CHAPTER VI

ABSOLUTE CONVERGENCE WHEN THE FUNCTION SATISFIES A BERNSTEIN TYPE HYPOTHESIS ONLY AT A POINT

§1. It can be observed that with $\beta = 1$ and with $F(n) = n$ for all $n \in N$, the hypothesis (1.57) in Theorem 32, Chapter V, resembles that in the Bernšteĭn's Theorem 3 — except for the fact that instead of the modulus of continuity $\omega(1/F(n), f, x_0)$ at a point x_0, $\omega^*(1/F(n))$ is considered in our theorem. We further continue investigating whether $\omega^*(1/F(n))$ can be replaced by $\omega(1/F(n), f, x_0)$. Here, we wish to point out that the modulus of continuity $\omega(1/F(n), f, x_0)$ at the point x_0, defined as in (1.48) with δ replaced by $1/F(n)$, does not satisfy the subadditivity property

$$\omega(t_1 + t_2, f, x_0) \leq \omega(t_1, f, x_0) + \omega(t_2, f, x_0)$$

and hence for it the property (iii) of $\omega^*(t)$ does not hold. Nevertheless, we are able to prove, in this concluding chapter of the present thesis, Theorem 33, 34 and 35 which involves Bernšteĭn type hypotheses on the function in terms of modulus of continuity or modulus of smoothness considered only at a point $x_0 \in [-\pi, \pi]$. We see that, since $\omega(\delta, f, x_0)$ or $\omega_\ell(\delta, f, x_0)$ does not satisfy property (iii) of $\omega^*(\delta)$, the only loss in this transition from $\omega^*(\delta)$ to $\omega(\delta, f, x_0)$ is that the constants

A and B can not be taken out from $\omega(A/F(n_k), f, x_o)$ and $\omega_\ell(B/F(n_k), f, x_o)$ respectively. However, this loss is not significant because the only effect of it is that the statements of our results are in terms of $\omega(A/F(n_k), f, x_o)$ or $\omega_\ell(B/F(n_k), f, x_o)$ instead of the good looking form $\omega(1/F(n_k), f, x_o)$ or $\omega(1/F(n_k), f, x_o)$ of the modulus of continuity or modulus of smoothness respectively. Interestingly, it follows from our Theorem 33 that Kennedy's Theorem 15 holds even if the set E having positive spread is replaced by a single point and the factor $\log n_k$ is suppressed from the gap hypothesis (1.22). This gives considerably more than the answer to the question raised by Kennedy [19]. Incidently, our this result betters the Theorems A and B (Chapter V) due to Chao and Izumi. Finally, we also study the condition on f which guarantees Theorem 15 at the critical index.

Observe that the method we used for proving Theorem 32 in Chapter V depends on the fact that Fourier coefficients of $f(x)$ are the same as those of $f(x)\,P(x)$ for any trigonometric polynomial $P(x)$ with constant term 1 and of degree less than $\min\{n_k - n_{k-1},\ n_{k+1} - n_k\}$. Here $P(x)$ is generally taken to be either the polynomial considered by Noble (e.g., refer : [23] , [19] , [20] , [22]) or the Fejér kernel (e.g., refer : [36] , [37] , [6]) or the Jackson kernel (e.g., refer : [6])

— all of which are very small outside a small subinterval of $[-\pi , \pi]$; thus making $|a_{n_k}|$ and $|b_{n_k}|$ to depend mainly on the local behaviour of $f(x)$. This technique is freely used later on for studying the properties of lacunary Fourier series (L). The proofs of our results in this chapter depend on a different method, namely, the Paley-Wiener method of approach to this kind of problems, which was also employed in proving the results of Chapter II.

§2. We need the following lemmas.

<u>LEMMA 1.</u> The modulus of continuity $\omega(t,f,x_o)$ of f at x_o, defined as in (1.48) with δ replaced by t, is such that $\omega(0,f,x_o)=0$, $\omega(t,f,x_o) > 0$ $(t > 0)$ and $\omega(t,f,x_o)$ is increasing.

