CHAPTER - V

ABSOLUTE CONVERGENCE OF A LACUNARY FOURIER SERIES AND A SERIES ASSOCIATED WITH IT FOR FUNCTIONS IN Lip(α,p) AND BOUNDED r^{th} VARIATION CLASS

1. In this chapter, we proceed to study the absolute convergence of lacunary Fourier series (L) for more general classes of functions than those considered by us in Chapter IV. In fact, we consider here, the class of functions of bounded r^{th} variation over arbitrary subinterval I of $[-\pi , \pi]$.

In order to explain the significance of the results proved by us in this chapter, we recall here (from Chapter IV) Theorem 4.E due to Noble and its generalization - Theorem 4.H due to Kennedy. We observe that, in Noble's theorem the generating function f is of bounded variation in I (over and above that it is in Lipα(I)). Replacing the condition of bounded variation by a less stringent condition of bounded r^{th} variation (r being a positive integer), S. M. Mazhar [22] obtained the following generalization of Noble's result.

THEOREM 5.A [22]. If

$$\text{(i)} \quad \lim_{k\to\infty} \frac{N_k}{\log n_k} = \infty , \quad \text{where} \tag{5.1}$$

$$N_k = \min\left\{n_{k+1} - n_k , n_k - n_{k-1}\right\} ,$$

(ii) $f \in Lip\alpha(I)$, $0 < \alpha < 1$, (5.2)

and (iii) f is of bounded r^{th} variation in I,

then the Fourier series (L) of f converges absolutely.

Mazhar also obtained the extensions of his above theorem similar to Theorems 4.F and 4.G.

Now, Kennedy's theorem 4.H requires that the generating function f is of bounded variation in I as well as in $Lip\alpha(I)$. However the gap condition taken by Kennedy is weaker than that of Noble. Replacing the condition of $Lip\alpha(I)$ in Kennedy's theorem by a weaker condition of $Lip(\alpha,p,I)$,we have studied the absolute convergence of (L) in Chapter IV. In this chapter, we propose to replace further the condition of bounded variation by a weaker condition of bounded r^{th} variation. In fact, we prove the following theorem.

<u>THEOREM 8</u>. If (i) $(n_{k+1} - n_k) \to \infty$ $(k \to \infty)$, (5.3)

(ii) $f \in Lip(\alpha,p,I)$ with $0 < \alpha \leq 1$; $p > 2$; (5.4)

$\alpha p > 1$,

and (iii) f is of bounded r^{th} variation in I, (5.5)

then the Fourier series (L) of f converges absolutely.

It may be observed that, our theorem 8 is a simultaneous generalization of Theorem 4.H due to Kennedy and

Theorem 5.A due to Mazhar. Theorem 4.H is generalized by replacing conditions (4.1) and (4.2) with that of the corresponding weaker conditions (5.4) and (5.5). At the same time, Theorem 5.A is generalized by replacing conditions (5.1) and (5.2) with that of the corresponding weaker conditions (5.3) and (5.4).

Further, it is interesting to note that, without the lacunarity condition (5.3) and with $r = 1$, $p = \infty$, $I = [-\pi, \pi]$, our Theorem 8 reduces to the classical result which gives a generalization of the well known theorem of Zygmund [2; p.161] on absolute convergence. At the same time, without the lacunarity condition (5.3) and with $r = 1$, $I = [-\pi, \pi]$, Theorem 8 reduces to the classical theorem which gives a generalization of the theorem due to Hardy and Littlewood [13] for $p > 2$. We also have the following extensions of Theorem 8.

<u>THEOREM 9</u>. If (i) $\{n_k\}$ satisfies (5.3),

(ii) $f \in \mathrm{Lip}(\alpha, p, I)$ with $0 < \alpha \leq 1$; $p > 2$,

and (iii) f satisfies (5.5),

then

$$\sum_{k=1}^{\infty} (|a_{n_k}|^{\beta} + |b_{n_k}|^{\beta}) < \infty \quad \text{for}$$

every β satisfying $2 > \beta > \dfrac{2(p-1)}{2p + \alpha p - 3}$.

<u>THEOREM 10</u>. Under the hypothesis of Theorem 9,

we have

$$\sum_{k=1}^{\infty} n_k^{\beta/2} \left(|a_{n_k}| + |b_{n_k}|\right) < \infty$$

for every $\beta < \frac{\alpha p - 1}{2(p - 1)}$.

