CHAPTER - III

LACUNARY TRIGONOMETRIC SERIES AND THE ORDER OF ITS COEFFICIENTS

1. In chapter II, we saw that, a good deal of work has been done in studying the set of necessary conditions for a given lacunary trigonometric series to become the Fourier series of a function which belongs to some Lipschitz class. But a glance at the literature of the theory of lacunary Fourier series reveals that much remains to be done for studying the sufficient conditions in this regard. In the present chapter, we shall study the sufficient conditions under which a given trigonometric series becomes the Fourier series of a function belonging to some Lipschitz class.

Here, we consider the sequence $\{n_k\}$ given by

$$n_k = \left[a^{k^r}\right], \text{ where } a > 1 \text{ and } 0 < r \leqslant 1. \quad (3.1)$$

It can be easily verified that, for

$$0 < r < 1, \quad \frac{n_{k+1}}{n_k} \to 1 \quad \text{as } k \to \infty$$

and for $r = 1$, the sequence $\{n_k\}$ in (3.1) satisfies Hadamard gap condition

$$\frac{n_{k+1}}{n_k} \geq \lambda > 1 \quad \text{for all sufficiently large}$$

values of k. Thus a sequence $\{n_k\}$ of the type described in (3.1) is less restrictive than a Hadamard sequence. It is also known that for such a sequence

$$(n_{k+1} - n_k) > A \frac{n_k}{k^{1-r}} \quad \text{for all sufficiently}$$

large values of k, where A is a positive constant independent of k.

Further, while considering the gap condition (3.1) for studying the estimates of the Fourier coefficients of functions in Lipα(P), we need the following result due to Chao [3].

<u>THEOREM 3.A</u> [3 ; Theorem 1]. If

(i) $f \in \text{Lip}\alpha(P)$, $\alpha > 0$,

and (ii) $(n_{k+1} - n_k) \geq A_1 F(n_k)$, where

$F(n_k) \uparrow \infty$ as $k \uparrow \infty$, $F(n_k) \leq n_k$ for all k

and A_1 is a positive constant, then

$$a_{n_k}, b_{n_k} = O\left(F(n_k)\right)^{-\alpha}; \quad k = 1,2,3,\ldots .$$

Using Theorem 3.A for the sequence in (3.1), we obtain :

If $f \in \text{Lip}\alpha(P)$ $(\alpha > 0)$ and $\{n_k\}$ satisfies the condition (3.1) then

$$a_{n_k}\ ,\ b_{n_k} = O\left(\frac{k^{(1-r)\alpha}}{n_k^{\alpha}}\right)\ ,\quad (k \to \infty)\ .$$

We now consider the converse problem and state the result proved by Kennedy [19] in this regard.

<u>THEOREM 3.B</u> [19 ; Theorem I]. If

(i) $\{n_k\}$ is a sequence of natural numbers satisfying the Hadamard gap condition (1.2) ,

and (ii) $$\sum_{k=1}^{\infty} (a_{n_k} \cos n_k x + b_{n_k} \sin n_k x) \tag{3.2}$$

is a trigonometric series with

$$a_{n_k},\ b_{n_k} = O(n_k^{-\alpha})\ ,\ 0 < \alpha < 1,\quad (k \to \infty)\ ,$$

then (3.2) is the Fourier series of a function belonging to $\mathrm{Lip}\alpha(E)$, where E is a set of positive measure in $[-\pi\ ,\ \pi]$.

While studying the order of magnitude of Fourier coefficients for more general gap condition than (1.2) M. Izumi and S. Izumi [15] and thereafter Chao [3] obtained the results similar to that of Theorem 3.B. In fact, M. Izumi and S. Izumi proved the following theorem.

THEOREM 3.C. [15 ; Corollary 1]. If

(i) $f \in \mathrm{Lip}\alpha(P)$ $(0 < \alpha < 1)$,

and (ii) f has the Fourier series with the Hadamard gap condition (1.2),

then $f \in \mathrm{Lip}\alpha$ in $[-\pi, \pi]$.

Now, our aim here is to replace the Hadamard gap condition (1.2) in Theorems 3.B and 3.C by a weaker gap condition (3.1). In fact, we prove the following theorem in this chapter.