<u>Proof</u>. Obviously $\omega(0,f,x_o)=0$ and for $t > 0$, $\omega(t,f,x_o)>0$. When $0 < t_1 < t_2$ then since

$$\{|f(x_o + h) - f(x_o)| : 0 \leq |h| \leq t_1\}$$

is a subset of

$$\{|f(x_o + h) - f(x_o)| : 0 \leq |h| \leq t_2\} ,$$

it follows that $\omega(t,f,x_o)$ is increasing.

<u>LEMMA 2.</u> Let $\delta = 8\pi / (C\, F(n_T))$ $(T \in N)$, $I = (x_o-\delta , x_o+\delta)$ and $f \in L^2(I)$ for some I. Put $n_o = 0$, $n_k = -n_{-k}$ $(k < o)$; $C_{n_o} = 0$, $C_{n_k} = \frac{1}{2}(a_{n_k} - ib_{n_k})$ $(k > o)$ and $C_{n_k} = \bar{C}_{n_{-k}}$ $(k < o)$.

If

$$(n_{k+1} - n_k) \geq 8\pi\delta^{-1} \quad \text{for all } k \tag{6.1}$$

then

$$\sum_{-\infty}^{\infty} |C_{n_k}|^2 \leq 8\,\delta^{-1} \int_I |f(x)|^2\, dx \;, \tag{6.2}$$

$$\sum_{|n_k| \geq n_T} |C_{n_k}|^2 \leq 64\, \left(\omega(A/F(n_T)\;,\; f\;,\; x_o)\right)^2 \tag{6.3}$$

or more generally

$$\sum_{|n_k| \geq n_T} |C_{n_k}|^2 \leq D\, \left(\omega_\ell(B/F(n_T)\;,\; f\;,\; x_o)\right)^2 \;, \tag{6.4}$$

where D is a constant and ℓ is an odd natural number.

Proof. The inequality (6.2) is infact the conclusion of Lemma 2, Chapter II. Now let $g(x)$ and $C^*_{n_k}$ be as in (2.6) and (2.7) respectively. Then proceeding as in the proof of Theorem 9, Chapter II, we get the inequality (2.11); and hence by (2.6) and (2.7) we obtain

$$4\sum_{-\infty}^{\infty} |C_{n_k}|^2 \sin^2|n_k|h \;\leq 8\,\delta^{-1} \int_I |f(x+h) - f(x-h)|^2 dx \;. \tag{6.5}$$

Integrating both the sides of (6.5) with respect to h over $(0\;,\; \pi/n_T)$ we get

$$4\sum_{-\infty}^{\infty}|c_{n_k}|^2\int_0^{\pi/n_T}\sin^2|n_k|h\,dh$$

$$\leq 8\,\frac{C\,F(n_T)}{8\pi}\int_0^{\pi/n_T}dh\left(\int_{x_0-8\pi/(C\,F(n_T))}^{x_0+8\pi/(C\,F(n_T))}|f(x+h)-f(x-h)|^2dx\right). \quad (6.6)$$

Now, we observe from (6.6) that $x\in[x_0-\delta\,,\,x_0+\delta]$ and hence we can put $x=x_0-\delta+\eta$, where $0\leq\eta\leq 2\delta$. Then, by Lemma 1, using $\pi/n_T\leq\pi/F(n_T)$ we obtain

$$|f(x+h)-f(x-h)|=|f(x_0-\delta+\eta+h)-f(x_0)+f(x_0)-f(x_0-\delta+\eta-h)|$$

$$\leq|f(x_0+\eta+h-\delta)-f(x_0)|+|f(x_0+\eta-h-\delta)-f(x_0)|$$

$$\leq\omega(\eta+\delta+h,\ f,\ x_0)+\omega(\eta+\delta+h\ ,\ f,\ x_0)$$

$$\leq 2\ \omega(3\delta+h\ ,\ f\ ,\ x_0)$$

$$\leq 2\ \omega\big(24\pi/(C.F(n_T))+\pi/F(n_T)\ ,\ f,\ x_0\big)$$

$$=2\ \omega\big(A/F(n_T),\ f,\ x_0\big),\quad A=24\pi/C+\pi. \qquad (6.7)$$

Using (6.7) and (*) of Chapter II, we get from (6.6)

$$4\sum_{|n_k|\geq n_T}|c_{n_k}|^2\cdot\frac{\pi}{4n_T}\leq 4\sum_{|n_k|\geq n_T}|c_{n_k}|^2\int_0^{\pi/n_T}\sin^2|n_k|h\,dh$$

$$\leq 8 \frac{C\,F(n_T)}{8\pi} \frac{\pi}{n_T} \frac{16\pi}{C\,F(n_T)} 4\left(\omega(A/F(n_T), f, x_o)\right)^2.$$