It is easy to see that, when $\beta = 1$ Theorem 9 reduces to Theorem 8 and when $\beta = 0$, Theorem 10 reduces to Theorem 8.

Further, it may be observed that, our study of the properties of lacunary Fourier series (L) depend mainly on two things - first, the localness and the type of the hypothesis to be satisfied by the underlying function and secondly - the kind of gaps in the Fourier series. Usually, when the hypothesis on the function is relaxed, the gap condition is strengthened to ensure the desired conclusion. However if we compare our theorems with Kennedy's theorem 4.H then we find that, it becomes possible for us to obtain Kennedy's conclusion by weakening both the conditions on a function even without strengthening the gap hypothesis $(n_{k+1} - n_k) \to \infty$. On the other hand, by comparing our theorems with Mazhar's theorems [22] , we find that Mazhar's conclusion can still hold even by weakening the gap hypothesis from (5.1) to (5.3) as well as the condition on f too.

2. In this chapter, we also discuss the convergence of (L), (L_1) and the absolute convergence of the series

$$\sum_{k=1}^{\infty} \left(\frac{S_{n_k} - S}{n_k} \right) , \tag{5.6}$$

where $S_{n_k} = \sum_{p=1}^{k} (a_{n_p} \cos n_p x + b_{n_p} \sin n_p x)$ and

S is an appropriate number independent of n_k .

The importance of the study of the series (5.6) as regards to its convergence and Cesàro summability was first recognized by Hardy and Littlewood [11] in the context of the general Fourier series. Later on it was studied by Zygmund [43; p.61] and several other researchers. Considering certain conditions on f, V. M. Shah [35; Chapter V] studied the convergence of (L), (L_1) and absolute convergence of (5.6). On account of our Theorems 1, 2 and 3 of Chapter II, it becomes possible to take more general conditions on the underlying function f than those taken by V. M. Shah. In fact, we prove the following theorems:

<u>THEOREM 11</u>. If (i) $\{n_k\}$ satisfies (5.3),

(ii) f satisfies (5.5),

and (iii) $f \in L^2(I)$, (5.7)

then the Fourier series (L) is convergent to $\frac{f(x+0) + f(x-0)}{2}$ at any point where this expression has a meaning; and the conjugate Fourier series (L_1) is convergent to $\bar{f}(x)$ whenever $\bar{f}(x)$ exists and when x is a point of the Lebesgue set.

<u>THEOREM 12.</u> If (i) $\{n_k\}$ satisfies (5.3),

(ii) f satisfies (5.5) and (5.7),

and (iii) $\sum_{k=1}^{\infty} \frac{\log n_k}{n_k}$ is convergent, (5.8)

then the series (5.6) is absolutely convergent.

<u>THEOREM 13.</u> If (i) $\{n_k\}$ satisfies (5.3),

(ii) $f \in Lip(\alpha,p,I)$, $0 < \alpha < 1$; $p \geq 2$, (5.9)

and (iii) $\sum_{k=1}^{\infty} \frac{k^{1-\alpha}}{n_k}$ is convergent, (5.10)

then the series (5.6) is absolutely convergent.

<u>THEOREM 14.</u> If (i) $\{n_k\}$ satisfies B_2 condition,

(ii) $f \in Lip(\alpha,p,E)$, $0 < \alpha < 1$; $p \geq 2$,

(E is a set of positive measure)

and (iii) the condition (5.10) holds,

then the series (5.6) is absolutely convergent.

Theorems analogous to Theorems 12, 13 and 14 can be stated for the conjugate Fourier series (L_1) also.

<u>REMARK</u>. V. M. Shah [35 ; p.92 , Theorem 25] has proved a Theorem similar to our theorems 13 and 14, by taking the conditions

(i) $(n_{k+1} - n_k) \to \infty$ as $k \to \infty$,

(ii) $f \in Lip\alpha(I)$, $0 < \alpha < 1$,

and (iii) $\sum \frac{1}{n_k^{\alpha}} < \infty$.

It may be observed that, in our theorem 13 condition (5.9) is certainly weaker than (ii). Since

$$\frac{k^{1-\alpha}}{n_k} < \frac{1}{n_k^{\alpha}} ,$$

our condition (5.10) is also weaker than the condition (iii).

