# Chapter 3

# Nonlinear Cauchy problem for abstract impulsive fractional quasilinear evolution equation with delay

The work discussed in chapter one is extended in chapter two by adding a delay condition on it and derived sufficient conditions for existence and uniqueness of mild solution of the integro-differential equations:

$$\begin{aligned} {}^cD^\beta x(t) &= A(t,x)x(t) + f(t, x(\phi(t)), Tx(t), Sx(t)) \quad t \neq t_k,\ k = 1, 2, \cdots, p \\ \Delta x(t_k) &= I_k(x(t_k)), \quad t = t_k,\ k = 1, 2, \cdots, p \\ x(0) &= x_0 - g(x), \end{aligned}$$

over the interval $[0, T_0]$ in a general Banach space $X$. Where, $A(t,x)$ is bounded quasi linear operator on $X$ and $f : [0, T_0] \times X \times X \times X \to X$, $T, S : X \to X$ are defined by $Tx(t) = \int_0^t h(t, s, x(\psi(s)))ds$ and $Sx(t) = \int_0^{T_0} k(t, s, x(\xi(s)))ds$, where $h : D_0 \times X \to X$, $D_0 = \{(t,s); 0 \leq s \leq t \leq T_0\}$ and $k : D_1 \times X \to X$, $D_1 = \{(t,s); 0 \leq t, s \leq T_0\}$ are the operators satisfying condition of the hypotheses, and an example shows application of the result.

## 3.1 Introduction

Many of the researchers taking interest in development in the theory of fractional differential equations because of its various applications in science and engineering [7, 8, 9, 10, 11, 12] this is due to its non local property [14] fractional differential equations are considered as an alternative model to nonlinear differential equations [13].Several researchers studied existence and uniqueness of the solutions of fractional order differential equations with classical condition using fixed point theory [11, 14, 15]. Existence results with nonlocal condition studied by N' Guerekata [18], Balachandran and Park [19].
The rapid development toward impulsive differential equations played important role in modeling of many problems [20]. Therefore, impulsive differential equations have been great interest to researchers. The existence and uniqueness of impulsive differential equations using fixed point theory studied by A. Anguraj, and M. M. Arjunan [39]. The existence result of impulsive fractional differential equations with classical conditions have been obtained by Benchohra and Slimani [40], Mophou [41], Ravichandran and Arjunan [42], Benchohra and Slimani [40]. Balachandran *et. al.* [21, 22] and Gao *et. al.* [43]. The systems in which past history of the state is required are modeled into Delay differential equations [45]. Existence and uniqueness of fractional impulsive differential equations with delay was initiated by K. Balachandran, S. Kiruthika and J. J. Trujillo [46].

## 3.2 Notations

(N1) $X$ = Banach space.

(N2) $\mathbb{R}_+ = [0, \infty)$

(N3) $C([0, T_0], X) = \{x : [0, T_0] \to X/x \, is \, continuous\}$ with norm $||x|| = sup_t ||x(t)||$

(N4) $PC([0, T_0], X) = \{x : [0, T_0] \to X; x \in C([t_{k-1}, t_k], X), \; and \, x(t_k^-) \, and \, x(t_k^+) \, exist,$

$k = 1, 2, \cdots, p \; with \; x(t_k^-) = x(t_k)\big\}$ with norm $||x||_{PC} = sup_{t\in[0,T_0]}||x(t)||$

(N5) $AC([0, T_0], X) = \big\{x : [0, T_0] \to X/x \; is \; absolutely \; continuous\big\}$ with norm $||x|| = sup_t||x(t)||$

(N6) $B(X) = \big\{A : X \to X/A \; is \; bouneded \; and \; linear\big\}$ with norm $||A||_{B(X)} = sup\{||A(y)||; y \in X, \; ||y|| \leq 1\}$

with preliminaries and properties, next section derived sufficient conditions for existence and uniqueness of solutions:

