CHAPTER - VI

SINE AND COSINE SERIES WITH QUASIMONOTONE COEFFICIENTS

6.1 INTRODUCTION :

In the preceding chapters, we have studied different types of summability methods. Most of the work carried out in the previous chapters was to obtain some relationship between different summabilities. Our present interest in this chapter is to study the series of Sine and Cosine for some special class of functions. It is known that, any function f on an interval of length 2π may be expressed uniquely as the sum of an odd and an even functions. Odd functions have Sine series and even functions have Cosine series. If the coefficients of these series decreases to zero, then the series

$$\frac{a_0}{2} + \sum a_k Coskx \tag{6.1.1}$$

$$\sum b_k Sinkx \tag{6.1.2}$$

will converge.

The monotone coefficients may be generalized by taking quasimonotone coefficients (see[21]). The uniform convergence of Sine series with quasimonotone coefficients is given by J.R.Nurcombe [46] as under:

THEOREM 19 [46]:

If (b_n) is positive and quasimonotone, then a necessary and sufficient condition either for the uniform convergence of $\sum b_n Sinnx$, or for the continuity of its sum function $f(x)$, is that $nb_n \to 0$.

Now, before giving the next result, we need the following definition.

Definition 13 [21]:

Let $f \in C[a,b]$ and $x, y \in [a,b]$. We say that $f \in Lip\alpha$, $0 < \alpha < 1$, if

$$|f(x) - f(y)| \leq k\,|x - y|^{\alpha}.$$

Lorentz [35] has proved the following Theorem for Sine and Cosine series with decreasing coefficients of some function belonging to the class Lipα. In fact, his result is as follows:

THEOREM 20:

Let

$$f(x) = \sum a_n Cosnx \,, \tag{6.1.3}$$

where $a_n \downarrow 0$. If $f(x) \in Lip\alpha$, $0 < \alpha < 1$, then

$$a_n = O\left(\frac{1}{n^{1+\alpha}}\right).$$

This is also valid for

$$g(x) = \sum a_n Sinnx. \tag{6.1.4}$$

The converse part of above Theorem is also proved by Lorentz.

6.2 MAIN RESULT :

In this chapter, we will prove a result on Sine and Cosine series with quasimonotone coefficients. In fact, we shall replace the condition of decreasing coefficients in Lorentz result by a weaker condition of quasimonotone coefficients. The condition of quasimonotone coefficients is weaker than that of positive decreasing sequence is seen in Chapter-I (see Remark under definition 11).

We first prove the following Lemma.

Lemma :

If (a_n) is quasimonotone sequence then the series $\sum a_n Cosnx$ is convergent.

Proof:

Let

$$\sum_{k=1}^{n} a_k Coskx = \sum_{k=1}^{n} \frac{a_k}{k^\beta} k^\beta Coskx$$

$$= \sum_{k=1}^{n} k^{\beta} \left(\frac{a_k}{k^{\beta}} Coskx \right).$$

By using Able's transformation, we have

$$\sum_{k=1}^{n} a_k Coskx = \sum_{k=1}^{n-1} \Delta\left(k^{\beta}\right)\left(\sum_{j=1}^{k} \frac{a_j}{j^{\beta}} Cosjx\right) + n^{\beta}\left(\sum_{j=1}^{n} \frac{a_j}{j^{\beta}} Cosjx\right)$$

$$\leq \text{M} \left[\sum_{k=1}^{n-1} \Delta\left(k^{\beta}\right) + n^{\beta}\right]$$

$$= \text{M} \left[\sum_{k=1}^{n-1} \left(k^{\beta} - (k+1)^{\beta}\right) + n^{\beta}\right]$$

$$= \text{M}.$$

Therefore $\sum_{k=1}^{\infty} a_k Coskx$ is convergent to some function, say $f(x)$.

Therefore, we can write

$$f(x) = \sum a_n Cosnx .$$

Now, we are in a position to state our result as under.

THEOREM M:

Let (a_n) be a quasi-monotone sequence of real numbers.

i.e. $\frac{a_n}{n^{\beta}} \downarrow 0$, for some $\beta \geq 0$.

