## CHAPTER - II

## THE ORDER OF MAGNITUDE OF FOURIER COEFFICIENTS OF A LACUNARY FOURIER SERIES

1. The study of the order of magnitude of Fourier coefficients of the lacunary Fourier series (L) began with the consideration of the hypothesis to be satisfied by the function on a subset of $[-\pi, \pi]$ instead of the whole interval $[-\pi, \pi]$. In this direction, Noble [23] proved the following result.

If the sequence $\{n_k\}$ satisfies the lacunarity condition

$$\liminf_{k \to \infty} \frac{N_k}{\log n_k} = \infty,$$

where $N_k = \min\{n_k - n_{k-1}, n_{k+1} - n_k\}$

and if the function $f$ satisfies the Lipschitz condition of order $\alpha$, $0 < \alpha < 1$, in a subinterval $I = \{x : |x - x_0| \leq \delta\}$ of $[-\pi, \pi]$, then

$$a_{n_k}, b_{n_k} = O(n_k^{-\alpha}) \quad (k \to \infty) \tag{2.1}$$

Further, he also proved that, if $f$ is of bounded variation in I, then

$$a_{n_k}, b_{n_k} = O(\frac{1}{n_k}) \quad (k \to \infty). \tag{2.2}$$

However, later on, it was pointed out by Ul'yanov [2 ; p.270] that, the above results of Noble also hold for slightly weaker gap condition

$$\liminf_{k \to \infty} \frac{N_k}{\log n_k} = B, \quad \text{where} \quad B \geq \frac{50}{\delta} .$$

P. B. Kennedy [17] employed more powerful method due to Paley-Wiener and proved the following result (Theorem 2.A) under a still weaker gap condition

$$(n_{k+1} - n_k) \to \infty \text{ as } k \to \infty \tag{2.3}$$

<u>THEOREM 2.A.</u> [17 ; Theorem V(ii)]. If

(i) $\{n_k\}$ satisfies (2.3)

and (ii) $f \in \mathrm{Lip}\alpha(I)$ $(0 < \alpha < 1)$,

then (2.1) holds.

Later on, Kennedy [18] showed that this result does not hold under the Fabry's gap condition

$$\frac{n_k}{k} \to \infty \text{ as } k \to \infty ,$$

which is weaker than the gap condition (2.3). However, the same author [19] has shown that, if a more stringent condition (1.2) due to Hadamard is taken in place of the lacunarity condition (2.3) then it becomes possible to replace the subinterval I in the above Theorem 2.A by a set E of positive measure. More precisely Kennedy proved the following theorem.

<u>THEOREM 2.B.</u>[19 ; Theorem I]. If

(i) $\{n_k\}$ satisfies the Hadamard gap condition

$$\frac{n_{k+1}}{n_k} \geq \lambda > 1 \quad \text{for all } k \in N \text{ , and}$$

(ii) $f \in \mathrm{Lip}\alpha(E)$, $0 < \alpha < 1$,

then (2.1) holds.

Now, here we intend to study the order of magnitude of Fourier coefficients of the lacunary Fourier series (L), by considering the classes of functions, namely $\mathrm{Lip}(\alpha,p,I)$ and $\mathrm{Lip}(\alpha,p,E)$ which are weaker than the classes $\mathrm{Lip}\alpha(I)$ and $\mathrm{Lip}\alpha(E)$, considered by Kennedy.

It is worthwhile to note that, though we are weakening a condition on the function, we do not ask for a gap condition stronger than that of Kennedy. More precisely, we prove the following theorems.

<u>THEOREM 1.</u> If (i) $\{n_k\}$ satisfies (2.3),

and (ii) $f \in \mathrm{Lip}(\alpha,p,I)$, $0 < \alpha < 1$ ; $p \geq 2$ , (2.4)

then (2.1) holds.

It may be observed that Theorem 1 generalizes Theorem 2.A. due to Kennedy.

THEOREM 2. If (i) $\{n_k\}$ satisfies the condition $B_2$ and (ii) $f \in Lip(\alpha,p,E)$ , $0 < \alpha < 1$ ; $p \geq 2$ , then (2.1) holds.