That is

$$\sum_{|n_k| \geq n_T} |C_{n_k}|^2 \leq 64 \left(\omega(A/F(n_T), f, x_o)\right)^2 .$$

The inequality (6.3) is thus established.

Further, if we let $g(x)$ and $C^*_{n_k}$ to be as in (2.6)' and (2.7)' respectively then proceeding as in the proof of Theorem 11, Chapter II, we get the inequality (2.11)'. Hence by (2.6)' and (2.7)' we obtain

$$2^{2\ell} \sum_{-\infty}^{\infty} |C_{n_k}|^2 \sin^{2\ell}|n_k|h$$
$$\leq 8\,\delta^{-1} \int_I \left| \sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} f(x+(2j-\ell)h)\right|^2 dx. \qquad (6.8)$$

Integrating both the sides of (6.8) with respect to h over $(0 , \pi/n_T)$, we get

$$2^{2\ell} \sum_{-\infty}^{\infty} |C_{n_k}|^2 \int_0^{\pi/n_T} \sin^{2\ell}|n_k|h \, dh$$

$$\leq 8 \frac{C\,F(n_T)}{8\pi} \int_0^{\pi/n_T} dh \left(\int_{x_o-8\pi/(C\,F(n_T))}^{x_o+8\pi/(C\,F(n_T))} \left| \sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} f(x+(2j-\ell)h)\right|^2 dx \right) \qquad (6.9)$$

Now, in view of (6.9) we have $x \in [x_o - \delta , x_o + \delta]$ and hence we can put $x = x_o - \delta + \eta$, where $0 \le \eta \le 2\delta$. Then, by Lemma 1, using $\pi/n_T \le \pi/F(n_T)$ and the fact that ℓ is an odd natural number, we obtain

$$\left| \sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} f(x + (2j - \ell)h) \right|$$

$$= \left| \sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} f(x_o + \eta - \delta + (2j - \ell)h) \right|$$

$$= \left| \sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} f\left(x_o + (2j-\ell)\left(h + (\eta - \delta)/(2j-\ell)\right)\right) \right|$$

$$\le \omega_\ell(h + \eta - \delta , f , x_o)$$

$$\le \omega_\ell(\delta + h , f , x_o)$$

$$\le \omega_\ell(8\pi/(C\, F(n_T)) + \pi/n_T , f , x_o)$$

$$\le \omega_\ell(8\pi/(C\, F(n_T)) + \pi/F(n_T) , f , x_o)$$

$$= \omega_\ell(B/F(n_T) , f , x_o) , \quad B = 8\pi/C + \pi . \tag{6.10}$$

Using (6.10) and (**) of Chapter II, alongwith (6.9), and proceeding analogously as in the proof of the inequality (6.3), the inequality (6.4) is proved. This completes the proof of the Lemma 2.

<u>Proof of Theorem 33</u>. Let $n_o = 0$, $n_k = -n_{-k}$ $(k < o)$;

$c_{n_o} = 0$, $c_{n_k} = \frac{1}{2}(a_{n_k} - i\, b_{n_k})$ $(k > o)$, $c_{n_k} = \bar{c}_{n_{-k}}$ $(k < o)$.

We assume without loss of generality that (6.1) holds. In view of (1.58) this can be achieved, if necessary, by adding to $f(x)$ a polynomial in $\exp(in_k x)$, a process which affects neither the hypothesis nor the conclusion of the theorem (It should be noted that for different δ, that is for different T, this polynomial may of course be different). Then Lemma 2 holds and putting

$$r_{n_T} = \sum_{|n_k| \geq n_T} |c_{n_k}|^2$$

in the inequality (6.3) we get

$$r_{n_T}^{\beta/2} \leq C \left(\omega\left(A/F(n_T) , f , x_o\right)\right)^{\beta} , \tag{6.11}$$

where C is some constant.

