CHAPTER - I

INTRODUCTION

1. The present thesis is devoted to the study of certain problems relating to the order of magnitude of Fourier coefficients, the absolute convergence and the absolute Cesàro summability of a lacunary Fourier series.

Let f be a 2π-periodic function, which is Lebesgue integrable over $[-\pi , \pi]$. A lacunary Fourier series corresponding to the function f is the trigonometric series

$$\sum_{k=1}^{\infty} (a_{n_k} \cos n_k x + b_{n_k} \sin n_k x) \qquad (L)$$

with an infinity of gaps (n_k , n_{k+1}), where $\{n_k\}$ $(k \in N)$ is a strictly increasing sequence of natural numbers satisfying some condition, called the lacunarity condition or gap condition or gap hypothesis, such that

$$(n_{k+1} - n_k) \to \infty \text{ as } k \to \infty ; \qquad (1.1)$$

and

$$a_{n_k} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cdot \cos n_k t \, dt ,$$

$$b_{n_k} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cdot \sin n_k t \, dt .$$

The numbers a_{n_k} and b_{n_k} are called the Fourier coefficients of the function f.

The series

$$\sum_{k=1}^{\infty} (b_{n_k} \cos n_k x - a_{n_k} \sin n_k x) \qquad (L_1)$$

is called the conjugate series of the series (L).

The function

$$\bar{f}(x) = -\frac{1}{2\pi} \int_{0}^{\pi} \frac{\Psi(t)}{\tan(t/2)} \, dt$$

$$= \lim_{\varepsilon \to 0} \left(-\frac{1}{2\pi} \int_{\varepsilon}^{\pi} \frac{\Psi(t)}{\tan(t/2)} \, dt \right),$$

where $\Psi(t) = f(x + t) - f(x - t)$, is called the conjugate function of the function f .

The theory of lacunary Fourier series has its origin in the construction of examples of functions having various pathological properties. The famous example of Weierstrass of a continuous and nowhere differentiable function (1872) is based upon the properties of the lacunary series

$$\sum_{n=1}^{\infty} (a^n \cos b^n x) , \quad \text{where} \quad 0 < a < 1 ,$$

b is an odd integer ≥ 3 and $ab > 1 + \frac{3\pi}{2}$ (refer: [10],[40]). In 1892 Hadamard [9] introduced the notion of lacunary Fourier series (L), with $\{n_k\}$ satisfying the lacunarity condition

$$\frac{n_{k+1}}{n_k} \geq \ell > 1 \quad \text{for all} \quad k \in N, \qquad (1.2)$$

known after his name as the Hadamard lacunarity condition for the study of functions that cannot be analytically continued beyond their circle of convergence (see also Fabry [4] ; Pólya [32]). F. Riesz [33] has used lacunary series to construct a continuous function of bounded variation whose Fourier coefficients are not of the order $o(\frac{1}{n})$. For such other examples refer [2 ; p.242] , [6] , [36] . In fact, the theory of lacunary Fourier series has always been not only one of the main tools for proving or disproving many instances of conjectures in analysis but also a source of interesting results in analysis since its very appearance in the year 1861.

The systematic study of the properties of lacunary Fourier series is started from the first decade of this century. This study can be divided into two parts. In the first one, the series (L) is considered as a series of almost independent random variables, or what is same as a series of almost independent functions, and the properties are studied through probabilistic methods by many well known mathematicians like Kolmogorov, Steinhaus, Kac, Marcinkiewicz, Zygmund and others. A survey of number of properties in this direction with an extensive bibliography can be found in [16] and in [8] . The second part roughly constitutes the study of the following general problem of Mandlebrojt [21] : assume that $\{n_k\}$ is given, suppose we know a property of a function given by (L) on an interval, or in a neighbourhood of a point, or on a perfect set without interior points; to what extent does it give information about (L) on the whole interval $[-\pi , \pi]$? This problem gives rise to several other problems including the problems of uniqueness or continuation of functions. To day number of properties of lacunary Fourier series are known which are connected to these problems and there are interesting applications of the same methods to a series of number theoretic problems. Here one can observe that in many of these properties, the sequence $\{n_k\}$ is assumed to satisfy the Hadamard lacunarity condition (1.2) and these properties distinguish themselves greatly from those of the Fourier series without gaps. For example, such a lacunary Fourier series converges almost everywhere (just as a Fourier series without gaps can diverge everywhere); if such a trigonometric series is summable by a method T^* in a set of positive measure then it is a Fourier series of a function belonging to L^p $(p > 1)$; such a series can not converge to zero in a set of positive measure unless all the coefficients are zero; properties of the function represented by such a series can be extended on the whole real line from a small interval (refer [2] and [44] also). On account of such remarkable features of lacunary Fourier series, one would like to go into further details of the series and to dig out more and more interesting properties as and when possible.

