CHAPTER III

ABSOLUTE CONVERGENCE OF FOURIER SERIES OF RESTRICTED Lip(α, p)-FUNCTIONS

In the present chapter we investigate the behaviour, as regards absolute convergence, of Fourier series of functions belonging to $Lip(\alpha, p)$ which also satisfy certain extra conditions. For this reason, these functions are called restricted $Lip(\alpha, p)$-functions.

We begin by giving a new proof of the following theorem of Min-Teh Cheng:[1)]

THEOREM-A. <u>Let</u> $0 < \alpha \leq 1$, $1 < p \leq 2$ <u>and</u> $h > 0$. <u>If</u> f <u>has the Fourier coefficients</u> a_n, b_n <u>and if</u>

$$(1) \quad \int_0^{2\pi} |f(x+h) - f(x)|^p dx = O\{h (\log h^{-1})^{-1-\alpha p}\},$$

<u>then</u>

$$(2) \quad \sum_{n=2}^{\infty} (|a_n| + |b_n|) \log^T n < \infty,$$

<u>for</u> $T < \alpha + p^{-1} - 1$. <u>Moreover,</u> (2) <u>may not hold for</u> $T = \alpha + p^{-1} - 1$.

Our proof of Theorem-A is comparatively shorter and more direct than that of Min-Teh Cheng. Our proof uses the Hausdorff-Young inequality and is essentially a

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1) Cheng [8]

modification of Benstein's method, whereas Min-Teh Cheng uses an inequality analogous to Hausdorff-Young inequality and relating to $Lip(\alpha, p)$ classes.

In addition to giving a new proof of Theorem-A, we prove three more theorems, namely Theorems 7, 8 and 9. Theorem 7 is a generalization of Theorem-A and Theorems 8 and 9 throw some light on the question of determining extra conditions satisfied by f so that (2) may hold for $T = \alpha + p^{-1} - 1$.

It will be observed that we have obtained such a condition under a more general situation in Theorem 9 which ensures the validity of (4) when $\beta = p(T+1)/(1+\alpha p)$.

<u>NEW PROOF OF THEOREM-A:</u> We have

$$f(x+h) - f(x-h) \sim 2\sum_{n=1}^{\infty} (-a_n \sin nx + b_n \cos nx) \sin nh.$$

Therefore, by Hausdorff-Young inequality, we get

$$\left(\sum_{n=1}^{\infty} |2\rho_n \sin nh|^{p'}\right)^{1/p'} \leq \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x+h) - f(x-h)|^p dx\right)^{1/p},$$

where $\rho_n^{p'} = |a_n|^{p'} + |b_n|^{p'}$ and p' is given by $\frac{1}{p} + \frac{1}{p'} = 1$.

Putting $h = \frac{\pi}{2N}$, we get from (1)

$$\left(\sum_{n=1}^{N} |\rho_n \sin nh|^{p'}\right)^{1/p'} \leq \left(\sum_{n=1}^{\infty} |\rho_n \sin nh|^{p'}\right)^{1/p'}$$

$$\leq \frac{1}{2}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|f(x+\tfrac{\pi}{2N})-f(x-\tfrac{\pi}{2N})\right|^p dx\right)^{1/p}$$

$$= O\left(\frac{N^{-1/p}}{(\log(\frac{N}{\pi}))^{(1+\alpha p)/p}}\right).$$

Taking $N = 2^\nu$ and taking into account only the terms with indices exceeding $\frac{1}{2}N$, we obtain

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} |\rho_n \sin nh|^{p'} = O\left(\frac{2^{-\nu p'/p}}{(\log\frac{2^\nu}{\pi})^{(1+\alpha p)p'/p}}\right).$$

Since $\sin\frac{n\pi}{2^{\nu+1}} > \frac{1}{\sqrt{2}}$, for $2^{\nu-1} < n \leq 2^\nu$, it follows that

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^{p'} = O\left(\frac{2^{-\nu p'/p}}{(\log\frac{2^\nu}{\pi})^{(1+\alpha p)p'/p}}\right).$$

Now, by Hölder's inequality,

$$\sum_{n=2^{\nu-1}-1}^{2^\nu} \rho_n \leq \left(\sum_{n=2^{\nu-1}+1}^{2^\nu}\rho_n^{p'}\right)^{1/p'}\left(\sum_{n=2^{\nu-1}+1}^{2^\nu} 1\right)^{1-1/p'}$$

$$= O\left(\frac{2^{-\nu/p}}{(\log\frac{2^\nu}{\pi})^{(1+\alpha p)/p}}\cdot 2^{\nu/p}\right)$$

$$= O\left(\frac{1}{\left(\log \frac{2^{\nu}}{\pi}\right)^{(1+\alpha p)/p}}\right);$$

and hence

$$\sum_{n=2^{\nu-1}+1}^{2^{\nu}} p_n \log^T n \leq \log^T 2^{\nu} \sum_{n=2^{\nu-1}+1}^{2^{\nu}} p_n$$

$$= O\left(\frac{\log^T 2^{\nu}}{\left(\log \frac{2^{\nu}}{\pi}\right)^{(1+\alpha p)/p}}\right).$$

