CHAPTER IV

ABSOLUTE CONVERGENCE OF LACUNARY FOURIER SERIES OF FUNCTIONS IN Lip(α,p) AND BOUNDED VARIATION CLASS

1. It is well known that S. Bernštein [2; p.154] and Antoni Zygmund [2; p.161] did the pioneering work in the study of absolute convergence of a Fourier series. The following theorem is due to Zygmund.

THEOREM 4.A. [2; p.161]. If

(i) $f \in \mathrm{Lip}\alpha$, $\alpha > 0$, in $[-\pi , \pi]$,

and (ii) f is of bounded variation in $[-\pi , \pi]$,

then the Fourier series of f converges absolutely.

This well known result of Zygmund is generalized by Z. Waraszkiewicz [39] and Zygmund himself [41] as under:

THEOREM 4.B. Under the conditions of Theorem 4.A, the series

$$\sum_{n=1}^{\infty} (|a_n|^{\beta} + |b_n|^{\beta}) < \infty \quad \text{for all}$$

$\beta > 2(\alpha + 2)^{-1}$, where a_n , b_n are the Fourier coefficients of f.

Theorem 4.B does not hold for $\beta = 2(\alpha + 2)^{-1}$.

We also have the following extension of Zygmund's theorem.

ON THE ABSOLUTE SUMMABILITY OF A LACUNARY FOURIER SERIES

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1 Let

$$\sum_{k=1}^{\infty} (a_{n_k} \cos n_k x + b_{n_k} \sin n_k x) \tag{1·1}$$

be the Fourier series of a 2π–periodic function $f \in L[-\pi, \pi]$ with an infinity of gaps (n_k, n_{k+1}), where $\{n_k\}$ $(k \in N)$ is a strictly increasing sequence of natural numbers. M. Izumi and S. I. Izumi [2] studied the order of magnitude of Fourier coefficients and the absolute convergence of the Fourier series (1·1) with some lacunae, when the function satisfies Lipschitz condition only at a point. In fact, they have proved the following theorems :

THEOREM A. [2; Theorem 1]. If

(i) $n_{k+1} - n_k > A n_k^{\beta}$ (A is a positive constant, $0 < \beta \leq 1$) (1·2)

and (ii) $f \in \text{Lip}\, \alpha$ at a point $x_0 \in (-\pi, \pi)$, $0 < \alpha < 1$, (1·3)

i.e., $| f(x_0 + t) - f(x_0) | = O(|t|^{\alpha})$ as $t \to 0$,

then

$$a_{n_k},\ b_{n_k} = O\left(\frac{1}{n_k^{\alpha\beta}}\right),\ k = 1, 2 \ldots\ldots \tag{1·4}$$

THEOREM B. [2; Theorem 2]. Let f satisfy the conditions of Theorem A with $0 < \beta < 1$. Then the series (1·1) converges absolutely when

$$\alpha > \min\left(\frac{1}{2\beta},\ \frac{1}{\beta} - 1\right).$$

These results are related to the earlier results due to Noble [5] and Kennedy [3, 4]. In fact, Theorem A generalizes a part of the theorem due to Kennedy [4; Theorem II] relating to the order of magnitude of Fourier coefficients. The particular case for $\gamma = 1$ and $r = 0$ of the following theorem C due to J. R. Patadia and V. M. Shah [6] is a generalization of the other part of Kennedy's theorem regarding the absolute convergence of (1·1) This generalization ensures the absolute convergence of (1·1) when $\alpha > \frac{1}{2}(\beta^{-1} - 1)$. Further, the theorem C provides us with a generalization of theorem B also.

THEOREM C. [6]. If f satisfies (1·3) and if

$$(n_{k+1} - n_k) > C\, n_k^{\beta}\, k^{r} \quad (0 < \beta < 1, r \geqslant 0) \tag{1·5}$$

where C is a positive constant, then

$$\sum_{k=1}^{\infty} (|a_{n_k}|^{\gamma} + |b_{n_k}|^{\gamma}) < \infty, \quad 0 < \gamma \leqslant 1 \tag{1·6}$$

when $\alpha\beta\gamma + \alpha\gamma r > \left(1 - \frac{\gamma}{2}\right)(1-\beta)$.