3. In order to prove our theorems we need Lemmas 2.1, 2.2 and 2.7. These lemmas are given in Chapter II.

<u>Proof of Theorem 8</u> :

Put $n_o = 0$, $n_k = -n_{-k}$ $(k < o)$,

$C_{n_o} = 0$, $C_{n_k} = \frac{1}{2}(a_{n_k} - ib_{n_k})$ $(k > 0)$,

$C_{n_k} = \bar{C}_{n_{-k}}$ $(k < 0)$. Therefore we can write (L) in the form

$$\sigma(f) = \sum_{k=-\infty}^{\infty} C_{n_k} e^{in_k x} .$$

We have

$$\sum_{k=-\infty}^{\infty} |c_{n_k}| \, s^{|n_k|} < \infty \qquad (0 < s < 1) \tag{5.11}$$

Put

$$f(s,x) = \sum_{-\infty}^{\infty} c_{n_k} \, e^{in_k x} \, s^{|n_k|} , \qquad (0 < s < 1)$$

for all real x. The existence of $f(s,x)$ is ensured by (5.11).

We obviously have

$$f(s,x) = \sum_{k=1}^{\infty} (a_{n_k} \cos n_k x + b_{n_k} \sin n_k x) s^{n_k} .$$

Now it follows from the condition (5.4) that $f \in \mathrm{Lip}(\alpha,2,I)$ and therefore by Lemma 2.7, $f \in L^2(I)$. Since $(n_{k+1} - n_k) \to \infty$, by Lemma 2.1, we have $f \in L^2[-\pi , \pi]$. Hence by a known theorem [43 ; p.87], it follows that

$$f(x) = L^2\text{-}\lim_{s \to 1} f(s,x) \qquad (|x| \leq \pi) \tag{5.12}$$

We assume that the integers j and K satisfy

$$n_K > \frac{2r\pi}{\delta} \text{ and } 0 \leq j \leq \frac{\delta n_K}{8\pi} \tag{5.13}$$

Without loss of generality, we assume that r is an even integer.

Take

$$
\begin{aligned}
& g_j(x) \\
& = f\left(x + \frac{2j\pi}{n_K} + \frac{r\pi}{2n_K}\right) - \binom{r}{1} f\left(x + \frac{2j\pi}{n_K} + \frac{r\pi}{2n_K} - \frac{\pi}{n_K}\right) \\
& + \binom{r}{2} f\left(x + \frac{2j\pi}{n_K} + \frac{r\pi}{2n_K} - \frac{2\pi}{n_K}\right) \\
& + \dots + (-1)^{r/2} \binom{r}{r/2} f\left(x + \frac{2j\pi}{n_K}\right) + \dots\dots \\
& - \binom{r}{r-1} f\left(x + \frac{2j\pi}{n_K} - \frac{r\pi}{2n_K} + \frac{\pi}{n_K}\right) \\
& + \binom{r}{r} f\left(x + \frac{2j\pi}{n_K} - \frac{r\pi}{2n_K}\right)
\end{aligned}
\tag{5.14}
$$

and

$$A_k(j) = C_{n_k}\left(2i \sin n_k \frac{\pi}{2n_K}\right)^r \cdot e^{\frac{in_k 2j\pi}{n_K}}.$$

Since

$|A_k(j)| \leq |C_{n_k}| 2^r$, it follows from (5.11) that

$$\sum_{k=-\infty}^{\infty} |A_k(j)| s^{|n_k|} < \infty \qquad (0 < s < 1) \tag{5.15}$$

Put

$$g_j(s,x) = \sum_{k=-\infty}^{\infty} A_k(j) s^{|n_k|} e^{in_k x} \qquad (0 < s < 1) \tag{5.16}$$

We then get the identity

$$g_j(s,x) = f(s,x + \frac{2j\pi}{n_K} + \frac{r\pi}{2n_K}) - \binom{r}{1} f(s,x + \frac{2j\pi}{n_K} + \frac{r\pi}{2n_K} - \frac{\pi}{n_K})$$

$$+ \binom{r}{2} f(s,x + \frac{2j\pi}{n_K} + \frac{r\pi}{2n_K} - \frac{2\pi}{n_K})$$

$$+ \dots + (-1)^{r/2} \binom{r}{r/2} f(s,x + \frac{2j\pi}{n_K}) + \dots$$

$$\dots - \binom{r}{r-1} f(s,x + \frac{2j\pi}{n_K} - \frac{r\pi}{2n_K} + \frac{\pi}{n_K})$$

$$+ \binom{r}{r} f(s,x + \frac{2j\pi}{n_K} - \frac{r\pi}{2n_K})$$

and from it together with (5.12) and (5.14) we obtain

$$g_j(x) = L^2\text{-}\lim_{s\to 1} g_j(s,x) \quad \text{in} \quad |x - x_0| \leq \delta/2 \tag{5.17}$$