## 3.3 Equation with classical condition

This section presents the study of the existence and uniqueness of the solution of impulsive fractional differential equation with classical condition. Consider the fractional quasilinear impulsive integro-differential equation of the form

$$\begin{aligned} {}^cD^\beta x(t) &= A(t, x)x(t) + f(t, x(\phi(t)), Tx(t), Sx(t)) \quad t \neq t_k, \; k = 1, 2, \cdots, p \\ \Delta x(t_k) &= I_k(x(t_k)), \quad t = t_k, \; k = 1, 2, \cdots, p \\ x(0) &= x_0 \end{aligned} \tag{3.3.1}$$

over the interval $[0, T_0]$, where $A(t, x)$ is bounded quasi linear operator on $X$ and $f : [0, T_0]\times X\times X\times X \to X$, $T, S : X \to X$ are defined by $Tx(t) = \int_0^t h(t, s, x(\psi(s)))ds$ and $Sx(t) = \int_0^{T_0} k(t, s, x(\xi(s)))ds$; where $h : D_0 \times X \to X, \; with$
$D_0 = \{(t, s); 0 \leq s \leq t \leq T_0\}$ and $k : D_1 \times X \to X, \; D_1 = \{(t, s); 0 \leq t, s \leq T_0\}$ are

continuous. The equation (3.3.1) is equivalent to the integral equation of the form

$$\begin{aligned} x(t) =& x_0 + \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} A(s, x(s)) x(s) ds \\ &+ \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t - s)^{\beta-1} A(s, x(s)) x(s) ds \\ &+ \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} f(s, x(\phi(s)), Tx(s), Sx(s)) ds \\ &+ \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t - s)^{\beta-1} f(s, x(\phi(s)), Tx(s), Sx(s)) ds + \sum_{0<t_k<t} Ix(t_k^-) \end{aligned} \tag{3.3.2}$$

The following conditions are assumed to show the existence and uniqueness of the solution (3.3.1).

(H1) $A : [0, T_0] \times X \to X$ is continuous bounded linear operator and there exists a positive constant $M$, such that $||A(t, x)x - A(t, y)y||_{B(X)} \leq M||x - y||$, for all $x, y \in X$.

(H2) $f : [0, T_0] \times X \times X \times X \to X$ is continuous and there exists positive constants $L_1, L_2$ and $L_3$, such that $||f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3)|| \leq L_1||x_1 - y_1|| + L_2||x_2 - y_2|| + L_3||x_3 - y_3||$ for all $x_1, x_2, x_3, y_1, y_2$ and $y_3$ in $X$.

(H3) $h : D_0 \times X \to X$ and $k : D_1 \times X \to X$ are continuous and there exists positive constants $H$ and $K$, such that $||h(t, s, x) - h(t, s, y)|| \leq H||x - y||$ and $||k(t, s, x) - k(t, s, y)|| \leq K||x - y||$ for all $x$ and $y$ in $X$.

(H4) The functions $I_k : X \to X$ are continuous and there exist positive constants $I_k^*$ for all $k = 1, 2, \cdots, p$, such that $||I_k x - I_k y|| \leq I_k^*||x - y||$ for all $x$ and $y$ in $X$.

Consider, $\gamma = \frac{T_0^\beta}{\Gamma\beta+1}$ and further assume that,

(H5) $q = \left\{ \gamma[(p + 1)[M + L_1 + T_0 H L_2 + T_0 K L_3]] + \sum I_k^* \right\} < 1$

Define $F : PC([0,T_0],X) \to PC([0,T_0],X)$ by

$$\begin{aligned}
Fx(t) =& x_0 + \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} A(s,x(s))x(s)ds \\
&+ \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} A(s,x(s))x(s)ds \\
&+ \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} f(s, x(\phi(s)), Tx(s), Sx(s))ds \\
&+ \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} f(s, x(\phi(s)), Tx(s), Sx(s))ds + \sum_{0<t_k<t} I_k x(t_k^-)
\end{aligned} \tag{3.3.3}$$

Then equation (3.3.2) has unique solution if $F$ defined by (3.3.3) has unique fixed point. This means $F$ is well defined bounded operator on $PC([0,T_0],X)$ and $F$ is contraction [23].

**Lemma 3.3.1.** *If the operators $A, f, T, S$ and $I_k$ for $k = 1, 2, \cdots, p$ are continuous then $F$ is bounded operator on $PC([0,T_0],X)$.*