It may be observed that Theorem 2 generalizes Theorem 2.B due to Kennedy.

Now, the study of the behaviour of Fourier coefficients is further done by Tomić [37] , Kennedy [20] , M. Izumi and S. Izumi [15] . These authors have replaced the set of positive measure by a single point. But in doing so, they have retained the Hadamard gap. The same problem is further studied by Tomić [38] , Jia-Arng Chao [3] and J. R. Patadia [27] , considering more general gap condition and a certain continuity condition on f only at a point.

Kennedy studied this problem, by considering the class of functions of bounded variation in I. The following result is due to Kennedy.

THEOREM 2.C [17 ; Theorem V(i)] . If

(i) $\{n_k\}$ satisfies (2.3),

and (ii) f is of bounded variation in I,

then (2.2) holds.

A glance at the literature of the theory of lacunary Fourier series reveals that there is a good scope for the study of the behaviour of the Fourier coefficients for the class of functions weaker than the class of functions of bounded variation. Hence, it will be quite interesting to probe further in this direction. Accordingly, we consider here the class of functions of bounded $r^{th}$ variation and study the behaviour of Fourier coefficients. More precisely, we prove the following theorem.

<u>THEOREM 3.</u> If (i) $\{n_k\}$ satisfies (2.3),

(ii) $f$ is of bounded $r^{th}$ variation in I, (2.5)

and (iii) $f \in L^2(I)$, (2.6)

then (2.2) holds.

It may be observed that the conditions (2.5) and (2.6) of Theorem 3 do not imply that the function $f$ is of bounded variation in I ; and in this manner, Theorem 3 generalizes Theorem 2.C due to Kennedy.

2. Before we proceed to prove our theorems, we need the following lemmas. Lemma 2.1 is a special case of a very general theorem due to Paley and Wiener [25 ; Theorem XLII']. Lemma 2.2 and Lemma 2.3 are due to Kennedy [17 ; Lemma 1 and Lemma 2]. Lemma 2.4 is proved, though not explicitly stated, by Zygmund [43 ; p.121] , taking Hadamard gap condition (1.2); but it is easy to see from the proof there

that we can as well take $\{n_k\}$ satisfying the condition $B_2$. Lemma 2.5 is due to J. R. Patadia [26] . Lemma 2.6, which is proved here, is an extension of the result given in the introduction of (the paper by Hardy and Littlewood) [14].

<u>LEMMA 2.1.</u> If $f \in L^2(I)$, where I is an interval, and if $(n_{k+1} - n_k) \to \infty$ in (L) then $f \in L^2[-\pi, \pi]$.

<u>LEMMA 2.2.</u> Let (i) $\{\lambda_k\}$ $(-\infty < k < \infty)$ be a sequence of real numbers satisfying $\lambda_{k+1} > \lambda_k$ $(-\infty < k < \infty)$, $\lambda_{k+1} - \lambda_k \to \infty$ $(|k| \to \infty)$, $\lambda_{k+1} - \lambda_k > 8\pi\delta^{-1}$,

(ii) $\{A_k\}$ $(-\infty < k < \infty)$ be a sequence of complex numbers such that $\sum_{-\infty}^{\infty} |A_k| \, s^{|\lambda_k|} < \infty$ $(0 < s < 1)$,

(iii) $\phi(s,x) = \sum_{-\infty}^{\infty} A_k \, s^{|\lambda_k|} \, e^{i\lambda_k x}$ $(0 < s < 1)$

for all real x,

(iv) $\phi(x) = L^2\text{-limit}_{s \to 1} \, \phi(s,x)$ $(|x-x_0| \leq \delta)$, where $x_0$ is fixed and $\delta > 0$.

