CHAPTER - VI

ABSOLUTE SUMMABILITY OF LACUNARY FOURIER SERIES AND ITS CONJUGATE SERIES

1. In this chapter, we discuss the absolute summability of lacunary Fourier series (L) and its conjugate series (L_1) by considering certain conditions on a function and on a gap under which the absolute convergence of (L) is not guaranteed. In order to appreciate the significance of the conditions considered by us in this regard, at the outset, it is desirable to recall some of the results on absolute convergence. Considering certain classes of functions either on an arbitrary subinterval or at a point of $[-\pi, \pi]$, the problem of absolute convergence of (L) is studied in great details by several mathematicians, under suitable lacunarity conditions. They have obtained very interesting results sharpening at each stage, the results obtained by the previous authors. The results due to M. Izumi and S. Izumi [15], Jia-Arng Chao [3], J. R. Patadia and V. M. Shah [28], and P. B. Kennedy [17] are mentioned in the present context.

M. Izumi and S. Izumi proved the following result on absolute convergence of (L).

THEOREM 6. A. [15; Theorem 2] . If

(i) $(n_{k+1} - n_k) > A\, n_k^{\beta}$, $0 < \beta < 1$, (6.1)

(A is a positive constant)

(ii) $f \in \mathrm{Lip}\alpha(P)$, $0 < \alpha < 1$, (6.2)

then the Fourier series (L) of f converges absolutely when $\alpha > \beta^{-1} - 1$.

Considering more general gap condition than Izumi, Chao obtained the following result in the year 1966.

THEOREM 6. B. [3] . If

(i) $(n_{k+1} - n_k) \geq A\, n_k^{\beta}\, k^{\gamma}$ $(0 < \beta < 1,\ \gamma \geq 0)$, . (6.3)

where A is a positive constant,

and (ii) f satisfies the condition (6.2),

then the Fourier series (L) of f converges absolutely when $\alpha\beta + \alpha\gamma + \beta > 1$.

Recently, in the year 1981, J. R. Patadia and V. M. Shah [28] obtained the following result on absolute convergence, which generalizes both the above results.

THEOREM 6.C. [28] If

(i) $\{n_k\}$ satisfies the condition (6.3)

and (ii) f satisfies the condition (6.2),

then

$$\sum_{k=1}^{\infty}(|a_{n_k}|^m + |b_{n_k}|^m) < \infty, \quad 0 < m \leq 1$$

when $\alpha\beta m + \alpha m\gamma > (1 - \frac{m}{2})(1 - \beta)$.

It can be observed that, the particular case of Theorem 6.C when $\gamma = 0$ and $m = 1$ provides us with a generalization of Theorem 6.A due to M. Izumi and S. Izumi. This generalization ensures the absolute convergence of (L) when $\alpha > \frac{1}{2}(\beta^{-1} - 1)$ under the conditions (6.1) and (6.2).

Now, it is quite natural to inquire into the behaviour of the lacunary Fourier series (L) of a function f in Lipα at a point, when $\alpha \leq \frac{1}{2}(\beta^{-1} - 1)$. In this regard, we have studied here the absolute summability of (L). In fact, we prove the following theorems.

<u>THEOREM 15</u>. If (i) $\{n_k\}$ satisfies (6.1) with some suitable constant A,

and (ii) f satisfies (6.2),

then the Fourier series (L) of f is absolutely summable $(c, \frac{1}{2})$

(i) for every $\alpha > 0$ if $\beta \geq \frac{3}{5}$; (6.4)

or (ii) for every $\alpha > \frac{3}{2\beta} - \frac{5}{2}$ if $\beta < \frac{3}{5}$. (6.5)

THEOREM 16. Under the hypothesis of theorem 15, the Fourier series (L) of f is absolutely summable (c,1) when $\alpha > \beta^{-1} - 2$.

REMARK 1. It can be observed that, the significance of the conclusion (6.5) in Theorem 15 could be visualized when $\beta > \frac{1}{2}$. Because, under this condition, we have

$$\frac{3}{2\beta} - \frac{5}{2} < \frac{1}{2}(\beta^{-1} - 1)$$

and consequently the absolute summability $(c, \frac{1}{2})$ of the series (L) is ensured for the range

$$\frac{3}{2\beta} - \frac{5}{2} < \alpha \leq \frac{1}{2}(\beta^{-1} - 1).$$

Similarly the significance of Theorem 16 could be visualized when $\beta > \frac{1}{3}$, as $\beta^{-1} - 2 < \frac{1}{2}(\beta^{-1} - 1)$ in this case.