*Proof.* Let a sequence $\{x_n\}$ be converges to $x$ in $PC([0,T_0],X)$.
Therefore $||x_n - x|| \to 0$ as $n \to \infty$. Consider,

$$\begin{aligned}
||Fx_n - Fx||_{PC} \leq& \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} ||A(s,x_n(s))x_n(s) - A(s,x(s))x(s)||ds \\
&+ \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} ||A(s,x_n(s))x_n(s) - A(s,x(s))x(s)||ds \\
&+ \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \{ \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} ||f(s, x_n(\phi(s)), Tx_n(s), S_n x(s)) \\
&- f(s, x(\phi(s)), Tx(s), Sx(s))||ds\} \\
&+ \frac{1}{\Gamma(\beta)} \{ \int_{t_k}^{t} (t-s)^{\beta-1} ||f(s, x_n(\phi(s)), Tx_n(s), S_n x(s)) \\
&- f(s, x(\phi(s)), Tx(s), Sx(s))||ds\} \\
&+ \sum_{0<t_k<t} ||I_k x_n(t_k^-) - I_k x(t_k^-)||
\end{aligned}$$

Assuming the continuity of $A, f, T, S$ and $I_k$ for $k = 1, 2, \cdots, p$ the right side of above expression tends to zero as $n \to \infty$. Therefore $F$ is continuous on $PC([0,T_0],X)$ and hence $F$ is bounded. $\Box$

Sufficient conditions for existence and uniqueness of the solution of equation (3.3.1).

**Theorem 3.3.2.** *If the hypotheses (H1)-(H5) are satisfied, then the fractional quasi-linear impulsive integro-differential equation (3.3.1) has unique solution in* $PC([0,T_0],X)$ *for* $0 < \beta \le 1$.

*Proof.* To show equation (3.3.1) has unique solution it is sufficient to show $F$ defined (3.3.3) is contraction. Let $x$ and $y$ in $PC([0,T_0],X)$ and consider,

$$
\begin{aligned}
||Fx - Fy||_{PC} \le \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} ||A(s,x(s))x(s) - A(s,y(s))y(s)||ds \\
+ \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} ||A(s,x(s))x(s) - A(s,y(s))y(s)||ds \\
+ \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \Big\{ \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} ||f(s,x(\phi(s)),Tx(s),Sx(s)) \\
- f(s,y(\phi(s)),Ty(s),Sy(s))||ds \Big\} \\
+ \frac{1}{\Gamma(\beta)} \Big\{ \int_{t_k}^{t} (t-s)^{\beta-1} ||f(s,x(\phi(s)),Tx(s),Sx(s)) \\
- f(t,y(\phi(s)),Ty(s),Sy(s))||ds \Big\} \\
+ \sum_{0<t_k<t} ||I_k x(t_k^-) - I_k y(t_k^-)||
\end{aligned}
$$

Using (H1)-(H4),

$$\begin{aligned}
||Fx - Fy||_{PC} \leq & \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} M||x-y||ds \\
& + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} M||x-y||ds \\
& + \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} \big\{L_1 + T_0 H L_2 + T_0 K L_3\big\} ||x-y||ds \\
& + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} ds \big\{L_1 + T_0 H L_2 + T_0 K L_3\big\} ||x-y||ds \\
& + \sum_{0<t_k<t} I_k^* ||x-y|| \\
& \leq \left\{ \frac{T_0^{\beta}}{\Gamma(\beta+1)} \big[(p+1)[M + L_1 + T_0 H L_2 + T_0 K L_3]\big] + \sum I_k^* \right\} ||x-y|| \\
& = \left\{ \gamma \big[(p+1)[M + L_1 + T_0 H L_2 + T_0 K L_3]\big] + \sum I_k^* \right\} ||x-y||
\end{aligned}$$

By (H5) , $||Fx - Fy||_{PC} \leq q||x-y||$ with $q < 1$. Hence by Banach fixed point theorem the equation (3.3.1) has unique solution. □

## 3.4 Equation with nonlocal condition

In this section, classical condition is replaced by a nonlocal condition for existence and uniqueness of solution of the impulsive fractional differential equation.

Consider the fractional quasilinear impulsive integro-differential equation of the form

$$\begin{aligned}
{}^cD^{\beta}x(t) &= A(t,x)x(t) + f(t, x(\phi(t)), Tx(t), Sx(t)) \quad t \neq t_k,\ k = 1, 2, \cdots, p \\
\Delta x(t_k) &= I_k(x(t_k)), \quad t = t_k,\ k = 1, 2, \cdots, p \\
x(0) &= x_0 - g(x)
\end{aligned} \tag{3.4.1}$$

over the interval $[0, T_0]$, where $A(t,x)$is bounded quasi linear operator on $X$ and $f : [0, T_0] \times X \times X \times X \to X$, $T, S : X \to X$ are defined by $Tx(t) = \int_0^t h(t, s, x(\psi(s)))ds$ and $Sx(t) = \int_0^{T_0} k(t, s, x(\xi(s)))ds$; where $h : D_0 \times X \to X,\ with$

$D_0 = \{(t,s); 0 \leq s \leq t \leq T_0\}$ and $k : D_1 \times X \to X,\ D_1 = \{(t,s); 0 \leq t, s \leq T_0\}$ are continuous and $g : X \to X$ is given function.

