

$$f_+(t) = \begin{cases} \dfrac{t^{\beta-1}}{\Gamma(\beta)}, & t > 0 \\ 0, & t < 0 \end{cases}$$

For Laplace transform of $f_+(t)$.

$$\begin{aligned} \mathcal{L}(f_+(t)) &= \int_0^\infty e^{-st} f_+(t)dt \\ &= \int_0^\infty e^{-st} \frac{t^{\beta-1}}{\Gamma(\beta)} dt \\ &= \frac{\Gamma(\beta)}{s^\beta} \end{aligned}$$

This means

$$\begin{aligned} \int_0^\infty e^{-st} f_+(t)dt &= \frac{\Gamma(\beta)}{s^\beta} \\ \frac{1}{\Gamma(\beta)} \int_0^\infty e^{-st} t^{\beta-1} dt &= s^{-\beta} \\ \int_0^\infty e^{-st} \frac{t^{\beta-1}}{\Gamma(\beta)} dt &= s^{-\beta} \end{aligned}$$

Consider $s = -i\omega$, $\omega \in \mathbb{R}$. Therefore by Dirichlet theorem, the integral converges if $0 < \beta < 1$ and then Fourier transform of $f_+(t)$ is

$$F(f_+(t)) = (-i\omega)^\beta$$

Now,

$$\begin{aligned} {}^cD^{-\beta}(f(t)) = I^\beta f(t) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)ds \\ &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)ds \\ &= \frac{t^{\beta-1}}{\Gamma(\beta)} * f(t) \end{aligned}$$

Therefore,

$$\begin{aligned}
F({}^cD^{-\beta}(f(t)) &= F(\frac{t^{\beta-1}}{\Gamma(\beta)} * f(t)) \\
&= F(\frac{t^{\beta-1}}{\Gamma(\beta)})F(f(t)) \\
&= (-i\omega)^{-\beta}F(f(t))
\end{aligned}$$

for $n-1 < \beta < n$, $n \in N$

$$\begin{aligned}
{}^cD^{\beta}f(t) &= \frac{1}{\Gamma(n-\beta)}\int_0^t (t-s)^{n-\beta-1}\frac{d^n f(s)}{dt^n}ds \\
F({}^cD^{\beta}f(t)) &= (-i\omega)^{-(n-\beta)}F(f^n(t)) \\
&= (-i\omega)^{-(n-\beta)}(-i\omega)^n F(f(t)) \\
&= (-i\omega)^{\beta}F(f(t))
\end{aligned}$$

□

Fourier transform is powerful tool for frequency domain analysis of linear dynamical system. Fourier transform of fractional derivatives has been used by H.Beyer and S. Kemplfe on the fractional calculus mode of viscoelastic behavior [36], and for analyzing the oscillation equation with a fractional order damping term $y''(t) + a\,{}^cD^{\beta}y(t) + by(t) = f(t)$ by S. Kemptel and L. Gaul [37]. For constructing global solutions of linear fractional differential equations with constant coefficients and implicitly by R. R. Nigmatullin arid Ya. E. Ryabov [38] for studying relaxation process in insulators.

1.8 Solution Representation

Consider the fractional differential equations in the sense of Caputo derivative of the form,

$$\begin{aligned}
{}^cD^{\beta}x(t) &= Ax(t) + f(t),\ 0 < \beta \leq 1, t \in [0, \infty), \qquad (1.8.1) \\
x(0) &= x_0
\end{aligned}$$

where A is $n \times n$ constant matrix and $x(t), f(t) \in \mathbb{R}^n$. Derivation of the solution of (1.8.1) by Laplace transform is,

$$x(t) = E_\beta(At^\beta)x_0 + \int_0^t (t-s)^\beta E_{\beta,\beta}(A(t-s)^\beta)f(s)ds$$