Then
$$\sum_{-\infty}^{\infty} |A_k|^2 \leq 8\delta^{-1} \int_{|x-x_0| \leq \delta} |\phi(x)|^2 \, dx \, .$$

<u>LEMMA 2.3</u> Let (i) $\{\lambda_k\}$ $(-\infty < k < \infty)$ be a sequence of real numbers satisfying $\lambda_{k+1} - \lambda_k > 16\pi\,\delta^{-1}$,

(ii) $\{A_k\}$ $(-\infty < k < \infty)$ be a sequence of complex numbers such that

$$\sum_{-\infty}^{\infty} |A_k| \, s^{|\lambda_k|} < \infty \qquad (0 < s < 1),$$

(iii) $\phi(s,x) = \sum_{-\infty}^{\infty} A_k \, s^{|\lambda_k|} \, e^{i\lambda_k x}$ $(0 < s < 1)$

for all real $x$,

(iv) $\phi(x) = L^2\text{-}\underset{s \to 1}{\text{limit}}\ \phi(s,x)$ $(|x-x_o| \leq \delta)$,

(v) The integers $j$ and $K$ satisfy

$$|\lambda_K| > 2\pi\,\delta^{-1}\ ; \quad 0 \leq j \leq \tfrac{1}{4}\,\pi^{-1}\,\delta\,|\lambda_K|,$$

(vi) $\phi_j(x) = \phi\left\{x + \frac{2j\pi}{|\lambda_K|}\right\} - \phi\left\{x + \frac{(2j-1)\pi}{|\lambda_K|}\right\}$

Then $$S_K = \sum_{\frac{1}{2}|\lambda_K| \leq |\lambda_k| \leq |\lambda_K|} |A_k|^2 \leqslant 8\,\delta^{-1} \int_{|x-x_o| \leq \delta/2} |\phi_j(x)|^2\,dx .$$

<u>LEMMA 2.4</u> Let $E \subset [-\pi, \pi]$ be a set of positive measure, $\{n_k\}$ satisfy the condition $B_2$ and $n_o = 0$, $n_k = -n_{-k}$ $(k < 0)$. Then there exists $\Upsilon \in N$ with the property : if $\{C_k\}$ $(k \in Z)$

is any sequence of complex numbers, then for $T > \gamma$ we have

$$|S_T| \leq \frac{|E|}{2} \sum_{-T}^{T} |c_k|^2 ,$$

where $S_T = \sum_{p,q} c_p \cdot \bar{c}_q \int_E \exp\left(i(n_p - n_q)x\right)dx$

in which the summation is over values of $p$ and $q$ such that $\gamma < |p| , |q| \leq T$ and $p \neq q$.

<u>LEMMA 2.5</u> Let $E$, $\{n_k\}$ and $\gamma$ be as in the Lemma 2.4. Put $c_0 = 0$, $c_k = \frac{1}{2}(a_{n_k} - ib_{n_k})$ $(k > 0)$, $c_k = \bar{c}_{-k}$ $(k < 0)$ and suppose

$$c_k = 0 \quad \text{for } |k| \leq \gamma. \text{ Then}$$

$$\sum_{-\infty}^{\infty} |c_k|^2 \leq \frac{2}{|E|} \int_E |f(x)|^2 dx .$$

<u>LEMMA 2.6</u> Let $1 \leq p < q$. If $f \in \mathrm{Lip}(\alpha,q,E)$ then $f \in \mathrm{Lip}(\alpha,p,E)$, where $E$ is a set of positive measure in $[-\pi , \pi]$.

<u>Proof of Lemma 2.6</u>

Since $f \in \mathrm{Lip}(\alpha,q,E)$ , it follows that

$$\int_E |f(x+h) - f(x)|^q dx = O(|h|^{\alpha q}) ,$$

uniformly for $x \in E$ , as $h \to 0$ through unrestricted real values.

Consider the conjugate indices $\frac{q}{p}$ and $\frac{q}{q-p}$ .