It can be seen that the above Theorems 15 and 16 are obtained under the gap condition (6.1). Now considering more general gap condition (6.3) than (6.1) we continue further our study on absolute summability. In this regard, we refer here the particular case of Theorem 6.C when m = 1, which ensures the absolute convergence of (L) when $\alpha\beta+\alpha\gamma > \frac{1}{2}(1-\beta)$ (under the conditions (6.2) and (6.3)). It may be noted that, when $\alpha\beta + \alpha\gamma = \frac{1}{2}(1 - \beta)$, then also the absolute convergence of (L) is obtained by J. R. Patadia and V. M. Shah [29], but under a little stronger condition on f than

Lipα(P). Consequently as before, we would like to inquire into the behaviour of the series (L) of a function in Lipα(P), when

$$\alpha\beta + \alpha\gamma \leq \tfrac{1}{2}(1 - \beta) .$$

In fact, we establish here the absolute summability (c,θ) $(0 < \theta \leq 1)$ of (L) in the following theorem.

<u>THEOREM 17</u>. If (i) $\{n_k\}$ satisfies the gap condition

(6.3) with some suitable constant A,

and (ii) f satisfies the condition (6.2),
then the lacunary Fourier series (L) of f is absolutely summable (c,θ) for $0 < \theta \leq 1$

when

$$\alpha > \max\left\{\frac{1 - \beta - \theta - \gamma\theta}{\beta + \gamma} , \frac{2 - 3\beta - \gamma + \beta\theta - \theta}{\beta + \beta\gamma}\right\} .$$

<u>REMARK 2</u>. It is easy to see that Theorems 15 and 16 are the particular cases of the above Theorem 17 when $\theta = \frac{1}{2}$, $\gamma = 0$; and $\theta = 1$, $\gamma = 0$ respectively.

<u>REMARK 3.</u> It is interesting to observe that when $\gamma = 1$, Theorem 17 gives the absolute summability (c,1) of the Fourier series (L) for every $\alpha > 0$; and that, when $\gamma = \frac{3}{2}$, we get the absolute summability $(c, \frac{1}{2})$ of (L) for every $\alpha > 0$.

Further, in this chapter, we also study the absolute summability of (L) and its conjugate Fourier series (L_1), by considering the following theorem on absolute convergence due to P. B. Kennedy [17 ; Theorem V(iv)] .

THEOREM 6.D. [17] If (i) $(n_{k+1} - n_k) \to \infty$ as $k \to \infty$, (6.6)

(ii) $f \in \mathrm{Lip}\alpha(I)$, $0 < \alpha \leq 1$, (6.7)

and (iii) f is of bounded variation in I, (6.8)

then the Fourier series (L) of f converges absolutely.

If $f \in \mathrm{Lip}\alpha(I)$ with $\alpha > \frac{1}{2}$, then also the absolute convergence of (L) is established by Kennedy.

Now, it is well known that the condition (6.7) only, with $0 < \alpha \leq \frac{1}{2}$ (together with (6.6)) is not sufficient to ensure the absolute convergence of (L) (refer [44; p.243]).

At the same time, the condition (6.8) only (together with (6.6)) does not guarantee the absolute convergence of (L) (refer [44 ; p.241]). Hence it is quite natural to inquire into the behaviour of (L), whenever the condition (6.7) or (6.8) only is satisfied. In this regard, V. M. Shah [35 ; Chapter VI] has studied the absolute summability (c,1) of (L) and (L_1). Here we intend to study the same problem of absolute summability under weaker conditions on a function than those considered by V. M. Shah. In fact, we consider a function either in $\mathrm{Lip}(\alpha,p)$ or in bounded r^{th} variation class over arbitrary subinterval I of $[-\pi , \pi]$. We prove the following theorems.