Finally,

$$\sum_{n=2}^{\infty} p_n \log^T n = \sum_{\nu=1}^{\infty} \sum_{n=2^{\nu-1}+1}^{2^{\nu}} p_n \log^T n$$

$$= O\left(\sum_{\nu=1}^{\infty} \frac{\log^T 2^{\nu}}{\left(\log \frac{2^{\nu}}{\pi}\right)^{(1+\alpha p)/p}}\right)$$

$$= O\left(\sum_{\nu=1}^{\infty} \nu^{T-(1+\alpha p)/p}\right)$$

$$= O(1),$$

for $T < \alpha + p^{-1} - 1$.

This completes the proof of the theorem.

We now proceed to give a generalization of Theorem-A in the form of the following

THEOREM 7. <u>If</u> $0 < \alpha \leq 1$, $1 < p \leq 2$, $h > 0$ <u>and</u>

$$(3)\qquad \int_0^{2\pi} |f(x+h) - f(x)|^p \, dx = O\left\{ h^{\delta} (\log h^{-1})^{-1-\alpha p} \right\},$$

<u>where</u> $\delta = 1 + p(1-\beta)/\beta$, <u>then</u>

$$(4)\qquad \sum_{n=2}^{\infty} \left(|a_n|^{\beta} + |b_n|^{\beta} \right) \log^{T} n < \infty,$$

<u>for all</u> $\beta > p(T+1)/(1+\alpha p)$. (4) <u>need not hold for</u> $\beta = p(T+1)/(1+\alpha p)$.

<u>PROOF</u>: Proceeding as in the proof of Theorem-A, we have from (3)

$$\left(\sum_{n=1}^{N} \left| \rho_n \sin \frac{n\pi}{2N} \right|^{p'} \right)^{1/p'} \leq \frac{1}{2} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(x + \frac{\pi}{2N}\right) - f\left(x - \frac{\pi}{2N}\right) \right|^p dx \right)^{1/p}$$

$$= O\left(\frac{N^{-\delta/p}}{\left(\log (N/\pi) \right)^{(1+\alpha p)/p}} \right).$$

Therefore

$$\sum_{n=2^{\nu-1}+1}^{2^{\nu}} \left| \rho_n \sin \frac{n\pi}{2^{\nu+1}} \right|^{p'} = O\left(\frac{2^{-\nu\delta p'/p}}{\left(\log \frac{2^{\nu}}{\pi} \right)^{(1+\alpha p)p'/p}} \right);$$

and hence

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^{p'} = O\left(\frac{2^{-\nu\delta p'/p}}{\left(\log \frac{2^\nu}{\pi}\right)^{(1+\alpha p)p'/p}}\right).$$

Now, by Hölder's inequality, we get

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^{\beta} \leq \left(\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^{p'}\right)^{\beta/p'} \left(\sum_{n=2^{\nu-1}+1}^{2^\nu} 1\right)^{1-\beta/p'}$$

$$= O\left(\frac{2^{-\nu\delta\beta/p}}{\left(\log \frac{2^\nu}{\pi}\right)^{\beta(1+\alpha p)/p}} \cdot 2^{\nu(1-\beta/p')}\right)$$

$$= O\left(\frac{1}{\left(\log \frac{2^\nu}{\pi}\right)^{\beta(1+\alpha p)/p}}\right);$$

and hence

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^{\beta} \log^T n \leq \log^T 2^\nu \sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^{\beta}$$

$$= O\left(\frac{\log^T 2^\nu}{\left(\log \frac{2^\nu}{\pi}\right)^{\beta(1+\alpha p)/p}}\right).$$

Therefore

$$\sum_{n=2}^{\infty} \rho_n^{\beta} \log^T n = \sum_{\nu=1}^{\infty} \sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^{\beta} \log^T n$$

$$= O\left(\sum_{\nu=1}^{\infty} \frac{\log^T 2^\nu}{\left(\log \frac{2^\nu}{\pi}\right)^{\beta(1+\alpha p)/p}} \right)$$

$$= O(1),$$

for $\beta > p(T+1)/(1+\alpha p)$.

We omit the proof of the second part of the theorem.

