CHAPTER - I

INTRODUCTION

§1. Let f be a 2π-periodic function which is Lebesgue integrable over $[-\pi, \pi]$. A lacunary Fourier series corresponding to the function f is the trigonometric series

$$\sum_{k=1}^{\infty} \left(a_{n_k} \cos n_k x + b_{n_k} \sin n_k x\right) \qquad \text{(L)}$$

with an infinity of gaps (n_k, n_{k+1}), where $\{n_k\}$ $(k \in N)$ is a strictly increasing sequence of natural numbers satisfying some condition, called the lacunarity condition or gap condition or gap hypothesis, such that

$$(n_{k+1} - n_k) \longrightarrow \infty \quad \text{as} \quad k \longrightarrow \infty \; ; \qquad (1.1)$$

and

$$a_{n_k} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cdot \cos n_k t \cdot dt \, ,$$

$$b_{n_k} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cdot \sin n_k t \cdot dt \, .$$

The numbers a_{n_k} , b_{n_k} are called the Fourier coefficients of the function f.

Throughout, the function under consideration is assumed to be 2π-periodic and Lebesgue integrable over $[-\pi , \pi]$.

The theory of lacunary Fourier series has its origin in the construction of examples of functions having various pathological properties. According to Weierstrass [38], Riemann told his students in 1861 that the function represented by the lacunary Fourier series $\sum_{n=1}^{\infty} (\sin n^2 x)/n^2$ is everywhere continuous but nowhere differentiable. As Weierstrass was not able to prove this (Hardy [14] has later on proved that this function has no finite differential coefficient for any irrational value of x), he gave (1872) his famous example of such a function which is represented again by the lacunary Fourier series, namely the series $\sum_{n=1}^{\infty}(a^n \cos b^n x)$, where $0 < a < 1$, b is an odd integer ≥ 3 and $ab > 1 + 3\pi/2$ (also refer : [14]). In 1892 Hadamard [13] introduced lacunary Fourier series (L), with $\{n_k\}$ satisfying the lacunarity condition

$$\frac{n_{k+1}}{n_k} \geq q > 1 \quad \text{for all} \quad k \in N , \qquad (1.2)$$

known after his name as the Hadamard lacunarity condition, for the study of functions that can not be analytically continued beyond their circle of convergence (see also Fabry [7] ; Pólya [27]). F. Riesz [28] has used lacunary series to construct a continuous function of bounded variation whose Fourier coefficients are not of order $o(1/n)$. For other such examples refer [2 ; P.242], [9], [33] . In fact, the theory of lacunary Fourier series has always been not only one of the main tools for proving or disproving many instances of conjectures in analysis but also a source of interesting results in analysis since its very appearance in 1861.

The systematic study of the properties of lacunary Fourier series started from the first decade of this century. This study can conveniently be divided into two parts. In the first one the series (L) is considered as a series of almost independent random variables, or what is same as a series of almost independent functions, and the properties are studied through probabilistic methods by many well known mathematicians like Kolmogorov, Steinhaus, Kac, Marcinkiewicz, Zygmund and others. A survey of number of properties in this direction with an extensive bibliography can be found in [16], [11] . The second part roughly constitutes the study of the following general problem of Mandlebrojt [21] : assume $\{n_k\}$ is given, suppose we know a property of a function given by (L) on an interval, or in the neighbourhood of a point, or on a perfect set without interior points; to what extent does it give information on (L) as a whole ? This problem gives rise to several other problems including the problems of uniqueness or continuation of functions. To day number of properties of lacunary Fourier series are known which are connected to these problems and there are interesting applications of the same methods to a series of number theoratic problems. Here one can observe that in many of these properties, the sequence $\{n_k\}$ is assumed to satisfy the Hadamard lacunarity condition (1.2) and these properties distinguish themselves greatly from those of the non-lacunary Fourier series. For example, such a lacunary Fourier series converges almost everywhere (just as a non-lacunary Fourier series can diverge everywhere); if such a trigonometric series is summable by a method T^* in a set of positive measure then it is a Fourier series of a function belonging to L^p $(p > 1)$; such a series can not converge to zero in a set of positive measure unless all the coefficients are zero; properties of the function represented by such a series can be extended to the whole real axis from a small interval (refer [2] and [40] also).

The study of the absolute convergence of lacunary Fourier series started with a paper of Fatou [8] in which he proved that if $\{n_k\}$ satisfies (1.2) with $q > 3$ then an everywhere convergent lacunary Fourier series (L) converges absolutely. Later on Sidon showed [2 ; P.246] that if (L) is a Fourier series of a bounded function with $\{n_k\}$ satisfying (1.2) then it converges absolutely. Here too the lacunarity condition is the Hadamard one.