<u>THEOREM 18</u>. If (i) $\{n_k\}$ satisfies the gap condition (6.6),

(ii) f is of bounded r^{th} variation in I, (6.9)

(iii) $f \in L^2(I)$, (6.10)

and (iv) $\sum_{k=1}^{\infty} \frac{k}{n_k^2}$ is convergent, (6.11)

then the Fourier series (L) and (L_1) are everywhere absolutely summable (c,1). } (6.12)

<u>THEOREM 19</u>. If (i) $\lim_{k \to \infty} \frac{n_{k+1} - n_k}{\log n_k} = B,\ B > 0,$ (6.13)

and (ii) f satisfies the conditions (6.9) and (6.10),

then the conclusion (6.12) holds.

<u>THEOREM 20</u>. If (i) $\{n_k\}$ satisfies the condition (6.6),

(ii) $f \in \mathrm{Lip}(\alpha, p, I)$ with $0 < \alpha \leq \frac{1}{2}$; $p \geq 2$,

and (iii) $\sum_{k=1}^{\infty} \left\{ \frac{n_1 + n_2 + \cdots + n_k}{n_k^2} \right\}$ is convergent, (6.14)

then the conclusion (6.12) holds.

<u>THEOREM 21</u>. If (i) $\{n_k\}$ satisfies the condition B_2 ,

(ii) $f \in \mathrm{Lip}(\alpha, p, E)$ with $0 < \alpha \leq \frac{1}{2}$; $p \geq 2$,

and (iii) the condition (6.14) holds,

then the conclusion (6.12) holds.

2. In order to prove our theorems we need the following lemmas. Lemma 6.1 is due to J. R. Patadia and V. M. Shah [29] and Lemma 6.2 is due to Chao [3 ; Theorem 1].

<u>LEMMA 6.1</u> If $\{n_k\}$ satisfies the gap condition (6.3) with $A > 2^M - 1$, M being a positive integer greater than δ, where $\delta = \frac{1+\gamma}{1-\beta}$, then

$$n_k \geq k^{\delta} \text{ for all } k \in N.$$

<u>LEMMA 6.2</u> Under the hypothesis of Theorem 6.B,

$$a_{n_k},\ b_{n_k} = O\left(\frac{1}{n_k^{\alpha\beta} \cdot k^{\gamma\alpha}}\right), \quad k = 1,2,3,\ldots.$$

<u>Proof of Theorem 15.</u> For a real number s, other than a negative integer, put $E_n^s = \binom{n+s}{n}$ where $n \in N$ and $E_0^s = 1$. Denoting the n^{th} Cesàro mean of order $\theta > 0$ by $\sigma_n^{\theta}(x)$ and replacing the absent terms in (L) by zeros, we have [7]

$$|\sigma_{n_k}^{\theta}(x) - \sigma_{n_k-1}^{\theta}(x)|$$

$$= \frac{1}{n_k E_{n_k}^{\theta}} \left| \sum_{p=1}^{k} E_{n_k - n_p}^{\theta-1} n_p \cdot (a_{n_p} \cos n_p x + b_{n_p} \sin n_p x) \right|$$

$$\leq \frac{1}{n_k E_{n_k}^{\theta}} \Big\{ |n_k(a_{n_k} \cos n_k x + b_{n_k} \sin n_k x)|$$

$$+ \left| \sum_{p=1}^{k-1} E_{n_k - n_p}^{\theta-1} n_p \cdot (a_{n_p} \cos n_p x + b_{n_p} \sin n_p x) \right| \Big\} \qquad (6.15)$$

Let $\theta = \frac{1}{2}$

Now

(i) $E_n^{\theta} \simeq \frac{n^{\theta}}{\Gamma(\theta+1)}$,

(ii) a_{n_k}, $b_{n_k} = O\left(\frac{1}{n_k^{\alpha\beta}}\right)$, $k = 1,2,3,\ldots,$

by taking $\gamma = 0$ in Lemma 6.2,

and (iii) $|n_k - n_p|$

$\geq |n_k - n_{k-1}|$ for $p = 1,2,3, \ldots, k-1$

$> A\, n_k^{\beta}$, by (6.1).