Suppose

$$f(x) = \sum a_n Cosnx .$$

If $f(x) \in Lip\alpha$, $0<\alpha<1$ and $0 \leq \beta < \frac{\alpha}{2}$, then

$$a_n = O\left(\frac{1}{n^{\alpha+1-2\beta}}\right).$$

This is also valid for

$$g(x) = \sum a_n Sinnx.$$

Remarks:

1. Here we observe that $na_n = \frac{n}{n^{1+\alpha-2\beta}}$

$$= \frac{1}{n^{\alpha-2\beta}} \rightarrow 0 \text{ as } \beta < \frac{\alpha}{2} \text{ i.e. } \alpha - 2\beta > 0.$$

2. If we put $\beta = 0$ in our Theorem M , then we get Theorem 20 due to Lorentz [35].

3. It is also interesting to note that the sufficient part of the above result can also be established under a slightly strong condition $\alpha - 2\beta > 1$.

If for a series (6.1.1), we take $a_n = O\left(\frac{1}{n^{\alpha+1-2\beta}}\right)$ with $\alpha - 2\beta > 1$, then (6.1.1) becomes a Fourier series as $\sum \frac{a_n}{n} < \infty$ (see [5], Vol.-II, page 201) and hence by using corollary (see [5], Vol.-II, page 217), we can say that $f \in Lip\alpha$, $0<\alpha<1$.

6.3 PROOF OF THEOREM M :

Since $f(x) \in Lip\alpha$, $0 < \alpha < 1$, it follows that

$$|f(x) - f(0)| < |x - 0|^{\alpha}. \tag{6.3.1}$$

Now,

$$\begin{aligned}
|f(x) - f(0)| &= \left|\sum_{k=1}^{\infty} a_k Coskx - \sum_{k=1}^{\infty} a_k Cosk(0)\right| \\
&= \left|\sum_{k=1}^{\infty} a_k Coskx - \sum_{k=1}^{\infty} a_k\right| \\
&= \left|\sum_{k=1}^{\infty} a_k (Coskx - 1)\right| \\
&= \left|\sum_{k=1}^{\infty} a_k (1 - Coskx)\right| \\
&= \left|\sum_{k=1}^{\infty} a_k \left(2Sin^2 \frac{kx}{2}\right)\right| \\
&= 2\sum_{k=1}^{\infty} a_k \left(Sin^2 \frac{kx}{2}\right).
\end{aligned}$$

Supposing $x = \dfrac{\pi}{n}$ and using (6.3.1) , we obtain

$$2\sum_{k=\left[\frac{n}{2}\right]}^{n} a_k Sin^2 \frac{kx}{2n} \le \frac{C\pi^{\alpha}}{n^{\alpha}}. \tag{6.3.2}$$

Now

$$\left[\frac{n}{2}\right] \le k \le n \quad \Rightarrow \quad \left[\frac{n}{2}\right] x \le kx \le nx$$

$$\Rightarrow \left[\frac{n}{2}\right]\frac{\pi}{2n} \leq \frac{k\pi}{2n} \leq \frac{\pi}{2}$$

$$\Rightarrow \left[\frac{n}{2}\right]\frac{\pi}{2n} \leq \frac{\pi}{4} \leq \frac{k\pi}{2n} \leq \frac{\pi}{2} \text{ ,}$$

and Sine function is increasing in $\left[0, \frac{\pi}{2}\right]$. Therefore, we get

$$Sin\frac{\pi}{4} \leq Sin\frac{k\pi}{2n}$$

i.e. $\frac{1}{\sqrt{2}} \leq Sin\frac{k\pi}{2n}$. (6.3.3)

Hence from (6.3.2), we have

$$\begin{aligned} \frac{C\pi^{\alpha}}{n^{\alpha}} &\geq 2\sum_{k=\left[\frac{n}{2}\right]}^{n} a_k Sin^2\frac{kx}{2n} \\ &\geq 2\sum_{k=\left[\frac{n}{2}\right]}^{n} a_k\left(\frac{1}{\sqrt{2}}\right)^2 \\ &\geq \sum_{k=\left[\frac{n}{2}\right]}^{n} a_k . \end{aligned} \quad (6.3.4)$$