Now, it is quite natural to inquire into the behaviour of the series (1·1) when $\alpha \leqslant \frac{1}{2}(\beta^{-1} - 1)$. In this regard, we propose to study the absolute summability of the series (1·1). Let us consider the gap condition

$$(n_{k+1} - n_k) > A n_k^{\beta} \quad (0 < \beta < 1) \tag{1·7}$$

where A is a positive constant to be selected as under. Suppose that M is a positive integer greater than δ, where $\delta = \frac{1}{1-\beta}$. Let $A > 2^M - 1$. We prove the following theorems.

THEOREM 1. If $f \in \text{Lip}\, \alpha$ at a point $x_0 \in (-\pi, \pi)$ and if $\{n_k\}$ satisfies (1·7), then the Fourier series (1·1) of f is absolutely summable $\left(C, \frac{1}{2}\right)$

$$\text{(i) for every } \alpha > 0 \text{ if } \beta \geqslant \frac{3}{5}; \tag{1·8}$$

$$\text{or (ii) for every } \alpha > \frac{3}{2\beta} - \frac{5}{2} \text{ if } \beta < \frac{3}{5}. \tag{1·9}$$

THEOREM 2. Under the hypothesis of theorem 1, the Fourier series (1·1) of f is absolutely summable (C, 1) when $\alpha > \beta^{-1} - 2$.

REMARK. The significance of the conclusion (1·9) in theorem 1 could be visualized when $\beta > \frac{1}{2}$. Because, under this condition, we have

$$\frac{3}{2\beta} - \frac{5}{2} < \frac{1}{2}(\beta^{-1} - 1)$$

and consequently the absolute summability $\left(C, \frac{1}{2}\right)$ of the series (1·1) is ensured for the range

$$\frac{3}{2\beta} - \frac{5}{2} < \alpha \leqslant \frac{1}{2}\left(\beta^{-1} - 1\right).$$

Similarly the significance of theorem 2 could be visualized when $\beta > \frac{1}{3}$,

as $\beta^{-1} - 2 < \frac{1}{2}\left(\beta^{-1} - 1\right)$ in this case.

2. We need the following lemma, due to J. R. Patadia and V. M. Shah [7], the proof of which is given here for the sake of completeness.

LEMMA. If $\{n_k\}$ satisfies (1·7) then

$$n_k \geqslant k^{\delta} \text{ for all } k \in N. \tag{2·1}$$

PROOF. Obviously $n_1 \geqslant 1^{\delta}$. Suppose $n_p \geqslant p^{\delta}$. Then, since $M > \delta$ and since $\delta = \frac{1}{1-\beta}$ implies $\delta\beta = \delta - 1$ we have by (1·7)

$$\begin{aligned}
n_{p+1} &> n_p + A\, n_p^{\beta} \\
&\geqslant p^{\delta} + A\, p^{\delta\beta} \\
&= p^{\delta}\left(1 + \frac{A}{p}\right) \\
&> p^{\delta}\left[1 + \frac{1}{p}\left(2^{M} - 1\right)\right] \\
&= p^{\delta}\left\{1 + \frac{1}{p}\left[\binom{M}{1} + \binom{M}{2} + \cdots\cdots + \binom{M}{M}\right]\right\} \\
&> p^{\delta}\left\{1 + \frac{1}{p}\left[\binom{M}{1} + \binom{M}{2}\frac{1}{p} + \binom{M}{3}\frac{1}{p^{2}} + \cdots\cdots + \binom{M}{M}\frac{1}{p^{M-1}}\right]\right\} \\
&= p^{\delta}\left(1 + \frac{1}{p}\right)^{M} \\
&> p^{\delta}\left(1 + \frac{1}{p}\right)^{\delta} \\
&= (p+1)^{\delta}
\end{aligned}$$

Hence the lemma by induction.