Consider the fractional differential equation in the sense of Riemann-Liouville,

$$\begin{aligned} &D^\beta x(t) = Ax(t) + f(t),\ 0 < \beta \leq 1, t \in [0, \infty) \\ &D^{\beta-1}x(t)|_{t=0} = x_0 \end{aligned} \tag{1.8.2}$$

where A is $n \times n$ constant matrix, $x(t), f(t) \in \mathbb{R}^n$ and D^β is Riemann-Liouville fractional derivative operator. Then the solution using Laplace transform is given by

$$x(t) = t^{\beta-1}E_{\beta,\beta}(At^\beta)x_0 + \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(A(t-s)^\beta)f(s)ds \tag{1.8.3}$$

1.9 Application of fractional derivative

Fractional derivative involves an integration and as integration defined on an interval so it is a non local operator. It can be also used for modeling systems with memory that is, if the system related to time, fractional derivative of a function $f(t)$ at some $t = t_1$ requires all past history means $f(t)$ is form $t = 0$ to $t = t_1$ and if system related to Space , fractional derivative of a function $f(x)$ at $x = x_1$ requires all non local $f(x)$ value. So fractional derivative can be used for modeling distributed parameter system.

Application of fractional calculus to real world problem is only four decades old even it has long history because due to high complexity and lack of physical and geometric interpretation of fractional derivative.

Some of applications of fractional differential equation are discussed here, as an illustration

1. Tautochrone Problem [2]

   The problem was to determine the shape of curve passing through the origin in a vertical frictionless plane along with a particle of mass 'm' fall in a time which is independent of the stating point. The integral equation

   $$\sqrt{2g}T = \int_0^{\eta} (\eta - y)^{-\frac{1}{2}} f'(y)dy \tag{1.9.1}$$

   where g is the acceleration due to gravity, (x_0, η) is initial position, $s = f(y)$ is equation of sliding curve, Able converting (1.9.1) to equivalent fractional integral equation

   $$\sqrt{2g}T = \Gamma(\frac{1}{2})^c D^{-\frac{1}{2})} f'(\eta)$$

2. Viscoelasticity [101]

   The most extensive application of fractional differential and integral operator is viscoelasticity field.

   The model of elasticity is

   $$E = \frac{\sigma(t)}{\epsilon(t)}$$

   and for Newtonian fluids (viscous liquid).

   $$\eta = \frac{\sigma(t)}{\frac{d\epsilon(t)}{dt}}$$

   where $\sigma(t)$ is stress and $\epsilon(t)$ is strain and E is modulus of the elasticity and η is velocity of material. Both equations can be rewrite in

   $$\sigma(t) = v\, {}^cD^{\beta}\, \epsilon(t)$$

   For $\beta = 0$ it will represent Hooke's law of elastic solid and for $\beta = 1$ it will represent viscous liquid. means it gives the sense of Viscoelasticity of the material which have behaviour in between elastic solid and viscous liquid. George William and Scott Blair introduced fractional derivative of stress $\sigma(t)$ and strain $\epsilon(t)$ given by fractional differential equations.

$$^{c}D^{\beta}\sigma(t) = \frac{\Gamma(1-\beta)}{\Gamma(1-2\beta)} t^{\beta}\sigma(t)$$

$$^{c}D^{\beta}\epsilon(t) = \Gamma(1+\beta)t^{\beta}\epsilon(t)$$

3. The fractional damped simple harmonic oscillation [102]

   The equation for damped simple harmonic oscillation with two initial value conditions is

$$y''(t) + by'(t) + w^2 y(t) = f(t), \qquad y(0) = c_0,\ y'(0) = c_1 \tag{1.9.2}$$

   in equation (1.9.2) replacing $y'(t)$ by fractional derivative of order β is described by

$$y''(t) + b\,{}^{c}D^{\beta}y(t) + w^2 y(t) = f(t), \qquad y(0) = c_0,\ y'(0) = c_1 \tag{1.9.3}$$

   where b, w_0, c_0 and c_1 are constant.

4. Anomalous Diffusion [103]

   Normaly in Fickian diffusion, the flow of particle is from high concentration to low concentration and concentration is given by Gaussian distribution. It has asymptotical mean-square displacement, is a linear function of time , $< x^2(t) > \propto t$.