The equivalent integral equation of (3.4.1) is given by

$$\begin{aligned} x(t) = & x_0 - g(x) + \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} A(s, x(s))x(s)ds \\ & + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} A(s, x(s))ds \\ & + \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} f(s, x(\phi(s)), Tx(s), Sx(s))ds \\ & + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} f(s, x(\phi(s)), Tx(s), Sx(s))ds + \sum_{0<t_k<t} Ix(t_k^-) \end{aligned} \tag{3.4.2}$$

The following hypotheses are assumed.

(H6) $g : X \to X$ is continuous and there exist a positive constant $g^*$, such that $||g(x) - g(y)|| \leq g^* ||x - y||$ for each $x$ and $y$ in $X$.

(H7) $q^* = g^* + \gamma\big[(p+1)[M + L_1 + T_0 H L_2 + T_0 K L_3]\big] + \sum I_k^* < 1.$

Define $G : PC([0, T_0], X) \to PC([0, T_0], X)$ by

$$\begin{aligned} Gx(t) = & x_0 - g(x) + \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} A(s, x(s))x(s)ds \\ & + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} A(s, x(s))x(s)ds \\ & + \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} f(s, x(\phi(s)), Tx(s), Sx(s))ds \\ & + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} f(s, x(\phi(s)), Tx(s), Sx(s))ds + \sum_{0<t_k<t} I_k x(t_k^-) \end{aligned} \tag{3.4.3}$$

**Lemma 3.4.1.** *If the operators $A, f, T, S$ and $I_k$ for $k = 1, 2, \cdots, p$ are continuous then $G$ is bounded operator on $PC([0, T_0], X)$.*

*Proof.* Let a sequence $\{x_n\}$ be converges to $x$ in $PC([0, T_0], X)$.
Therefore $||x_n - x|| \to 0$ as $n \to \infty$. Consider,

$$\begin{aligned}
||Gx_n - Gx||_{PC} \leq\ & ||g(x_n(s)) - g(x(s))|| \\
& + \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} ||A(s, x_n(s))x_n(s) - A(s, x(s))x(s)|| ds \\
& + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t - s)^{\beta-1} ||A(s, x_n(s))x_n(s) - A(s, x(s))x(s)|| ds \\
& + \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \Big\{ \int_{t_k-1}^{t_k} (t_k - s)^{\beta-1} ||f(s, x_n(\phi(s)), Tx_n(s), Sx_n(s)) \\
& - f(s, x(\phi(s)), Tx(s), Sx(s))|| ds \Big\} \\
& + \frac{1}{\Gamma(\beta)} \Big\{ \int_{t_k}^{t} (t - s)^{\beta-1} ||f(s, x_n(\phi(s)), Tx_n(s), Sx_n(s)) \\
& - f(s, x(\phi(s)), Tx(s), Sx(s))|| ds \Big\} \\
& + \sum_{0<t_k<t} ||I_k x_n(t_k^-) - I_k x(t_k^-)||
\end{aligned}$$

So by continuity of $A, f, T, S, g$ and $I_k$ for $k = 1, 2, \cdots, p$ the right side of above expression tends to zero as $n \to \infty$. Therefore $G$ is continuous on $PC([0, T_0], X)$ and hence $G$ is bounded. □

The sufficient conditions are derived as under for existence and uniqueness of the solution of equation (3.4.1).

**Theorem 3.4.2.** *If the hypotheses (H1)-(H4) and (H6)-(H7) are satisfied, then the fractional quasi-linear impulsive integro-differential equation (3.4.1) has unique solution in $PC([0, T_0], X)$ for $0 < \beta \leq 1$.*

