Chapter 2

Existence and uniqueness of non-local Cauchy problem for fractional differential equation on Banach space

The impulsive fractional quasiliner integro-differential equation

$$\begin{aligned} {}^cD^\beta x(t) &= A(t,x)x(t) + f(t,x(t),Tx(t),Sx(t)) \quad t \neq t_k,\ k = 1,2,\cdots,p \\ \Delta x(t_k) &= I_k(x(t_k)), \quad t = t_k,\ k = 1,2,\cdots,p \end{aligned}$$

Where, $A(t,x)$ is bounded quasi-linear operator on Banach space X and $f : [0,T_0] \times X \times X \times X \to X$, $T, S : X \to X$ are defined by $Tx(t) = \int_0^t h(t,s,x(s))ds$ and $Sx(t) = \int_0^{T_0} k(t,s,x(s))ds$, $h : D_0 \times X \to X,\ D_0 = \{(t,s); 0 \leq s \leq t \leq T_0\}$ and $k : D_1 \times X \to X,\ D_1 = \{(t,s); 0 \leq t, s \leq T_0\}$ are continuous, with local condition $x(0) = x_0$ and nonlocal condition $x(0) = x_0 - g(x)$ over the interval $[0,T_0]$ in X, is considered in this chapter. Existence and uniqueness of mild solution of the problem is established . Inclusion of Fredholm integral operator S in the equation is more relavent in modelling of many physical phenomena arises in field of viscoelasticity.

An example is added to illustrate the efficacy of the method.

2.1 Introduction

There are various problems in science and engineering like seepage flow in porous media [7], anomalous diffusion [8, 9], and transport [10], the nonlinear oscillations of earthquake [11], fluid dynamics traffic model [12] etc are well modeled in fractional differential equations. This is because of its non local property [14] which mean, that the next state of the system depends upon its entire historical states. Due to this nonlocal property fractional dynamical systems are considered as an alternative model for highly nonlinear integer order systems [13]. Using fixed point theory existence and uniqueness of fractional dynamical semilinear and nonlinear systems have been studied by Delbosco and Rodino [11], Cheng and Guozhu [14] and El-Borai [15]. Byszewski [16] initiated study of existence of nonlocal Cauchy problem using fixed point theorem and extended it by various researchers [17]. The study of existence and uniqueness of mild solution of nonlocal Cauchy problem of fractional dynamical systems with fixed point theory was initiated by N' Guerekata [18] followed by Balachandran and Park [19].

On the other hand, rapid development of impulsive differential equations played very important role in modeling of many problems of population dynamics, chemical technology and biotechnology [20]. This motivates many researchers to study existence and uniqueness solutions of the impulsive differential equations [39]. Determination of sufficient conditions for existence and uniqueness of mild solution of impulsive fractional Cauchy problem with classical conditions with the use of fixed point theory and semigroup theory given by Benchohra and Slimani [40], Mophou [41], Ravichandran and Arjunan [42]. Sufficient conditions for the mild solution for semilinear impulsive fractional nonlocal Cauchy problem using various fixed point theorem was studied by Benchohra and Slimani [40]. Balachandran et. al. [21, 22] and Gao et. al. [43] studied existence and uniqueness of mild solution of impulsive fractional nonlocal quasilinear Cauchy problem with nonlocal conditions. Motivated by the work of Balachandran et. al. [21].

2.2 Notations

(N1) X = Banach space.

(N2) $\mathbb{R}_+ = [0, \infty)$

(N3) $C([0,T_0],X) = \{x : [0,T_0] \to X/x \; is \; continuous\}$ with norm $||x|| = sup_t||x(t)||$

(N4) $PC([0,T_0],X) = \{x : [0,T_0] \to X; x \in C([t_{k-1},t_k],X), \; and \; x(t_k^-) \; and \; x(t_k^+) \; exist,$ $k = 1,2,\cdots,p \; with \; x(t_k^-) = x(t_k)\}$ with norm $||x||_{PC} = sup_{t\in[0,T_0]}||x(t)||$

(N5) $AC([0,T_0],X) = \{x : [0,T_0] \to X/x \; is \; absolutely \; continuous\}$ with norm $||x|| = sup_t||x(t)||$

(N6) $B(X) = \{A : X \to X/A \; is \; bouneded \; and \; linear\}$ with norm $||A||_{B(X)} = sup\{||A(y)||; y \in X, \; ||y|| \leq 1\}$

Using these definitions and properties, sufficient conditions for existence and uniqueness of solutions are derived as follows:

2.3 Equation with classical condition

This section deals with the study of the existence and uniqueness of the solution of impulsive fractional differential equation with classical condition of the form

$$\begin{aligned} {}^cD^\beta x(t) &= A(t,x)x(t) + f(t,x(t),Tx(t),Sx(t)) \quad t \neq t_k, \; k = 1,2,\cdots,p \\ \Delta x(t_k) &= I_k(x(t_k)), \quad t = t_k, \; k = 1,2,\cdots,p \\ x(0) &= x_0 \end{aligned} \tag{2.3.1}$$

over the interval $[0,T_0]$, where $A(t,x)$ is bounded linear operator on Banach space X and $f : [0,T_0] \times X \times X \times X \to X$, $T,S : X \to X$ are defined by $Tx(t) = \int_0^t h(t,s,x(s))ds$ and $Sx(t) = \int_0^{T_0} k(t,s,x(s))ds$; where $h : D_0 \times X \to X, \; with$ $D_0 = \{(t,s); 0 \leq s \leq t \leq T_0\}$ and $k : D_1 \times X \to X, \; D_1 = \{(t,s); 0 \leq t,s \leq T_0\}$

are continuous. This type of nonlinear equations arise in many physical situations, like mathematical problems concerned with heat flow in materials with viscoelastic problems [44].
Applying fractional integral operator both side of the equation (2.3.1), we get

$$\begin{aligned} x(t) = x_0 + & \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} A(s, x(s)) x(s) ds \\ & + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t - s)^{\beta-1} A(s, x(s)) ds \\ & + \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} f(s, x(s), Tx(s), Sx(s)) ds \\ & + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t - s)^{\beta-1} f(s, x(s), Tx(s), Sx(s)) ds + \sum_{0<t_k<t} Ix(t_k^-). \end{aligned}$$

The following conditions are assumed to show the existence and uniqueness of the solution (2.3.1).

(H1) $A : [0, T_0] \times X \to X$ is continuous bounded linear operator and there exists a positive constant M, such that $||A(t, x) - A(t, y)||_{B(X)} \leq M||x - y||$, for all $x, y \in X$.

(H2) $f : [0, T_0] \times X \times X \times X \to X$ is continuous and there exists positive constants L_1, L_2 and L_3, such that $||f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3)||$ $\leq L_1||x_1 - y_1|| + L_2||x_2 - y_2|| + L_3||x_3 - y_3||$ for all x_1, x_2, x_3, y_1, y_2 and y_3 in X.

(H3) $h : D_0 \times X \to X$ and $k : D_1 \times X \to X$ are continuous and there exists positive constants H and K, such that $||h(t, s, x) - h(t, s, y)|| \leq H||x - y||$ and $||k(t, s, x) - k(t, s, y)|| \leq K||x - y||$ for all x and y in X.

(H4) The functions $I_k : X \to X$ are continuous and there exist positive constants I_k^* for all $k = 1, 2, \cdots, p$, such that $||I_k x - I_k y|| \leq I_k^*||x - y||$ for all x and y in X.

Consider, $\gamma = \frac{T_0^\beta}{\Gamma\beta+1}$ and further assume that,

(H5) $q = \gamma\big[(p+1)[M + L_1 + T_0HL_2 + T_0KL_3]\big] + \sum I_k^*$, *with* $q < 1$

Define $F : PC([0,T_0],X) \to PC([0,T_0],X)$ by

$$\begin{aligned}
Fx(t) = x_0 &+ \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} A(s,x(s))x(s)ds \\
&+ \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} A(s,x(s))x(s)ds \\
&+ \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} f(s,x(s),Tx(s),Sx(s))ds \\
&+ \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} f(s,x(s),Tx(s),Sx(s))ds + \sum_{0<t_k<t} I_k x(t_k^-).
\end{aligned} \tag{2.3.2}$$

Then equation (2.3.1) has unique solution if F defined by (2.3.2) has unique fixed point. This means F is well defined bounded operator on $PC([0,T_0],X)$ and F is contraction [23].

Lemma 2.3.1. *If the operators* A, f, T, S *and* I_k *for* $k = 1, 2, \cdots, p$ *are continuous then* F *is bounded operator on* $PC([0,T_0],X)$.