Then by Hölder's inequality, we have

$$\int_E 1 \cdot \left\{ |f(x+h) - f(x)|^p \right\} dx$$

$$\leq \left( \int_E 1^{\frac{q}{q-p}} dx \right)^{\frac{q-p}{q}} \cdot \left( \int_E |f(x+h)-f(x)|^{p \cdot \frac{q}{p}} dx \right)^{\frac{p}{q}}$$

$$\leq (2\pi)^{\frac{q-p}{q}} \left( \int_E |f(x+h) - f(x)|^q dx \right)^{\frac{p}{q}}$$

$$= 0 \left( |h|^{\alpha q} \right)^{\frac{p}{q}}$$

$$= 0 \left( |h|^{\alpha p} \right) ,$$

uniformly for $x \in E$, as $h \to 0$ through unrestricted real values.

Hence $f \in Lip(\alpha,p,E)$. This proves lemma 2.6.

<u>LEMMA 2.7</u> If $f \in Lip(\alpha,p,E)$, $p \geq 1$ ; $0 < \alpha < 1$ , then $f \in L^p(E)$ , where $E$ is a set of positive measure in $[-\pi , \pi]$.

<u>Proof of Lemma 2.7</u>. We select a (small) positive $h$ such that $-\pi \leq x < x + h \leq \pi$.

We write $E_h = E \cap [0,h]$ .

Since $E$ and $[0,h]$ are measurable , $E_h$ is measurable. Let $|E_h|$ denote the measure of $E_h$ . Then $0 \leq |E_h| \leq h$ .

Now, for $x \in E$, we take

$$\phi_h = \phi_h(x) = \frac{1}{|E_h|} \int_{E_h} f(x+u)\, du \quad .$$

This $\phi_h$ is continuous.

We have $f - \phi_h = \frac{1}{|E_h|} \int_{E_h} \left(f(x) - f(x+u)\right) du$ .

Hence, by Hölder's inequality,

$$|f - \phi_h| \leq \frac{1}{|E_h|} \int_{E_h} |f(x+u) - f(x)|\, du$$

$$\leq \left\{ \int_{E_h} |f(x+u)-f(x)|^p du \right\}^{\frac{1}{p}} \cdot \left\{ \int_{E_h} \left(\frac{1}{|E_h|}\right)^q du \right\}^{\frac{1}{q}} ,$$

where $\frac{1}{p} + \frac{1}{q} = 1$

$$= \left\{ \int_{E_h} |f(x+u)-f(x)|^p du \right\}^{\frac{1}{p}} \frac{1}{|E_h|} \left(|E_h|\right)^{\frac{1}{q}}$$

Therefore

$$|f-\phi_h| \leq \left(\frac{1}{|E_h|}\right)^{1 - \frac{1}{q}} \left\{ \int_{E_h} |f(x+u) - f(x)|^p\, du \right\}^{\frac{1}{p}}$$

$$= \left(\frac{1}{|E_h|}\right)^{\frac{1}{p}} \left\{ \int_{E_h} |f(x+u) - f(x)|^p\, du \right\}^{\frac{1}{p}} .$$

This gives

$$|f-\phi_h|^p \leq \frac{1}{|E_h|} \int_{E_h} |f(x+u) - f(x)|^p\, du \; .$$

Therefore,

$$\int_E |f-\phi_h|^p dx \leq \frac{1}{|E_h|} \int_E \left\{ \int_{E_h} |f(x+u) - f(x)|^p du \right\} dx$$

$$= \frac{1}{|E_h|} \int_{E_h} du \int_E |f(x+u) - f(x)|^p dx ,$$

by Tonelli's theorem

$$= \frac{A_1}{|E_h|} \int_{E_h} |u|^{\alpha p} du$$

$$\leq \frac{A_1 h^{\alpha p}}{|E_h|} \cdot |E_h| \quad \text{as } |u| < h$$

$$< A_2 , \text{ as } h \to 0 , \text{ where } A_1 , A_2 \text{ are constants.}$$

Since $\phi_h$ is continuous, it follows that

$$f \in L^p(E) .$$

This completes the proof of Lemma 2.7.

It may be observed that Lemma 2.6 and Lemma 2.7 will hold for $E = I$ also.

<u>Proof of Theorem 1.</u> Since, by Lemma 2.6, a function of Lip $(\alpha,q)$ belongs to Lip $(\alpha,p)$ for $1 \leq p < q$, it is sufficient to prove the theorem for the class Lip $(\alpha,2)$ in I.