We assume that

$$(n_{k+1} - n_k) > 16\pi\delta^{-1} \quad \text{for all} \quad k \tag{5.18}$$

In view of (5.3), this can be achieved, if necessary, by adding to $f(x)$ a polynomial in $\exp(in_k x)$, a process which affects neither the hypothesis nor the conclusion of the theorem.

Thus from (5.15), (5.16), (5.17) and (5.18), we see that all the conditions of Lemma 2.2 are satisfied.

Hence

$$\sum_{k=-\infty}^{\infty} |A_k(j)|^2 \leq 16\delta^{-1} \int_{I_1 : |x-x_0| \leq \delta/2} |g_j(x)|^2 dx \tag{5.19}$$

But

$$|A_k(j)| = |C_{n_k}| \, |2i|^r \, \left|\sin \frac{n_k\pi}{2n_K}\right|^r \left|e^{\frac{in_k 2j\pi}{n_K}}\right|$$

$$= |C_{n_k}| 2^r \sin^r \frac{n_k\pi}{2n_K} \cdot 1$$

for every j satisfying (5.13).

Therefore (5.19) gives,

$$\sum_{\frac{1}{2}n_K}^{n_K} 2^{2r} \left(\sin \frac{n_k\pi}{2n_K} \right)^{2r} |C_{n_k}|^2 \leq \sum_{k=-\infty}^{\infty} |A_k(j)|^2$$

$$\leq 16\delta^{-1} \int_{I_1} |g_j(x)|^2 dx \tag{5.20}$$

Since

$\frac{1}{2}n_K \leq |n_k| \leq n_K$, it follows that

$$\left(\sin \frac{\pi n_k}{2n_K}\right)^{2r} \geq \left(\frac{1}{2}\right)^r .$$

Therefore (5.20) reduces to

$$\sum_{\frac{1}{2}n_K}^{n_K} |C_{n_k}|^2 \leq \frac{16\delta^{-1}}{2^r} \int_{I_1} |g_j(x)|^2 dx \tag{5.21}$$

Let $p' = p - 1$. As $p > 2$, $p' > 1$.

Let q be a real number such that

$$\frac{1}{p'} + \frac{1}{q} = 1 .$$

Now (5.21) gives

$$\left\{ \sum_{\frac{1}{2}n_K}^{n_K} |c_{n_k}|^2 \right\}^q \leq \left(\frac{16}{\delta 2^r} \right)^q \left\{ \int_{I_1} |g_j(x)|^2 dx \right\}^q .$$

Taking summation over j, we get

$$\sum_{0 \leq j \leq \left[\frac{\delta n_K}{8\pi}\right]} \left\{ \sum_{\frac{1}{2}n_K}^{n_K} |c_{n_k}|^2 \right\}^q \leq \sum_{0 \leq j \leq \left[\frac{\delta n_K}{8\pi}\right]} \left(16\delta^{-1} \, 2^{-r} \right)^q \left\{ \int_{I_1} |g_j(x)|^2 dx \right\}^q .$$

From this we get

$$\left\{ \sum_{\frac{1}{2}n_K}^{n_K} |c_{n_k}|^2 \right\}^q \leq \frac{(16\delta^{-1} \, 2^{-r})^q}{\left[\frac{\delta n_K}{8\pi}\right]} \sum_j \left\{ \int_{I_1} |g_j(x)|^2 dx \right\}^q$$

$$\leq \frac{C}{n_K} \sum_j \left\{ \int_{I_1} |g_j(x)|^2 dx \right\}^q , \qquad (5.22)$$

where C is some constant depending on δ and r. Note that C denotes a positive constant which is different at different occurances.