§2. Noble [23] observed in 1954 that very little attention has been paid to the effect of a weaker gap condition than the Hadamard's. He considers well known theorems of Bernšteĭn [2 ; P.154, Theorem 1] and Zygmund [2 ; P.161, Corollary 1] for the absolute convergence of non-lacunary Fourier series and certain results [2 ; P.269] concerning the order of magnitude of Fourier coefficients; and shows that a certain gap condition weaker than the Hadamard's enables us to replace, in these theorems, the hypothesis concerning $f(x)$ on the whole interval $[-\pi , \pi]$ by the same hypothesis on an arbitrary small subinterval of $[-\pi , \pi]$. Since many a times a property of a function is known only locally, the problem considered by Noble can be set (on the line of Mandlebrojt's problem) like this: If the fulfilment of some property of a function f on the whole interval $[-\pi , \pi]$ implies certain conclusions concerning the non-lacunary Fourier series $\sigma(f)$ of f then what lacunae in $\sigma(f)$ guarantees the same conclusions when the property is fulfilled only locally ? Several mathematicians including Noble [23] , Kennedy [17 ; 18 ; 19 ; 20] , Tomić [36 ; 37] , M.Izumi and S.I.Izumi [15] , Chao [6] and others have studied this problem in the recent years and have bettered the earlier results or obtained new results by considering weaker and weaker gap conditions and also by considering the property of a function either on a subset of positive measure or at an arbitrary fixed point of $[-\pi, \pi]$.

We note that the above referred theorems of Bernšteĭn and Zygmund have been generalized later on by Bernšteĭn [2; P.154, Theorem 2], Szàsz [2 ; P.155] and Stečkin [2 ; P.155, P.196] - all of which give sufficiency conditions for the absolute convergence of non-lacunary Fourier series. In view of these generalizations, it will be interesting to carry on investigations of the problem studied by Noble by considering the property of a function in terms of its modulus of continuity or quadratic modulus of continuity or modulus of smoothness of order ℓ ($\ell \in N$) or quadratic modulus of smoothness of order ℓ or L^2- trigonometric best approximation. The present thesis is the outcome of researches carried out by the author, mainly regarding these investigations.

This chapter aims at providing the introduction to the subject matter of the thesis through the recent developments regarding the concerned aspects of the problem. The statements of all the results we have proved in this thesis are included in this chapter only. It can be noted that all the lacunarity conditions considered by us in our results are weaker than the Hadamard's gap condition (1.2).

§3. In order to state the problem more precisely and to give an account of the results obtained, it will be convenient to introduce some definitions and notations at this stage.

Let $x_0 \in [-\pi , \pi]$ be an arbitrary fixed point, and δ be an arbitrary positive real number such that $I = [x_0 - \delta , x_0 + \delta]$ is a subinterval of $[-\pi , \pi]$. Note that $x_0 = 0$ and $\delta = \pi$ gives $I = [-\pi , \pi]$. Let $\omega(u , f)$ $(u > 0 , u \in R)$ denote the modulus of continuity of the function f over $[-\pi , \pi]$ defined by

$$\omega(u , f) = \sup_{\substack{0 \leq |t| \leq u \\ x \in [-\pi, \pi]}} \left\{ \left| f(x + t) - f(x) \right| \right\}. \tag{1.3}$$

We say that f satisfies a Lipschitz condition of order α, $0 < \alpha \leq 1$, in $[-\pi , \pi]$ and write $f \in \mathrm{Lip}\, \alpha [-\pi , \pi]$ if there exists a constant M, depending only on f, such that $\omega(u , f) \leq M \cdot u^{\alpha}$. The modulus of continuity of f over I, denoted by $\omega(\delta , f , I)$, and the class $\mathrm{Lip}\, \alpha(I)$ of functions satisfying the Lipschitz condition in I are defined in a similar way replacing $[-\pi , \pi]$ by I. If f is a function of bounded variation over I (respectively over $[-\pi , \pi]$) then we write it as $f \in BV(I)$ (respectively $f \in BV[-\pi , \pi]$).

As is referred above, Noble [23] studied the gap condition

$$\underset{k \to \infty}{\text{limit inf.}} \frac{N_k}{\log n_k} = \infty, \; N_k = \min\{n_{k+1} - n_k \,,\, n_k - n_{k-1}\} \quad (1.4)$$

which is weaker than the Hadamard lacunarity condition (1.2) and deduced the results concerning the order of magnitude of Fourier coefficients and the absolute convergence of the lacunary Fourier series (L) with $\{n_k\}$ satisfying (1.4). In fact, his theorems concerning the absolute convergence are as follows:

<u>THEOREM 1</u>. (Noble). If (L) is a Fourier series of f with $\{n_k\}$ satisfying (1.4) then

(a). $f \in \text{Lip}\,\alpha(I)$, $\alpha > 1/2$, implies

$$\sum_{k=1}^{\infty} \left(|a_{n_k}| + |b_{n_k}| \right) < \infty \; ; \qquad (1.5)$$

(b). $f \in \text{Lip}\,\alpha(I)$, $\alpha > 0$, and $f \in BV(I)$ implies (1.5).

<u>THEOREM 2</u>. (Noble).(a). If in the hypothesis of Theorem 1(a) α is restricted by $0 < \alpha < 1$ and $\beta > 2/(2\alpha + 1)$ then

$$\sum_{k=1}^{\infty} \left(|a_{n_k}|^{\beta} + |b_{n_k}|^{\beta} \right) < \infty \; ; \qquad (1.6)$$

(b). If in the hypothesis of Theorem 1(b) α is restricted by $0 < \alpha < 1$ and $\beta > 2/(\alpha + 2)$ then (1.6) holds.