Proof. Let a sequence $\{x_n\}$ be converges to x in $PC([0,T_0],X)$.
Therefore $||x_n - x|| \to 0$ as $n \to \infty$. Consider,

$$\begin{aligned}
||Fx_n - Fx||_{PC} \leq &\frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} ||A(s,x_n(s))x_n(s) - A(s,x(s))x(s)||ds \\
&+ \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} ||A(s,x_n(s))x_n(s) - A(s,x(s))x(s)||ds \\
&+ \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} ||f(s,x_n(s),Tx_n(s),Sx_n(s)) \\
&- f(s,x(s),Tx(s),Sx(s))||ds \\
&+ \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} ||f(s,x_n(s),Tx_n(s),Sx_n(s)) - f(s,x(s),Tx(s),Sx(s))||ds \\
&+ \sum_{0<t_k<t} ||I_k x_n(t_k^-) - I_k x(t_k^-)||
\end{aligned}$$

Assuming the continuity of A, f, T, S and I_k for $k = 1, 2, \cdots, p$ the right side of above expression tends to zero as $n \to \infty$. Therefore F is continuous on $PC([0, T_0], X)$ and hence F is bounded. □

For derivation of sufficient conditions for existence and uniqueness of the solution of equation (2.3.1).

Theorem 2.3.2. *If the hypotheses (H1)-(H5) are satisfied, then the fractional quasi-linear impulsive integro-differential equation (2.3.1) has unique solution in* $PC([0, T_0], X)$ *for* $0 < \beta \leq 1$.

Proof. To show equation (2.3.1) has unique solution it is sufficient to show F defined (2.3.2) is contraction. Let x and y in $PC([0, T_0], X)$ and consider,

$$\begin{aligned}
||Fx - Fy||_{PC} \leq & \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} ||A(s, x(s))x(s) - A(s, y(s))y(s)|| ds \\
& + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t - s)^{\beta-1} ||A(s, x(s))x(s) - A(s, y(s))y(s)|| ds \\
& + \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \Big\{ \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} ||f(s, x(s), Tx(s), Sx(s)) \\
& - f(s, y(s), Ty(s), Sy(s))|| ds \Big\} \\
& + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t - s)^{\beta-1} ||f(s, x(s), Tx(s), Sx(s)) - f(s, y(s), Ty(s), Sy(s))|| ds \\
& + \sum_{0<t_k<t} ||I_k x(t_k^-) - I_k y(t_k^-)||
\end{aligned}$$

By hypotheses (H1)-(H4),

$$\begin{aligned}
||Fx - Fy||_{PC} &\leq \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta-1} M||x-y||ds + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} M||x-y||ds \\
&+ \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta-1} \{L_1 + T_0HL_2 + T_0KL_3\}||x-y||ds \\
&+ \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} ds \{L_1 + T_0HL_2 + T_0KL_3\}||x-y||ds + \sum_{0<t_k<t} I_k^*||x-y|| \\
&\leq \left\{ \frac{T_0^\beta}{\Gamma(\beta+1)} \big[(p+1)[M + L_1 + T_0HL_2 + T_0KL_3]\big] + \sum I_k^* \right\} ||x-y|| \\
&= \left\{ \gamma \big[(p+1)[M + L_1 + T_0HL_2 + T_0KL_3]\big] + \sum I_k^* \right\} ||x-y||
\end{aligned}$$

And by hypotheses (H5), $||Fx - Fy||_{PC} \leq q||x-y||$ with $q < 1$. Hence by Banach fixed point theorem [97] the equation (2.3.1) has unique solution. □

2.4 Equation with non-local condition

This section determines sufficient conditions for existence of mild solution of fractional impulsive system with non-local condition of the form

$$\begin{aligned}
{}^cD^\beta x(t) &= A(t,x)x(t) + f(t, x(t), Tx(t), Sx(t)) \quad t \neq t_k, \ k = 1, 2, \cdots, p \\
\Delta x(t_k) &= I_k(x(t_k)), \quad t = t_k, \ k = 1, 2, \cdots, p \\
x(0) &= x_0 - g(x)
\end{aligned} \tag{2.4.1}$$

over the interval $[0, T_0]$, where $A(t,x)$ is bounded linear operator on Banach space X and $f : [0, T_0] \times X \times X \times X \to X$, $T, S : X \to X$ are defined by $Tx(t) = \int_0^t h(t, s, x(s))ds$ and $Sx(t) = \int_0^{T_0} k(t, s, x(s))ds$; where $h : D_0 \times X \to X, \ with$ $D_0 = \{(t,s); 0 \leq s \leq t \leq T_0\}$ and $k : D_1 \times X \to X, \ D_1 = \{(t,s); 0 \leq t, s \leq T_0\}$ are continuous and $g : X \to X$ is given function.

