CHAPTER VI

A THEOREM ON THE ALMOST EVERYWHERE CONVERGENCE OF FOURIER SERIES

The aim of this chapter is to prove a theorem on the almost everywhere convergence of Fourier series. The theorem is similar to a theorem of R.P.Boas[1)] which in turn is a generalization of a theorem due to A. Beurling[2)] on the absolute convergence of Fourier series. Also, it is different in character from those of Kolmogorov and Seliverstov,[3)] Plessner[4)] and G.H.Hardy and J.E.Littlewood.[5)] In establishing this theorem we have used the methods of Chapter V (Theorem 13), and hence the justification for the inclusion of this chapter in the thesis. Our theorem is as follows:

THEOREM 17. If f and g are even functions of class L_2, each of period 2π, with Fourier cosine coefficients c_n and g_n, if f is a contraction of g, and if $|g_n| \leq \gamma_n$, where $\{\gamma_n\}$ is a sequence of positive real numbers such that

$$(1) \quad \sum_{\nu=1}^{\infty} \frac{\log \nu}{\nu^2} \left\{ \sum_{k=1}^{\nu} k^2 \gamma_k^2 \right\}^{1/2} + \sum_{\nu=1}^{\infty} \frac{\log \nu}{\nu} \left\{ \sum_{k=\nu+1}^{\infty} \gamma_k^2 \right\}^{1/2} < \infty,$$

then the Fourier series of f converges almost everywhere.

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1) Boas [5] 2) Beurling [4]
3) Kolmogorov and Seliverstov [16], [17]
4) Plessner [21]
5) Hardy and Littlewood [14]

PROOF: Since

$$f \sim \frac{1}{2} c_0 + \sum_{\nu=1}^{\infty} c_\nu \cos \nu x,$$

it follows that

$$f(x+h) - f(x-h) \sim -2 \sum_{\nu=1}^{\infty} c_\nu \sin \nu h \sin \nu x .$$

Put

$$T_n(x) \equiv \sum_{k=1}^{n} (-c_k \log k) \sin kx .$$

Then, since $T_n(x)$ is a trigonometrical polynomial,

$$\int_0^{2\pi} \{ f(x+h) - f(x-h) \} T_n(x)\, dx$$

$$= 2 \sum_{\nu=1}^{\infty} \left\{ -c_\nu \int_0^{2\pi} T_n(x) \sin \nu x \, dx \right\} \sin \nu h$$

$$= 2\pi \sum_{\nu=1}^{n} c_\nu^2 \log \nu \cdot \sin \nu h .$$

Since $\sin x \geq \frac{2x}{\pi}$, for $0 < x \leq \frac{\pi}{2}$, putting $h = \frac{\pi}{2n}$, we get

$$\frac{1}{n} \sum_{\nu=1}^{n} \nu c_\nu^2 \log \nu \leq \sum_{\nu=1}^{n} c_\nu^2 \log \nu \cdot \sin \frac{\nu \pi}{2n}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left\{f\left(x+\frac{\pi}{2n}\right)-f\left(x-\frac{\pi}{2n}\right)\right\}T_n(x)\,dx$$

$$(2) \qquad \leq \left[\frac{1}{2\pi}\int_0^{2\pi}\left\{f\left(x+\frac{\pi}{2n}\right)-f\left(x-\frac{\pi}{2n}\right)\right\}^2 dx\right]^{1/2} \times$$

$$\times \left[\frac{1}{2\pi}\int_0^{2\pi} T_n^2(x)\,dx\right]^{1/2},$$

by Cauchy-Schwarz inequality.

Now

$$\frac{1}{2\pi}\int_0^{2\pi} T_n^2(x)\,dx = \sum_{\nu=1}^{n} c_\nu^2 (\log \nu)^2$$

$$(3) \qquad \leq (\log n)^2 \sum_{\nu=1}^{n} c_\nu^2$$

$$\leq A(\log n)^2, \quad \text{for } f \in L_2,$$

where A is a constant not necessarily the same in all occurrences.