In 1956, Kennedy [17] proved Theorem 1 under a less restrictive gap hypothesis (1.1). Theorem 2 under the gap condition (1.1) can easily be deduced on the same line. He then raises the question whether the Theorem 1(a) holds under a still less stringent gap condition

$$\frac{n_k}{k} \longrightarrow \infty \quad \text{as} \quad k \longrightarrow \infty \tag{1.7}$$

and answers it in negative [18] .

Theorems 1(b) and 2(b) are generalized also by relaxing the hypothesis on the function — maintaining gap condition the same. This result is due to S. M. Mazhar [22] who shows that the condition of bounded variation in these theorems can be replaced by a weaker condition of bounded r^{th} variation. Goyal [12] then shows that Theorem 2(b), at the critical index, that is, when $\beta = 2/(\alpha + 2)$, holds under the weaker gap condition (1.1) and when function is of bounded second variation — provided the function satisfies a generalized Lipschitz condition.

It is evident that Noble's Theorem 1 and 2, and the subsequent results mentioned above, are gap analogues of the theorems of Berštein [2 ; P.154, Theorem 1] , Zygmund [2 ; P.161, Corollary 1] and Szàsz [39 ; P.137] . These theorems of Bernštein and Zygmund are later on generalized by the authors themselves [2 ; P.154 and P.160] as under.

THEOREM 3. (Bernšteĭn). If

$$\sum_{n=1}^{\infty} \frac{\omega(1/n,f)}{\sqrt{n}} < \infty \tag{1.8}$$

then the Fourier series $\sigma(f)$ of f converges absolutely.

THEOREM 4. (Zygmund). If $f(x)$ is of bounded variation and

$$\sum_{n=1}^{\infty} \frac{(\omega(1/n,f))^{1/2}}{n} < \infty$$

then the Fourier series $\sigma(f)$ of f converges absolutely.

In 1968, Bojanić and Tomić [4] proved the following 'small gap' analogue of the Theorems 3 and 4.

THEOREM 5. (Bojanić and Tomić). Let $I = [-\delta, \delta]$, (L) be the Fourier series of f with $\{n_k\}$ satisfying the 'small gap' $(n_{k+1} - n_k) \geq 4\pi/\delta$, and $N(x) = \sum_{n_k \leq x} 1$. If

$$\int_1^{\infty} \omega(1/t,f,I)\cdot\left(N^{1/2}(t)/t\right)\cdot dt < \infty$$

or

$$\int_I |df(t)| < \infty \text{ and } \int_1^{\infty} \omega^{1/2}(1/t,f,I)\cdot\left(N^{1/2}(t)/t^{3/2}\right)\cdot dt < \infty$$

then (1.5) holds.

This theorem generalizes Kennedy's results. We also refer here to the generalization of Kennedy's results due to Se-Tin fan [34] who considers the same gap condition (1.1) but the hypothesis on the function in a bit general but complicated way. The results are merely stated, without proof, together with some two dimensional analogous results.

We observe that Theorems 3 and 4 are further generalized by Szàsz and Stečkin [2 ; P.155] .

THEOREM 6. (Szàsz). If

$$\sum_{n=1}^{\infty} \frac{\omega^{(2)}(1/n,f)}{\sqrt{n}} < \infty \quad , \tag{1.9}$$

where $\omega^{(2)}(1/n,f)$ is the quadratic modulus of continuity of f over $[-\pi , \pi]$ given by

$$\omega^{(2)}(1/n,f)= \sup_{0\leq h\leq 1/n} \left\{ \left(\int_{-\pi}^{\pi} |f(x+h) - f(x-h)|^2 dx \right)^{1/2} \right\}, \tag{1.10}$$

then the Fourier series $\sigma(f)$ of a function $f \in L^2[-\pi , \pi]$ converges absolutely.

THEOREM 7. (Stečkin). If

$$\sum_{n=1}^{\infty} \frac{E_n^{(2)}(f)}{\sqrt{n}} < \infty ,$$

where $E_n^{(2)}(f)$ is L^2-trigonometric best approximation to f over $[-\pi , \pi]$ given by

$$E_n^{(2)}(f) = \min_{T_n} \left\{ \left(\int_{-\pi}^{\pi} |f(x) - T_n(x)|^2 dx \right)^{1/2} \right\} \tag{1.11}$$

in which $T_n(x)$ is a trigonometric polynomial of order not higher than n, then the Fourier series $\sigma(f)$ of $f \in L^2[-\pi, \pi]$ converges absolutely.

It may be noted that both these theorems are equivalent and are included in the following more general theorem due to Steĉkin [2 ; P.196].

THEOREM 8. (Steĉkin). If for a given increasing sequence $\{n_k\}$ of natural numbers, we have

$$\sum_{k=1}^{\infty} \frac{\omega^{(2)}(1/n_k , f)}{\sqrt{k}} < \infty , \tag{1.12}$$

where $\omega^{(2)}(1/n_k , f)$ is as in (1.10) with $1/n$ replaced by $1/n_k$, then (1.5) holds for the Fourier series $\sigma(f)$ of $f \in L^2[-\pi, \pi]$.

Observe that with $n_k = k$ for all k, Theorem 8 is Theorem 6.