Fatou [5], in the year 1906, began the study of the absolute convergence of lacunary Fourier series (L) by proving that, if $\{n_k\}$ satisfies (1.2) with $\lambda > 3$ than an everywhere convergent lacunary Fourier series (L) converges absolutely. Later on Sidon [2 ; p. 246] showed that if (L) is a Fourier series of just a bounded function with $\{n_k\}$ satisfying (1.2) then it converges absolutely. We observe that Fatou and Sidon both took Hadamard lacunarity condition.

2. Noble [23] observed in 1954 that very little attention has been paid to the effect of a gap condition

weaker than that of Hadamard. He considered the well known theorems of Bernšteǐn [2 ; p. 154, Theorem 1] and Zygmund [2; p.161, Corollary 1] for the absolute convergence of a Fourier series without gaps and certain results [2 ; p.269] concerning the order of magnitude of Fourier coefficients; and showed that a certain gap condition weaker than the Hadamard's enables us to replace, in these theorems, the hypothesis concerning $f(x)$ from the whole interval $[-\pi , \pi]$ to an arbitrary small subinterval of $[-\pi , \pi]$. Since many a times a property of a function is known only locally, the problem considered by Noble can be set (on the line of Mandlebrojt's problem) like this : If the fulfilment of some property of a function f on the whole interval $[-\pi , \pi]$ implies certain conclusions concerning the Fourier series $\sigma(f)$ (without gaps) of f then what lacunae in $\sigma(f)$ guarantees the same conclusions when the property is fulfilled only locally ? Several mathematicians including Noble [23], Kennedy [17, 18, 19, 20], Tomić [37 , 38], M. Izumi and S. I. Izumi [15], Chao [3], J. R. Patadia and V. M. Shah [28] and others have studied this problem in the recent years and have improved upon the earlier results or obtained new results by considering weaker and weaker gap conditions and also by considering the property of a function either on a subset of positive measure or at an arbitrary fixed point of $[-\pi , \pi]$. However, a sharp observation to the above study indicates that, the underlying function, considered so far, is either of bounded variation or it satisfies Lipschitz condition of order $\alpha > 0$. This suggests that there is a good scope of studying the problems of lacunary Fourier series by considering the classes of functions weaker than the class of functions of bounded variation and the Lipschitz class of order $\alpha > 0$.

Obrechkoff [24], considered a class of functions of bounded r^{th} variation in the year 1924. Later on similar but not identical definition of such a class has been given by Mazhar [22]. It can be noted that, this class of functions of bounded r^{th} variation is more general than the class of functions of bounded variation. On the other hand, in the year 1928, Hardy and Littlewood [12] introduced yet another new class of functions, called $Lip(\alpha,p)$ class, $\alpha > 0$; $p \geq 1$, which also happens to be more general than the Lipschitz class of order $\alpha > 0$. Hardy and Littlewood then studied the order of magnitude of Fourier coefficients and the absolute convergence of the Fourier series of a function belonging to such a class (refer [12] and [13]). Their result on absolute convergence generalizes Zygmund's theorem [2 ; p. 161] on absolute convergence. In view of these generalizations, it will be quite interesting to carry out further investigations of the problem (considered by Noble and Kennedy) by taking the underlying function f either in Lip(α,p) class or in bounded r^{th} variation class, and studying the order of magnitude of Fourier coefficients, the absolute convergence and the absolute summability (c,θ) ($0 < \theta \leq 1$) of a lacunary Fourier series. The present thesis is the outcome of researches carried out by the author, mainly in these directions.