Put $n_o = 0$ , $n_k = - n_{-k}$ $(k < 0)$ ; $C_o = 0$ ,

$$C_k = \tfrac{1}{2}(a_{n_k} - ib_{n_k}) \quad (k > 0), \; C_k = \bar{C}_{-k} \quad (k < 0).$$

Therefore we can write (L) in the form

$$\sigma(f) = \sum_{k=-\infty}^{\infty} C_k \, e^{in_k x} \tag{2.7}$$

We assume that, the integers j and K satisfy

$$n_K > \frac{2\pi}{\delta} \quad \text{and } 0 \leq j \leq \tfrac{1}{4}\pi^{-1} \, \delta n_K \tag{2.8}$$

$$\text{Take } f_j(x) = f(x + \frac{2j\pi}{n_K}) - f(x + \frac{(2j-1)\pi}{n_K}) \tag{2.9}$$

Since $a_{n_k}, b_{n_k} \to 0$ as $k \to \infty$, $\{|C_k|\}$ is a bounded sequence ; and hence we have

$$\sum_{-\infty}^{\infty} |C_k| \, s^{|n_k|} < \infty \quad (0 < s < 1). \tag{2.10}$$

$$\text{Put } f(s,x) = \sum_{-\infty}^{\infty} C_k \, s^{|n_k|} \, e^{in_k x}, \; (0 < s < 1) \tag{2.11}$$

for all real x. The existence of $f(s,x)$ is ensured by (2.10).

We get

$$f(s,x) = \sum_{k=1}^{\infty} \tfrac{1}{2}(a_{n_k} - ib_{n_k}) \, (\cos n_k x + i\sin n_k x) s^{n_k}$$

$$+ \sum_{k=1}^{\infty} \tfrac{1}{2}(a_{n_k} + ib_{n_k}) \, (\cos n_k x - i\sin n_k x) s^{n_k}$$

$$= \sum_{k=1}^{\infty} (a_{n_k} \cos n_k x + b_{n_k} \sin n_k x) s^{n_k}.$$

By (2.4) and lemma 2.7, we have $f \in L^2(I)$.

Since (2.3) holds, Lemma 2.1 gives $f \in L^2[-\pi, \pi]$. Hence by a known theorem [43 ; p.87] it follows that

$$f(x) = L^2\text{-}\lim_{s \to 1} f(s,x) \qquad (|x| \leq \pi). \tag{2.12}$$

Without loss of generality, we assume that

$$n_{k+1} - n_k > 16\pi\, \delta^{-1} \quad \text{for all} \quad k. \tag{2.13}$$

In view of (2.3) this can be achieved, if necessary, by adding to $f(x)$ a polynomial in $\exp(in_k x)$, a process which affects neither the hypothesis nor the conclusion of the theorem.

Thus from (2.9), (2.10), (2.11), (2.12) and (2.13), we see that all the conditions of Lemma 2.3 are satisfied. Hence

$$S_K = \sum_{\frac{1}{2}n_K \leq |n_k| \leq n_K} |C_k|^2 \leq 8\delta^{-1} \int_{I_1 : |x-x_0| \leq \delta/2} |f_j(x)|^2 dx .$$

We observe that $S_K \geq |C_K|^2$

This gives $|C_K|^2 \leq 8\delta^{-1} \int_{I_1} |f_j(x)|^2 dx$ .

But condition (2.4) gives

$$\int_{I_1} |f_j(x)|^2 dx = O\left( \left(\frac{\pi}{n_K}\right)^{2\alpha} \right).$$

Hence it follows that,

$$|C_K|^2 = O\left(\frac{1}{n_K^{2\alpha}}\right).$$

Thus $|C_K| = O\left(\frac{1}{n_k^{\alpha}}\right)$ as $k \to \infty$.

Therefore (2.1) holds and this completes the proof of Theorem 1.

<u>Proof of Theorem 2</u>. As mentioned in the proof of Theorem 1, it is sufficient to prove the theorem for the class $Lip(\alpha,2)$ in E.