   The model of diffusion equation is given by

$$\frac{\partial \phi(x,t)}{\partial t} = D\frac{\partial^2 \phi(x,t)}{\partial x^2} \tag{1.9.4}$$

But Some process are an exception of Fickian diffusion like Photocopy machine and laser printer. In these type of systems moment of holes and electrons in the semiconductor inside them is not a normal Gaussian diffusion, such type of diffusion is known as Anomalous Diffusion. Here the asymptotical mean-square displacement is not a linear function of time,

$< x^2(t) > \propto t^\beta$, $\beta \neq 1$

For $\beta < 1$, it gives sub diffusion means, the movement of the particles is slow. and for $\beta > 1$, it gives super diffusion, means the movement of the particles is fast. The Fractional diffusion equation model is given by

$$\frac{\partial^\beta \phi(x,t)}{\partial t} = D\frac{\partial^2 \phi(x,t)}{\partial x^2} \tag{1.9.5}$$

Movement of contaminants in groundwater, Spread of pollutants from environmental accident and Diffusion of Proteins across cell membranes are examples of sub diffusion. While Flight of sea birds (Albatrosses), movement of spider monkey and Spread of pollutants in the sea are examples of super diffusion.

5. Modeling of inductor [104]

   As all the iron-core inductors have a core loss, this lose is either neglected in the the convectional 'integer order' model or by arranging the external circuit in the model, the core loss will be considered. But then the resultant model is very bulky and it is not trustworthy. So Schafer and Kruger [104] propose the fractional order model of inductor. This model is very compact, easy to handle and relivable. The voltage-current relationship is given by

$$V(t) = L_\beta \, {}^cD^\beta \, i(t) \tag{1.9.6}$$

   parameter β and L_β are the measures of core loss in the inductor.

6. Mathematical analysis of dengue fever [107]

In 2009, at Cape Verde islands there were dengue fever outbreak, considering real statistical data the classical mathematical model form by Hiroshi Nishiura [105], it consists liner differential equation with five independent functions $\mathcal{A}_h(t)$ as susceptible humans, $\mathcal{A}_m(t)$ as susceptible female mosquitoes. $\mathcal{B}_h(t)$ as infected humans, $\mathcal{B}_m(t)$ as infected female mosquitoes, $\mathcal{C}_h(t)$ as humans after recovery, the mathematical model takes the following form

$$\begin{aligned}
\mathcal{A}'_h(t) &= \mu_h(N_h(t) - \mathcal{A}_h(t)) - \frac{\alpha_h b}{N_h(t)+m}\mathcal{A}_h(t)\mathcal{B}_m(t)\\
\mathcal{B}'_h(t) &= \frac{\alpha_h b}{N_h(t)+m}\mathcal{A}_h(t)\mathcal{B}_m(t) - (\mu_h+\gamma)\mathcal{B}_h(t)\\
\mathcal{C}'_h(t) &= \gamma\mathcal{B}_h(t) + \mu_h\mathcal{C}_h(t)\\
\mathcal{A}'_m(t) &= A - \frac{\alpha_h b}{N_h(t)+m}\mathcal{A}_h(t)\mathcal{B}_m(t) - \mu_m\mathcal{A}_m(t)\\
\mathcal{B}'_m(t) &= \frac{\alpha_m b}{N_h(t)+m}\mathcal{A}_m(t)\mathcal{B}_h(t) - \mu_m\mathcal{B}_m(t)
\end{aligned} \tag{1.9.7}$$

with $N_h(t) = \mathcal{A}_h(t) + \mathcal{B}_h(t) + \mathcal{C}_h(t)$ and $N_m(t) = \mathcal{A}_m(t) + \mathcal{B}_m(t)$ for all t. Further be noted that $N'_h(t) = \mathcal{A}'_h(t) + \mathcal{B}'_h(t) + \mathcal{C}'_h(t)$=0 , $N'_m(t) = \mathcal{A}'_m(t) + \mathcal{B}'_m(t) = A - \mu_m N_m(t)$ with $A = \mu_m N_m(t)$ is constant and $m = 0$ caused of for mosquitoes human blood being the only one source.

