CHAPTER II

ABSOLUTE CONVERGENCE WHEN THE FUNCTION SATISFIES SOME HYPOTHESES ON A SUBINTERVAL

§1. In this chapter we propose to prove Theorems 9 to 14; which give sufficiency conditions for the absolute convergence of the lacunary Fourier series (L) with $\{n_k\}$ satisfying the gap condition (1.1). These conditions are of this type : for all functions possessing quadratic modulus of continuity or quadratic modulus of smoothness or L^2-trigonometric best approximations (all considered only on a subinterval of $[-\pi, \pi]$) tending sufficiently rapidly to zero, it is possible to secure the absolute convergence of their Fourier series (L).

We have seen in Chapter I that sufficiency conditions for the absolute convergence of the lacunary Fourier series (L), under different lacunarity conditions and when the underlying function belongs to Lip α(I) or BV(I) or possesses a modulus of continuity over I satisfying a certain condition, are studied by Noble [23] , Kennedy [17 ; 19] , Mazhar [22] , Bojanić and Tomić [4] and others. Noble's method is a kind of modification of that due to S. Bernštein [3] . This modification is based on the fact that

$$a_{n_k} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot P(x)\cdot \cos n_k x\cdot dx$$

(with a similar formula for b_{n_k}) for any trigonometric polynomial $P(x)$ with constant term 1 and of degree less than $\min\{n_k - n_{k-1}\ ,\ n_{k+1} - n_k\}$. He shows that given $n \in N$ (large enough) and $\delta > 0$, one can always choose such a $P(x)$ so as to be small enough in $[-\pi\ ,\ -\delta] \cup [\delta\ ,\ \pi]$; thus making $|a_{n_k}|$ and $|b_{n_k}|$ to depend mainly on the behaviour of $f(x)$ over $[-\delta\ ,\ \delta]$. Kennedy [17] has employed the more powerful method of approach to this kind of problems - developed originally by Paley and Wiener [24]. The method employed by Bojanić and Tomić [4] is completely different from these and it depends on various estimates of

$$W_r = \frac{1}{n_r}\sum_{k=1}^{r} n_k\left(|a_{n_k}| + |b_{n_k}|\right) \quad (r = 1,2,\ldots\ldots).$$

We observe that while the Noble's method is freely used later on, for studying the properties of lacunary Fourier series, the method due to Paley and Wiener, employed by Kennedy, is still not used to their best. We employ this method for proving our results in this chapter.

§2. We need the following lemmas. Lemma 1 is a special case of a very general theorem due to Paley and Wiener [24 ; Theorem XLII$'$]. Lemma 2 is a simple consequence of the more general lemma quoted by Kennedy [17 ; Lemma 1].

LEMMA 1. If $f \in L^2(J)$, where J is an interval, and if $(n_{k+1} - n_k) \longrightarrow \infty$ in its Fourier series (L) then $f \in L^2[-\pi, \pi]$.

LEMMA 2. Let $f \in L^2(I)$ and (L) be its Fourier series. Put $n_o = 0$, $n_k = -n_{-k}$ $(k < o)$; $C_{n_o} = 0$, $C_{n_k} = \frac{1}{2}(a_{n_k} - i\, b_{n_k})$ $(k > o)$ and $C_{n_k} = \overline{C}_{n_{-k}}$ $(k < o)$. If $\{n_k\}$ satisfies (1.2) and

$$(n_{k+1} - n_k) \geq 8\pi\, \delta^{-1} \quad \text{for all } k \tag{2.1}$$

then

$$\sum_{-\infty}^{\infty} |C_{n_k}|^2 \leq 8\, \delta^{-1} \int_I |f(x)|^2\, dx . \tag{2.2}$$

Proof of Lemma 2. Since a_{n_k}, $b_{n_k} \longrightarrow 0$ as $k \longrightarrow \infty$, $\{|C_{n_k}|\}$ is a bounded sequence; and hence we have

$$\sum_{-\infty}^{\infty} |C_{n_k}|\, r^{|n_k|} < \infty \qquad (0 < r < 1). \tag{2.3}$$