Assume that, the conditions of Lemma 2.5 are satisfied. Put $n_o = 0$ and $n_k = -n_{-k}$ $(k < 0)$. Therefore we can write (L) in the form (2.7).

Put $g(x) = f(x + \frac{\pi}{2n_K}) - f(x - \frac{\pi}{2n_K})$ (2.14)

and $C_k^* = 2i\, C_k \sin n_k \frac{\pi}{2n_K}$ (2.15)

Then $|C_k^*| \leq 2|C_k|$ and hence by (2.10)

we have $\sum_{-\infty}^{\infty} |C_k^*|\, s^{|n_k|} < \infty \quad (0 < s < 1).$ (2.16)

Put $g(s,x) = \sum_{-\infty}^{\infty} C_k^*\, s^{|n_k|}\, e^{in_k x} \quad (0 < s < 1)$

for all real x.

The existence of $g(s,x)$ is ensured by (2.16).

Now if we put

$$\phi(s,x) = \sum_{-\infty}^{\infty} C_k\, s^{|n_k|}\, e^{in_k x} \quad (0 < s < 1),$$

then we get the identity

$$g(s,x) = \phi(s, x + \frac{\pi}{2n_K}) - \phi(s, x - \frac{\pi}{2n_K}) . \tag{2.17}$$

Also by a corollary to Zygmund's theorem [2;p.241] $f \in L^2[-\pi , \pi]$ and hence by a known theorem [43;p.87], it follows that

$$f(x) = L^2\text{-}\lim_{s\to 1} \phi(s,x) \quad (|x| \le \pi). \tag{2.18}$$

Therefore from (2.14), (2.17) and (2.18), we obtain

$$g(x) = L^2\text{-}\lim_{s\to 1} g(s,x) , \quad (|x| \le \pi ).$$

We now apply Lemma 2.5 with $C_k$ and $f(x)$ replaced by $C_k^*$ and $g(x)$ respectively to get

$$\sum_{-\infty}^{\infty} |C_k^*|^2 \le \frac{2}{|E|} \int_E |g(x)|^2 dx .$$

From this we obtain $|C_k^*|^2 \le \frac{2}{|E|} \int_E |g(x)|^2 dx.$

Therefore (2.15) gives

$$|C_K|^2 \le \frac{1}{2|E|} \int_E |g(x)|^2 dx. \tag{2.19}$$

Now, by (2.14) and Minkowski's inequality, we have

$$\left\{ \int_E |g(x)|^2 dx \right\}^{\frac{1}{2}}$$

$$\leq \left\{ \int_E \left| f(x + \frac{\pi}{2n_K}) - f(x) \right|^2 dx \right\}^{\frac{1}{2}} + \left\{ \int_E \left| f(x - \frac{\pi}{2n_K}) - f(x) \right|^2 dx \right\}^{\frac{1}{2}}$$

$$= O\left(\frac{\pi}{2n_K}\right)^{\alpha} + O\left(\frac{\pi}{2n_K}\right)^{\alpha}, \text{ by the hypothesis } f \in \mathrm{Lip}\,(\alpha, p, E),\ p \geq 2,$$

$$= O(n_K^{-\alpha}).$$

This together with (2.19), gives $|C_K|^2 = O\left(\frac{1}{n_K^{2\alpha}}\right)$.

Thus $|C_{n_k}| = O\left(\frac{1}{n_k^{\alpha}}\right)$ as $k \to \infty$.

This completes the proof of Theorem 2.

REMARK : It can be observed that under the conditions of Theorems 1 and 2 the absolute convergence of the Fourier series (L) of f can be established for $\alpha > \frac{1}{2}$, by using (2.1).

Proof of Theorem 3.