Since $2 = 1 + 1$

$$= (1 + \frac{1}{p'}) + \frac{1}{q}$$

$$= \frac{p'+1}{p'} + \frac{1}{q} ,$$

an application of Hölder's inequality gives,

$$\left\{ \int_{I_1} |g_j(x)|^2 dx \right\}^q = \left\{ \int_{I_1} |g_j(x)|^{\frac{p'+1}{p'} + \frac{1}{q}} dx \right\}^q$$

$$\leq \left[\left(\int_{I_1} |g_j(x)|^{\frac{p'+1}{p'} \cdot p'} dx \right)^{\frac{1}{p'}} \cdot \left(\int_{I_1} |g_j(x)|^{\frac{1}{q} \cdot q} \right)^{\frac{1}{q}} \right]^q$$

$$= \left[\left(\int_{I_1} |g_j(x)|^p dx \right)^{\frac{q}{p'}} \cdot \left(\int_{I_1} |g_j(x)| dx \right) \right].$$

This combined with (5.22) gives

$$\left\{ \sum_{\frac{1}{2}n_K}^{n_K} |c_{n_k}|^2 \right\}^q \leq \frac{C}{n_K} \sum_j \left\{ \left(\int_{I_1} |g_j(x)|^p dx \right)^{\frac{q}{p'}} \cdot \int_{I_1} |g_j(x)| dx \right\} \quad (5.23)$$

Now condition (5.4) and Minkowski's inequality give

$$\left[\int_{I_1} |g_j(x)|^p dx \right]^{\frac{1}{p}}$$

$$= \left[\int_{I_1} \Big| \left\{ f(y) - \binom{r}{1} f(y - \tfrac{\pi}{n_K}) + \binom{r}{2} f(y - \tfrac{2\pi}{n_K}) \right. \right.$$

$$+ \dots + (-1)^{r/2} \binom{r}{r/2} f(y - \tfrac{r\pi}{2n_K}) + \dots$$

$$- \binom{r}{r-1} f(y - \frac{r\pi}{n_K} + \frac{\pi}{n_K}) + \binom{r}{r} f(y - \frac{r\pi}{n_K}) \Big\} \Big|^p dx \Big]^{\frac{1}{p}},$$

where $y = x + \frac{2j\pi}{n_K} + \frac{r\pi}{2n_K}$

$$= \Big[\int_{I_1} \Big| \Big\{ \Big(f(y) - f(y - \frac{\pi}{n_K}) \Big) - \Big(\binom{r}{1} - 1 \Big) \Big(f(y - \frac{\pi}{n_K}) - f(y - \frac{2\pi}{n_K}) \Big)$$

$$+ \Big(\binom{r}{2} - \binom{r}{1} + 1 \Big) \Big(f(y - \frac{2\pi}{n_K}) - f(y - \frac{3\pi}{n_K}) \Big) - \dots$$

$$\dots - \Big(f(y - \frac{r\pi}{n_K} + \frac{\pi}{n_K}) - f(y - \frac{r\pi}{n_K}) \Big) \Big\} \Big|^p dx \Big]^{\frac{1}{p}}$$

$$= \Big[\int_{I_1} \Big| \Big\{ a_o \Big(f(y) - f(y - \frac{\pi}{n_K}) \Big) - a_1 \Big(f(y - \frac{\pi}{n_K}) - f(y - \frac{2\pi}{n_K}) \Big)$$

$$+ a_2 \Big(f(y - \frac{2\pi}{n_K}) - f(y - \frac{3\pi}{n_K}) \Big) - \dots$$

$$- a_{r-1} \Big(f(y - \frac{r\pi}{n_K} + \frac{\pi}{n_K}) - f(y - \frac{r\pi}{n_K}) \Big) \Big\} \Big|^p dx \Big]^{\frac{1}{p}},$$

where $a_o = 1$, $a_1 = \binom{r}{1} - 1$,

$a_2 = \binom{r}{2} - \binom{r}{1} + 1$, ..., $a_{r-1} = 1$ are

constants.