The equivalent integral equation of (2.4.1) is given by

$$
\begin{aligned}
x(t) = x_0 - g(x) &+ \frac{1}{\Gamma(\beta)}\Big\{ \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} A(s, x(s))x(s)ds \\
&+ \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} A(s, x(s))ds \Big\} \\
&+ \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} f(s, x(s), Tx(s), Sx(s))ds \\
&+ \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} f(s, x(s), Tx(s), Sx(s))ds + \sum_{0<t_k<t} Ix(t_k^-)
\end{aligned}
$$

The following hypotheses are to be assumed,

(H6) $g : X \to X$ is continuous and there exist a positive constant g^*, such that $||g(x) - g(y)|| \leq g^*||x - y||$ for each x and y in X.

(H7) $q^* = g^* + \gamma\big[(p+1)[M + L_1 + T_0HL_2 + T_0KL_3]\big] + \sum I_k^* < 1.$

Define $G : PC([0, T_0], X) \to PC([0, T_0], X)$ by

$$
\begin{aligned}
Gx(t) = x_0 - g(x) &+ \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} A(s, x(s))x(s)ds \\
&+ \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} A(s, x(s))x(s)ds \\
&+ \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} f(s, x(s), Tx(s), Sx(s))ds \\
&+ \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} f(s, x(s), Tx(s), Sx(s))ds + \sum_{0<t_k<t} I_k x(t_k^-)
\end{aligned}
\tag{2.4.2}
$$

Lemma 2.4.1. *If the operators A, f, T, S and I_k for $k = 1, 2, \cdots, p$ are continuous then G is bounded operator on $PC([0, T_0], X)$.*

Proof. Let a sequence $\{x_n\}$ be converges to x in $PC([0, T_0], X)$.

Therefore $||x_n - x|| \to 0$ as $n \to \infty$. Consider,

$$
\begin{aligned}
||Gx_n - Gx||_{PC} \leq\ & ||g(x_n(s)) - g(x(s))|| \\
& + \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} ||A(s, x_n(s))x_n(s) - A(s, x(s))x(s)||ds \\
& + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t - s)^{\beta-1} ||A(s, x_n(s))x_n(s) - A(s, x(s))x(s)||ds \\
& + \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \Big\{ \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} ||f(s, x_n(s), Tx_n(s), Sx_n(s)) \\
& - f(s, x(s), Tx(s), Sx(s))||ds \Big\} \\
& + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} \Big\{ (t - s)^{\beta-1} ||f(s, x_n(s), Tx_n(s), Sx_n(s)) \\
& - f(s, x(s), Tx(s), Sx(s))||ds \Big\} \\
& + \sum_{0<t_k<t} ||I_k x_n(t_k^-) - I_k x(t_k^-)||
\end{aligned}
$$

By assumption of continuity of A, f, T, S, g and I_k for $k = 1, 2, \cdots, p$ the right side of above expression tends to zero as $n \to \infty$. Therefore G is continuous on $PC([0, T_0], X)$ and hence G is bounded. □

The sufficient conditions are derived as under for existence and uniqueness of the solution of equation (2.4.1).

Theorem 2.4.2. *If the hypotheses (H1)-(H4) and (H6)-(H7) are satisfied, then the fractional quasi-linear impulsive integro-differential equation (2.4.1) has unique solution in* $PC([0, T_0], X)$ *for* $0 < \beta \leq 1$.

Proof. To show equation (2.4.1) has unique solution it is sufficient to show G defined in (2.4.2) is contraction. Let x and y in $PC([0,T_0],X)$ and consider,

$$\begin{aligned}
||Gx-Gy||_{PC} \le\ & ||g(x)-g(y)|| \\
&+ \frac{1}{\Gamma(\beta)}\sum_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-s)^{\beta-1}||A(s,x(s))x(s)-A(s,y(s))y(s)||ds \\
&+ \frac{1}{\Gamma(\beta)}\int_{t_k}^{t}(t-s)^{\beta-1}||A(s,x(s))x(s)-A(s,y(s))y(s)||ds \\
&+ \frac{1}{\Gamma(\beta)}\sum_{0<t_k<t}\Big\{\int_{t_k-1}^{t_k}(t_k-s)^{\beta-1}||f(s,x(s),Tx(s),Sx(s)) \\
&- f(s,y(s),Ty(s),Sy(s))||ds\Big\} \\
&+ \frac{1}{\Gamma(\beta)}\int_{t_k}^{t}\Big\{(t-s)^{\beta-1}||f(s,x(s),Tx(s),Sx(s)) \\
&- f(s,y(s),Ty(s),Sy(s))||ds\Big\} \\
&+ \sum_{0<t_k<t}||I_kx(t_k^-)-I_ky(t_k^-)||
\end{aligned}$$