This chapter aims at providing the introduction to the subject matter of the thesis through the recent developments regarding the concerned aspects of the problem. It can be noted that the lacunarity conditions considered by us in our results are always weaker than the Hadamard's gap condition (1.2).

3. Before we discuss the details of the results established by us, it is desirable to introduce some definitions and notations at this stage.

Let $x_0 \in [-\pi , \pi]$ be an arbitrary fixed point and δ be an arbitrary positive real number such that $I = [x_0 - \delta , x_0 + \delta]$ becomes a subinterval of $[-\pi , \pi]$. Note that $x_0 = 0$ and $\delta = \pi$ gives $I = [-\pi , \pi]$.

We say that $f \in Lip(\alpha,p,I)$; $0 < \alpha \leq 1$, $p \geq 1$; if

$$\left\{ \int_I |f(x+h)-f(x)|^p dx \right\}^{\frac{1}{p}} = O(|h|^{\alpha}), \text{ as } h \to 0 ;$$

and $f \in Lip\alpha(I)$ if

$$|f(x+h) - f(x)| = O(|h|^{\alpha}) ,$$

uniformly for $x + h$, $x \in I$, as $h \to 0$.

Let $E \subset [-\pi , \pi]$ be a set of positive measure and $|E|$ be its measure. We say that $f \in Lip(\alpha,p,E)$; $0 < \alpha \leq 1$, $p \geq 1$; or $f \in Lip\alpha(E)$ according as

$$\left\{ \int_E |f(x+h) - f(x)|^p dx \right\}^{\frac{1}{p}} = O(|h|^{\alpha})$$

or

$$|f(x+h)- f(x)|= O(|h|^{\alpha}) ,$$

uniformly for $x \in E$, as $h \to 0$ through unrestricted real values, respectively.

It is known that [13] a function of $Lip\alpha$ belongs to $Lip(\alpha,p)$ for every $p \geq 1$. The class $Lip\alpha$ may be regarded roughly as the limit of $Lip(\alpha,p)$ as $p \to \infty$. This shows that the class $Lip(\alpha,p)$ is weaker than the class $Lip\alpha$.

A function f defined in an interval $[a,b]$ is said to be of bounded r^{th} variation (r being a positive integer), if for every partition

$$P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$$

of $[a,b]$, with $x_0, x_1, x_2, \ldots, x_n$ in arithmetic progression, we have

$$\sum_{i=0}^{n-r} | \Delta^r f(x_i) | \leq M, \text{ where } M \text{ is a constant}$$

and

$$\Delta^q f(x_i) = \Delta^{q-1} f(x_{i+1}) - \Delta^{q-1} f(x_i),$$

$$\Delta f(x_i) = f(x_{i+1}) - f(x_i).$$

We now consider some of the important properties of the functions of bounded r^{th} variation, with which we are closely connected.

(a) It is obvious that every function of bounded variation is also of bounded r^{th} variation; but the converse is not true. This can be seen by considering the well known continuous non-differentiable function of Weierstrass [10], namely

$$f(x) = \sum_{n=1}^{\infty} b^{-n} \cos b^n x, \quad b > 1. \qquad (1.3)$$

This f satisfies the condition

$$|f(x+h) + f(x-h) - 2f(x)| = O(h), \text{ as } h \to 0 ,$$

uniformly in x and therefore it is of bounded second variation [42]. But f being nowhere differentiable function, it is not of bounded variation.

(b) It can also be observed that, on account of continuity of f (given in (1.3)') in any closed interval, we have $f \in L^2(I)$. This shows that, there exists a function in $L^2(I)$ which is of bounded r^{th} variation $(r \geq 2)$ in I but not of bounded variation in I.

(c) As against the fact that, a function of bounded variation is bounded, a function of bounded r^{th} variation is so much so weak that it is not even bounded. This can be observed from the following example :

We define f in $[0, 1]$ as

$$f(x) = \begin{cases} n, & \text{when } x = \frac{1}{\sqrt{n}} \text{ for } n \text{ an integer which is not a perfect square,} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly f is unbounded in $[0, 1]$. Also for any partition $P = \{0 = x_0, x_1, x_2, \ldots, x_n = 1\}$, with $x_0, x_1, x_2, \ldots, x_n$ are in arithmetic progression, it follows that the points x_0 , x_1 , ..., x_n are rational numbers. This gives $f(x_i) = 0$, $i = 0,1,2,....$, which in turn gives

$$\sum_{i=0}^{n-r} | \Delta^r f(x_i) | = 0 .$$

Therefore f is of bounded r^{th} variation in $[0,1]$.