Then, using (6.11) and (1.59) instead of (2.31) and (1.19) respectively and proceeding analogously as in the proof of Theorem 12, Chapter II, this theorem is proved.

<u>Proof of Theorem 34</u>. Applying the inequality (6.4) instead of the inequality (6.3), replacing $\omega\left(A/F(n_T), f , x_o\right)$ by $\omega_l\left(B/F(n_T), f , x_o\right)$ throughout and proceeding

analogously as in the proof of Theorem 33, this theorem is proved.

Remark. The inequalities (6.3) and (6.4) generalize the inequalities (3.19) and (3.20) respectively, of Lemma 3, Chapter III (also refer : Remark, made at the end of the proof of Lemma 3, Chapter III). Of course it being meaningless to talk of quadratic modulus of continuity at a point, we have to consider either the modulus of continuity or the modulus of smoothness at a point; and then we observe that in the inequalities (3.19) and (3.20) the gap condition involved is the condition B_2 and the quadratic modulus of continuity on a subset E of positive measure, while in the inequalities (6.3) and (6.4) we consider the gap condition (1.58) but the modulus of continuity is considered only at a point (while studying (3.19) and (3.20), we keep in mind that $\omega^{(2)}(1/n_k, f, E) \leq C\,\omega(1/n_k, f, E)$). It may be noted here that taking $F(n_k) = n_k$ for all k, the gap condition (1.58) gives rise to the Hadamard gap condition which satisfies the condition B_2 ; meaning by , there are sequences satisfying the condition B_2 as well as the condition (1.58).

Proof of Theorem 35. Taking $F(n_k) = n_k$ for all $k \in N$, we see that this theorem follows immediately from Theorem 34.

Remark. With modulus of continuity or modulus of smoothness considered on a subset E of positive measure instead of at a point x_o, and with the gap condition as the condition B_2, Theorem 33 and 34 are equivalent to our Theorem 21.

§3. Observe that with $F(n_k) = n_k^{\gamma} k^{\theta}$ $(0 < \gamma < 1,\ \theta \geq 0)$, the gap condition (1.58) reduces to the gap condition

$$n_{k+1} - n_k > C\, n_k^{\gamma} k^{\theta} \quad (0 < \gamma < 1,\ \theta \geq 0)\ , \tag{6.12}$$

and with $\theta = 0$ (6.12) then reduces to the gap condition

$$n_{k+1} - n_k > C\, n_k^{\gamma} \quad (0 < \gamma < 1). \tag{6.13}$$

The gap condition (6.13) is weaker than the gap condition (1.22) (considered by Kennedy) with $\log n_k$ suppressed.

In this section we propose to prove the following corollaries.

<u>COROLLARY 1</u>. If $f \in \mathrm{Lip}\,\alpha(P)$ $(0 < \alpha < 1)$ and if $\{n_k\}$ satisfies the gap condition (6.12) then for the Fourier series (L) of f, we have (1.6) provided $\alpha\beta\gamma + \alpha\beta\theta > (1 - \beta/2)(1 - \gamma)$.

<u>COROLLARY 2</u>. If $\{n_k\}$ satisfies the gap condition (6.12) with $C > S$ and if

$$|f(x_o \pm h) - f(x_o)| = \frac{O(h^{\alpha})}{\left(\ell_1(h)\cdot \ell_2(h) \cdot \ldots \cdot \ell_m^{1+\varepsilon}(h)\right)^{(2\alpha\gamma+2\alpha\theta+1-\gamma)/2(1-\gamma)}} \tag{6.14}$$

then (1.6) holds in case $\alpha\beta\gamma + \alpha\beta\theta = (1 - \beta/2)(1 - \gamma)$, where $m \in N$, $\varepsilon > 0$; for $h > 0$, $\ell_1(h), \ell_2(h), \ldots, \ell_m(h)$ are as in (1.23) and S is the sum of the series

$$\delta + \frac{\delta(\delta - 1)}{2!}\frac{1}{k} + \frac{\delta(\delta - 1)(\delta - 2)}{3!}\frac{1}{k^2} + \ldots\ldots\ldots$$

in which $\delta = (1 + \theta)/(1 - \beta)$ and $k \in N$.