We see that any kind of gap analogues of these Theorems 6, 7 and 8 are not available in the literature concerning the absolute convergence of lacunary Fourier series. In Chapter II of the present thesis we propose to prove that if $\{n_k\}$ satisfies (1.1) and (L) is a Fourier series of f then (1.5) holds, that is, (L) converges absolutely, even when the hypotheses in Theorems 6 and 7 are satisfied only in a subinterval of $[-\pi, \pi]$. More precisely, we prove the following theorems.

THEOREM 9. If

$$\sum_{n=1}^{\infty} \frac{\omega^{(2)}(1/n,f,I)}{\sqrt{n}} < \infty \tag{1.13}$$

and if $\{n_k\}$ satisfies (1.1), then (1.5) holds for the Fourier series (L) of $f \in L^2(I)$, where $\omega^{(2)}(1/n,f,I)$ is the quadratic modulus of continuity of the function f over an interval I defined by

$$\omega^{(2)}(1/n,f,I) = \sup_{0\le h\le 1/n} \left\{ \left(\int_I |f(x+h)-f(x-h)|^2 dx \right)^{1/2} \right\}. \tag{1.14}$$

THEOREM 10. Theorem 9 holds if (1.13) is replaced by the equivalent condition

$$\sum_{n=1}^{\infty} \frac{E_n^{(2)}(f,I)}{\sqrt{n}} < \infty \quad , \tag{1.15}$$

where $E_n^{(2)}(f,I)$ is the trigonometric best approximation to f in the space $L^2(I)$ given by

$$E_n^{(2)}(f,I) = \inf_{T_n} \left\{ \left(\int_I |f(x) - T_n(x)|^2 dx \right)^{1/2} \right\} \tag{1.16}$$

in which $T_n(x)$ is as in Theorem 7.

We then generalize Theorem 9 by considering the higher order differences of f in the following theorem.

THEOREM 11. Theorem 9 holds if (1.13) is replaced by the more general condition

$$\sum_{n=1}^{\infty} \frac{\omega_\ell^{(2)}(1/n,f,I)}{\sqrt{n}} < \infty , \tag{1.17}$$

where $\omega_\ell^{(2)}(1/n,f,I)$ is the quadratic modulus of smoothness of order ℓ ($\ell \in N$) over an interval I given by

$$\omega_\ell^{(2)}(1/n,f,I) = \sup_{0\le h\le 1/n} \left\{ \left(\int_I \left| \sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} f(x+(2j-\ell)h) \right|^2 dx \right)^{\frac{1}{2}} \right\}. \tag{1.18}$$

Further, we prove the following generalizations of these theorems [25].

<u>THEOREM 12</u>. If

$$\sum_{k=1}^{\infty} \frac{\left(\omega^{(2)}(1/n_k , f , I)\right)^{\beta}}{k^{\beta/2}} < \infty \quad (0<\beta\le 1) \tag{1.19}$$

and if $\{n_k\}$ satisfies (1.1) then (1.6) holds for the Fourier series (L) of $f \in L^2(I)$, where $\omega^{(2)}(1/n_k,f,I)$ is as in (1.14) with $1/n$ replaced by $1/n_k$.

<u>THEOREM 13</u>. Theorem 12 holds if (1.19) is replaced by the more general condition

$$\sum_{k=1}^{\infty} \frac{\left(\omega_\ell^{(2)}(1/n_k, f , I)\right)^{\beta}}{k^{\beta/2}} < \infty , \tag{1.20}$$

where $\omega_\ell^{(2)}(1/n_k,f,I)$ is as in (1.18) with $1/n$ replaced by $1/n_k$.

THEOREM 14. Theorem 12 holds if (1.19) is replaced by the condition

$$\sum_{k=1}^{\infty} \frac{\left(E_{n_k}^{(2)}(f\ ,\ I)\right)^{\beta}}{k^{\beta/2}} < \infty\ , \qquad (1.21)$$

where $E_{n_k}^{(2)}(f\ ,\ I)$ is as in (1.16) with $1/n$ replaced by $1/n_k$.

Observe that Theorems 12, 13 and 14 sharpen Theorems 9, 11 and 10 respectively.

The absolute convergence of a lacunary Fourier series (L), when the underlying function satisfies some hypothesis in a certain subset E of $[-\pi\ ,\ \pi]$, not necessarily a subinterval, was studied first by Kennedy [19]. He proved the following theorem.

THEOREM 15. (Kennedy). If $f \in \mathrm{Lip}\ \alpha(E)$ [1)] , $\alpha > 0$, and $E \subset [-\pi\ ,\ \pi]$ has positive spread [2)] and if $\{n_k\}$ satisfies

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1) refer [19] for the definition. It has also been shown there that there does exist function satisfying a Lipschitz condition in a set of positive measure without satisfying such a condition in an interval.

2) refer [19] for the definition. For example, it has been mentioned there that every set dense in some subinterval of $[-\pi\ ,\ \pi]$ as well as a subset of positive measure has a positive spread. There also exists countable nowhere dense set which has positive spread.

the gap condition

$$\lim_{k \to \infty} \frac{n_{k+1} - n_k}{n_k^{\gamma} \cdot \log n_k} = \infty \text{ (γ is independent of k and $0 < \gamma < 1$)} \tag{1.22}$$

then (1.5) holds for the Fourier series (L) of f provided $2\alpha\gamma + \gamma > 1$, that is, provided $\alpha > \frac{1}{2}(\gamma^{-1} - 1)$.