For physical meaning of the model the rest of biological parameters and initial value conditions is provided in [106]. and hence the model (1.9.7) is reduce to system of three equations after removing third and fifth equations in it.

S. Qureshi and A. Atangana [107] convert the model in to fractional differential equations in Caputo sense as

$$\begin{aligned}
{}^{c}D^{\beta_h}\mathcal{A}_h(t) &= \mu_h^{\beta_h}(N_h(t)-\mathcal{A}_h(t)) - \frac{\alpha_h b^{\beta_h}}{N_h(t)}\mathcal{A}_h(t)\mathcal{B}_m(t) \\
{}^{c}D^{\beta_h}\mathcal{B}_h(t) &= \frac{\alpha_h b^{\beta_h}}{N_h(t)+m}\mathcal{A}_h(t)\mathcal{B}_m(t) - (\mu_h^{\beta_h}+\gamma^{\beta_h})\mathcal{B}_h(t) \\
{}^{c}D^{\beta_m}\mathcal{A}_m(t) &= \mu_h^{\beta_m}(N_m(t)-\mathcal{A}_m(t)) - \frac{\alpha_m b^{\beta_m}}{N_m(t)}\mathcal{A}_m(t)\mathcal{B}_h(t)
\end{aligned} \tag{1.9.8}$$

Then by analysis on it they conclude that, fractional order mathematical model perform better than integer order model related to the epidemic of dengue virus.

1.10 Research work

In this thesis, qualitative property like existence and uniqueness of the solution of fractional order differential equations are studied. And fractional order differential equation with and without impulses governing physical phenomena is also considered.
This thesis consist of Six chapters:

1.10.1 Chapter One

The impulsive fractional quasiliner integro-differential equation

$$\begin{aligned}
{}^{c}D^{\beta}x(t) &= A(t,x)x(t) + f(t,x(t),Tx(t),Sx(t)) \quad t \neq t_k,\ k=1,2,\cdots,p \\
\Delta x(t_k) &= I_k(x(t_k)), \quad t=t_k,\ k=1,2,\cdots,p
\end{aligned}$$

Where, $A(t,x)$ is bounded quasi-linear operator on Banach space X,
$f:[0,T_0]\times X\times X\times X\to X$, $T,S:X\to X$ are defined by $Tx(t)=\int_0^t h(t,s,x(s))ds$ and $Sx(t)=\int_0^{T_0} k(t,s,x(s))ds$; with $h:D_0\times X\to X,\ D_0=\{(t,s);0\le s\le t\le T_0\}$ and $k:D_1\times X\to X,\ D_1=\{(t,s);0\le t,s\le T_0\}$ are continuous, with local

condition $x(0) = x_0$ and nonlocal condition $x(0) = x_0 - g(x)$ over the interval $[0, T_0]$, is considered in this chapter. Existence and uniqueness of mild solution of the problem is established . Inclusion of Fredholm integral operator S in the equation is more relevant in modelling of many physical phenomena arises in field of viscoelasticity. An example is added to illustrate the efficacy of the method.

1.10.2 Chapter Two

The work discussed in chapter one is extended in chapter two by adding a delay condition on it and derived sufficient conditions for existence and uniqueness of mild solution of the integro-differential equations:

$$\begin{aligned}
{}^cD^\beta x(t) &= A(t,x)x(t) + f(t, x(\phi(t)), Tx(t), Sx(t)) \quad t \neq t_k,\ k = 1, 2, \cdots, p \\
\Delta x(t_k) &= I_k(x(t_k)), \quad t = t_k,\ k = 1, 2, \cdots, p \\
x(0) &= x_0 - g(x),
\end{aligned}$$

over the interval $[0, T_0]$ in a Banach space X. Where, $A(t,x)$ is bounded quasi linear operator on X and $f : [0, T_0] \times X \times X \times X \to X$, $T, S : X \to X$ are defined by $Tx(t) = \int_0^t h(t, s, x(\psi(s)))ds$ and $Sx(t) = \int_0^{T_0} k(t, s, x(\xi(s)))ds$;
$h : D_0 \times X \to X$, $D_0 = \{(t, s); 0 \leq s \leq t \leq T_0\}$ and
$k : D_1 \times X \to X$, $D_1 = \{(t, s); 0 \leq t, s \leq T_0\}$ are the operators satisfying condition of the hypotheses.