Now, let us write

$$\sum_{k=\left[\frac{n}{2}\right]}^{n} a_k = \sum_{k=\left[\frac{n}{2}\right]}^{n} \frac{a_k}{k^{\beta}} k^{\beta} \text{ , for some } \beta \geq 0.$$

We put $\left[\frac{n}{2}\right] = m$ and $\frac{a_k}{k^{\beta}} = H_k$. Then

$$\sum_{k=\left[\frac{n}{2}\right]}^{n} \frac{a_k}{k^\beta} k^\beta = \sum_{k=m}^{n} H_k\, k^\beta$$

$$\geq H_n \sum_{k=m}^{n} k^\beta$$

$$\geq H_n m^\beta \sum_{k=m}^{n} 1$$

$$\geq H_n m^\beta (n-m+1)$$

$$\geq \frac{a_n}{n^\beta}\left[\frac{n}{2}\right]^\beta \left(n-\left[\frac{n}{2}\right]+1\right)$$

Since

$$\left(n+1-\left[\frac{n}{2}\right]\right) \geq \frac{n}{4} \,\forall\, n \quad \text{and}$$

$$\left[\frac{n}{2}\right]^\beta \geq \frac{1}{n^\beta} \,\forall\, n > 1.$$

Therefore, by (6.3.4) we have

$$\frac{C\pi^\alpha}{n^\alpha} \geq \frac{a_n}{n^\beta}\left(\frac{n}{4}\right)\left(\frac{1}{n^\beta}\right).$$

$$\therefore \quad a_n \leq \frac{4C\pi^\alpha n^{2\beta}}{n^{\alpha+1}}$$

$$\therefore \quad a_n = O\left(\frac{1}{n^{\alpha+1-2\beta}}\right).$$

Here $na_n \to o$. Therefore it follows from Theorem 19 [46] that the Sine series with quasimonotone coefficients will converge uniformly. Hence term by term integration of the series is justified. This gives

$$\int_0^x g(t)dt = \int_0^x \left(\sum_{k=1}^{\infty} a_k S\operatorname{int}\right) dt$$

$$= \int_0^x a_1 Sinxdx + \int_0^x a_2 Sin2xdx + \text{-------} + \int_0^x a_n Sinnxdx + \text{-----}$$

$$= a_1(1 - Cosx) + \frac{a_2}{2}(1 - Cos2x) + \text{------} + \frac{a_n}{n}(1 - Cosnx) + \text{-----}$$

$$= \sum_{n=1}^{\infty} \frac{a_n}{n}(1 - Cosnx)$$

$$= 2\sum_{n=1}^{\infty} \frac{a_n}{n} Sin^2 \frac{nx}{2}.$$

Since $g(x) \in Lip\alpha$, it follows that

$$\max_{0 \le t \le x} |g(t)| = \max_{0 \le t \le x} |g(t) - g(0)| \le C\, x^{\alpha+1}.$$

Therefore,

$$C\, x^{\alpha+1} \ge 2\sum_{n=1}^{\infty} \frac{a_n}{n} Sin^2 \frac{nx}{2}$$

$$\ge 2\sum_{k=\left[\frac{n}{2}\right]}^{n} \frac{a_k}{k} Sin^2 \frac{kx}{2}$$

$$\ge 2\sum_{k=\left[\frac{n}{2}\right]}^{n} \frac{a_k}{k},$$ by ((6.3.3) and (6.3.4))

$$\ge \left(\frac{a_n}{n^{\beta}}\right)\left(\frac{n}{4}\right)\left(\frac{1}{n}\right)^{\beta},$$ by (6.3.5).

Therefore, we have

$$\therefore \quad a_n \le \frac{2C\pi^{\alpha} n^{2\beta}}{n^{\alpha+1}}$$

$$\therefore$$

$$\therefore \quad a_n = O\left(\frac{1}{n^{\alpha+1-2\beta}}\right).$$

This completes the proof of theorem M.