Kennedy then conjectures that this theorem remains true if the factor $\log n_k$ is suppressed from the gap condition (1.22). He further says that he is unable to decide whether this theorem remains true at the critical index, that is, when $\alpha = \frac{1}{2}(\gamma^{-1} - 1)$. Studying this V. M. Shah [31] proves that Theorem 15 holds at the critical index provided the function f satisfies the generalized Lipschitz condition of the form

$$|f(x+h)-f(x)| = \frac{O(h^{\alpha})}{\ell_1(h) \cdot \ell_2(h) \cdots\cdots \ell_m^{1+\varepsilon}(h)} \quad (\alpha>0,\ \varepsilon>0) \tag{1.23}$$

in E as $h \to +0$ through unrestricted real values, where $m \in N$ and $\ell_1(h) = \log(e + 1/h)$, $\ell_2(h) = \log\log(e^e + 1/h)$, ...

Observe that these theorems are a kind of Bernšteǐn type theorems [2 ; P.154, Theorem 1] when the function satisfies the hypothesis on a subset of $[-\pi , \pi]$ of positive spread.
In Chapter III of the present thesis we propose to prove certain theorems concerning the absolute convergence of lacunary Fourier series (L) when the underlying function satisfies Szàsz or

Stečkin type hypothesis only in a subset E of $[-\pi, \pi]$ of positive measure. To state the results obtained by us, we need the following definition.

DEFINITION. [2 ; P.248]. A strictly increasing sequence $\{n_k\}$ ($k \in N$) of natural numbers is said to satisfy the condition B_2 if $\sup_n P_2(n)$ is finite, where $P_2(n)$ denotes the number of different representations of an integer n in the form

$$n = \varepsilon_1 n_{k_1} + \varepsilon_2 n_{k_2} \quad (\varepsilon_1, \varepsilon_2 = \pm 1;\ n_{k_1}, n_{k_2} \in \{n_k\}).$$

We prove the following theorems [26].

THEOREM 16. If

$$\sum_{n=1}^{\infty} \frac{(\omega^{(2)}(1/n, f, E))^{\beta}}{n^{\beta/2}} < \infty \qquad (0 < \beta \leq 1) \qquad (1.24)$$

and if $\{n_k\}$ satisfies the condition B_2 then (1.6) holds for the Fourier series (L) of f, where $\omega^{(2)}(1/n, f, E)$ is as in (1.14) with I replaced by E.

Considering the higher order differences of f, we prove the following generalization.

THEOREM 17. Theorem 16 holds when (1.24) is replaced by the more general condition

$$\sum_{n=1}^{\infty} \frac{(\omega_{\ell}^{(2)}(1/n, f, E))^{\beta}}{n^{\beta/2}} < \infty, \qquad (1.25)$$

where the quadratic modulus of smoothness $\omega_\ell^{(2)}(1/n, f, E)$ of order ℓ defined over E is as in (1.18) with I replaced by E.

Taking then the hypothesis on the function in terms of modulus of continuity or modulus of smoothness, we prove the following theorem [26].

<u>THEOREM 18</u>. Theorem 16 holds when (1.24) is replaced either by

$$\sum_{n=1}^{\infty} \frac{(\omega(1/n\ ,\ f\ ,\ E))^{\beta}}{n^{\beta/2}} < \infty\ , \tag{1.26}$$

or more generally by

$$\sum_{n=1}^{\infty} \frac{(\omega_\ell(1/n\ ,\ f\ ,\ E))^{\beta}}{n^{\beta/2}} < \infty\ , \tag{1.27}$$

where $\omega(1/n, f, E)$ is the modulus of continuity of f over E given by

$$\omega(1/n, f, E) = \sup_{\substack{0 \leq h \leq 1/n \\ x \in E}} \left\{ |f(x+h) - f(x-h)| \right\} \tag{1.28}$$

and $\omega_\ell(1/n, f, E)$ is the modulus of smoothness of order ℓ of f over E given by

$$\omega_\ell(1/n, f, E) = \sup_{\substack{0 \leq h \leq 1/n \\ x \in E}} \left\{ \left| \sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} f(x+(2j-\ell)h) \right| \right\}. \tag{1.29}$$

Further, the following general results are obtained. For this purpose, we sharpen the inequalities obtained by us (Chapter III, Lemma 2) while proving the above theorems.

THEOREM 19. Theorem 16 holds if (1.24) is replaced by the condition

$$\sum_{k=1}^{\infty} \frac{\left(\omega^{(2)}(1/n_k\ ,\ f\ ,\ E)\right)^{\beta}}{k^{\beta/2}} < \infty, \tag{1.30}$$

where $\omega^{(2)}(1/n_k\ ,\ f\ ,\ E)$ is as in (1.14) with I and 1/n replaced by E and $1/n_k$ respectively.

THEOREM 20. Theorem 16 holds if (1.24) is replaced by the condition

$$\sum_{k=1}^{\infty} \frac{\left(\omega_{\ell}^{(2)}(1/n_k\ ,\ f\ ,\ E)\right)^{\beta}}{k^{\beta/2}} < \infty\ , \tag{1.31}$$

where $\omega_{\ell}^{(2)}(1/n_k\ ,\ f\ ,\ E)$ is as in (1.18) with I and 1/n replaced by E and $1/n_k$ respectively.