Proof of Theorem 1. For a real number γ, other than a negative integer, put $E_n^{\gamma} = \binom{n+\gamma}{n}$ where $n \in N$ and $E_0^{\gamma} = 1$. Denoting the n^{th} Cesàro mean of order $\theta > 0$ by $\sigma_n^{\theta}(x)$ and replacing the absent terms in (1·1) by zeros, we have [1]

$$\left| \sigma_{n_k}^{\theta}(x) - \sigma_{n_k - 1}^{\theta}(x) \right|$$

$$= \frac{1}{n_k E_{n_k}^{\theta}} \left| \sum_{p=1}^{k} E_{n_k - n_p}^{\theta - 1} \cdot n_p \left(a_{n_p} \cos n_p x + b_{n_p} \sin n_p x \right) \right|$$

$$\leqslant \frac{1}{n_k E_{n_k}^{\theta}} \left\{ \left| n_k \left(a_{n_k} \cos n_k x + b_{n_k} \sin n_k x \right) \right| \right.$$

$$\left. + \left| \sum_{p=1}^{k-1} E_{n_k - n_p}^{\theta - 1} \cdot n_p \cdot \left(a_{n_p} \cos n_p x + b_{n_p} \sin n_p x \right) \right| \right\} \qquad (2\cdot2)$$

Let $\theta = \frac{1}{2}$.

Now

(i) $E_n^{\theta} \backsimeq \dfrac{n^{\theta}}{\Gamma(\theta+1)}$,

(ii) $a_{n_k}, b_{n_k} = O\left(\dfrac{1}{n_k^{\alpha\beta}}\right)$, $k = 1, 2, 3, \ldots\ldots$, by Theorem A,

and (iii) $|n_k - n_p|$

$\geqslant |n_k - n_{k-1}|$ for $p = 1, 2, 3, \ldots\ldots\ldots, k-1$

$> A\, n_k^{\beta}$, by (1·7).

Hence, from (2·1) and (2·2), we obtain

$$\left| \sigma_{n_k}^{\theta}(x) - \sigma_{n_k - 1}^{\theta}(x) \right|$$

$$= O(1)\,\frac{1}{n_k n_k^{\theta}}\left\{ n_k\left(n_k^{-\alpha\beta} + n_k^{-\alpha\beta}\right) + \sum_{p=1}^{k-1} \frac{1}{|n_k - n_p|^{1-\theta}}\, n_p\left(n_p^{-\alpha\beta} + n_p^{-\alpha\beta}\right)\right\}$$

$$= O(1)\,\frac{1}{n_k^{1+\theta}}\left\{ n_k^{1-\alpha\beta} + \frac{1}{n_k^{\beta(1-\theta)}}\cdot k\cdot n_k^{1-\alpha\beta}\right\}$$

$$= O(1)\left\{\frac{1}{n_k^{\theta+\alpha\beta}} + \frac{k}{n_k^{\beta-\beta\theta+\alpha\beta+\theta}}\right\}$$

$$= O(1)\left\{\frac{1}{k^{\theta\delta+\alpha\beta\delta}} + \frac{1}{k^{\delta\beta-\delta\beta\theta+\alpha\beta\delta+\theta\delta-1}}\right\}$$

$$= O(1)\left[\frac{1}{k^{\frac{2\alpha\beta+1}{2(1-\beta)}}} + \frac{1}{k^{\frac{3\beta+2\alpha\beta-1}{2(1-\beta)}}}\right], \text{ as } \delta = \frac{1}{1-\beta} \text{ and } \theta = \frac{1}{2}.$$

But $\alpha > \frac{1}{2\beta} - 1$ implies $\frac{2\alpha\beta+1}{2(1-\beta)} > 1$ and

$\alpha > \frac{3}{2\beta} - \frac{5}{2}$ implies $\frac{2\alpha\beta+3\beta-1}{2(1-\beta)} > 1$.

Hence, in order to establish the convergence of

$\sum_{k=1}^{\infty} \left| \sigma^{\theta}_{n_k}(x) - \sigma^{\theta}_{n_k - 1}(x) \right|$, it is sufficient to have

$$\alpha > \max\left\{\frac{1}{2\beta} - 1, \frac{3}{2\beta} - \frac{5}{2}\right\},$$

which is ensured in the following.

Case (i) Let $\beta \geqslant \frac{3}{5}$.

