CHAPTER IV

CONVERGENCE AND SUMMABILITY WHEN THE FUNCTION SATISFIES A CERTAIN CONTINUITY CONDITION ONLY AT A POINT

§1. Our study of the properties of lacunary Fourier series (L) depend mainly on two things — first, the localness and the type of the hypothesis to be satisfied by the underlying function and the second is the kind of gaps in the Fourier series. The results of the previous two chapters reveal the connection between these two things — in a sense that when the hypothesis is relaxed from subinterval to subset of positive measure, the gap condition is strengthened from (1.1) to the condition B_2. Now, from this chapter onwards, we consider the hypothesis on a function only at a point and study the convergence, summability and the order of magnitude of Fourier coefficients of lacunary Fourier series. We have mentioned in Chapter I that first results regarding such study are Theorems 23 and 24 due to Masako Satô and that in the proof of Theorem 24 the conclusion of Theorem 23 is used. This means in the hypothesis of Theorem 24 γ is required to be conditioned by $0 < \gamma < \min\{1 - \alpha , (2 - \alpha)/3\}$. That is, geometrically speaking, α and γ must be chosen in such a way that the point (α , γ) lies in the interior of the triangular region bounded by $\alpha = 1/2$, $\alpha + \gamma = 1$ and $\gamma = 0$. In this chapter we propose to prove Theorems 25, 26 and 27 — all of which are proved only under the less restrictive[1] hypothesis of Theorem 23. In fact, in the case of the absolute convergence (Theorem 25), we have been able to exhibit that the above triangular region, for the choice of (α, γ), can be enlarged to the quadrilateral bounded by $\gamma = 0$, $\alpha + \gamma = 1$, $\alpha + 3\gamma = 2$ and $2\alpha + \gamma = 1$. In case (α, γ) lies outside this region, we

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1) Since

$$\gamma/2 < \alpha - a \leq (2 - \alpha - \gamma)/4 \quad \text{implies}$$

$$\text{implies } 2 - \alpha - 3\gamma > 4\alpha - 4a - 2\gamma$$

we have

$$2/(2 - \alpha - 3\gamma) < 1/(2\alpha - 2a - \gamma) \;;$$

and hence (1.40) implies (1.35).
Also, using Cauchy's inequality and (1.42), we get (1.37) as under:

$$\frac{1}{h^\gamma} \int_0^{h^\gamma} |f(t) - f(t \pm h)|\,dt = \int_0^{h^\gamma} 1 \cdot \left(|f(t) - f(t \pm h)|/h^\gamma \right) dt$$

$$\leq h^{\gamma/2} \left(\frac{1}{h^{2\gamma}} \int_0^{h^\gamma} |f(t) - f(t \pm h)|^2\,dt \right)^{1/2}$$

$$= O(h^\alpha).$$

Similarly (1.43) implies (1.38).

It now follows that hypothesis of Theorem 24 is more restrictive than that of Theorem 23.

study the convergence almost everywhere and the absolute summability (C,θ) , $\theta \geq 1/2$, of lacunary Fourier series (L) in Theorems 26 and 27.

§2. We need the following lemma.

LEMMA. If $0 < \alpha < 1$, $0 < \gamma < (2-\alpha)/3$ and if $\{n_k\}$ satisfies (1.36) then

$$n_k \geq C \cdot k^{\delta} \tag{4.1}$$

for all $k \geq 5$ and for any $\delta < 2/(1-\gamma)$, where C is some constant such that $0 < C < 1$.