We need the following lemma due to Chao [6 ; Proof of Theorem 2]. We have already used its part (a) in Chapter V and since its part (b) is not proved there precisely, we prove it here for the sake of completeness.

LEMMA 3. If $\{n_k\}$ satisfies (6.12) then

(a). $n_k > A\, k^{\delta}$ for any $\delta < (1 + \theta)/(1 - \gamma)$ and for all sufficiently large k, where A is some constant ;

(b). $n_k \geq k^{\delta}$ for $\delta = (1 + \theta)/(1 - \gamma)$ and for all k provided $C > S$, where S is as in Corollary 2.

Proof of Lemma 3. We prove (b) by induction. Since $n_1 \in N$, $n_1 \geq 1^{\delta}$. Assume that $n_k \geq k^{\delta}$ for any $k \in N$. Then, since $\delta = (1 + \theta)/(1 - \gamma)$ implies $\gamma\delta + \theta = \delta - 1$, and $C > S$, we get from (6.12)

$$n_{k+1} > n_k + C\, n_k^{\gamma}\, k^{\theta}$$
$$\geq k^{\delta} + C\, k^{\delta\gamma+\theta}$$

$$\geq k^{\delta} + k^{\delta-1}\left(\delta + \frac{\delta(\delta-1)}{2!}\,\frac{1}{k} + \frac{\delta(\delta-1)(\delta-2)}{3!}\,\frac{1}{k^2} + \cdots\cdots\right)$$

$$= (k+1)^{\delta} .$$

Therefore (b) is proved.

<u>Proof of Corollary 1.</u> We have

$$\alpha\beta\gamma + \alpha\beta\theta > (1 - \beta/2)\ (1 - \gamma) .$$

Hence

$$2\alpha\beta\gamma + 2\alpha\beta\theta + 2\alpha\beta\gamma\theta - 2\alpha\beta\theta\cdot\gamma > (2 - \beta)\ (1 - \gamma)$$

which means

$$2\alpha\beta\gamma(1 + \theta) > (2 - \beta)\ (1 - \gamma) - 2\alpha\beta\theta\ (1 - \gamma).$$

Therefore

$$(1 + \theta)/(1-\gamma) > (2 - \beta - 2\alpha\beta\theta)/(2\alpha\beta\gamma).$$

Then choose δ such that

$$(1 + \theta)/(1 - \gamma) > \delta > (2 - \beta - 2\alpha\beta\theta)/(2\alpha\beta\gamma). \qquad (6.15)$$

Now, since

(1) $\{n_k\}$ satisfies (6.12),

(2) $f \in \mathrm{Lip}\,\alpha(P)$ implies $\omega(\delta\ , f\ , x_o) = O(\delta^{\alpha})$

and

(3) the inequality (6.15) implies $\alpha\beta\gamma\delta + \alpha\beta\theta + \beta/2 > 1$,

we get by Lemma 3(a)

$$\sum_{k=1}^{\infty} \frac{\left(\omega(A/F(n_k), f, x_o)\right)^{\beta}}{k^{\beta/2}} = \sum_{k=1}^{\infty} \frac{O(1)}{\left(F(n_k)\right)^{\alpha\beta} \cdot k^{\beta/2}}$$

$$= \sum_{k=1}^{\infty} \frac{O(1)}{(n_k^{\gamma} k^{\theta})^{\alpha\beta} k^{\beta/2}}$$

$$= \sum_{k=1}^{\infty} \frac{O(1)}{k^{\alpha\beta\gamma\delta + \alpha\beta\theta + \beta/2}}$$

$$= O(1).$$

In view of the fact that $f \in \text{Lip}\,\alpha(P)$ implies f is continuous at x_0, which gives f is bounded in some neighbourhood I of x_0 and hence that $f \in L^2(I)$, we can apply Theorem 3 to complete the proof of the corollary.