*Proof.* To show equation (3.4.1) has unique solution it is sufficient to show $G$ defined in (3.4.3) is contraction. Let $x$ and $y$ in $PC([0,T_0],X)$ and consider,

$$
\begin{aligned}
||Gx-Gy||_{PC} \leq\ & ||g(x)-g(y)|| \\
& + \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k-s)^{\beta-1}||A(s,x(s))x(s)-A(s,y(s))y(s)||ds \\
& + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1}||A(s,x(s))x(s)-A(s,y(s))y(s)||ds \\
& + \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \Big\{ \int_{t_k-1}^{t_k} (t_k-s)^{\beta-1}||f(s,x(\phi(s)),Tx(s),Sx(s)) \\
& - f(s,y(\phi(s)),Ty(s),Sy(s))||ds \Big\} \\
& + \frac{1}{\Gamma(\beta)} \Big\{ \int_{t_k}^{t} (t-s)^{\beta-1}||f(s,x(\phi(s)),Tx(s),Sx(s)) \\
& - f(t,y(\phi(s)),Ty(s),Sy(s))||ds \Big\} \\
& + \sum_{0<t_k<t} ||I_k x(t_k^-) - I_k y(t_k^-)||
\end{aligned}
$$

BY (H1)-(H4) and (H6) the result is,

$$
\begin{aligned}
||Gx-Gy||_{PC} \leq\ & g^*||x-y|| \\
& + \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k-s)^{\beta-1}M||x-y||ds + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1}M||x-y||ds \\
& + \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_k-1}^{t_k} (t_k-s)^{\beta-1}\big\{L_1+T_0HL_2+T_0KL_3\big\}||x-y||ds \\
& + \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1}ds\big\{L_1+T_0HL_2+T_0KL_3\big\}||x-y||ds + \sum_{0<t_k<t} I_k^*||x-y|| \\
\leq\ & \Big\{ g^* + \frac{T_0^{\beta}}{\Gamma(\beta+1)}\big[(p+1)[M+L_1+T_0HL_2+T_0KL_3]\big] + \sum I_k^* \Big\}||x-y|| \\
=\ & \Big\{ g^* + \gamma\big[(p+1)[M+L_1+T_0HL_2+T_0KL_3]\big] + \sum I_k^* \Big\}||x-y||
\end{aligned}
$$

From (H7), $||Gx-Gy||_{PC} \leq q^*||x-y||$ with $q^*<1$. Hence, by Banach fixed point theorem [97]the equation (3.4.1) has unique solution. □

## 3.5 Example

Consider the following fractional integro-differential equation with the impulsive condition,

$$
\begin{aligned}
{}^{c}D^{\beta}x(t) &= \frac{1}{10}sinx(t)x(t) + \frac{1}{(t+3)^4}\frac{|x(sint)|}{1+|x(sint)|} + \frac{1}{10}\int_0^t se^{\frac{-x(coss)}{4}} + \frac{1}{20}\int_0^1 (t-s)x^2 ds \\
\Delta x(\frac{1}{2}) &= \frac{|x(\frac{1}{2}^-)|}{18+|x(\frac{1}{2}^-)|} \\
x(0) &= x_0 - \frac{x}{18}
\end{aligned}
\tag{3.5.1}
$$

where $\beta = \frac{1}{2}$ over the interval $[0,1]$. Since, $A(t,x) = \frac{1}{10}sinxI$ therefore $||A(t,x)x - A(t,y)y|| \leq \frac{1}{10}||sinxIx - sinyIy|| \leq \frac{1}{20}||x-y||$,
$Tx(t) = \frac{1}{10}\int_0^t se^{\frac{-x(coss)}{4}}$ $||Tx - Ty|| \leq \frac{1}{10}\int_0^t s||e^{\frac{-x(sins)}{4}} - e^{\frac{-y(sins)}{4}}|| \leq \frac{1}{40}||x-y||$,
and $||Sx - Sy|| \leq \frac{1}{20}\int_0^1 |(t-s)||x^2 - y^2|||ds \leq \frac{1}{40}||x-y||$
and $\frac{1}{(t+3)^4}||\frac{|x(sint)|}{1+|x(sint)|} - \frac{|y(sint)|}{1+|y(sint)|}|| \leq \frac{1}{81}||x-y||$
therefore $q^* = g^* + \gamma\big[(p+1)[M + L_1 + T_0HL_2 + T_0KL_3]\big] + \sum I_k^* < 1$. Therefore by existence theorem the given system has unique solution in the interval $[0,1]$.

## 3.6 Remark

1. This method suggest not only the existence and uniqueness about the solution but it also suggest method to find approximate solution of impulsive fractional differential equations (3.3.1) and (3.4.1).

2. This condition is not necessary condition this means equations (3.3.1) and (3.4.1) may have solution if one of the (H1) to (H7) not satisfied.