After having given the necessary background, we now turn to the problem studied by us in the thesis.

In chapter II of the thesis, we study the order of magnitude of Fourier coefficients of the lacunary Fourier series (L), by considering certain classes of functions either on an arbitrary subinterval or on an arbitrary subset of $[-\pi , \pi]$ of positive measure, under suitable lacunarity conditions. The classes of functions considered by us in this respect are —

- The class of functions belonging to $Lip(\alpha,p,I)$,
- The class of functions belonging to $Lip(\alpha,p,E)$ and
- The class of functions of bounded r^{th} variation over I.

These classes are known to be weaker than the corresponding class of functions belonging to $Lip\alpha(I)$

- The class of functions belonging to $Lip\alpha(E)$ and
- The class of functions of bounded variation over I.

In order to explain the significance of the results established by us in this chapter, it is desirable to recall briefly the developments that have been taken place regarding the study of the behaviour of Fourier coefficients.

Kennedy [17 ; Theorem V(ii)] proved the following Theorem.

THEOREM 1. A (Kennedy). If

(i) $\{n_k\}$ satisfies the gap condition (1.1), and

(ii) $f \in Lip\alpha(I)$ $(0 < \alpha < 1)$, then

$$a_{n_k} , b_{n_k} = O(n_k^{-\alpha}) \quad (k \to \infty) \tag{1.4}$$

Further, replacing the subinterval I in Theorem 1.A by a set E of positive measure, Kennedy [19 ; Theorem I] proved the following theorem.

THEOREM 1.B (Kennedy). If

(i) $\{n_k\}$ satisfies the Hadamard gap condition (1.2), and

(ii) $f \in Lip\alpha(E)$, $0 < \alpha < 1$,

then (1.4) holds.

Kennedy [17 ; Theorem V(i)] also considered the class of functions of bounded variation over I for studying the same problem and proved the following result.

<u>THEOREM 1. C</u> (Kennedy). If

(i) $\{n_k\}$ satisfies (1.1), and

(ii) f is of bounded variation in I,

then

$$a_{n_k} , b_{n_k} = O(n_k^{-1}) \quad (k \to \infty) \qquad (1.5)$$

All these theorems of Kennedy are related to the earlier results due to Noble [23] .

It may be observed that, without the lacunarity condition and with $I = [-\pi , \pi]$, $E = [-\pi , \pi]$, the above Theorems 1.A , 1.B and 1.C are well known classical results [1 ; p. 71, 215].

Now, considering more general classes of functions than those considered by Kennedy, we propose to prove the following theorems in chapter II.

<u>THEOREM 1</u>. If (i) $\{n_k\}$ satisfies (1.1), and

(ii) $f \in Lip(\alpha,p,I)$, $0 < \alpha < 1$; $p \geq 2$,

then (1.4) holds.

It may be observed that Theorem 1 generalizes Theorem 1.A due to Kennedy.

To state the next result obtained by us, we need the following definition.

DEFINITION. [2 ; p.248]. A strictly increasing sequence $\{n_k\}$ $(k \in N)$ of natural numbers is said to satisfy the condition B_2 if $\sup_n P_2(n)$ is finite, where $P_2(n)$ denotes the number of different representations of an integer n in the form

$$n = \epsilon_1 . n_{k_1} + \epsilon_2 . n_{k_2} \quad (\epsilon_1 , \epsilon_2 = \pm 1, n_{k_1} , n_{k_2} \in \{n_k\}).$$

Note that gap condition B_2 is weaker than Hadamard's gap condition (1.2).

THEOREM 2. If (i) $\{n_k\}$ satisfies the condition B_2 and

(ii) $f \in \mathrm{Lip}\,(\alpha,p,F)$, $0 < \alpha < 1$; $p \geq 2$,

then (1.4) holds.

It may be observed that Theorem 2 generalizes Theorem 2.B.