If $\frac{3}{5} \leqslant \beta < \frac{2}{3}$ then

$$\frac{1}{2\beta} - 1 < \frac{3}{2\beta} - \frac{5}{2} \leqslant 0 \leqslant \alpha.$$

If $\frac{2}{3} \leqslant \beta$ then

$$\frac{3}{2\beta} - \frac{5}{2} \leqslant \frac{1}{2\beta} - 1 < 0 < \alpha.$$

Case (ii) Let $\beta < \frac{3}{5}$ and $\alpha > \frac{3}{2\beta} - \frac{5}{2}$.

Then certainly $\beta < \frac{2}{3}$, which shows that

$$\frac{1}{2\beta} - 1 < \frac{3}{2\beta} - \frac{5}{2} < \alpha$$

This proves that (1·1) is absolutely summable $\left(C, \frac{1}{2}\right)$.

Proof of Theorem 2. Let $\theta = 1$. Then by using Theorem A and (2·1), we obtain from (2·2),

$$\left| \sigma_{n_k}(x) - \sigma_{n_k - 1}(x) \right|$$

$$= \frac{1}{n_k (n_k + 1)} \left| \sum_{p=1}^{k} n_p \left(a_{n_p} \cos n_p x + b_{n_p} \sin n_p x \right) \right|$$

$$= O(1) \frac{1}{n_k^2} \sum_{p=1}^{k} n_p^{(1-\alpha\beta)}$$

$$= O(1) \frac{1}{n^2_k} : \frac{k}{n_k^{\alpha\beta - 1}}$$

$$= O(1) \left\{ \frac{k}{k^{\delta(\alpha\beta+1)}} \right\}$$

$$= O(1) \left\{ \frac{1}{k^{\alpha\beta\delta + \delta - 1}} \right\}$$

$$= O(1) \left\{ \frac{1}{k^{\frac{\alpha\beta + \beta}{1 - \beta}}} \right\}, \text{ as } \delta = \frac{1}{1-\beta}$$

Since $\alpha > \beta^{-1} - 2$, it follows that $\frac{\alpha\beta + \beta}{1-\beta} > 1$.

Hence

$$\sum_{k=1}^{\infty} \left| \sigma_{n_k}(x) - \sigma_{n_k - 1}(x) \right| < \infty,$$

which implies the absolute summability (C, 1) of (1·1). This completes the proof of the theorem.

The author is thankful to Professor V.M. Shah for his help and guidance in the preparation of this paper.

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THEOREM 4.C. Under the conditions of Theorem 4.A, the series

$$\sum_{n=1}^{\infty} n^{\beta/2} (|a_n| + |b_n|) < \infty \text{ for}$$

every $\beta < \alpha$.

Considering more general class of functions belonging to Lip(α,p) than that of Lipα, Hardy and Littlewood [13] obtained the following generalization of Zygmund's theorem 4.A.

THEOREM 4.D. [13] If (i) $f \in$ Lip(α,p), $0 < \alpha \leq 1$; $\alpha p > 1$ in $[-\pi , \pi]$; and (ii) f is of bounded variation over $[-\pi , \pi]$, then the Fourier series of f converges absolutely.

This theorem reduces to Zygmund's theorem when $p = \infty$.

We now look to the relevant development in the theory of lacunary Fourier series, by considering the paper due to Noble, which was published in the year 1954. Noble [23] studied a lacunarity condition which enabled him to deduce results of general character concerning the behaviour of the Fourier coefficients and the absolute convergence of the lacunary Fourier series (L) under the assumption that the corresponding function f has certain property e.g. being of bounded variation or belonging to Lipα, in a arbitrary small subinterval of the interval of periodicity. Noble's lacunarity condition makes it possible to relax restrictions on the behaviour of f. The following theorems due to Noble are mentioned in this regard.

THEOREM 4.E. [23 ; Theorem 5] . If

(i) $\lim_{k \to \infty} \frac{N_k}{\log n_k} = \infty$, $N_k = \min \{n_{k+1} - n_k , n_k - n_{k-1}\}$,

(ii) $f \in \mathrm{Lip}\alpha(I)$, $0 < \alpha < 1$, (4.1)

and (iii) f is of bounded variation in I, (4.2)

then

$$\sum_{k=1}^{\infty} (|a_{n_k}| + |b_{n_k}|) < \infty \qquad (4.3)$$

THEOREM 4.F. [23] . Under the hypothesis of Theorem 4.E,

$$\sum_{k=1}^{\infty} (|a_{n_k}|^{\beta} + |b_{n_k}|^{\beta}) < \infty ,$$

whenever $\beta > \frac{2}{\alpha + 2}$.