The following two theorems give sufficient conditions under which (2) is valid for $T = \alpha + p^{-1} - 1$ and (4) holds for $\beta = p(T+1)/(1+\alpha p)$

THEOREM 8. <u>If</u> $0 < \alpha \leq 1$, $1 < p \leq 2$, $h > 0$, $\varepsilon > 0$ <u>and</u>

(5) $$\int_0^{2\pi} |f(x+h) - f(x-h)|^p dx = O\left(h (\log h^{-1})^{-(1+\alpha p)} (\log\log h^{-1})^{-(1+\varepsilon)p}\right),$$

<u>then</u> (2) <u>holds for</u> $T = \alpha + p^{-1} - 1$.

THEOREM 9. <u>If</u> $0 < \alpha \leq 1$, $1 < p \leq 2$, $h > 0$, $\varepsilon > 0$ <u>and</u>

(6) $$\int_0^{2\pi} |f(x+h) - f(x)|^p dx = O\left(h^{\delta} (\log h^{-1})^{-(1+\alpha p)} (\log\log h^{-1})^{-(1+\varepsilon)p/\beta}\right),$$

then (4) holds for $\beta = p(T+1)/(1+\alpha p)$.

PROOF: Again, as in the proof of Theorem-A, we can obtain, by using (5),

$$\left(\sum_{n=1}^{N} \left| \rho_n \sin \frac{n\pi}{2N} \right|^{p'} \right)^{1/p'} \leq \frac{1}{2} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(x+\tfrac{\pi}{2N}) - f(x-\tfrac{\pi}{2N}) \right|^p dx \right)^{1/p}$$

$$= \left(\frac{N^{-1/p}}{(\log \frac{N}{\pi})^{(1+\alpha p)/p} (\log\log \frac{N}{\pi})^{1+\varepsilon}} \right);$$

and hence

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^{p'} = O\left(\frac{2^{-\nu p'/p}}{(\log \frac{2^\nu}{\pi})^{(1+\alpha p)p'/p} (\log\log \frac{2^\nu}{\pi})^{(1+\varepsilon)p'}} \right).$$

Now, applying Holder's inequality,

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n \leq \left(\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^{p'} \right)^{1/p'} \left(\sum_{n=2^{\nu-1}+1}^{2^\nu} 1 \right)^{1-1/p'}$$

$$= O\left(\frac{2^{-\nu/p}}{(\log \frac{2^\nu}{\pi})^{(1+\alpha p)/p} (\log\log \frac{2^\nu}{\pi})^{1+\varepsilon}} \cdot 2^{\nu/p} \right).$$

Therefore

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n \log^T n \le \log^T 2^\nu \sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n$$

$$= O\left(\frac{\log^T 2^\nu}{\left(\log \frac{2^\nu}{\pi}\right)^{(1+\alpha p)/p}\left(\log\log\frac{2^\nu}{\pi}\right)^{1+\varepsilon}}\right)$$

$$= O\left(\frac{1}{\nu \log^{1+\varepsilon}\nu}\right),$$

putting $T = \alpha + p^{-1} - 1$.

Hence

$$\sum_{n=2}^{\infty} \rho_n \log^T n = \sum_{\nu=1}^{\infty} \sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n \log^T n$$

$$= O\left(\sum_{\nu=1}^{\infty} \frac{1}{\nu \log^{1+\varepsilon}\nu}\right)$$

$$= O(1).$$

This proves Theorem 8.

To prove Theorem 9, we obtain with the help of (6)

$$\left(\sum_{n=1}^{N} \left|\rho_n \sin\frac{n\pi}{2N}\right|^{p'}\right)^{1/p'} \le \frac{1}{2}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(x+\frac{\pi}{2N}\right)-f\left(x-\frac{\pi}{2N}\right)\right|^p dx\right)^{1/p}$$

$$= O\left(\frac{N^{-\delta/\beta}}{(\log\frac{N}{\pi})^{(1+\alpha p)/p}(\log\log\frac{N}{\pi})^{(1+\varepsilon)/\beta}}\right).$$

Therefore

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^{p'} = O\left(\frac{2^{-\nu\delta p'/p}}{(\log\frac{2^\nu}{\pi})^{(1+\alpha p)p'/p}(\log\log\frac{2^\nu}{\pi})^{(1+\varepsilon)p'/\beta}}\right).$$