THEOREM 3. If (i) $\{n_k\}$ satisfies (1.1),

(ii) f is of bounded r^{th} variation in I and

(iii) $f \in L^2(I)$,

then (1.5) holds.

It has been already remarked earlier that a function f of bounded variation in I is always of bounded r^{th} variation in I and that such $f \in L^2(I)$ also, but there exists a function of bounded r^{th} variation ($r \geq 2$) in

$L^2(I)$, which is not of bounded variation in I. In view of this remark, our Theorem 3 generalizes Theorem 1.C due to Kennedy.

Further, it is interesting to observe that, without the lacunarity conditions and with $I = [-\pi, \pi]$, $E = [-\pi, \pi]$ our Theorems 1 and 2 are reduced to the classical result due to Hardy and Littlewood [12 ; Lemma 11] for $p \geq 2$.

Having studied the sufficient conditions for the order of magnitude of Fourier coefficients of the lacunary Fourier series (L), we propose to study in Chapter III, the nature of trigonometric series when the behaviour of its coefficients under certain gap condition is known. In fact, the question before us is:

Given a trigonometric series (L). Under what conditions on a_{n_k}, b_{n_k} and under what lacunae does the series (L) become the Fourier series of a function belonging to some Lipschitz class of order $\alpha > 0$. This problem is studied by P. B. Kennedy [19 ; Theorem I], M. Izumi and S. Izumi [15] and Jia-Arng Chao [3]. In fact, Kennedy proved the following theorem in this regard.

<u>THEOREM 1.D</u> (Kennedy). If

(i) $\{n_k\}$ is a sequence of natural numbers satisfying (1.2), and

(ii) $a_{n_k}, b_{n_k} = O(n_k^{-\alpha})$, $0 < \alpha < 1$, $(k \to \infty)$

then the trigonometric series (L) is the Fourier series of a function which belongs to Lipα in some set of positive measure.

Here, we study a theorem analogous to the Theorem 1.D for a less restrictive sequence $\{n_k\}$ defined by

$$n_k = [a^{k^r}], \text{ where } a > 1 \text{ and } 0 < r \leq 1 \qquad (1.6)$$

([ ] denotes the integral part).

It is known that, if $\{n_k\}$ satisfies (1.6) and $f \in$ Lip$\alpha(P)$, that is, f satisfies Lipschitz condition of order $\alpha > 0$ at a point x_0, namely,

$$|f(x_0 + t) - f(x_0)| = O(|t|^\alpha) \text{ as } t \to 0,$$

then the Fourier coefficients

$$a_{n_k}, b_{n_k} = O\left(\frac{k^{(1-r)\alpha}}{n_k^\alpha}\right) \quad (k \to \infty).$$

The following converse theorem [31] is proved in this chapter.

<u>THEOREM 4.</u> If (i) $\{n_k\}$ is a given increasing sequence of natural numbers satisfying

$n_k = [a^{k^r}]$, where $a > 1$; $0 < r < 1$, and

(ii) a_{n_k} , $b_{n_k} = O\left(\frac{k^{(1-r)\alpha}}{n_k^{\alpha}}\right)$, $0 < \alpha < 1$,

then the trigonometric series (L) is the Fourier series of a function which belongs to Lipα in $[-\pi , \pi]$.

It may be observed that, when $r = 1$, our theorem matches with Theorem 1.D due to Kennedy and also with Corollary 1 due to M. Izumi and S. Izumi [15] .

In the remaining chapters of the thesis, our aim is to study the convergence aspect of the lacunary Fourier series (L). In Chapters IV and V, we have studied the absolute convergence and in Chapter VI, we have studied the absolute summability (c,θ) $(0 < \theta \leq 1)$ of a lacunary Fourier series (L).

In order to study the absolute convergence of (L), we examine the convergence of the series

$$\sum_{k=1}^{\infty} (|a_{n_k}| + |b_{n_k}|) ,$$

under suitable conditions on the underlying function f and on the gap. Several mathematicians such as Szidon [2; p. 246] , Noble [23] , Kennedy [17 ; 18 ; 19], Mazhar [22], Masako Satô [34] and Chao [3] have studied this problem imposing various conditions on the function as well as on the gap. More particularly we refer here the papers due to P. B. Kennedy [17] and S. M. Mazhar [22] .