From (H1)-(H4) and (H6) the result is,

$$\begin{aligned}
||Gx-Gy||_{PC} \le\ & g^*||x-y|| + \frac{1}{\Gamma(\beta)}\sum_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-s)^{\beta-1}M||x-y||ds \\
&+ \frac{1}{\Gamma(\beta)}\int_{t_k}^{t}(t-s)^{\beta-1}M||x-y||ds \\
&+ \frac{1}{\Gamma(\beta)}\sum_{0<t_k<t}\int_{t_k-1}^{t_k}(t_k-s)^{\beta-1}\big\{L_1+T_0HL_2+T_0KL_3\big\}||x-y||ds \\
&+ \frac{1}{\Gamma(\beta)}\int_{t_k}^{t}(t-s)^{\beta-1}ds\big\{L_1+T_0HL_2+T_0KL_3\big\}||x-y||ds + \sum_{0<t_k<t}I_k^*||x-y|| \\
\le\ & \Big\{g^*\frac{T^\beta}{\Gamma(\beta+1)}\big[(p+1)[M+L_1+T_0HL_2+T_0KL_3]\big] + \sum I_k^*\Big\}||x-y|| \\
=\ & \Big\{g^* + \gamma\big[(p+1)[M+L_1+T_0HL_2+T_0KL_3]\big] + \sum I_k^*\Big\}||x-y||
\end{aligned}$$

And by (H7) , $||Gx-Gy||_{PC} \le q^*||x-y||$ with $q^*<1$. Hence by Banach fixed point theorem [97] the equation (2.4.1) has unique solution. □

2.5 Example

Consider the following fractional integro-differential equation with the impulsive condition,

$$
\begin{aligned}
{}^{c}D^{\beta}x(t) &= \frac{1}{9}cosx(t)x(t) + \frac{1}{(t+3)^4}\frac{|x|}{1+|x|} + \frac{1}{9}\int_0^t se^{\frac{-x(s)}{4}} + \frac{1}{9}\int_0^1 (t-s)e^{-x(s)}ds,\\
\Delta x(\frac{1}{2}) &= \frac{|x(\frac{1}{2}^{-})|}{18+|x(\frac{1}{2}^{-})|}\\
x(0) &= x_0 - \frac{x}{18}
\end{aligned}
\tag{2.5.1}
$$

where $0 < \beta < 1$ over the interval $[0,1]$. Since, $A(t,x) = \frac{1}{9}cosxI$ therefore $||A(t,x)x - A(t,y)y|| \leq \frac{1}{9}||(cosxI)x - (cosyI)y|| \leq \frac{1}{9}||x-y||$,
$||Tx - Ty|| \leq \frac{1}{9}\int_0^t s||e^{\frac{-x(s)}{4}} - e^{\frac{-y(s)}{4}}|| \leq \frac{1}{36}||x-y||$,
$||Sx - Sy|| \leq \frac{1}{9}\int_0^1 |(t-s)||e^{-x(s)} - e^{-y(s)}|||ds \leq \frac{1}{18}||x-y||$,
$||f(t,x,Tx,Sx) - f(t,y,Ty,Sy)|| \leq \frac{7}{48}||x-y||$, $||g(x) - g(y)|| \leq \frac{1}{18}||x-y||$ and $q^* = g^* + \gamma\big[(p+1)[M + L_1 + T_0HL_2 + T_0KL_3]\big] + \sum I_k^* = \frac{1}{18} + \gamma\frac{29}{72} + \frac{1}{18}$. Choose $\beta = \frac{1}{2}$ then $q^* = 0.57$ which is less than 1. Therefore by existence theorem the given system has unique solution in the interval $[0,1]$.

2.6 Remark

1. This method suggest not only the existence and uniqueness about the solution but it also suggest method to find approximate solution of impulsive fractional differential equations (3.3.1) and (3.4.1).

2. This condition is not necessary condition this means equations (3.3.1) and (3.4.1) may have solution if one of the (H1) to (H7) not satisfied.

2.7 Conclusion

The system taken by Balachandran et. al. [21] is a special case of the system taken in this chapter because of inclusion of the nonlinear Fredholm operator in the system which is more relevant in many physical situations.