Applying Hölder's inequality, we have

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^{\beta} \leq \left(\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^{p'}\right)^{\beta/p'}\left(\sum_{n=2^{\nu-1}+1}^{2^\nu} 1\right)^{1-\beta/p'}$$

$$= O\left(\frac{2^{-\nu\delta\beta/p}}{(\log\frac{2^\nu}{\pi})^{(1+\alpha p)\beta/p}(\log\log\frac{2^\nu}{\pi})^{1+\varepsilon}}\cdot 2^{\nu(1-\beta/p')}\right).$$

$$= O\left(\frac{1}{(\log\frac{2^\nu}{\pi})^{(1+\alpha p)\beta/p}(\log\log\frac{2^\nu}{\pi})^{1+\varepsilon}}\right);$$

and hence

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^{\beta}\log^{T} n \leq \log^{T} 2^\nu \sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^{\beta}$$

$$= O\left(\frac{\log^T 2^\nu}{\left(\log \frac{2^\nu}{\pi}\right)^{(1+\alpha p)\beta/p}\left(\log\log\frac{2^\nu}{\pi}\right)^{1+\varepsilon}}\right).$$

Therefore

$$\sum_{n=2}^{\infty} \rho_n^{\beta} \log^T n = \sum_{\nu=1}^{\infty} \sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^{\beta} \log^T n$$

$$= O\left(\sum_{\nu=1}^{\infty} \frac{\log^T 2^\nu}{\left(\log \frac{2^\nu}{\pi}\right)^{(1+\alpha p)\beta/p}\left(\log\log\frac{2^\nu}{\pi}\right)^{1+\varepsilon}}\right)$$

$$= O\left(\sum_{\nu=1}^{\infty} \frac{1}{\nu \log^{1+\varepsilon} \nu}\right)$$

$$= O(1),$$

by putting $\beta = p(T+1)/(1+\alpha p)$.

In fact, the method of proofs which we have developed above can be used to prove each and every one of the results proved above in their more general forms. Thus we have

THEOREM 10. Let $0 < \alpha \le 1$, $1 < p \le 2$, $\varepsilon > 0$ and $h > 0$.

(1) If

$$\int_0^{2\pi} |f(x+h) - f(x)|^p dx =$$

$$= O\left(h\,(\log_1 h^{-1})^{-p}(\log_2 h^{-1})^{-p}\dots(\log_{k-1} h^{-1})^{-p}(\log_k h^{-1})^{-(1+\alpha p)}\right),$$

where $\log_1 h^{-1} = \log h^{-1}$ and $\log_n h^{-1} = \log \log_{n-1} h^{-1}$,

then

$$\sum_{n=2}^{\infty} \left(|a_n| + |b_n|\right) \log^{T} n < \infty ,$$

for $T < \alpha + p^{-1} - 1$, **but not for** $T = \alpha + p^{-1} - 1$.[1]

(ii) **If**

$$\int_0^{2\pi} |f(x+h) - f(x)|^p dx =$$

$$= O\left(h^{\delta}(\log_1 h^{-1})^{-p}(\log_2 h^{-1})^{-p}\dots(\log_{k-1} h^{-1})^{-p}(\log_k h^{-1})^{-(1+\alpha p)}\right),$$

where $\delta = 1 + p(1-\beta)/\beta$, **then**

$$\sum_{n=2}^{\infty} \left(|a_n|^{\beta} + |b_n|^{\beta}\right) \log^{T} n < \infty ,$$

for $\beta > p(T+1)/(1+\alpha p)$, **but not necessarily for** $\beta = p(T+1)/(1+\alpha p)$.

(iii) **If**

$$\int_0^{2\pi} |f(x+h) - f(x)|^p dx =$$

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1) Cheng, loc. cit, Theorem 3.

$$= O\left(h(\log_1 h^{-1})^{-p}(\log_2 h^{-1})^{-p}\cdots(\log_{k-1} h^{-1})^{-p}(\log_k h^{-1})^{-(1+\alpha p)}(\log_{k+1} h^{-1})^{-(1+\varepsilon)p}\right),$$

then

$$\sum_{n=2}^{\infty}\left(|a_n|+|b_n|\right)\log^{T} n < \infty,$$

for $T = \alpha + p^{-1} - 1.$

(iv) **If**

$$\int_0^{2\pi} |f(x+h)-f(x)|^p\,dx =$$

$$= O\left(h^{\delta}(\log_1 h^{-1})^{-p}(\log_2 h^{-1})^{-p}\cdots(\log_{k-1} h^{-1})^{-p}(\log_k h^{-1})^{-(1+\alpha p)}(\log_{k+1} h^{-1})^{-(1+\varepsilon)p/\beta}\right),$$

where $\delta = 1 + p(1-\beta)/\beta$, **then**

$$\sum_{n=2}^{\infty}\left(|a_n|^{\beta}+|b_n|^{\beta}\right)\log^{T} n < \infty,$$

for $\beta = p(T+1)/(1+\alpha p).$