THEOREM 4. If (i) $\{n_k\}$ is a given increasing sequence of natural numbers satisfying

$$n_k = [a^{k^r}], \text{ where } a > 1;\ 0 < r < 1 \tag{3.3}$$

and (ii) $$a_{n_k}, b_{n_k} = O\left(\frac{k^{(1-r)\alpha}}{n_k^{\alpha}}\right),\ 0 < \alpha < 1, \tag{3.4}$$

then the trigonometric series (3.2) is the Fourier series of a function belonging to Lip α in $[-\pi, \pi]$.

It may be observed that, when $r = 1$, our theorem matches with Theorem 3.B due to Kennedy as well as with Theorem 3.C due to M. Izumi and S. Izumi.

2. In order to prove the theorem, we need the following lemma due to Chao [3].

LEMMA 3.1 If $0 < \alpha r < 1$ then

$$\sum_{p=1}^{K} n_p^{1-\alpha r}\, p^{-1} = O(n_K^{1-\alpha r}).$$

Proof of Theorem 4: It follows, from the gap hypothesis (3.3) that, given any positive integer m, we have

$$n_p > p^m \text{ for all sufficiently large values of } p. \quad (3.5)$$

We choose $m > 1 + \frac{1}{\alpha(1-r)}$. (3.6)

Then certainly $m > \frac{1 + \alpha - r\alpha}{\alpha}$ (3.7)

Now

$$\begin{aligned}
&|a_{n_k} \cos n_k x + b_{n_k} \sin n_k x| \\
&\leq (|a_{n_k}| + |b_{n_k}|), \quad x \in [-\pi, \pi] \\
&\leq \frac{A\, k^{(1-r)\alpha}}{n_k^{\alpha}} \\
&\leq \frac{A\, k^{(1-r)\alpha}}{k^{m\alpha}} \\
&= \frac{A}{k^{m\alpha - \alpha + r\alpha}},
\end{aligned}$$

and $\sum_{k=1}^{\infty} \frac{A}{k^{m\alpha - \alpha + r\alpha}}$ is convergent as

$$m > \frac{1 + \alpha - r\alpha}{\alpha} \text{ by } (3.7)$$

Hence, by Weierstrass M-test, the series (3.2) converges uniformly to some function f in $[-\pi, \pi]$ and that it is a Fourier series of f. Therefore

$$f(x) = \sum_{k=1}^{\infty} (a_{n_k} \cos n_k x + b_{n_k} \sin n_k x), \quad x \in [-\pi, \pi].$$

In order to show that $f \in \text{Lip}\,\alpha r$, $0 < \alpha < 1$; $0 < r < 1$

in $[-\pi , \pi]$, we have to show that

$$f(x+h) - f(x) = O(|h|^{\alpha r}) \text{ uniformly for } x \in [-\pi , \pi]$$

as $h \to 0$.

It follows from (3.5) that there exists a positive integer K_1 such that $n_p > p^m$ for all $p \geq K_1$. (3.8)

Consider any $K > K_1$, and then the interval $\left(\frac{1}{n_{K+1}} , \frac{1}{n_K} \right]$. Take a real number h so small that

$$|h| \in \left(\frac{1}{n_{K+1}} ; \frac{1}{n_K} \right]. \tag{3.9}$$

Now $|f(x+h)-f(x)| \leq A \sum_{p=1}^{\infty} |n_p^{-\alpha} p^{(1-r)\alpha} \sin \frac{n_p h}{2}|$,

where $A > 0$ is a constant, independent of x and h.

$$= A\left\{\sum_{p=1}^{K} + \sum_{p=K+1}^{\infty}\right\}|n_p^{-\alpha} p^{(1-r)\alpha} \sin \frac{n_p h}{2}|$$

$$= A(S + T) \text{, say} \tag{3.10}$$

Since $|\sin y| \leq |y|$ for all real y,

$$S = \sum_{p=1}^{K} |n_p^{-\alpha} p^{(1-r)\alpha} \sin \frac{n_p h}{2}|$$

$$\leq \sum_{p=1}^{K} n_p^{-\alpha} p^{(1-r)\alpha} \left|\frac{n_p h}{2}\right|$$

$$= \frac{1}{2}|h|\left\{\sum_{p=1}^{K_1} + \sum_{p=K_1}^{K}\right\} n_p^{(1-\alpha)} \cdot p^{(1-r)\alpha} \tag{3.11}$$

But, by (3.6)

$$m > 1 + \frac{1}{\alpha(1 - r)}$$

Therefore $p^{m(\alpha-\alpha r)} \geq p^{(1-r)\alpha+1}$

This together with (3.8) gives

$$n_p^{\alpha-\alpha r} \geq p^{(1-r)\alpha+1} \qquad \text{for all } p \geq K_1 .$$