1.10.3 Chapter Three

The sufficient conditions for existence and uniqueness of mild and classical solution of fractional order impulsive integro-differential equations of the following form is established in this chapter. And also derived conditions in which mild and classical solution are congruence.

$$
\begin{aligned}
{}^cD^\beta x(t) &= Ax(t) + f(t, x(t), Tx(t), Sx(t)) \quad t \neq t_k,\ k = 1, 2, \cdots, p \\
\Delta x(t_k) &= I_k(x(t_k)), \quad t = t_k,\ k = 1, 2, \cdots, p \\
x(0) &= x_0
\end{aligned}
$$

over the interval $[0, T_0]$ in a Banach space X. Here, A is bounded linear operator on X and $f : [0, T_0] \times X \times X \times X \to X$,$T, S : X \to X$ are defined by $Tx(t) = \int_0^t h(t, s, x(s))ds$ and $Sx(t) = \int_0^{T_0} k(t, s, x(s))ds$,
where $h : D_0 \times X \to X,\ D_0 = \{(t, s); 0 \leq s \leq t \leq T_0\}$ and $k : D_1 \times X \to X$,
$D_1 = \{(t, s); 0 \leq t, s \leq T_0\}$ are the operators satisfying condition of the hypotheses.

1.10.4 Chapter Four

This chapter, includes existence and uniqueness of mild and classical solution of generalized fractional order impulsive evolution equations

$$
\begin{aligned}
{}^cD^\beta x(t) &= Ax(t) + g_k(t, x(t)) \quad t \in (t_{k-1}, t_k)\ k = 1, 2, \cdots, p \\
\Delta x(t_k) &= I_k(x(t_k)), \quad t = t_k,\ k = 1, 2, \cdots, p \\
x(0) &= x_0
\end{aligned}
$$

in which perturbing force is different after every impulse over the interval $[0, T_0]$ on a Banach space X. A is bounded linear operator on X and $I_k : [0, T_0] \times X \to X$ for $k = 1, 2, \cdots, p+1$. Also established conditions in which mild and classical solutions are congruent.

1.10.5 Chapter Five

Existence of mild solution for non-instantaneous impulses fractional order integro-differential equations with local and nonlocal conditions in Banach space X is established in this paper.

$$
\begin{aligned}
{}^cD^\beta x(t) &= Ax(t) + f\left(t, x(t), \int_0^t a(t,s,x(s))ds\right),\ t \in [s_k, t_{k+1}),\ k = 1,2,\cdots,p \\
x(t) &= I_k(k, x(t)),\ t \in [t_k, s_k)
\end{aligned}
$$

with local condition $x(0) = x_0$ and nonlocal condition $x(0) = x_0 + h(x)$ over the interval $[0, T_0]$ in X. Here $A : X \to X$ is linear operator, $Kx = \int_0^t a(t,s,x(s))ds$ is nonlinear Volterra integral oprator on X, $f : [0,T_0] \times X \times X \to X$ is nonlinear function and $I_k : [0,T_0] \times X$ are set of nonlinear functions applied in the interval $[t_k, s_k)$ for all $i = 1, 2, \cdots, p$.

1.10.6 Chapter Six

This chapter established sufficient conditions for the mild solution of the fractional hybrid equations:

$$
\begin{aligned}
{}^cD^\beta \big[x(t) - f(t, x(t))\big] &= g(t, x(t)) \\
x(0) &= h(x)
\end{aligned}
$$

with nonlocal condition over the interval $[0, T_0]$ on partially ordered Banach space X. The nonlocal condition in this equation is more relevant in many physical phenomena in physics.