If we put

$$\phi(r,x) = \sum_{-\infty}^{\infty} C_{n_k}\, r^{|n_k|} \exp(i n_k x) \quad (0 < r < 1) \tag{2.4}$$

for all real x, then its existence is assured by (2.3) and we get

$$\phi(r,x) = \sum_{k=1}^{\infty} \frac{1}{2} (a_{n_k} - ib_{n_k}) (\cos n_k x + i\sin n_k x)\, r^{n_k}$$

$$+ \sum_{k=1}^{\infty} \frac{1}{2} (a_{n_k} + ib_{n_k}) (\cos n_k x - i\sin n_k x)\, r^{n_k}$$

$$= \sum_{k=1}^{\infty} (a_{n_k} \cos n_k x + b_{n_k} \sin n_k x)\, r^{n_k} . \qquad (2.4)'$$

Since $f \in L^2(I)$ and (1.2) holds, therefore $f \in L^2[-\pi, \pi]$ by Lemma 1 ; and hence by a known theorem [39 ; P.87] it follows that

$$f(x) = L^2 - \lim_{r \to 1} \phi(r,x) \qquad (|x| \le \pi) . \qquad (2.5)$$

Now, on account of (2.1), (2.3), (2.4) and (2.5), we can apply the lemma quoted by Kennedy [17 ; Lemma 1] to obtain the inequality (2.2). This completes the proof of Lemma 2.

<u>LEMMA 3</u>. For $s \in R$, $s > 0$, if $s \le |n_k|$ then

$$\int_0^{\pi/s} \sin^2 |n_k| h \, dh > \frac{\pi}{4s} ,$$

or more generally

$$\int_0^{\pi/s} \sin^{2\ell} |n_k| h \, dh > \frac{\pi}{2^{\ell+1}} \frac{1}{s} ,$$

where $\ell \in N$.

<u>Proof of Lemma 3.</u>

$$\int_0^{\pi/s} \sin^2 |n_k| h \, dh = \frac{1}{|n_k|} \int_0^{(|n_k|/s)\pi} \sin^2 t \, dt$$

$$> \frac{1}{s} \frac{1}{1 + [|n_k|/s]} \int_0^{[|n_k|/s]\pi} \sin^2 t \, dt$$

$$= \frac{1}{s} \frac{[|n_k|/s]}{1 + [|n_k|/s]} \int_0^{\pi} \sin^2 t \, dt$$

$$\geq \frac{1}{s} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4s} \, ,$$

using

$$1 \leq [|n_k|/s] \leq |n_k|/s < [|n_k|/s] + 1 \quad ,$$

where $[\ ]$ denotes the integral part.

Also,

$$\int_0^{\pi/s} \sin^{2\ell} |n_k| h \, dh = \frac{1}{|n_k|} \int_0^{(|n_k|/s)\pi} \sin^{2\ell} t \, dt$$

$$> \frac{1}{s} \frac{[|n_k|/s]}{1 + [|n_k|/s]} \int_0^{\pi} \sin^{2\ell} t \, dt$$

$$> \frac{1}{s} \cdot \frac{1}{2} \cdot \frac{2\ell - 1}{2\ell} \cdot \frac{2\ell - 3}{2\ell - 2} \cdots \cdots \frac{1}{2}\pi$$

$$> \frac{\pi}{2^{\ell+1} s} .$$

This completes the proof of Lemma 3.

Proof of Theorem 9. Put $n_o = 0$, $n_k = -n_{-k}$ $(k < o)$; $C_{n_o} = 0$, $C_{n_k} = \frac{1}{2}(a_{n_k} - ib_{n_k})$ $(k > o)$ and $C_{n_k} = \bar{C}_{n_{-k}}$ $(k < o)$. We assume throughout, without loss of generality, that (2.1) holds. In view of (1.1) this can be achieved, if necessary, by adding to $f(x)$ a polynomial in $\exp(in_k x)$, a process which affects neither the hypothesis nor the conclusion of the theorem.