Therefore, on multiplying both the sides by

$n_p^{1-\alpha}\, p^{-1}$, we get $n_p^{1-\alpha r} \cdot p^{-1} \geq n_p^{1-\alpha} \cdot p^{(1-r)\alpha}$ (3.12)

Thus $\sum_{p=K_1}^{K} n_p^{(1-\alpha)}\, p^{(1-r)\alpha} \leq \sum_{p=K_1}^{K} n_p^{1-\alpha r} \cdot p^{-1}$ (3.13)

Also, by using Archimedean property, we have

$$\sum_{p=1}^{K_1} n_p^{(1-\alpha)} \cdot p^{(1-r)\alpha} \leq N \sum_{p=1}^{K_1} n_p^{1-\alpha r} \cdot p^{-1} ,$$

for some positive integer N. (3.14)

Hence, using (3.13) and (3.14), the relation (3.11) gives

$$S \leq \frac{N|h|}{2} \sum_{p=1}^{K} (n_p^{(1-\alpha r)} \cdot p^{-1})$$

$$= O(|h|\, n_K^{1-\alpha r}) \text{ , by Lemma 3.1.}$$

Therefore, by using (3.9) we have

$$S = O(|h|^{\alpha r}) \qquad (3.15)$$

On the other hand,

$$T = \sum_{p=K+1}^{\infty} \left| n_p^{-\alpha} \cdot p^{(1-r)\alpha} \left(\sin \frac{n_p h}{2}\right) \right.$$

$$\leq \sum_{p=K+1}^{\infty} \left| n_p^{-\alpha} \cdot p^{(1-r)\alpha} \right| \qquad (3.16)$$

But by (3.12)

$$n_p^{-\alpha r} \cdot p^{-1} \geq n_p^{-\alpha} \cdot p^{(1-r)\alpha} .$$

Therefore (3.16) becomes,

$$T \leq \sum_{p=K+1}^{\infty} n_p^{-\alpha r} \cdot p^{-1}$$

$$= \sum_{\gamma=0}^{\infty} \left\{ \sum_{p=2^{\gamma}(K+1)}^{2^{\gamma+1}(K+1)-1} \left(n_p^{-\alpha r} \cdot p^{-1}\right) \right\}$$

$$\leq \sum_{\gamma=0}^{\infty} \left\{ n_{2^{\gamma}(K+1)}^{-\alpha r} \cdot \left(2^{\gamma}(K+1)\right)^{-1} \sum_{p=2^{\gamma}(K+1)}^{2^{\gamma+1}(K+1)-1} 1 \right\}$$

$$= \sum_{\gamma=0}^{\infty} \left\{ n_{2^{\gamma}(K+1)}^{-\alpha r} \left(2^{\gamma}(K+1)\right)^{-1} \cdot 2^{\gamma}(K+1) \right\}$$

$$= \sum_{\gamma=0}^{\infty} n_{2^{\gamma}(K+1)}^{-\alpha r}$$

$$= n_{K+1}^{-\alpha r}\left\{1 + \left(\frac{n_{2(K+1)}}{n_{K+1}}\right)^{-\alpha r}\right.$$

$$\left. + \left(\frac{n_{2^2(K+1)}}{n_{K+1}}\right)^{-\alpha r} + \dots\dots\dots \right\} \qquad (3.17)$$

Now it is known [3 ; p.311] that, by some modifications,

$$\frac{n_{2k}}{n_k} \geq \ell > 1 \quad \text{for all} \quad k .$$

Therefore (3.17) becomes,

$$T \leq n_{K+1}^{-\alpha r}\left\{1 + \ell^{-\alpha r} + \ell^{-2\alpha r} + \dots\dots \right\} .$$

The series inside the bracket is a geometric series with common ratio less than 1, as $\ell > 1$ and therefore it is convergent. This shows that

$$T = O(n_{K+1}^{-\alpha r})$$

$$= O(|h|)^{\alpha r}) , \quad \text{by} \quad (3.9) . \qquad (3.18)$$

Hence, from (3.10), (3.15) and (3.18), we have

$$|f(x+h)-f(x)| = O(|h|^{\alpha r}) \quad \text{uniformly for every}$$

$$x \in [-\pi , \pi], \text{ as } h \to 0 ,$$

and therefore $f \in \mathrm{Lip}\,\alpha r$ in $[-\pi , \pi]$.

This completes the proof of thèorem 4.