$$\leq \left[\int_{I_1} \left|a_o\left(f(y) - f(y - \frac{\pi}{n_K})\right)\right|^p dx\right]^{\frac{1}{p}}$$

$$+ \left[\int_{I_1} \left|a_1\left(f(y - \frac{\pi}{n_K}) - f(y - \frac{2\pi}{n_K})\right)\right|^p dx\right]^{\frac{1}{p}}$$

$$+ \ldots + \left[\int_{I_1} \left|a_{r-1}\left(f(y - \frac{r\pi}{n_K} + \frac{\pi}{n_K}) - f(y - \frac{r\pi}{n_K})\right)\right|^p dx\right]^{\frac{1}{p}} .$$

$$= O(\frac{\pi}{n_K})^{\alpha} + O(\frac{\pi}{n_K})^{\alpha} + \ldots + O(\frac{\pi}{n_K})^{\alpha}$$

(r times)

$$= O(n_K^{-\alpha})$$

Thus

$$\int_{I_1} |g_j(x)|^p dx = O(n_K^{-\alpha p}).$$

Also, for $|x-x_o| \leq \delta/2$ and integers j and K satisfying (5.13), the intervals

$$(y - \frac{i\pi}{n_K} \ , \ y - \frac{i\pi}{n_K} + \frac{\pi}{n_K}) \ , \ i = 1,2, \ldots, r$$

are non-overlapping subintervals of $\left[x_o - \delta \ , \ x_o + \delta\right]$. Therefore, if V is the total r^{th} variation of f in I, then it follows from (5.5) that

$$\sum_j |g_j(x)| \leq V \quad , \quad (|x - x_o| \leq \frac{1}{2}\delta).$$

Therefore (5.23) becomes

$$\left\{ \sum_{\frac{1}{2}n_K}^{n_K} |c_{n_k}|^2 \right\}^q \leq \frac{C}{n_K} \cdot n_K^{-\frac{\alpha pq}{p'}} \cdot V.\delta$$

$$= C \cdot \frac{1}{n_K^{(1+\frac{\alpha pq}{p'})}} ,$$

which in turn gives

$$\sum_{\frac{1}{2}n_K}^{n_K} |c_{n_k}|^2 \leq C \cdot \frac{1}{n_K^{(\frac{1}{q}+\frac{\alpha p}{p'})}} \tag{5.24}$$

Let m be a positive integer. Either the set of integers k for which

$2^m < |n_k| \leq 2^{m+1}$ is empty, or

there is a member of this set, say $K = K(m)$, which has largest modulus; and in the later case, the set is included in the set of k for which

$$\tfrac{1}{2}n_K \leq |n_k| \leq n_K .$$

Thus in either case,

$$\sum_{2^m<|n_k|\leq 2^{m+1}} |c_{n_k}|^2 \leq \sum_{\frac{1}{2}n_K\leq|n_k|\leq n_K} |c_{n_k}|^2 = O(n_K^{-\frac{1}{q}-\frac{\alpha p}{p'}}) \quad , \text{ by (5.24)}$$

$$= O\left(2^{m\left(-\frac{1}{q}-\frac{\alpha p}{p'}\right)}\right) \tag{5.25}$$

Also, by (5.18), the number of terms in the summation

$$\sum_{2^m}^{2^{m+1}} 1 \text{ is of } O(2^m) .$$

Hence by Cauchy's inequality we get,

$$\sum_{2^m}^{2^{m+1}} |c_{n_k}|.1 \leq \left\{ \sum_{2^m}^{2^{m+1}} |c_{n_k}|^2 \right\}^{\frac{1}{2}} \cdot \left\{ \sum_{2^m}^{2^{m+1}} 1 \right\}^{\frac{1}{2}}$$

$$= O\left\{ 2^{\left(-\frac{1}{q}-\frac{\alpha p}{p'}\right)\frac{m}{2}} \cdot 2^{\frac{m}{2}} \right\}$$

$$= O\left\{ 2^{\frac{m}{2}\left(1-\frac{1}{q}-\frac{\alpha p}{p'}\right)} \right\}$$

$$= O\left\{ 2^{\frac{m}{2}\left(\frac{1-\alpha p}{p'}\right)} \right\} \tag{5.26}$$

and hence

$$\sum_{k=-\infty}^{\infty} |c_{n_k}| \leq C \sum_{m=1}^{\infty} 2^{\frac{m}{2}\left(\frac{1-\alpha p}{p'}\right)} \tag{5.27}$$

Since $\alpha p > 1$, the series on the right of (5.27), is convergent.

Hence

$$\sum_{k=1}^{\infty} (|a_{n_k}| + |b_{n_k}|) < \infty .$$

This completes the proof of Theorem 8.

Proof of Theorem 9.