Proof of Corollary 2. For simplicity we take $m = 2$. Since $\{n_k\}$ satisfies (6.12) with $C > S$, Lemma 3(b) holds. Therefore, observing that the hypothesis $\alpha\beta\gamma + \alpha\beta\theta = (1 - \beta/2)(1 - \gamma)$ implies

$$(1 + \theta)/(1 - \gamma) = \delta = (2 - \beta - 2\alpha\beta\theta)/2\alpha\beta\gamma ,$$

and the equation (6.14) implies

$$\omega(A/F(n_k), f, x_0) = \frac{O(1)\,(A/F(n_k))^{\alpha}}{\left(\ell_1(A/F(n_k))\cdot \ell_2^{1+\varepsilon}(A/F(n_k))\right)^{(2\alpha\gamma+2\alpha\theta+1-\gamma)/2(1-\gamma)}} ,$$

we obtain

$$\sum_{k=1}^{\infty} \frac{(\omega(A/F(n_k), f, x_0))^{\beta}}{k^{\beta/2}} = \sum_{k=1}^{\infty} \left(\frac{O(1)\,(1/n_k^{\gamma} k^{\theta})^{\alpha\beta}}{k^{\beta/2}\cdot \ell_1(A/n_k^{\gamma} k^{\theta})\,\ell_2^{1+\varepsilon}(A/n_k^{\gamma} k^{\theta})} \right)$$

$$= \sum_{k=1}^{\infty}\left(\frac{O(1)}{k^{\alpha\beta\gamma\delta + \alpha\beta\theta + \beta/2}\left(\ell_1(A/k^{\delta\gamma+\theta})\,\ell_2^{1+\varepsilon}(A/k^{\delta\gamma+\theta})\right)}\right)$$

$$= \sum_{k=1}^{\infty}\left(\frac{O(1)}{k\,\log^k\,\log^{1+\varepsilon} k}\right)$$

$$= O(1).$$

Therefore applying Theorem 33, the corollary is proved.

Remark 1. First taking $\beta = 1$, and then taking $\beta = 1$, $\theta = 0$, these corollaries give following four statements.

(I) If $f \in \mathrm{Lip}\,\alpha(P)$ $(0 < \alpha < 1)$ and if $\{n_k\}$ satisfies (6.12) then the Fourier series (L) of f converges absolutely provided $2\alpha\gamma + 2\alpha\theta + \gamma > 1$.

(II) If $f \in \mathrm{Lip}\,\alpha(P)$ $(0 < \alpha < 1)$ and if $\{n_k\}$ satisfies (6.13) then the Fourier series (L) of f converges absolutely if $2\alpha\gamma + \gamma > 1$, that is, if $\alpha > \frac{1}{2}(\gamma^{-1} - 1)$.

(III) Statement (I) holds at the critical index, that is, when $2\alpha\gamma + 2\alpha\theta + \gamma = 1$ if

$$|f(x_0 \pm h) - f(x_0)| = \frac{O(h^{\alpha})}{\ell_1(h)\cdot\ell_2(h)\;\ldots\ldots\;\ell_m^{1+\varepsilon}(h)} \qquad (**)$$

(IV) In case $\alpha = \frac{1}{2}\left(\gamma^{-1} - 1\right)$, statement (II) holds provided (**) holds.

Observe that statement (I) (respectively (II)) sharpens Theorem A (respectively Theorem B) due to Chao (respectively due to Izumi) stated at the end of Chapter V ; while statement (III) (respectively (IV)) gives a condition on f which guarantees these theorems at the critical index.

Remark 2. Kennedy has raised two questions in his paper [19] concerning Theorem 15 stated in Chapter I. First, whether the factor $\log n_k$ can be suppressed from the gap condition (1.22); and second, whether Theorem 15 holds at the critical index or breaks down. Izumi's Theorem B throws some light on Kennedy's first question; in fact, the factor $\log n_k$ is indeed suppressed and moreover the set E of positive spread is replaced by a single point, but the range of α is limited to a smaller region. It may be noted that our statement (II) gives exact affirmative answer to Kennedy's question — even when the set E having positive spread is replaced by a single point.

Concerning Kennedy's second question, it can be seen from our statement (IV) that the answer is again affirmative — even with the factor $\log n_k$ suppressed from (1.22) and with the hypothesis on a function only at a point — provided the hypothesis is in terms of generalized Lipschitz condition (**) instead of in terms of the Lipschitz condition.