THEOREM 21. Theorem 16 holds if (1.14) is replaced either by

$$\sum_{k=1}^{\infty} \frac{\left(\omega(1/n_k\ ,\ f\ ,\ E)\right)^{\beta}}{k^{\beta/2}} < \infty\ , \tag{1.32}$$

or more generally by

$$\sum_{k=1}^{\infty} \frac{\left(\omega_{\ell}(1/n_k\ ,\ f\ ,\ E)\right)^{\beta}}{k^{\beta/2}} < \infty\ , \tag{1.33}$$

where $\omega(1/n_k\ ,\ f\ ,\ E)$ and $\omega_{\ell}(1/n_k\ ,\ f\ ,\ E)$ are as in (1.28) and (1.29) respectively with 1/n replaced by $1/n_k$.

Note that Theorems 19, 20 and 21 are the sharpened versions of Theorems 16, 17 and 18 respectively. Among all these, Theorem 18 is the most general theorem. Finally, in this chapter, we prove the following theorem.

THEOREM 22. Theorem 16 holds if (1.14) is replaced by the condition

$$\sum_{k=1}^{\infty} \frac{\left(E_{n_k}^{(2)}(f, E)\right)^{\beta}}{k^{\beta/2}} < \infty , \qquad (1.34)$$

where $E_{n_k}^{(2)}(f, E)$ is as in (1.16) with I and n replaced by E and n_k respectively.

In the rest of the chapters of the present thesis we propose to study the problem under consideration, when the function satisfies some hypothesis only at a point $x_0 \in [-\pi, \pi]$. The first results regarding such a study concerns the order of magnitude of Fourier coefficients and the absolute convergence of the lacunary Fourier series (L) when the function satisfies some continuity condition only at a point. These results are due to Masako Satô [29 ; 30] . In fact, he proved the following theorems.

THEOREM 23. [29 ; P.404]. Let $0 < \alpha < 1$ and $0 < \gamma < \min\{1-\alpha , (2-\alpha)/3\}$. If (L) is a Fourier series of f with $\{n_k\}$ satisfying

$$k^{2/(2-\alpha-3\gamma)} < n_k < e^{2k/(2+\alpha+\gamma)} , \qquad (1.35)$$

$$|n_{k\pm1} - n_k| > 4e\cdot k\cdot n_k^{\gamma} \quad ; \tag{1.36}$$

and if f satisfies

$$\frac{1}{h^{\gamma}}\int_0^{h^{\gamma}} |f(t) - f(t \pm h)|\,dt = O(h^{\alpha}) \tag{1.37}$$

and

$$\frac{1}{\tau}\int_0^{\tau} |f(t) - f(t \pm h)|\,dt = O(1), \text{ unif. in } \tau \geq h^{\gamma} \ , \tag{1.38}$$

then

$$a_{n_k}\ ,\ b_{n_k} = O(n_k^{-\alpha}) \ . \tag{1.39}$$

<u>THEOREM 24</u>. [30 ; P.509]. Let $1/2 < a < \alpha < 1$, $0 < \gamma < (2 - \alpha)/3$ and $\gamma/2 < \alpha - a \leq (2 - \alpha - \gamma)/4$. If (L) is a Fourier series of f with $\{n_k\}$ satisfying

$$k^{1/(2\alpha-2a-\gamma)} < n_k < e^{2k/(2+\alpha+\gamma)}, \tag{1.40}$$

$$|n_{k\pm1} - n_k| > 4e\cdot k\cdot n_k^{\gamma} \quad ; \tag{1.41}$$

and if f satisfies

$$\frac{1}{h^{\gamma}}\int_0^{h^{\gamma}} |f(t) - f(t \pm h)|^2\,dt = O(h^{2\alpha}) \text{ as } h\to 0 \ , \tag{1.42}$$

and

$$\frac{1}{\tau}\int_0^{\tau}|f(t) - f(t \pm h)|^2 dt = O(1), \text{ unif. in } \tau > h^{\gamma}, \qquad (1.43)$$

then (1.5) holds.

We observe that hypothesis of Theorem 24 is more restrictive than the hypothesis of Theorem 23 and in the proof of Theorem 24 the conclusion (1.39) of Theorem 23 is used. Hence it is natural to ask whether Theorem 24 holds only under the less restrictive hypothesis of Theorem 23. We prove in Chapter IV of the present thesis that this is indeed the case even if $\alpha > (1 - \gamma)/2$. Our result obviously generalizes Theorem 24. When $\alpha \leq (1 - \gamma)/2$ we study the convergence almost everywhere and the absolute summability (C,θ), $\theta \geq 1/2$, of the lacunary Fourier series (L). More precisely we prove the following theorems.

<u>THEOREM 25</u>. Under the hypothesis of Theorem 23, if

$$0 < (1 - \gamma)/2 < \alpha < 1 \qquad (1.44)$$

then (1.5) holds.

<u>THEOREM 26</u>. Under the hypothesis of Theorem 23, if

$$0 < (1 - \gamma)/4 < \alpha < 1 \qquad (1.45)$$

then (L) converges almost everywhere.