In the year 1956, Kennedy gave the following theorem on absolute convergence of (L).

<u>THEOREM 1.E</u> (Kennedy). If

(i) $\{n_k\}$ satisfies (1.1),

(ii) $f \in \text{Lip}\alpha(I)$, $0 < \alpha < 1$, (1.7)

and (iii) f is of bounded variation in I, (1.8)

then

$$\sum_{k=1}^{\infty} (|a_{n_k}| + |b_{n_k}|) < \infty \qquad (1.9)$$

It may be observed that, without the lacunarity condition (1.1) and with $I = [-\pi , \pi]$, Kennedy's theorem reduces to the well known theorem due to Zygmund [2 ; p.161] on absolute convergence. Thus, we can see that, Kennedy studied the absolute convergence of lacunary Fourier series (L), when the hypothesis in the Zygmund's theorem is satisfied only in a subinterval I of $[-\pi , \pi]$.

Now a look at the hypothesis of Theorem 1.E leads to the natural question as to whether condition (1.7) or (1.8) on the function f can be replaced by the weaker conditions, maintaining of course, the same gap condition (1.1). Investigating this aspect, we study in Chapter IV, the absolute convergence of lacunary Fourier series, replacing the condition (1.7) in Kennedy's theorem by a weaker condition of Lip(α,p,I). In fact, we have proved the following theorems in this chapter.

<u>THEOREM 5</u>. If (i) $\{n_k\}$ satisfies (1.1),

(ii) $f \in \text{Lip}(\alpha,p,I)$ with $0 < \alpha \leq 1$; $p > 2$; $\alpha p > 1$,

and (iii) f satisfies (1.8),

then (1.9) holds.

It may be observed that, without the lacunarity condition (1.1) and with $I = [-\pi , \pi]$, our Theorem 5 reduces to the Theorem due to Hardy and Littlewood [13] for $p > 2$.

<u>THEOREM 6</u>. If (i) $\{n_k\}$ satisfies (1.1),

(ii) $f \in \text{Lip}(\alpha,p,I)$ with $0 < \alpha \leq 1$; $p > 2$,

and (iii) f satisfies (1.8),

then

$$\sum_{k=1}^{\infty} (|a_{n_k}|^{\beta} + |b_{n_k}|^{\beta}) < \infty \quad \text{for}$$

every β satisfying $2 > \beta > \dfrac{2(p-1)}{2p + \alpha p - 3}$.

It may be observed that, when $\beta = 1$, Theorem 6 reduces to Theorem 5.

THEOREM 7. Under the hypothesis of Theorem 6,

$$\sum_{k=1}^{\infty} n_k^{\beta/2}(|a_{n_k}| + |b_{n_k}|) < \infty \quad \text{for}$$

every $\beta < \frac{\alpha p - 1}{2(p - 1)}$.

It may be observed that, when $\beta = 0$, Theorem 7 reduces to Theorem 5.

Further, in Chapter V, we continue our investigation by replacing both the conditions (1.7) and (1.8) in Kennedy's theorem by the corresponding weaker conditions of $\mathrm{Lip}(\alpha,p,I)$ and bounded r^{th} variation over I. In this regard, we refer the following theorem due to S. M. Mazhar [22].

THEOREM 1.F (Mazhar). If

(i) $$\lim_{k\to\infty} \frac{N_k}{\log n_k} = \infty , \tag{1.10}$$

where

$$N_k = \min\{(n_{k+1} - n_k), \quad (n_k - n_{k-1})\} ,$$

(ii) f satisfies (1.7),

and (iii) f is of bounded r^{th} variation in I, (1.11)

then the Fourier series (L) of f converges absolutely.

It can be observed that, Mazhar weakened the condition (1.8) of bounded variation by taking the condition (1.11) of bounded r^{th} variation in I, but in doing so he took the gap condition (1.10) which is stronger than (1.1).

In fact, Mazhar's aim was to generalize Noble's theorem [23 ; Theorem 5] on absolute convergence. While our aim, in Chapter V, is to generalize Kennedy's theorem, maintaining the gap condition (1.1). Consequently, theorems Proved here will generalize Mazhar's theorem as also the results of Chapter IV. Theorems proved in this chapter are as under :

THEOREM 8. Theorem 5 holds if the condition (1.8) is replaced by the condition (1.11).