Proof. We prove the Lemma by induction. Choose C , $0 < C < 1$, such that $n_k \geq C\, k^{\delta}$ for $k = 5$. Assume $n_p \geq C\, p^{\delta}$ for some $p \geq 5$. Then, using (1.36) and observing that $C < C^{\gamma}$, we get

$$\begin{aligned} n_{p+1} &> n_p + 4\, e\, p\, n_p^{\gamma} \\ &> C\, p^{\delta} + 4\, e\, C^{\gamma}\, p^{1+\delta\gamma} \\ &> C\, p^{\delta} \left(1 + 4\, e\, p^{1+\delta\gamma-\delta}\right) . \end{aligned} \tag{4.2}$$

Since

(i) $\delta < 2/(1-\gamma)$ implies $1 + \delta\gamma - \delta > -1$,

(ii) $0 < \gamma < (2-\alpha)/3$ implies $2 < 2/(1-\gamma) < 6$

and

(iii) $p \geq 5$ implies

$$(1 + 1/p)^6 = 1 + (1/p)\left(6 + 15/p + 20/p^2 + 15/p^3 + 6/p^4 + 1/p^5\right)$$

$$\leq 1 + (1/p)\left(6 + 3 + 0{\cdot}8 + 15/5^3 + 6/5^4 + 1/5^5\right)$$

$$< 1 + 4\, e/p\,,$$

we obtain

$$C(p + 1)^\delta < C\, p^\delta (1 + 1/p)^6$$

$$< C\, p^\delta (1 + 4\, e/p)$$

$$< C\, p^\delta \left(1 + 4\, e p^{1+\delta\Upsilon-\delta}\right)\,. \tag{4.3}$$

It follows from (2.2) and (2.3) that $n_{p+1} \geq C(p + 1)^\delta$ and hence the Lemma.

<u>Proof of Theorem 25</u>. We have $2\alpha/(1 - \Upsilon) > 1$ by (1.44). Choose S such that $2\alpha/(1 - \Upsilon) > S > 1$ and put $\delta = S/\alpha$. Then, since $\alpha\delta > 1$, using (1.39) and applying the inequality (4.1) of the Lemma we get

$$\sum_{k=1}^{\infty} \left(|a_{n_k}| + |b_{n_k}| \right) = O(1) \sum_{k=1}^{\infty} 1/n_k^\alpha$$

$$= O(1) \sum_{k=1}^{\infty} 1/k^{\alpha\delta}$$

$$= O(1)$$

and hence the theorem.

Proof of Theorem 26. We have $4\alpha/(1-\gamma) > 1$ by (1.45). Choose S such that $4\alpha/(1-\gamma) > S > 1$ and put $\delta = S/2\alpha$. Then, since $2\alpha\delta > 1$, using (1.39) and applying the inequality (4.1) of the Lemma we get

$$\sum_{k=1}^{\infty} (a_{n_k}^2 + b_{n_k}^2) = O(1) \sum_{k=1}^{\infty} 1/k^{2\alpha\delta}$$

$$= O(1)$$

which implies that $f \in L^2[-\pi, \pi]$. Therefore, by Carleson's theorem [5], the Fourier series (L) of f converges almost everywhere and hence the theorem.

Proof of Theorem 27. For a real number t, other than a negative integer, put

$$E_n^t = \binom{n+t}{n} \quad \text{when } n \in N \text{ and } E_0^t = 1 .$$

Denoting the n^{th} Cesàro mean of order $\theta > 0$ by $\sigma_n^\theta(x)$, replacing the absent terms in (L) by zeros and considering the equality quoted by T. M. Flett [10 ; P.115], we get

$$n_k \left| \sigma_{n_k}^{\theta}(x) - \sigma_{n_k-1}^{\theta}(x) \right|$$

$$= \frac{1}{E_{n_k}^{\theta}} \left| \sum_{p=1}^{k} E_{n_k - n_p}^{\theta-1} \cdot n_p \cdot (a_{n_p} \cos n_p x + b_{n_p} \sin n_p x) \right| . \quad (4.4)$$