<u>THEOREM 27</u>. Under the hypothesis of Theorem 23, if

$$0 < (1 - 5\gamma)/4 < \alpha < 1 \qquad (1.46)$$

then (L) is absolutely summable (C,1/2) and if no additional condition is imposed on α then (L) is absolutely summable (C,1).

In Chapter V of the present thesis we study the order of magnitude of Fourier coefficients and then the absolute convergence of the lacunary Fourier series (L), with $\{n_k\}$ satisfying the lacunarity condition

$$\min\{n_{k+1} - n_k , n_k - n_{k-1}\} > C\cdot F(n_k) , \qquad (1.47)$$

where $F(n_k)$ increases to ∞ as $k \longmapsto \infty$, $F(n_k) \leq n_k$ for all $k \in N$ and $C > 0$ is a constant, when the function satisfies a certain continuity condition only at a point x_0. The continuity condition considered by us is in terms of a function $\omega^*(t)$, where

(i) $\omega^*(0) = 0$, $\omega^*(t) > 0$ as $t \longmapsto +0$,

(ii) $\omega^*(t)$ is an almost increasing function in the neighbourhood of zero, that is to say,

$$\omega^*(t_1) \leq A\, \omega^*(t_2) \quad \text{as} \quad 0 < t_1 < t_2 \longrightarrow 0$$

and

(iii) there exists $\alpha > 0$ such that $\omega^*(t)\, t^{-\alpha}$ is an almost decreasing function, that is to say,

$$\omega^*(t_1)\, t_1^{-\alpha} > B\, \omega^*(t_2)\, t_2^{-\alpha} \quad \text{when} \quad 0 < t_1 < t_2 ,$$

in which A and B are constants. In order to explain the significance of the results established in this chapter, it is desirable to recall briefly the developments that have taken place during recent years regarding the study of the behaviour of Fourier coefficients.

Tomić [36] has proved the following theorem in 1962.

THEOREM 28. (Tomić). If (L) is a Fourier series of f with $\{n_k\}$ satisfying the Hadamard gap condition (1.2) and if the modulus of continuity $\omega(\delta , f , x_0)$ of $f(x)$ at the point x_0 , defined by

$$\omega(\delta , f , x_0) = \sup_{0\leq|h|\leq\delta} \left\{|f(x_0+h)-f(x_0)| \right\}, \qquad (1.48)$$

satisfies $\omega(\delta , f , x_0) = O(\delta^{-\alpha})$ $(0 < \alpha \leq 1)$,then we have

$$a_{n_k} , b_{n_k} = O(n_k^{-\theta}) , \quad \theta = \alpha/(2 + \alpha). \qquad (1.49)$$

This theorem is related to earlier results due to Noble [23] and Kennedy [17 ; 19] . Kennedy [20] then sharpens this estimation (1.49) to

$$a_{n_k} , b_{n_k} = O(1) \left(\log n_k / n_k \right)^{\alpha} \qquad (1.50)$$

keeping hypothesis the same and posing the question if one can possibly suppress the factor $(\log n_k)^{\alpha}$ in (1.50). The affirmative answer was given by J. P. Kahane, M. Izumi and S. I. Izumi [16 ; P.210] . Here we also refer to a paper by Se , Tin-fan (or Hsieh, Ting-fan) [35] who not only gives affirmative answer but also proves [35 ; Theorem 1] that under the hypothesis of Theorem 28 we have $a_{n_k} , b_{n_k} \neq o(n_k^{-\alpha})$.

Tomić [37] then proves, utilising the process given in his note [36] , the following theorem which generalizes the above referred result due to J. P. Kahane, M. Izumi and S. I. Izumi.

THEOREM 29. If $\{n_k\}$ satisfies the Hadamard gap condition (1.2) and

$$f(x_0 \pm t) - f(x_0) = O(1) \cdot \omega^*(t) \quad \text{as } t \longrightarrow +0 ,$$

where $\omega^*(t)$ is as above, then for the Fourier series (L) of f we have

$$a_{n_k} , b_{n_k} = O(1)\, \omega^*(1/n_k) \quad , \text{ if } 0 < \alpha < 1 \text{ in (iii);}$$

$$= O(1) \cdot \log n_k \cdot \omega^*(1/n_k), \text{ if } \alpha = 1 \text{ in (iii).} \quad (1.51)$$

On the other hand M. Izumi and S. I. Izumi have proved the following theorem under the gap condition

$$n_{k+1} - n_k > A\, n_k^{\gamma} \quad (A \text{ is a constant and } 0 < \gamma \leq 1) \quad (1.52)$$

which is weaker than the Hadamard gap (1.2), when the function $f \in \text{Lip}\, \alpha(P)$, that is, f satisfies the α-Lipschitz condition at a point x_0 , namely,

$$|f(x_0 + t) - f(x_0)| \leq A\, |t|^{\alpha} \quad \text{as } t \longrightarrow 0 . \quad (1.53)$$

THEOREM 30. (Izumi). If $\{n_k\}$ satisfies the gap condition (1.52) and $f \in \text{Lip}\, \alpha(P)$ then for the Fourier series (L) of f we have

$$a_{n_k} , b_{n_k} = O(n_k^{-\alpha\gamma}). \quad (1.54)$$

This theorem is a simultaneous generalization of the Theorem 28 and the result due to Kennedy [19 ; Theorem 2]. Chao [6] further generalizes this theorem and proves that if $\{n_k\}$ satisfies (1.47) and $f \in \mathrm{Lip}\,\alpha(P)$ then we have $a_{n_k}, b_{n_k} = O\left((F(n_k))^{-\alpha}\right)$.