THEOREM 4.G. [23] . Under the hypothesis of Theorem 4.E,

$$\sum_{k=1}^{\infty} n_k^{\beta/2} (|a_{n_k}| + |b_{n_k}|) < \infty \text{ for every } \beta < \alpha .$$

In 1956, Kennedy [17] used more powerful methods due to Paley and Wiener [25] to give a simple proof of Noble's theorems under less restricted gap hypothesis. He proved the following theorem.

<u>THEOREM 4.H.</u> [17 ; Theorem V (iv)]. If

(i) $(n_{k+1} - n_k) \to \infty$ as $k \to \infty$ (4.4)

and (ii) f satisfies the conditions (4.1) and (4.2), then (4.3) holds.

Theorems 4.F and 4.G , under the gap condition (4.4), can be easily deduced on the same line.

Thus, we observe that, Kennedy studied the absolute convergence of the lacunary Fourier series (L), under the gap condition (4.4), when the hypothesis in the Zygmund's theorem 4.A is satisfied only in a subinterval of $[-\pi , \pi]$.

In this chapter, we wish to carry out a study of the absolute convergence of a lacunary Fourier series (L), keeping in mind the Hardy-Littlewood Theorem 4.D. The nature of this study is analogous to the one which was carried out by Kennedy in case of Zygmund's Theorem 4.A. In fact, we prove the following theorems.

<u>THEOREM 5.</u> If (i) $\{n_k\}$ satisfies (4.4),

(ii) $f \in Lip(\alpha, p, I)$ with $0 < \alpha \leq 1$; $p > 2$; $\alpha p > 1$,

and (iii) f satisfies (4.2), then the Fourier series (L) of f converges absolutely.

It may be observed that the above Theorem 5 generalizes Theorem 4.H due to Kennedy. We also have the following two extensions of Theorem 5.

THEOREM 6. If (i) $\{n_k\}$ satisfies (4.4),

(ii) $f \in Lip(\alpha,p,I)$ with $0 < \alpha \leq 1$; $p > 2$,

and (iii) f satisfies (4.2),

then

$$\sum_{k=1}^{\infty} (|a_{n_k}|^{\beta} + |b_{n_k}|^{\beta}) < \infty \quad \text{for}$$

every β satisfying $2 > \beta > \frac{2(p-1)}{2p + \alpha p - 3}$.

It may be observed that, when $\beta = 1$, Theorem 6 reduces to Theorem 5.

THEOREM 7. Under the hypothesis of Theorem 6,

$$\sum_{k=1}^{\infty} n_k^{\beta/2} (|a_{n_k}| + |b_{n_k}|) < \infty \quad \text{for}$$

every $\beta < \frac{\alpha p - 1}{2(p-1)}$.

It may be observed that, for $\beta = 0$, Theorem 7 reduces to Theorem 5.

2. In order to establish our theorems, we need Lemma 2.1 and Lemma 2.3. These lemmas are given in Chapter II.

Proof of Theorem 5.

Put $n_0 = 0$, $n_k = -n_{-k}$ $(k < 0)$

$C_{n_0} = 0$, $C_{n_k} = \frac{1}{2}(a_{n_k} - ib_{n_k})$ $(k > 0)$

$C_{n_k} = \bar{C}_{n_{-k}}$ $(k < 0)$. Therefore we can write (L) in the form

$$\sigma(f) = \sum_{k=-\infty}^{\infty} C_{n_k} e^{in_k x} .$$

We assume that the integers j and K satisfy

$$n_K > \frac{2\pi}{\delta} \text{ and } 0 \leq j \leq \frac{1}{4}\pi^{-1}\delta n_K \tag{4.5}$$