Now, put

$$g(x) = f(x + h) - f(x - h) \tag{2.6}$$

and

$$C^*_{n_k} = 2i\, C_{n_k} \sin n_k h . \tag{2.7}$$

Then $|C^*_{n_k}| \leq 2|C_{n_k}|$ and hence by (2.3) we have

$$\sum_{-\infty}^{\infty} |C^*_{n_k}|\, r^{|n_k|} < \infty \qquad (0 < r < 1). \tag{2.8}$$

Put

$$g(r,x) = \sum_{-\infty}^{\infty} C^*_{n_k}\, r^{|n_k|} \exp(in_k x) \qquad (0 < r < 1) \tag{2.9}$$

for all real x; then its existence is assured by (2.8) and we have

$$\phi(r,x+h) - \phi(r,x-h)$$

$$= \sum_{-\infty}^{\infty} C_{n_k} r^{|n_k|} \left(\exp\left(in_k(x+h)\right) - \exp\left(in_k(x-h)\right)\right)$$

$$= \sum_{-\infty}^{\infty} C_{n_k} r^{|n_k|} \exp(in_k x)\, 2i \sin n_k h$$

$$= g(r,x).$$

This identity, together with (2.5) and (2.6) gives

$$g(x) = L^2 - \lim_{r\to 1} g(r,x) \qquad (|x| \le \pi). \tag{2.10}$$

It follows from (2.1), (2.8), (2.9), (2.10) and Lemma 2 that

$$\sum_{-\infty}^{\infty} |C^{*}_{n_k}|^2 \le 8\, \delta^{-1} \int_I |g(x)|^2\, dx\,. \tag{2.11}$$

Hence by (2.6) and (2.7) we get

$$4 \sum_{k,T} |C_{n_k}|^2 \sin^2 |n_k| h \le 8\, \delta^{-1} \int_I |f(x-h) - f(x-h)|^2 dx \tag{2.12}$$

in which the summation is over values of k such that

$$n_T/2 \le |n_k| \le n_T \qquad (T \in N).$$

Integrating both the sides of (2.12) with respect to h over $(0\,,\, 2\pi/n_T)$, we get

$$\sum_{k,T} |c_{n_k}|^2 \int_0^{2\pi/n_T} \sin^2 |n_k| h \, dh$$

$$\leq 2\,\delta^{-1} \int_0^{2\pi/n_T} dh \left(\int_I |f(x+h) - f(x-h)|^2 dx \right). \quad (2.13)$$

Now, by Lemma 3, taking $s = (n_T / 2)$ in it, we get

$$\int_0^{2\pi/n_T} \sin^2 |n_k| h \cdot dh > \frac{\pi}{2\, n_T} \quad \text{as } |n_k| \geq (n_T / 2).$$

Then from (2.13) we get

$$\frac{\pi}{2\, n_T} \sum_{k,T} |c_{n_k}|^2 \leq 2\,\delta^{-1}\, \frac{2\pi}{n_T} \left(\omega^{(2)}(2\pi / n_T , f , I) \right)^2.$$

Therefore

$$\sum_{k,T} |c_{n_k}|^2 \leq C(\delta) \left(\omega^{(2)}(1 / n_T , f , I) \right)^2 , \quad (2.14)$$

using $\omega^{(2)}(\lambda/n , f , I) \leq C(\lambda) \cdot \omega^{(2)}(1/n , f , I)$ for $\lambda > 0$, where $C(\delta)$ and $C(\lambda)$ are constants depending on δ and λ respectively.

Now let p be a positive integer. Either the set of integers k for which $2^p < |n_k| \leq 2^{p+1}$ is empty or there is a member of this set, say $T = T(p)$, which has largest modulus, and in the later case the set is included in the set of k for which $n_T / 2 \le |n_k| \le n_T$. Thus, in either case we have

$$\sum_{2^p < |n_k| \le 2^{p+1}} |c_{n_k}|^2 \le \sum_{k,T} |c_{n_k}|^2$$

$$\le C(\delta) \left(\omega^{(2)}(1/n_T , f , I) \right)^2$$

$$\le C(\delta) \left(\omega^{(2)}(1/2^p , f , I) \right)^2.$$

Therefore by Cauchy's inequality

$$\sum_{2^p < |n_k| \le 2^{p+1}} |c_{n_k}| \le C_1(\delta) \cdot 2^{p/2} \, \omega^{(2)}(1/2^p , f , I) ,$$

and hence

$$\sum_{-\infty}^{\infty} |c_{n_k}| \le C_1(\delta) \sum_{p=1}^{\infty} \omega^{(2)}(1/2^p , f , I) \cdot 2^{p/2} , \qquad (2.15)$$

where $C_1(\delta) = (C(\delta))^{1/2}$. But

$$\omega^{(2)}(1/n , f , I) / \sqrt{n} \ge 0$$

for all n and form a decreasing sequence, hence on account of (1.13) and Cauchy's condensation test, the series on the right hand side of (2.15) converges. Therefore

$$\sum_{-\infty}^{\infty} |c_{n_k}| < \infty$$

giving (1.5) and hence the Theorem.