We observe that under the Hadamard gap hypothesis (1.2), Theorem 29 is the most general theorem available in the literature and in so much as weaker gap conditions are concerned the result due to Chao is general most known theorem. We prove the following theorem which is a kind of bridge between these two theorems, thus generalizing all the above theorems. We do not know whether our result is the best possible.

THEOREM 31. If

$$f(x_0 \pm t) - f(x_0) = O(1)\,\omega^*(t) \text{ as } t \to +0\,, \qquad (1.55)$$

where $\omega^*(t)$ is as in Theorem 29 and if $\{n_k\}$ satisfies (1.47) then

$$a_{n_k}\,,\; b_{n_k} = O(1)\,\omega^*(1/F(n_k))\,. \qquad (1.56)$$

We also prove in this chapter the following theorem concerning the absolute convergence of lacunary Fourier series (L), with $\{n_k\}$ satisfying the gap condition (1.47), when the function satisfies a certain hypothesis in terms of $\omega^*(t)$. In fact, our theorem is as follows.

THEOREM 32. If $\{n_k\}$ satisfies (1.47) , f satisfies (1.55) and

$$\sum_{n=1}^{\infty} \frac{\left(\omega^*(1/F(n))\right)^{\beta}}{n^{\beta/2}(F(n))^{1-\beta}} < \infty \quad (0 < \beta \leq 1) , \qquad (1.57)$$

then (1.6) holds for the Fourier series (L) of f.

We then show that with $\omega^*(t) = t^{\alpha}$, $0 < \alpha < 1$, the estimation (1.56) gives the result concerning absolute convergence due to Chao [6 ; Theorem 2] and Theorem 32 gives that due to M. Izumi and S. I. Izumi [15 ; Theorem 2].

A look at the hypothesis of Theorem 32 leads to the natural question whether $\omega^*(1/F(n))$ can be replaced by the modulus of continuity $\omega(1/F(n) , f , x_0)$ of f considered at a point x_0 in (1.57). Investigating this question, we prove in Chapter VI of the present thesis that the absolute convergence of lacunary Fourier series (L) is assured when the function satisfies Bernšteĭn type condition in terms of either the modulus of continuity or the modulus of smoothness of order ℓ considered only at a point x_0 . The gap condition considered by us is

$$(n_{k+1} - n_k) > C \cdot F(n_k) , \qquad (1.58)$$

where $F(n_k)$ and C are as in (1.47). Let $\delta = 8\pi/(C\, F(n_T))$, where T is a natural number, and put $I = [x_0 - \delta , x_0 + \delta]$. We then prove the following theorems.

<u>THEOREM 33</u>. If

$$\sum_{k=1}^{\infty} \frac{\left(\omega(A/F(n_k) , f , x_0)\right)^{\beta}}{k^{\beta/2}} < \infty \quad (0 < \beta \leq 1) \quad (1.59)$$

and if $\{n_k\}$ satisfies (1.58) then (1.6) holds for the Fourier series (L) of $f \in L^2(I)$ (for some I), where $\omega(A/F(n_k), f, x_0)$ is the modulus of continuity of f at the point x_0 defined as in (1.48) with δ replaced by $A/F(n_k)$, $A = 24\pi/C + \pi$.

<u>THEOREM 34</u>. Theorem 33 holds if (1.59) is replaced by the condition

$$\sum_{k=1}^{\infty} \frac{\left(\omega_\ell(B/F(n_k)\,,\ f\,,\ x_0)\right)^\beta}{k^{\beta/2}} < \infty\,, \qquad (1.60)$$

where $\omega_\ell(B/F(n_k),\ f\ ,\ x_0)$ is the modulus of smoothness of order ℓ of f at the point x_0, defined by

$$\omega_\ell(B/F(n_k), f, x_0) = \sup_{0 \le h \le B/F(n_k)} \left\{ \left| \sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} f(x_0 + (2j-\ell)h) \right| \right\}$$

in which $B = 8\pi/C + \pi$ and ℓ is an odd natural number.

The following theorem follows from Theorem 34.

<u>THEOREM 35</u>. If

$$\sum_{k=1}^{\infty} \frac{\left(\omega_\ell(B/n_k\,,\ f\,,\ x_0)\right)^\beta}{k^{\beta/2}} < \infty$$

and if $\{n_k\}$ satisfies the Hadamard gap condition (1.2) then (1.6) holds for the Fourier series (L) of $f \in L^2(I)$ (for some I).

We then show that Kennedy's Theorem 15 holds even if the set E is replaced by a single point and $\log n_k$ is suppressed from the gap condition (1.22); thus giving something more than the answer to the question raised by him. Finally, before concluding the chapter, we also investigate his second question about the validity of his theorem at the critical index and obtain a condition on f guaranteeing it.