Take $f_j(x) = f(x + \frac{2j\pi}{n_K}) - f(x + \frac{(2j-1)\pi}{n_K})$. (4.6)

We have $\sum_{-\infty}^{\infty} |C_{n_k}| s^{|n_k|} < \infty$, $(0 < s < 1)$. (4.7)

Put $f(s,x) = \sum_{-\infty}^{\infty} C_{n_k} s^{|n_k|} e^{in_k x}$, $(0 < s < 1)$ (4.8)

for all real x. The existence of f(s,x) is ensured by (4.7). We obviously have

$$f(s,x) = \sum_{k=1}^{\infty} (a_{n_k} \cos n_k x + b_{n_k} \sin n_k x) s^{n_k} .$$

It follows from (4.2) that f is bounded in I. Therefore $f \in L^2(I)$. Since $n_{k+1} - n_k \to \infty$, it follows from Lemma 2.1

that $f \in L^2[-\pi, \pi]$.

Hence by a known theorem [43; p.87] it follows that

$$f(x) = L^2\text{-}\lim_{s \to 1} f(s,x) , \quad (|x| \leq \pi). \tag{4.9}$$

Without loss of generality, we assume that

$$n_{k+1} - n_k > 16\pi\delta^{-1} \text{ for all } k \tag{4.10}$$

In view of (4.4), this can be achieved, if necessary, by adding to $f(x)$ a polynomial in $\exp(in_k x)$, a process which affects neither the hypothesis nor the conclusion of the theorem.

Thus from (4.6), (4.7), (4.8), (4.9) and (4.10) we see that all the conditions of Lemma 2.3 are satisfied. Hence

$$\sum_{\frac{1}{2}n_K \leq |n_k| \leq n_K} |c_{n_k}|^2 \leq 8\delta^{-1} \int_{I_1 : |x-x_0| \leq \delta/2} |f_j(x)|^2 dx . \tag{4.11}$$

Let $p' = p - 1$. As $p > 2$, $p' > 1$.

Let q be a real number such that

$\frac{1}{p'} + \frac{1}{q} = 1.$

Now (4.11) gives,

$$\left\{ \sum_{\frac{1}{2} n_K}^{n_K} |c_{n_k}|^2 \right\}^q \leq (8\delta^{-1})^q \left\{ \int_{I_1} |f_j(x)|^2 dx \right\}^q .$$

Taking summation over j, we get,

$$\sum_{0 \leq j \leq \left[\frac{\delta n_K}{4\pi}\right]} \left\{ \sum_{\frac{1}{2} n_K}^{n_K} |c_{n_k}|^2 \right\}^q \leq \sum_{0 \leq j \leq \left[\frac{\delta n_K}{4\pi}\right]} (8\delta^{-1})^q \left\{ \int_{I_1} |f_j(x)|^2 dx \right\}^q .$$

From this we get,

$$\left\{ \sum_{\frac{1}{2} n_K}^{n_K} |c_{n_k}|^2 \right\}^q \leq \frac{(8\delta^{-1})^q}{\left[\frac{\delta n_K}{4\pi}\right]} \sum_j \left\{ \int_{I_1} |f_j(x)|^2 dx \right\}^q$$

$$\leq \frac{C}{n_K} \sum_j \left\{ \int_{I_1} |f_j(x)|^2 dx \right\}^q , \qquad (4.12)$$

where C is some constant depending on δ.

Note that, C denotes a positive constant which is different at different occurances.

Since $2 = 1 + 1$

$$= \left(1 + \frac{1}{p'}\right) + \frac{1}{q}$$

$$= \frac{p' + 1}{p'} + \frac{1}{q} ,$$

an application of Hölder's inequality gives

$$\left\{ \int_{I_1} |f_j(x)|^2 dx \right\}^q = \left\{ \int_{I_1} |f_j(x)|^{\frac{p'+1}{p'} + \frac{1}{q}} dx \right\}^q$$

$$\leq \left\{ \left(\int_{I_1} |f_j(x)|^{\frac{p'+1}{p'} \cdot p'} dx \right)^{\frac{1}{p'}} \cdot \left(\int_{I_1} |f_j(x)|^{\frac{1}{q} \cdot q} dx \right)^{\frac{1}{q}} \right\}^q$$

$$= \left\{ \int_{I_1} |f_j(x)|^p dx \right\}^{\frac{q}{p'}} \left\{ \int_{I_1} |f_j(x)| dx \right\}.$$