Using (5.25), this theorem can be proved as per Theorem 6 of Chapter IV.

Proof of Theorem 10.

Using (5.26), this theorem can be proved as per Theorem 7 of Chapter IV.

Proof of Theorem 11.

If S_n are the partial sums and σ_n are the arithmetic means of order n for the series

$$u_o + u_1 + u_2 + \dots + u_n + \dots\dots ,$$

then

$$S_n - \sigma_n = \frac{u_1 + 2u_2 + \dots + nu_n}{n+1}$$

In case of a lacunary series, where in calculating Fejér sums it is necessary to replace the absent terms by zeros, we have,

$$S_{n_k} - \sigma_{n_k} = \frac{n_1 u_{n_1} + n_2 u_{n_2} + \dots + n_k u_{n_k}}{n_k + 1} .$$

Now, we take

$$u_{n_k} = a_{n_k} \cos n_k x + b_{n_k} \sin n_k x$$ in case of the series (L)

and

$u_{n_k} = b_{n_k} \cos n_k x - a_{n_k} \sin n_k x$ in case of the series (L_1).

Under the hypothesis of the theorem, we have by Theorem 3 of Chapter II,

$$a_{n_k} = O(\frac{1}{n_k}) \ , \ b_{n_k} = O(\frac{1}{n_k})$$

Therefore $u_{n_k} = O(\frac{1}{n_k})$

and hence,

$$n_k \, u_{n_k} = O(1).$$

This gives,

$$|S_{n_k} - \sigma_{n_k}| \leq \frac{A\,k}{n_k + 1}$$

$$\leq \frac{A\,k}{n_k} \ ,$$ where A is an absolute constant.

But $\frac{k}{n_k} \to 0$ as $k \to \infty$, whenever $n_{k+1} - n_k \to \infty$.

Therefore $|S_{n_k} - \sigma_{n_k}| \to 0$.

Now, it is known that the Fourier series (L) is summable (c,1) to $\frac{f(x+0) + f(x-0)}{2}$ for every value of x for which this expression has a meaning. i.e.

$$\sigma_{n_k} \to \frac{f(x+0) + f(x-0)}{2} .$$

Hence $S_{n_k} \rightarrow \frac{f(x+0) + f(x-0)}{2}$ for every value of x for which this expression has a meaning.

It is also known that the series (L_1) is summable (c,1) to $\bar{f}(x)$ for every value of x for which $\bar{f}(x)$ exists and when x is a point of the Lebesgue set. Hence by the same argument as used above, (L_1) converges to $\bar{f}(x)$ whenever it exists, and when x is a point of the Lebesgue set.

Proof of Theorem 12:

Again by Theorem 3 of Chapter II

$$u_{n_k} = O\left(\frac{1}{n_k}\right).$$

Therefore,

$$\left| \frac{S_{n_k} - S}{n_k} \right| = \left| \frac{u_{n_1} + u_{n_2} + \ldots + u_{n_k} - S}{n_k} \right| \tag{5.28}$$

$$\leq \frac{A\left(\frac{1}{n_1} + \frac{1}{n_2} + \ldots + \frac{1}{n_k}\right) + |S|}{n_k},$$

where A is an absolute constant.

This gives $\left| \frac{S_{n_k} - S}{n_k} \right| = O\left(\frac{\log n_k}{n_k}\right)$.

Hence the absolute convergence of the series (5.6) follows from (5.8).

Proof of Theorems 13 and 14:

Under the hypothesis of the theorems, we have

$$u_{n_k} = O\left(\frac{1}{n_k^{\alpha}}\right), \text{ by Theorems}$$

1 and 2 of Chapter II.

Therefore, using (5.28) we get

$$\left|\frac{S_{n_k} - S}{n_k}\right| \leq \frac{A\left(\frac{1}{n_1^{\alpha}} + \frac{1}{n_2^{\alpha}} + \ldots + \frac{1}{n_k^{\alpha}}\right) + |S|}{n_k}$$

$$\leq \frac{A\left(\frac{1}{1^{\alpha}} + \frac{1}{2^{\alpha}} + \ldots + \frac{1}{k^{\alpha}}\right) + |S|}{n_k}$$

$$= O\left(\frac{k^{1-\alpha}}{n_k}\right).$$

Hence the absolute convergence of the series (5.6) follows from (5.10).