THEOREM 9. Theorem 6 holds if the condition (1.8) is replaced by the condition (1.11).

THEOREM 10. Theorem 7 holds if the condition (1.8) is replaced by the condition (1.11).

Further, in Chapter V, we also study the convergence of (L), (L_1) and the absolute convergence of the series

$$\sum_{k=1}^{\infty} \left(\frac{S_{n_k} - S}{n_k} \right) , \tag{1.12}$$

where

$$S_{n_k} = \sum_{p=1}^{k} (a_{n_p} \cos n_p x + b_{n_p} \sin n_p x)$$

and S is an appropriate number independent of n_k. The convergence of the series (1.12) was first studied by Hardy and Littlewood [11] and thereafter it was studied by Zygmund [43 ; p.61] and so many others. We, in fact, prove here, the following theorems in this regard, by taking more general conditions than those considered by V. M. Shah [35 ; Chapter V].

THEOREM 11. Under the conditions of Theorem 3, the Fourier series (L) is convergent to $\frac{f(x+0)+f(x-0)}{2}$ at any point where this expression has a meaning; and the conjugate Fourier series (L_1) is convergent to $\bar{f}(x)$ whenever $\bar{f}(x)$ exists and x is a point of the Lebesgue set.

THEOREM 12. Suppose the hypothesis of Theorem 3 is satisfied. If, in addition, the series

$$\sum_{k=1}^{\infty} \frac{\log n_k}{n_k} \text{ is convergent,}$$

then the series (1.12) is absolutely convergent.

THEOREM 13. Suppose the hypothesis of Theorem 1 is satisfied. If, in addition, the series

$$\sum_{k=1}^{\infty} \frac{k^{1-\alpha}}{n_k} \text{ is convergent,} \tag{1.13}$$

then the series (1.12) is absolutely convergent.

THEOREM 14. Suppose the hypothesis of Theorem 2 is satisfied. If, in addition, the condition (1.13) holds, then the series (1.12) is absolutely convergent.

Finally, in the last chapter of the thesis, i.e. in chapter VI, we study the absolute summability (c,θ) of the lacunary Fourier series (L) and its conjugate series (L_1) by considering certain conditions on a function and on a gap under which the absolute convergence of (L) is not guaranteed. While studying this aspect, we have kept in mind the results on absolute convergence due to J. R. Patadia and V. M. Shah [28] and P. B. Kennedy [17 ; Theorem V(iv)]. Recently in the year 1981, J. R. Patadia and V. M. Shah proved the following theorem.

THEOREM 1.G (J. R. Patadia and V. M. Shah). If

(i) $(n_{k+1} - n_k) \geq A n_k^{\beta} k^{\gamma}$ $(0 < \beta < 1, \ \gamma \geq 0)$, (1.14)

where A is a positive constant,

and (ii) $f \in \mathrm{Lip}\alpha(P)$, $0 < \alpha < 1$, (1.15)

then

$$\sum_{k=1}^{\infty} (|a_{n_k}|^m + |b_{n_k}|^m) < \infty, \quad 0 < m \leq 1,$$

when $\alpha\beta m + \alpha m\gamma > (1 - \frac{m}{2})(1 - \beta)$.

It can be observed that Theorem 1.G extends the regions of absolute convergence obtained by M. Izumi and S. Izumi [15 ; Theorem 2] and Chao [3 ; Theorem 2] . This can be observed by considering the following particular cases of Theorem 1.G. For $m = 1$ and $\gamma = 0$ Theorem 1.G ensures the absolute convergence of (L) when $\alpha > \frac{1}{2}(\beta^{-1} - 1)$, under the gap condition

$$(n_{k+1} - n_k) \geq An_k^{\beta} \quad (0 < \beta < 1), \tag{1.16}$$

(A is a positive constant)

At the same time, for $m = 1$ Theorem 1.G ensures the absolute convergence of (L) when

$$\alpha\beta + \alpha\gamma > \frac{1}{2}(1 - \beta) ,$$

under the gap condition (1.14). Here it is worthwhile to note that, when

$$\alpha\beta + \alpha\gamma = \frac{1}{2}(1 - \beta)$$

then also the absolute convergence of (L) is established by J. R. Patadia and V. M. Shah [29] . But in that case, the condition taken on f happens to be stronger than Lipα(P).