Let $\theta = 1$. Using (1.39) and the inequality (4.1) of the Lemma, we obtain from (2.4)

$$\left| \sigma_{n_k}(x) - \sigma_{n_k - 1}(x) \right|$$

$$= \frac{1}{n_k(n_k + 1)} \left| \sum_{p=1}^{k} n_p \cdot \left(a_{n_p} \cos n_p x + b_{n_p} \sin n_p x \right) \right|$$

$$= O(1) \quad \frac{1}{n_k^2} \sum_{p=1}^{k} n_p^{1-\alpha}$$

$$= O(1) \frac{k}{n_k^2 \cdot n_k^{\alpha - 1}}$$

$$= O(1) \cdot \frac{1}{k^{\alpha\delta + (\delta - 1)}} \quad . \tag{4.5}$$

Since $2/(1 - \gamma) > 2$ and (4.1) holds for any $\delta < 2/(1 + \gamma)$, we choose δ such that $2 < \delta < 2/(1 - \gamma)$. This together with (4.5) gives

$$\sum_{k=1}^{\infty} \left| \sigma_{n_k}(x) - \sigma_{n_k - 1}(x) \right| < \infty$$

which implies the absolute summability (C,1) of the lacunary Fourier series (L).

Let $\theta = 1/2$. Becuase $(2\alpha + 1)/(1 - \gamma) > 1$, and (1.46) implies

$$2(2\alpha + \gamma + 1) \; / \; 3(1 - \gamma) > 1 \; ,$$

we choose S such that

$$1 < S < \min\left\{\frac{2\alpha + 1}{1 - \gamma}, \frac{2(2\alpha + \gamma + 1)}{3(1 - \gamma)}\right\}$$

and put

$$\delta = \max.\left\{2S/(2\alpha + 1)\ ,\ 3S/(2\alpha + \gamma + 1)\right\}.$$

This choice of δ and S gives

$$\delta < 2/(1 - \gamma)$$

and

$$(2\alpha + 1)\delta/2 > 1\ ;\ (2\alpha + \gamma + 1)\delta/2 - 1/2 > 1. \qquad (4.6)$$

Since

(i) $E_n^\theta \simeq n^\theta$,

(ii) (1.39) implies $n_p\left(|a_{n_p}| + |b_{n_p}|\right) = O(n_p^{1-\alpha})$

and

(iii) (1.36) implies $|n_k - n_p| > 4\,e\,k\,n_k^\gamma$ for $p = 1, 2, \ldots\ldots, k-1$,

we obtain from (4.4) and the inequality (4.1) of the Lemma

$$\left|\sigma_{n_k}^\theta(x) - \sigma_{n_k-1}^\theta(x)\right|$$

$$= \frac{O(1)}{n_k^{1+\theta}}\left(n_k^{1-\alpha} + \sum_{p=1}^{k-1}\left(\frac{n_p^{1-\alpha}}{|n_k - n_p|^{1-\theta}}\right)\right)$$

$$= O(1)\left(\frac{1}{n_k^{\alpha+\theta}} + \frac{k}{n_k^{\alpha+\theta}|4\,e\,k\,n_k^\gamma|^{1-\theta}}\right)$$

$$= O(1)\left(1/k^{\delta(\alpha+\theta)} + 1/k^{\delta(\alpha+\theta+\gamma-\gamma\theta)-\theta}\right)$$

$$= O(1)\left(k^{-(2\alpha+1)\delta/2} + k^{-(2\alpha+\gamma+1)\delta/2+1/2}\right) .$$

Hence, using (4.6) we get

$$\sum_{k=1}^{\infty} \left| \sigma_{n_k}^{\theta}(x) - \sigma_{n_k-1}^{\theta}(x) \right| < \infty$$

which implies absolute summability (C , 1/2) of the lacunary Fourier series (L).

This completes the proof of the theorem.

Completing the study of the convergence and summability of the lacunary Fourier series (L) under the hypothesis of Theorem 23 of Masako Satô, we observe that much work is done later on concerning the order of magnitude of Fourier coefficients of (L) — considering hypothesis on the function of course only at a point. We proceed to study, in the next chapter, the order of magnitude of Fourier coefficients and hence the absolute convergence of (L).