This combined with (4.12) gives

$$\left\{ \sum_{\frac{1}{2}n_K}^{n_K} |c_{n_k}|^2 \right\}^q \leq \frac{C}{n_K} \sum_j \left\{ \left(\int_{I_1} |f_j(x)|^p dx \right)^{\frac{q}{p'}} \cdot \int_{I_1} |f_j(x)| dx \right\}. \tag{4.13}$$

Now the condition $f \in Lip(\alpha, p, I)$ implies that

$$\int_{I_1} |f_j(x)|^p dx = \int_{I_1} \left| f\left(x + \frac{2j\pi}{n_K}\right) - f\left(x + \frac{(2j-1)\pi}{n_K}\right) \right|^p dx$$

$$= O\left(\left(\frac{\pi}{n_K} \right)^{\alpha p} \right),$$

and, for $|x-x_0| \leq \frac{1}{2}\delta$ and integers j, K satisfying (4.5), the intervals

$$\left(x + (2j-1)\frac{\pi}{n_K}, \; x + \frac{2j\pi}{n_K} \right)$$

are non-overlapping subintervals of $[x_0 - \delta, x_0 + \delta]$. Therefore, if V is the total variation of f in $|x-x_0| \leq \delta$, we have

$$\sum_j |f_j(x)| \leq V \qquad \left(|x - x_0| \leq \tfrac{1}{2}\delta \right).$$

Therefore (4.13) becomes

$$\Big\{ \sum_{\frac{1}{2}n_K}^{n_K} |c_{n_k}|^2 \Big\}^q \leq \frac{C}{n_K} \left(\frac{\pi}{n_K}\right)^{\frac{\alpha p . q}{p'}} . V . \delta$$

$$\doteq C \frac{1}{n_K^{(1+\frac{\alpha p . q}{p'})}} ,$$

which in turn gives,

$$\sum_{\frac{1}{2}n_K}^{n_K} |c_{n_k}|^2 \leq C \frac{1}{n_K^{(\frac{1}{q} + \frac{\alpha p}{p'})}} . \tag{4.14}$$

Let m be a positive integer. Either the set of integers k for which

$2^m < |n_k| \leq 2^{m+1}$ is empty, or there is a member of this set, say $K = K(m)$, which has largest modulus; and in the later case the set is included in the set of k for which

$$\tfrac{1}{2}n_K \leq |n_k| \leq n_K .$$

Thus, in either case,

$$\sum_{2^m < |n_k| \leq 2^{m+1}} |c_{n_k}|^2 \leq \sum_{\frac{1}{2}n_K \leq |n_k| \leq n_K} |c_{n_k}|^2 = O(n_K^{(-\frac{1}{q} - \frac{\alpha p}{p'})}), \text{ by } (4.14)$$

$$= O(2^{m(-\frac{1}{q} - \frac{\alpha p}{p'})}). \tag{4.15}$$

Also, by (4.10), the number of terms in the summation

$\sum_{2^m}^{2^{m+1}} 1$ is of $O(2^m)$.

Hence by Cauchy's inequality we get,

$$\sum_{2^m}^{2^{m+1}} |c_{n_k}|\cdot 1 \leq \left\{\sum_{2^m}^{2^{m+1}} |c_{n_k}|^2\right\}^{\frac{1}{2}} \cdot \left\{\sum_{2^m}^{2^{m+1}} 1\right\}^{\frac{1}{2}}$$

$$= O\left\{2^{(-\frac{1}{q} - \frac{\alpha p}{p'})\frac{m}{2}} \cdot 2^{\frac{m}{2}}\right\}$$

$$= O\left\{2^{\frac{m}{2}(1 - \frac{1}{q} - \frac{\alpha p}{p'})}\right\}$$

$$= O\left\{2^{\frac{m}{2}(\frac{1 - \alpha p}{p'})}\right\} \qquad (4.16)$$

and hence

$$\sum_{k=-\infty}^{\infty} |c_{n_k}| \leq C \sum_{m=1}^{\infty} 2^{\frac{m}{2}(\frac{1 - \alpha p}{p'})} . \qquad (4.17)$$

Since $\alpha p > 1$, the series on the right of (4.17) is convergent. This completes the proof of Theorem 5.