Hence, from (6.15), we obtain

$$|\sigma_{n_k}^{\theta}(x) - \sigma_{n_k - 1}^{\theta}(x)|$$

$$= O(1) \frac{1}{n_k\, n_k^{\theta}} \left\{ n_k \left(n_k^{-\alpha\beta} + n_k^{-\alpha\beta}\right) + \sum_{p=1}^{k-1} \frac{1}{|n_k - n_p|^{1-\theta}}\, n_p \left(n_p^{-\alpha\beta} + n_p^{-\alpha\beta}\right) \right\}$$

$$= O(1) \frac{1}{n_k^{1+\theta}} \left\{ n_k^{1-\alpha\beta} + \frac{1}{n_k^{\beta(1-\theta)}}\; k\; n_k^{1-\alpha\beta} \right\}$$

$$= O(1) \left\{ \frac{1}{n_k^{\theta+\alpha\beta}} + \frac{k}{n_k^{\beta-\beta\theta+\alpha\beta+\theta}} \right\}$$

$$= O(1)\left\{\frac{1}{k^{\theta\delta + \alpha\beta\delta}} + \frac{1}{k^{\delta\beta - \delta\beta\theta + \alpha\beta\delta + \theta\delta - 1}}\right\},$$

by Lemma 6.1

$$= O(1)\left\{\frac{1}{k^{\frac{2\alpha\beta + 1}{2(1-\beta)}}} + \frac{1}{k^{\frac{3\beta+2\alpha\beta-1}{2(1-\beta)}}}\right\},$$

as $\delta = \frac{1}{1-\beta}$ and $\theta = \frac{1}{2}$.

But $\alpha > \frac{1}{2\beta} - 1$ implies $\frac{2\alpha\beta + 1}{2(1-\beta)} > 1$ and

$\alpha > \frac{3}{2\beta} - \frac{5}{2}$ implies $\frac{2\alpha\beta + 3\beta - 1}{2(1-\beta)} > 1.$

Hence, in order to establish the convergence of

$$\sum_{k=1}^{\infty} \left| \sigma^{\theta}_{n_k}(x) - \sigma^{\theta}_{n_k - 1}(x) \right|,$$ it is sufficient

to have

$$\alpha > \max\left\{\frac{1}{2\beta} - 1, \frac{3}{2\beta} - \frac{5}{2}\right\},$$

which is ensured in the following.

Case (i) Let $\beta \geq \frac{3}{5}$.

If $\frac{3}{5} \leq \beta < \frac{2}{3}$ then

$$\frac{1}{2\beta} - 1 < \frac{3}{2\beta} - \frac{5}{2} \leq 0 < \alpha .$$

If $\frac{2}{3} \leq \beta$ then

$$\frac{3}{2\beta} - \frac{5}{2} \leq \frac{1}{2\beta} - 1 < 0 < \alpha .$$

Case (ii). Let $\beta < \frac{3}{5}$ and $\alpha > \frac{3}{2\beta} - \frac{5}{2}$.

Then certainly $\beta < \frac{2}{3}$, which shows that

$$\frac{1}{2\beta} - 1 < \frac{3}{2\beta} - \frac{5}{2} < \alpha .$$

This proves that (L) is absolutely summable $(c, \frac{1}{2})$.

<u>Proof of Theorem 16.</u>

Let $\theta = 1$. Then by using Lemma 6.1 and Lemma 6.2 for $\gamma = 0$, we obtain from (6.15),

$$| \sigma_{n_k}(x) - \sigma_{n_k - 1}(x) |$$

$$= \frac{1}{n_k(n_k + 1)} \left| \sum_{p=1}^{k} n_p (a_{n_p} \cos n_p x + b_{n_p} \sin n_p x) \right|$$

$$= O(1) \frac{1}{n_k^2} \sum_{p=1}^{k} n_p^{(1-\alpha\beta)}$$

$$= O(1) \frac{1}{n_k^2} \cdot \frac{k}{n_k^{\alpha\beta - 1}}$$

$$= O(1) \left\{ \frac{k}{k^{\delta(\alpha\beta + 1)}} \right\}$$

$$= O(1) \left\{ \frac{1}{k^{\alpha\beta\delta+\delta-1}} \right\}$$

$$= O(1) \left\{ \frac{1}{k^{\frac{\alpha\beta + \beta}{1 - \beta}}} \right\}, \text{ as } \delta = \frac{1}{1 - \beta} .$$

Since $\alpha > \beta^{-1} - 2$, it follows that $\frac{\alpha\beta + \beta}{1 - \beta} > 1$.