Put $n_0 = 0$, $n_k = -n_{-k}$ $(k < 0)$,

$C_{n_0} = 0$, $C_{n_k} = \frac{1}{2}(a_{n_k} - ib_{n_k})$ $(k > 0)$,

$C_{n_k} = \bar{C}_{n_{-k}}$ $(k < 0)$. Therefore we can write

(L) in the form

$$\sigma(f) = \sum_{k=-\infty}^{\infty} c_{n_k} e^{in_k x} .$$

We have $\sum_{k=-\infty}^{\infty} |c_{n_k}| s^{|n_k|} < \infty \quad (0 < s < 1)$ (2.20)

Put $f(s,x) = \sum_{-\infty}^{\infty} c_{n_k} e^{in_k x} s^{|n_k|}, \quad (0 < s < 1)$

for all real x. The existence of $f(s,x)$ is ensured by (2.20).

We obviously have

$$f(s,x) = \sum_{k=1}^{\infty} (a_{n_k} \cos n_k x + b_{n_k} \sin n_k x) s^{n_k} .$$

Since the conditions (2.6) and (2.3) hold, Lemma 2.1 gives $f \in L^2[-\pi, \pi]$.

Hence, by a known Theorem [43;p.87], it follows that

$$f(x) = L^2\text{-limit}_{s \to 1} f(s,x) \quad (|x| \leq \pi) \tag{2.21}$$

We assume that the integers j and K satisfy

$$n_K > \frac{2r\pi}{\delta} \quad \text{and } 0 \leq j \leq \frac{\delta n_K}{8\pi} \tag{2.22}$$

Without loss of generality, we assume that r is an even integer.

Take $g_j(x)$

$$= f(x + \frac{2j\pi}{n_K} + \frac{r\pi}{2n_K}) - \binom{r}{1} f(x + \frac{2j\pi}{n_K} + \frac{r\pi}{2n_K} - \frac{\pi}{n_K})$$

$$+ \binom{r}{2} f(x + \frac{2j\pi}{n_K} + \frac{r\pi}{2n_K} - \frac{2\pi}{n_K}) + \cdots\cdots$$

$$+ (-1)^{r/2} \binom{r}{r/2} \cdot f(x + \frac{2j\pi}{n_K}) + \cdots\cdots$$

$$- \binom{r}{r-1} f(x + \frac{2j\pi}{n_K} - \frac{r\pi}{2n_K} + \frac{\pi}{n_K})$$

$$+ \binom{r}{r} . f(x + \frac{2j\pi}{n_K} - \frac{r\pi}{2n_K}) \qquad (2.23)$$

and $A_k(j) = C_{n_k}(2i \sin \frac{n_k\pi}{2n_K})^r e^{\frac{in_k 2j\pi}{n_K}}$.

Since $|A_k(j)| \leq |C_{n_k}| 2^r$, it follows from (2.20) that

$$\sum_{k=-\infty}^{\infty} |A_k(j)| s^{|n_k|} < \infty \quad (0 < s < 1) \qquad (2.24)$$

Put $g_j(s,x) = \sum_{k=-\infty}^{\infty} A_k(j) s^{|n_k|} e^{in_k x}$, $(0 < s < 1)$ (2.25)

We then get the identity

$$g_j(s,x) = f(s,x + \frac{2j\pi}{n_K} + \frac{r\pi}{2n_K})$$

$$- \binom{r}{1} f(s,x + \frac{2j\pi}{n_K} + \frac{r\pi}{2n_K} - \frac{\pi}{n_K})$$

$$+ \binom{r}{2} f(s, x + \frac{2j\pi}{n_K} + \frac{r\pi}{2n_K} - \frac{2\pi}{n_K})$$

$$+ \ldots + (-1)^{r/2} \binom{r}{r/2} f(s, x + \frac{2j\pi}{n_K}) +$$

$$\ldots - \binom{r}{r-1} f(s, x + \frac{2j\pi}{n_K} - \frac{r\pi}{2n_K} + \frac{\pi}{n_K})$$

$$+ \binom{r}{r} f(s, x + \frac{2j\pi}{n_K} - \frac{r\pi}{2n_K})$$

and from it together with (2.21) and (2.23) we obtain

$$g_j(x) = L^2\text{-}\lim_{s \to 1} g_j(s,x) \quad \text{in}$$

$$|x - x_o| \leq \delta/2 \qquad (2.26)$$

We assume that

$$(n_{k+1} - n_k) > 16\pi\,\delta^{-1} \quad \text{for all} \quad k \qquad (2.27)$$

In view of the gap condition (2.3), this can be achieved, if necessary, by adding to f(x) a polynomial in $\exp(in_k x)$, a process which affects neither the hypothesis nor the conclusion of the theorem.