Remark. With $I = [-\pi , \pi]$ and without the gap condition (1.1), Theorem 9 is Theorem 6 due to Szàsz.

Proof of Theorem 10. We have [1 ; Appendix §7]

$$E_n^{(2)}(f , I) < C_1 \cdot \omega^{(2)}(1/n , f , I)$$

and

$$\omega^{(2)}(1/n, f , I) \leq C_2 \frac{1}{n} \sum_{k=0}^{n-1} E_k^{(2)}(f , I),$$

where C_1 and C_2 are constants depending on $x_0 - \delta$, $x_0 + \delta$ and the norm of f. We also have [2 ; P.160] : if $u_n \geq 0$, $v_n \geq 0$; $\{u_n\}$ and $\{v_n\}$ both are decreasing as $n \longrightarrow \infty$, $u_n < C v_n$ and $v_n < (C/n) \sum_{k=0}^{n-1} u_k$, then

$$\sum_{n=1}^{\infty} (u_n / \sqrt{n}) \quad \text{and} \quad \sum_{n=1}^{\infty} (v_n / \sqrt{n})$$

both converge or diverge together. Since $\omega^{(2)}(1/n , f , I)$ and $E_n^{(2)}(f , I)$ are non negative and both form a decreasing sequence, in view of these quoted results Theorem 10 is obviously proved.

<u>Proof of Theorem 11.</u> Let $\{n_k\}$ $(k \in Z)$ and $\{C_{n_k}\}$ $(k \in Z)$ be as in the proof of Theorem 9. Assume again that (2.1) holds. Put

$$g(x) = \sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} f(x + (2j-\ell)h) \qquad (2.6)'$$

and

$$C_{n_k}^{*} = C_{n_k} \exp(-in_k \ell h) \left(\exp(2in_k h) - 1\right)^{\ell}$$

$$= 2^{\ell} C_{n_k} \exp(-in_k \ell h)\ (-1)^{\ell} \exp\left(i\ell(n_k h - \pi/2)\right) \sin^{\ell} n_k h. \qquad (2.7)'$$

Then $|C_{n_k}^{*}| \leq 2^{\ell} |C_{n_k}|$ and hence by (2.3) we have

$$\sum_{-\infty}^{\infty} |C_{n_k}^{*}|\ r^{|n_k|} < \infty \qquad (0 < r < 1). \qquad (2.8)'$$

Put

$$g(r,x) = \sum_{-\infty}^{\infty} C_{n_k}^{*}\ r^{|n_k|} \exp(in_k x) \qquad (0 < r < 1) \qquad (2.9)'$$

for all real x, then its existence is assured by $(2.8)'$ and we have

$$\sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} \phi(r, x + (2j-\ell)h)$$

$$= \sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} \left(\sum_{-\infty}^{\infty} C_{n_k}\ r^{|n_k|} \exp\left(in_k(x+(2j-\ell)h)\right)\right)$$

$$= \sum_{-\infty}^{\infty} C_{n_k} \cdot r^{|n_k|} \cdot \exp(in_k x - in_k \ell h) \cdot \left(\sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} \exp(in_k j2h) \right)$$

$$= \sum_{-\infty}^{\infty} C_{n_k} \cdot r^{|n_k|} \cdot \exp(in_k x) \cdot \exp(-i\ell n_k h) \left(\exp(2in_k h) - 1 \right)^{\ell}$$

$$= \sum_{-\infty}^{\infty} C^{*}_{n_k} \, r^{|n_k|} \exp(in_k x)$$

$$= g(r, x).$$

This together with (2.5) and (2.6)' gives

$$g(x) = L^2\text{—}\lim_{r \to 1} g(r, x) \qquad (|x| \leq \pi). \qquad (2.10)'$$