Also, since f is a contraction of g, we have by Parseval's relation

$$\int_0^{2\pi}\left\{f\left(x+\frac{\pi}{2n}\right)-f\left(x-\frac{\pi}{2n}\right)\right\}^2 dx$$

$$\leq \int_0^{2\pi}\left\{g\left(x+\frac{\pi}{2n}\right)-g\left(x-\frac{\pi}{2n}\right)\right\}^2 dx$$

$$(4) \qquad = \sum_{k=1}^{\infty} g_k^2 \sin^2\frac{k\pi}{2n}$$

$$\leq \sum_{k=1}^{\infty} \gamma_k^2 \sin^2\frac{k\pi}{2n}.$$

Therefore from (2), (3) and (4), we get

$$\frac{1}{n}\sum_{\nu=1}^{n} \nu c_\nu^2 \log\nu \leq A \log n \left\{\sum_{k=1}^{\infty} \gamma_k^2 \sin^2\frac{k\pi}{2n}\right\}^{1/2}.$$

Now if we put

$$\mu_n = \frac{1}{n}\sum_{\nu=1}^{n} \nu c_\nu^2 \log\nu ,$$

then we obtain, by partial summation,

$$\sum_{\nu=1}^{n} c_\nu^2 \log\nu = \mu_n + \sum_{\nu=1}^{n-1} \frac{\mu_n}{\nu+1}$$

$$\leq A \log n \left\{\sum_{k=1}^{\infty} \gamma_k^2 \sin^2\frac{k\pi}{2n}\right\}^{1/2} +$$

$$+ A \sum_{\nu=1}^{n-1} \frac{\log \nu}{\nu+1} \left\{ \gamma_k^2 \sin^2 \frac{k\pi}{2\nu} \right\}^{1/2}$$

$$= A(S_1 + S_2), \text{ say.}$$

The boundedness of S_1, and S_2, as $N \to \infty$, can be shown with the help of (1) by using an argument parallel to that of Boas.[1] In fact, we have

$$S_1^2 = (\log n)^2 \left(\sum_{k=1}^{n} + \sum_{k=n+1}^{\infty} \right)$$

$$\leq (\log n)^2 \sum_{k=1}^{n} \frac{k^2 \gamma_k^2 \pi^2}{4n^2} + (\log n)^2 \sum_{k=n+1}^{\infty} \gamma_k^2$$

$$= \frac{\pi^2 (\log n)^2}{4n^2} \sum_{k=1}^{n} k^2 \gamma_k^2 + (\log n)^2 \sum_{k=n+1}^{\infty} \gamma_k^2$$

$$= T_1 + T_2, \text{ say.}$$

Now writing

$$\left\{ \sum_{k=1}^{\nu} k^2 \gamma_k^2 \right\}^{1/2} \equiv B_\nu,$$

we obtain from the first series of (1)

$$\sum_{\nu=n}^{2n} \frac{\log \nu}{\nu^2} B_\nu \geq B_n \log n \sum_{\nu=n}^{2n} \frac{1}{\nu^2}$$

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1) Boas, loc. cit.

$$\geq (A B_n \log n)/n \,;$$

and hence $B_n^2(\log n)^2/n^2 \to 0$, as $n \to \infty$. This proves, in particular, that $T_1 = O(1)$. For T_2, we see that the second series in (1) has decreasing terms which must be $O(\frac{1}{\nu})$, and hence $T_2 = O(1)$.

Again

$$S_2 = \sum_{\nu=1}^{n-1} \frac{\log \nu}{\nu+1} \left\{ \sum_{k=1}^{\nu} + \sum_{k=\nu+1}^{\infty} \right\}^{1/2}$$

$$\leq \sum_{\nu=1}^{n-1} \frac{\log \nu}{\nu+1} \left\{ \sum_{k=1}^{\nu} \gamma_k^2 \sin^2 \frac{k\pi}{2\nu} \right\}^{1/2} +$$

$$+ \sum_{\nu=1}^{n-1} \frac{\log \nu}{\nu+1} \left\{ \sum_{k=\nu+1}^{\infty} \gamma_k^2 \sin^2 \frac{k\pi}{2\nu} \right\}^{1/2}$$

$$\leq \sum_{\nu=1}^{n-1} \frac{\log \nu}{\nu+1} \left\{ \sum_{k=1}^{\nu} \frac{k^2 \gamma_k^2 \pi^2}{4\nu^2} \right\}^{1/2} + \sum_{\nu=1}^{n-1} \frac{\log \nu}{\nu+1} \left\{ \sum_{k=\nu+1}^{\infty} \gamma_k^2 \right\}^{1/2}$$

$$\leq A \sum_{\nu=1}^{n-1} \frac{\log \nu}{\nu^2} \left\{ \sum_{k=1}^{\nu} k^2 \gamma_k^2 \right\}^{1/2} + A \sum_{\nu=1}^{n-1} \frac{\log \nu}{\nu} \left\{ \sum_{k=\nu+1}^{\infty} \gamma_k^2 \right\}^{1/2}$$

$$= O(1), \text{ from } (1).$$

The proof is complete.