Hence

$$\sum_{k=1}^{\infty} \left| \sigma_{n_k}(x) - \sigma_{n_k - 1}(x) \right| < \infty,$$

which implies the absolute summability (c,1) of (L).

This completes the proof of Theorem 16.

<u>Proof of Theorem 17.</u>

Under the conditions of the theorem,

$$a_{n_k}, \; b_{n_k} = O\left(\frac{1}{n_k^{\alpha\beta} \cdot k^{\gamma\alpha}}\right), \; k = 1,2,3,\ldots,$$

(by Lemma 6.2).

Also $E_n^{\theta} \simeq \frac{n^{\theta}}{\Gamma(\theta+1)}$, $(0 < \theta \leq 1)$,

and $|n_k - n_p| \geq |n_k - n_{k-1}|$ for $p = 1,2,3,\ldots,k-1$

$$\geq A\, n_k^{\beta} \cdot k^{\gamma}, \text{ by } (6.3).$$

Therefore, the relation (6.15), given in the proof of Theorem 15, becomes

$$\left| \sigma_{n_k}^{\theta}(x) - \sigma_{n_k - 1}^{\theta}(x) \right|$$

$$= O(1) \frac{1}{n_k\, n_k^{\theta}} \left\{ n_k\, n_k^{-\alpha\beta}\, k^{-\gamma\alpha} + \sum_{p=1}^{k-1} \frac{1}{(n_k - n_p)^{1-\theta}} \cdot n_p \cdot n_p^{-\alpha\beta} \cdot p^{-\gamma\alpha} \right\}$$

$$= O(1)\ \frac{1}{n_k^{1+\theta}}\left\{ n_k^{1-\alpha\beta}\ k^{-\gamma\alpha} + \left(\frac{1}{n_k^{\beta}\ k^{\gamma}}\right)^{1-\theta} \sum_{p=1}^{k-1} \frac{n_p^{1-\alpha\beta}}{p^{\gamma\alpha}} \right\}$$

$$= O(1)\ \frac{1}{n_k^{1+\theta}}\left\{ n_k^{1-\alpha\beta}\ k^{-\gamma\alpha} + \left(\frac{1}{n_k^{\beta}\ k^{\gamma}}\right)^{1-\theta} \cdot\ k.n_k^{1-\alpha\beta} \right\},$$

as $\frac{1}{p^{\gamma\alpha}} \leq 1$ and $n_p^{1-\alpha\beta} \leq n_k^{1-\alpha\beta}$, $0 < \alpha, \beta < 1$.

Therefore

$$\left| \sigma_{n_k}^{\theta}(x) - \sigma_{n_k -1}^{\theta}(x) \right|$$

$$= O(1)\left\{ \frac{1}{n_k^{\theta+\alpha\beta}\ k^{\gamma\alpha}} + \frac{1}{n_k^{\theta+\beta-\beta\theta+\alpha\beta}\ k^{\gamma-\gamma\theta-1}} \right\}$$

$$= O(1)\left\{ \frac{1}{k^{\delta(\theta+\alpha\beta)+\gamma\alpha}} + \frac{1}{k^{\delta(\theta+\beta-\beta\theta+\alpha\beta)+\gamma-\gamma\theta-1}} \right\}, \text{ by Lemma 6.1}$$

$$= O(1)\left\{ \frac{1}{k^{\frac{\theta+\alpha\beta+\gamma\theta+\alpha\gamma}{1-\beta}}} + \frac{1}{k^{\frac{\theta+2\beta-\beta\theta+\alpha\beta+\alpha\beta\gamma+\gamma-1}{1-\beta}}} \right\}, \qquad (6.16)$$

as $\delta = \frac{1+\gamma}{1-\beta}$.

Finally, since

$$\alpha > \frac{1-\beta-\theta-\gamma\theta}{\beta+\gamma}, \text{ it follows that}$$

$$\frac{\theta+\alpha\beta+\gamma\theta+\alpha\gamma}{1-\beta} > 1 ;$$

and since $\alpha > \frac{2 - 3\beta - \gamma + \beta\theta - \theta}{\beta + \beta\gamma}$, we have

$$\frac{\theta + 2\beta - \beta\theta + \alpha\beta + \alpha\beta\gamma + \gamma - 1}{1 - \beta} > 1 .$$

Hence, from (6.16) we obtain

$$\sum_{k=1}^{\infty} \left| \sigma_{n_k}^{\theta}(x) - \sigma_{n_k -1}^{\theta}(x) \right| < \infty ,$$

which implies the absolute summability (c,θ) of (L). This completes the proof of Theorem 17.