<u>Proof of Theorem 6.</u>

The proof of this theorem resembles the proof of Theorem 5.

Let $\gamma = \dfrac{2(p - 1)}{2p + \alpha p - 3}$.

It is clear that $0 < \beta < 2$.

We observe, from (4.15), that

$$\sum_{2^m}^{2^{m+1}} |c_{n_k}|^2 = O\left(2^{m(-\frac{1}{q} - \frac{\alpha p}{p'})}\right).$$

Applying Hölder's inequality, we obtain

$$\sum_{2^m}^{2^{m+1}} |c_{n_k}|^\beta \leq \left(\sum_{2^m}^{2^{m+1}} |c_{n_k}|^2\right)^{\beta/2} \cdot \left(\sum_{2^m}^{2^{m+1}} 1\right)^{1 - \frac{\beta}{2}}$$

$$= O\left\{2^{m(-\frac{1}{q} - \frac{\alpha p}{p'})\frac{\beta}{2}} \cdot \left(2^m\right)^{1 - \beta/2}\right\}$$

$$= O\left\{2^{m(-\frac{\beta}{2q} - \frac{\alpha p \beta}{2p'} + 1 - \frac{\beta}{2})}\right\}$$

Since $p' = p - 1$ and $\frac{1}{p'} + \frac{1}{q} = 1$,

it follows that,

$$\sum_{2^m}^{2^{m+1}} |c_{n_k}|^\beta = O\left(2^{m(1 - \beta/\gamma)}\right)$$

Hence

$$\sum_{-\infty}^{\infty} |c_{n_k}|^\beta \leq C \sum_{m=1}^{\infty} 2^{m(1-\beta/\gamma)}.$$

Since $\beta > \gamma$, we have $1 - \frac{\beta}{\gamma} < 0$ and therefore, the series

$$\sum_{-\infty}^{\infty} |c_{n_k}|^{\beta} \text{ is convergent.}$$

Hence

$$\sum_{k=1}^{\infty} \left(|a_{n_k}|^{\beta} + |b_{n_k}|^{\beta} \right) < \infty .$$

This completes the proof of Theorem 6.

<u>Proof of Theorem 7.</u>

By (4.16), we obtain

$$\sum_{2^m}^{2^{m+1}} |c_{n_k}| = O\left\{ 2^{\frac{m}{2}\left(\frac{1 - \alpha p}{p'}\right)} \right\} .$$

Hence

$$\sum_{2^m}^{2^{m+1}} n_k^{\beta/2} |c_{n_k}| \leq (2^{m+1})^{\beta/2} \sum_{2^m}^{2^{m+1}} |c_{n_k}|$$

$$= O\left\{ 2^{(m+1)\beta/2} \cdot 2^{\frac{m}{2}\left(\frac{1 - \alpha p}{p'}\right)} \right\}$$

$$= O\left\{ 2^{\frac{m}{2}\left(\frac{1 - \alpha p}{p - 1} + \beta\left(1 + \frac{1}{m}\right)\right)} \right\}$$

$$= O\left\{2^{\frac{m}{2}\left(\frac{1-\alpha p}{p-1}+2\beta\right)}\right\}.$$

Thus

$$\sum_{k=1}^{\infty} n_k^{\beta/2}\,|c_{n_k}| \le \sum_{m=1}^{\infty} 2^{\frac{m}{2}\left(\frac{1-\alpha p}{p-1}+2\beta\right)}$$

Since $\beta < \frac{\alpha p - 1}{2(p-1)}$, it follows that

$\frac{1-\alpha p}{p-1} + 2\beta < 0$ and therefore

$$\sum_{k=1}^{\infty} n_k^{\beta/2}\,(|c_{n_k}| < \infty$$

i.e. $\sum n_k^{\beta/2}\,(|a_{n_k}| + |b_{n_k}|) < \infty$.

This proves Theorem 7.