Now put $A_k^* = \sum_j A_k(j)$ , $(-\infty < k < \infty)$

This is a finite summation and therefore (2.24) gives

$$\sum_{k=-\infty}^{\infty} |A_k^*| \, s^{|n_k|} < \infty \; , \; (0 < s < 1) \qquad (2.28)$$

Put $g^*(s,x) = \sum_{k=-\infty}^{\infty} A_k^* \, s^{|n_k|} \, e^{in_k x}$ , $(0 < s < 1)$

$$= \sum_j g_j(s,x) \qquad (2.29)$$

Therefore, from (2.26), we get

$$L^2\text{-limit}_{s \to 1} \; g^*(s,x) = \sum_j g_j(x)$$

$$= g^*(x), \quad \text{say}, \qquad (2.30)$$

over the range $|x-x_0| \leq \delta/2$ .

Thus from (2.27), (2.28), (2.29) and (2.30), we see that all the conditions of Lemma 2.2 are satisfied. Therefore

$$|A_K^*|^2 \leq \sum_{-\infty}^{\infty} |A_k^*|^2 \leq 16\delta^{-1} \int_{I_1 : |x-x_0| \leq \frac{\delta}{2}} |g^*(x)|^2 \, dx \qquad (2.31)$$

But $A_K^* = \sum_j A_K(j)$

$$= \sum_j C_{n_K} \, (2i)^r$$

$$= (-1)^{r/2} \; 2^r \, C_{n_K} \left[ \frac{\delta n_K}{8\pi} \right],$$

where $[\;\;]$ denotes the integral part.

Therefore

$$|A_K^*|^2 \geq (-1)^r \, 2^{2r} \, (a_{n_K}^2 + b_{n_K}^2) \; \frac{\delta^2 n_K^2}{64\pi^2} \cdot C ,$$

where $C$ is some positive constant.

This gives,

$$\frac{C\delta^2 \, n_K^2 \, a_{n_K}^2}{\pi^2 \, 2^{6-2r}} \leq |A_K^*|^2 \qquad (2.32)$$

On the other hand, for $|x - x_o| \leq \delta/2$ and integers j and K satisfying (2.22), the intervals

$$(y - \frac{i\pi}{n_K} , \; y - \frac{i\pi}{n_K} + \frac{\pi}{n_K}) ,$$

$$i = 1,2,\ldots,r \text{ where } y = x + \frac{2j\pi}{n_K} + \frac{r\pi}{2n_K} ,$$

are non-overlapping subintervals of $[x_o - \delta , \; x_o + \delta]$. Therefore, if V is the total $r^{th}$ variation of f in I, then it follows from the condition (2.5) that

$$\sum_j |g_j(x)| \leq V , \; (|x - x_o| \leq \frac{\delta}{2} ).$$

Thus $|g^*(x)| \leq \sum_j |g_j(x)| \leq V$ , which gives

$$|g^*(x)|^2 \leq V^2 \qquad (2.33)$$

Hence from (2.31), (2.32) and (2.33), we have

$$\frac{C \, \delta^2 \, n_K^2 \, a_{n_K}^2}{\pi^2 \; 2^{6-2r}} \leq 16 \, \delta^{-1} \int_{I_1} V^2 \, dx$$

$$= 16V^2 .$$

This gives

$$a_{n_K}^2 = O\left(\frac{1}{n_K^2}\right),$$

which in turn implies

$$a_{n_k} = O\left(\frac{1}{n_k}\right) \qquad (k \to \infty).$$

Similarly, we get $b_{n_k} = O\left(\frac{1}{n_k}\right) \qquad (k \to \infty).$

This proves Theorem 3.