In view of this situation, it is quite natural to inquire into the behaviour of the lacunary Fourier series (L) of a function in Lipα(P) when $\alpha \leq \frac{1}{2}(\beta^{-1} - 1)$ and more generally when $\alpha\beta + \alpha\gamma \leqslant \frac{1}{2}(1 - \beta)$, by taking the gap conditions (1.16) and (1.14) respectively. Hence, in this direction, the following theorems are proved in this chapter.

<u>THEOREM 15</u>[30]. If (i) $\{n_k\}$ satisfies (1.16) with some suitable constant A, and (ii) f satisfies (1.15), then the Fourier series (L) of f is absolutely summable

$(c , \frac{1}{2})$

(i) for every $\alpha > 0$ if $\beta \geq \frac{3}{5}$;

or (ii) for every $\alpha > \frac{3}{5\beta} - \frac{5}{2}$ if $\beta < \frac{3}{5}$.

THEOREM 16 [30]. Under the hypothesis of Theorem 15, the Fourier series (L) of f is absolutely summable $(c,1)$ when

$$\alpha > \beta^{-1} - 2 .$$

THEOREM 17. If (i) $\{n_k\}$ satisfies the gap condition (1.14) with some suitable constant A, and (ii) f satisfies condition (1.15), then the Fourier series (L) of f is absolutely summable (c,θ) $(0 < \theta \leq 1)$ when

$$\alpha > \max \left\{ \frac{1 - \beta - \theta - \gamma\theta}{\beta + \gamma} , \frac{2 - 3\beta - \gamma + \beta\theta - \theta}{\beta + \beta\gamma} \right\} .$$

It may be observed that Theorem 17 is a generalized form of Theorems 15 and 16.

It is also interesting to observe that when $\gamma = 1$, Theorem 17 gives the absolute summability $(c,1)$ of (L) for every $\alpha > 0$; and that, when $\gamma = \frac{3}{2}$, we get the absolute summability $(c, \frac{1}{2})$ of (L) for every $\alpha > 0$.

We further continue our study on absolute summability by considering Theorem 1.E on absolute convergence due to P. B. Kennedy [17] . Here it is known that, it does not become possible to dropout, either the condition (1.7) or (1.8) to ensure the absolute convergence of (L). Keeping this fact in mind, the behaviour of (L) and its conjugate series (L_1) is studied by V. M. Shah [35 ; Chapter VI]. Here we study the same problem, by taking a set of conditions on a function weaker than those considered by V. M. Shah. This study becomes possible on account of our Theorems 1, 2 and 3 of Chapter II. In fact, we prove the following theorems in this chapter.

THEOREM 18. Suppose the hypothesis of Theorem 3 is satisfied. If, in addition, the series

$$\sum_{k=1}^{\infty} \frac{k}{n_k^2} \text{ is convergent,}$$

then the Fourier series (L) and (L_1) are everywhere absolutely summable (c,1). $\quad$ (1.17)

THEOREM 19. If

(i) $\displaystyle \varliminf_{k \to \infty} \frac{n_{k+1} - n_k}{\log n_k} = B, \; B > 0,$

(ii) f is of bounded r^{th} variation in I, and

(iii) $f \in L^2(I)$,

then the conclusion (1.17) holds.

THEOREM 20. If (i) $\{n_k\}$ satisfies (1.1),

(ii) $f \in \mathrm{Lip}(\alpha,p,I)$ with $0 < \alpha \leq \frac{1}{2}$; $p \geq 2$,

and (iii) $$\sum_{k=1}^{\infty} \left\{ \frac{n_1 + n_2 + \dots + n_k}{n_k^2} \right\} \text{ is convergent,} \qquad (1.18)$$

then the conclusion (1.17) holds.

THEOREM 21. If (i) $\{n_k\}$ satisfies the gap condition B_2 ,

(ii) $f \in \mathrm{Lip}(\alpha,p,E)$ with $0 < \alpha \leq \frac{1}{2}$; $p \geq 2$,

and (iii) the condition (1.18) holds,

then the conclusion (1.17) holds.