It follows from (2.1), (2.8)', (2.9)', (2.10)' and the Lemma 2 that

$$\sum_{-\infty}^{\infty} |C^{*}_{n_k}|^2 \leq 8\, \delta^{-1} \int_I |g(x)|^2 dx. \qquad (2.11)'$$

Hence by (2.6)' and (2.7)' we obtain

$$2^{2\ell} \sum_{k;T} |C_{n_k}|^2 \sin^{2\ell} |n_k| h$$

$$\leq 8\, \delta^{-1} \int_I \left| \sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} f(x + (2j-\ell)h) \right|^2 dx, \qquad (2.12)'$$

where $\sum_{k,T}$ has the same meaning as that in the inequality (2.12). Further, by Lemma 3, taking $s = (n_T / 2)$ in it, we have

$$\int_0^{2\pi/n_T} \sin^{2\ell}|n_k|h \, dh > \frac{\pi}{2^{\ell} \cdot n_T} \quad \text{as } |n_k| \geq (n_T / 2).$$

Using this alongwith (2.11)' , replacing $\omega^{(2)}(1/n , f , I)$ by $\omega_\ell^{(2)}(1/n , f , I)$ in the proof of Theorem 9 and proceeding analogously, this theorem is proved.

Remark. With $\ell = 1$, Theorem 11 is Theorem 9. With $I = [-\pi , \pi]$ and without the gap condition (1.1) Theorem 10 is Theorem 7 due to Stečkin.

For proving the rest of the results of this chapter we need the following lemmas. Lemma 4 is due to Stečkin [32 ; Lemma 2].

LEMMA 4. If $u_n \geq 0$ for all $n \in N$, $u_n \not\equiv 0$ and if the function $F(u)$ is concave, increasing and such that $F(0)=0$, then

$$\sum_{n=1}^{\infty} F(u_n) \leq 2 \sum_{n=1}^{\infty} F\left(\frac{u_n + u_{n+1} + \cdots\cdots\cdots}{n} \right).$$

LEMMA 5. Under the hypothesis of our Lemma 2 we have

$$\sum_{|n_k| \geq n_p} |c_{n_k}|^2 \leq C(\delta) \left(\omega^{(2)}(1/n_p , f , I) \right)^2, \qquad (2.16)$$

or more generally

$$\sum_{|n_k| \geq n_p} |c_{n_k}|^2 \leq C(\delta) \left(\omega_\ell^{(2)} (1/n_p , f , I) \right)^2 , \qquad (2.17)$$

and

$$\sum_{|n_k| \geq n_p} |c_{n_k}|^2 \leq C(\delta) \left(E_{n_p}^{(2)} (f , I) \right)^2 , \qquad (2.18)$$

where $C(\delta)$ is some constant depending on δ.

<u>Proof of Lemma 5</u>. Let $g(x)$ and $C_{n_k}^*$ be as in (2.6) and (2.7) respectively. Then proceeding as in the proof of Theorem 9 we shall get the inequality (2.11). Using the definition of $g(x)$ and $C_{n_k}^*$, we obtain from (2.11)

$$4 \sum_{-\infty}^{\infty} |c_{n_k}|^2 \sin^2 |n_k| h \leq 8 \delta^{-1} \int_I |f(x+h) - f(x-h)|^2 dx. \qquad (2.19)$$

Integrating both the sides of (2.19) with respect to h over $(0 , \pi/n_p)$, $p \in N$, we get

$$4 \sum_{-\infty}^{\infty} |c_{n_k}|^2 \int_0^{\pi/n_p} \sin^2 |n_k| h \, dh$$

$$\leq 8 \delta^{-1} \int_0^{\pi/n_p} dh \left(\int_I |f(x+h) - f(x-h)|^2 dx \right). \qquad (2.20)$$

Now, by Lemma 3 , taking $s = n_p$ in it, we have

$$\int_0^{\pi/n_p} \sin^2 |n_k| h \, dh > \frac{\pi}{4n_p} \quad \text{if } |n_k| \geq n_p . \qquad (*)$$

Therefore, from (2.20), using

$$\omega^{(2)}(\lambda/n , f , I) \leq C(\lambda) \ \omega^{(2)}(1/n , f , I)$$

($\lambda > 0$, $C(\lambda)$ is a constant depending on λ), we get

$$\frac{\pi}{n_p} \sum_{|n_k| \geq n_p} |C_{n_k}|^2 \leq 8 \ \delta^{-1} \frac{\pi}{n_p} \left(\omega^{(2)}(\pi / n_p , f , I) \right)^2$$

$$\leq 8 \ \delta^{-1} \frac{\pi}{n_p} \cdot C \cdot \left(\omega^{(2)}(1/n_p , f , I) \right)^2$$

from which (2.16) follows.