<u>Proof of Theorem 18.</u>

Denoting the n^{th} Cesàro mean of order 1 by $\sigma_n(x)$ and replacing the absent terms in (L) and (L_1) by zèros, we have

$$\left| \sigma_{n_k}(x) - \sigma_{n_k -1}(x) \right|$$

$$= \frac{1}{n_k(n_k + 1)} \left| \sum_{p=1}^{k} n_p \, u_{n_p} \right| , \text{ where}$$

$u_{n_p} = (a_{n_p} \cos n_p x + b_{n_p} \sin n_p x)$ in case of the series (L)

$= (b_{n_p} \cos n_p x - a_{n_p} \sin n_p x)$ in case of the series (L_1).

On account of conditions (6.6), (6.9) and (6.10) we have, by Theorem 3 of Chapter II,

$$a_{n_k} , b_{n_k} = O(\frac{1}{n_k}) \qquad (k \to \infty)$$

This gives

$$n_p \, u_{n_p} = O(1).$$

Therefore,

$$\left| \sigma_{n_k}(x) - \sigma_{n_k - 1}(x) \right|$$

$$= O(1) \frac{1}{n_k^2} \sum_{p=1}^{k} 1$$

$$= O(1) \frac{k}{n_k^2} \qquad (6.17)$$

Hence the convergence of $\sum_{k=1}^{\infty} \left| \sigma_{n_k}(x) - \sigma_{n_k - 1}(x) \right|$ follows from the convergence of $\sum_{k=1}^{\infty} \frac{k}{n_k^2}$.

This completes the proof of theorem 18.

<u>Proof of Theorem 19</u>.

The lacunarity condition (6.13) implies that $(n_{k+1} - n_k) \to \infty$ and therefore, as discussed in the proof of Theorem 18 , we have from (6.17)

$$\left| \sigma_{n_k}(x) - \sigma_{n_k - 1}(x) \right| = O\left(\frac{k}{n_k^2}\right) .$$

Also the gap condition (6.13) gives

$$n_k - n_{k-1} > c_1 \log n_k \geq c_1 \log k \quad , \quad c_1 > 0 \; ,$$

$$(k = 1,2,3,\ldots.) \; ;$$

and hence $n_k > n_1 + c_1 \sum_{p=2}^{k} \log p$

$$> c_1 \int_2^k \log t \, dt$$

$$> c\, k \log k\,, \quad (c > 0) \qquad (6.18)$$

$$(k = 1,2,3,\ldots\ldots)$$

Therefore using (6.17) and (6.18) we obtain,

$$|\sigma_{n_k} - \sigma_{n_k - 1}| = O\left(\frac{1}{k \log^2 k}\right)$$

Since $\sum_{k=2}^{\infty} \frac{1}{k \log^2 k}$ is convergent, our Theorem 19 follows.

Proof of Theorems 20 and 21.

Under the conditions of our theorems,

$$a_{n_k}\,,\ b_{n_k} = O\left(\frac{1}{n_k^{\alpha}}\right) \qquad (k \to \infty)\,,$$

by Theorems 1 and 2 of Chapter II.

This gives $n_p\, u_{n_p} = O(n_p^{1-\alpha})$.

Therefore,

$$|\sigma_{n_k}(x) - \sigma_{n_k - 1}(x)|$$

$$= O(1) \frac{1}{n_k^2} \sum_{p=1}^{k} n_p^{(1-\alpha)}$$

$$= O(1)\ \frac{1}{n_k^2} \sum_{p=1}^{k} \frac{n_p}{n_1^{\alpha}}$$

$$= O(1) \sum_{p=1}^{k} \frac{n_p}{n_k^2}$$

Hence the convergence of $\sum_{k=1}^{\infty} \left| \sigma_{n_k}(x) - \sigma_{n_k - 1}(x) \right|$

follows from the condition (6.14) of our theorems.

This completes the proof.