Further, if we let $g(x)$ and $C^*_{n_k}$ to be as in (2.6)' and (2.7)' respectively then proceeding as in the proof of Theorem 11 we shall get the inequality (2.11)'. Again, by Lemma 3, taking $s = n_p$ in it, we get

$$\int_0^{\pi/n_p} \sin^{2\ell} |n_k| h \, dh > \frac{\pi}{2^{\ell+1} n_p} \qquad \text{if } |n_k| \geq n_p . \qquad (**)$$

Using this alongwith (2.11)' and proceeding analogously as as in the proof of (2.16), the inequality (2.17) is proved.

Finally, let

$$T_{n_p}(x) = \sum_{-n_p}^{n_p} \alpha_m \cdot \exp(imx) \qquad (2.21)$$

be any trigonometric polynomial of order not higher than n_p $(p \in N)$. If we write

$$T_{n_p}(r,x) = \sum_{-n_p}^{n_p} \alpha_m \, r^{|m|} \exp(imx) \qquad (0 < r < 1) \qquad (2.22)$$

for all real x, then since $T_{n_p} \in L^2[-\pi , \pi]$ it obviously follows that

$$T_{n_p}(x) = L^2 \text{-}\lim_{r \to 1} T_{n_p}(r , x) \qquad (|x| \leq \pi). \qquad (2.23)$$

Put

$$g(x) = f(x) - T_{n_p}(x) \qquad (2.24)$$

and

$$A_k = \begin{cases} -\alpha_k & , \text{ if } k \neq n_s \text{ and } |k| \leq n_p , \\ C_{n_s} - \alpha_{n_s} & , \text{ if } k = n_s \text{ and } |k| \leq n_p ; \\ 0 & , \text{ if } k \neq n_s \text{ and } |k| > n_p , \\ C_{n_s} & , \text{ if } k = n_s \text{ and } |k| > n_p . \end{cases} \qquad (2.25)$$

Then, for $|k| > n_p$, $|A_k| = 0$ if $k \neq n_s$ and $|A_k| = |C_{n_s}|$ if $k = n_s$. Hence by (2.3) we have

$$\sum_{-\infty}^{\infty} |A_k| \, r^{|k|} < \infty \qquad (0 < r < 1), \qquad (2.26)$$

and if we put

$$g(r,x) = \sum_{-\infty}^{\infty} A_k \, r^{|k|} \exp(ikx) \qquad (0 < r < 1) \qquad (2.27)$$

for all real x, then its existence is assured by (2.26) and we get

$$\phi(r,x) - T_{n_p}(r,x) = \sum_{-\infty}^{\infty} C_{n_k} \, r^{|n_k|} \exp(in_k x) - \sum_{-n_p}^{n_p} \alpha_m \, r^{|m|} \exp(imx)$$

$$= \sum_{-\infty}^{\infty} A_k \, r^{|k|} \exp(ikx)$$

$$= g(r,x).$$

This, together with (2.5), (2.23) and (2.24), implies

$$g(x) = L^2 - \lim_{r \to 1} g(r,x) \qquad (|x| \le \pi). \qquad (2.28)$$

Now, on account of (2.1), (2.26), (2.27) and (2.28) we can apply the lemma quoted by Kennedy [17 ; Lemma 1] to obtain

$$\sum_{-\infty}^{\infty} |A_k|^2 \le 8 \, \delta^{-1} \int_I |g(x)|^2 \, dx \, . \qquad (2.29)$$

Therefore, from (2.24) and the definition (2.25) of A_k , we get

$$\sum_{|n_k| \geq n_p} |C_{n_k}|^2 \leq C(\delta) \int_I |f(x) - T_{n_p}(x)|^2 \, dx . \tag{2.30}$$

Since (2.30) holds for arbitrary trigonometric polynomial $T_{n_p}(x)$ of order not higher than n_p, we get the inequality (2.18). This completes the proof of Lemma 5.

<u>Proof of Theorem 12.</u> Let $n_o = 0$, $n_k = - n_{-k}$ $(k < o)$;

$C_{n_o} = 0$, $C_{n_k} = \frac{1}{2}(a_{n_k} - ib_{n_k})$ $(k > o)$ and $C_{n_k} = \bar{C}_{n_{-k}}$ $(k < o)$.

We assume throughout, without loss of generality, that (2.1) holds. In view of (1.1), this can be achieved, if necessary, by adding to f(x) a polynomial in $\exp(in_k x)$, a process which affects neither the hypothesis nor the conclusion of the theorem. Then putting

$$r_{n_p} = \sum_{|n_k| \geq n_p} |C_{n_k}|^2$$

in the inequality (2.16) of Lemma 5, we get

$$r_{n_p}^{\beta/2} \leq C \left(\omega^{(2)}(1/n_p , f , I)\right)^{\beta}, \tag{2.31}$$

where C is some constant depending on δ.
Now, applying the Lemma 4 with $u_k = |C_{n_k}|^2$ $(k \in Z)$ and

and $F(u) = u^{\beta/2}$, we obtain using (2.31)

$$\sum_{-\infty}^{\infty} |c_{n_k}|^{\beta} = \sum_{|k|=1}^{\infty} |c_{n_k}|^{\beta}$$

$$= 2 \sum_{k=1}^{\infty} F(|c_{n_k}|^2)$$

$$\leq 4 \sum_{k=1}^{\infty} F(r_{n_k} / k)$$

$$= 4 \sum_{k=1}^{\infty} (r_{n_k} / k)^{\beta/2}$$

$$= 4 \sum_{k=1}^{\infty} (r_{n_k}^{\beta/2} / k^{\beta/2})$$

$$\leq C \cdot \sum_{k=1}^{\infty} (\omega^{(2)}(1/n_k , f , I))^{\beta} / k^{\beta/2}$$

$$< \infty$$

on account of (1.19). Therefore (1.6) holds and this completes the proof of Theorem 12.

Remark. With $\beta = 1$, $I = [-\pi , \pi]$, without the lacunarity condition and taking $\{n_k\}$ as an arbitrary sequence of natural numbers, this theorem reduces to Theorem 8 due to Stečkin. Thus, Theorem 12 tells us that if $\{n_k\}$ satisfies (1.1) and if (L) is a Fourier series of f then (L) converges absolutely even when the hypothesis in the Stečkin's Theorem 8 is satisfied only in a subinterval of $[-\pi , \pi]$.

Proof of Theorems 13 and 14. Applying the inequalities (2.17) and (2.18) instead of the inequality (2.16) in the proof of Theorem 12 and proceeding analogously, we get Theorems 13 and 14 respectively.

Remark. With $\ell = 1$, Theorem 13 gives Theorem 12.

Note. Since $\omega^{(2)}(\delta , f , I)$ and $\omega_{\ell}^{(2)}(\delta , f , I)$ are non-decreasing functions of δ, we have

$$\omega^{(2)}(1/n_k , f , I) \leq \omega^{(2)}(1/k , f , I)$$

and

$$\omega_{\ell}^{(2)}(1/n_k , f , I) \leq \omega_{\ell}^{(2)}(1/k , f , I)$$

for all $k \in N$. Therefore it may be noted that Theorems 12 and 13 generalize Theorems 9 and 11 respectively.

Further, the set of all the trigonometric polynomials of order not higher than n_k contains the set of all the trigonometric polynomials of order not higher than k. Hence

$$\left\{\left(\int_I |f(x) - T_k(x)|^2 dx\right)^{1/2}\right\} \subset \left\{\left(\int_I |f(x) - T_{n_k}(x)|^2 dx\right)^{1/2}\right\}$$

giving $E_{n_k}^{(2)}(f , I) \leq E_k^{(2)}(f , I)$ for all $k \in N$. This shows that Theorem 14 generalizes Theorem 10.

It is quite natural to ask now whether the subinterval I in our results can be replaced by a subset E of $[-\pi , \pi]$ of positive measure. We investigate this problem in the next chapter.