CHAPTER-I

INTRODUCTION

1.1 A BACKGROUND:

The present thesis entitled " **A STUDY OF DIFFERENT SUMMABILITY METHODS OF AN INFINITE SERIES** " is an outcome of the investigations of the author into the study of various summability methods and its applications. We propose to give in this chapter a survey of the summability and resume of relevant results obtained by various authors on different summability methods of an infinite series which will provide sufficient background for later chapters.

Let (a_n) be a sequence of real numbers. The expression of the form

$$a_0 + a_1 + a_2 + ------ + a_n + ----- \qquad (1.1.1)$$

is called an infinite series and the sequence (s_n) defined by

$$s_n = a_0 + a_1 + a_2 + -------- + a_n$$

is called the n^{th} partial sum of the series. The series (1.1.1) is said to be convergent if $\lim_{n\to\infty} s_n$ exists and the value of $\lim_{n\to\infty} s_n$ is called the sum of the series. Thus if a given series is convergent then it is possible to find its sum. If a series is not convergent then it is said to be divergent. A question arises: Is it possible to assign a sum to a non-convergent series? . It is well known that the notion of summability has provided a very effective affirmative answer to this natural question.

The theory of summability has a long illustrious history of more than 100 years with contributions from celebrated Mathematicians including Fejer [25], Lebesgue [34], Hardy [27] and Borel [6]. Results from summability theory have been found to be useful in various other areas of Mathematics and related sciences. We make a note of few of them:

The problem of analytic continuation has been studied by various summability methods and its history goes back to 1910. Important contributions in this area includes work of Borel for Borel summability [6], results due to Okada [47] covering Nörlund summability, Agnew's work on Hausdorff summability [3] and Knopp's paper on Euler summability [30]. Infact, Knopp [30] and Peyrimhoff [48] have successfully addressed the problem of exact summability regions for meromorphic function related to Hausdorff summability.

Mathematicians have worked to find summability tests for singular points of analytic function. In 1965, King [29] obtained necessary and sufficient conditions under which z = 1 becomes singular point of the function given by $f(z)=\sum_{n=0}^{\infty} a_n z^n$. There are more contributions by Titchmarsh [53], Hille [28] and others. Hartmann [26] has further extended King's and other results in 1972.

We know the Prime number theorem which states " $\pi(x)$ is asymptotic to $\frac{x}{\log x}$ ". This theorem was conjectured by C.F.Gauss in 1792 and was independently proved by Hadamard and Poussin in 1896. Summability theory has helped to provide easier proofs of this theorem. In fact, Lambert summability methods and Wiener's Tauberian theorem are being used to prove this theorem and a good reference for this is Peyerimhoff [48].

Nörlund method of summability finds applications in the convergence problems of power series in complex analysis, e.g. it is known that [31] a regular as well as non-regular Nörlund methods (N, p_n) sum a given power series $f(z)=\sum_{n=0}^{\infty} a_n z^n$ with $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{1}{R}$, $R>0$ at most at countably many points outside the disc of convergence, and these points can only accumulate on $|Z| = R$.

Now we get back to our discussion on summability. The notion of convergence was instrumented to the development of various summability methods. Likewise later on, the concept of the absolute summability was developed from the notion of absolute convergence. There are various methods due to Abel, Borel, Cesàro. Euler, Nörlund, Riemann, Riesz and many more. Some of the most familiar methods of summability with which we shall be concerned here are Cesàro, Nörlund and Riesz summability methods.

The Nörlund summability was first introduced by Voronoi in 1902, but not much work was carried out until 1919 when Nörlund introduced the same notion independently. Since then this summability has been initially used and is known as Nörlund summability. In 1949, Hardy [27] observed the usefulness of the concept of Nörlund summability in correlating the same with Cesàro summability as well as Riesz summability. The summability studied by Hardy to obtain these correlation are denoted by $(\bar{N}, p_n)$ and $|\bar{N}, p_n|$. In 1983, Hüseyin Bor [8] generalized $|\bar{N}, p_n|$ summability and introduced the concept of absolute $(\bar{N}, p_n)$ summability of order k, $k \geq 1$, which is denoted by $|\bar{N}, p_n|_k$. He has also remarked that $|\bar{N}, p_n|_k$ summability is more general than those of absolute Cesàro summability $|C,1|_k$ of order one with index k and absolute Riesz summability $|R, p_n|$ of order one. Further Hüseyin Bor [14] also extended $|\bar{N}, p_n|_k$ summability to $|\bar{N}, p_n, \gamma|_k$ summability by introducing $\gamma \geq 0$. On the other hand, in the year 1992, W.T.Sulaiman [50] introduced yet another new summability by introducing a sequence (ϕ_n). This summability is denoted by $|\bar{N}, p_n, \phi_n|_k$ and is more general than $|\bar{N}, p_n|_k$ summability.

Hüseyin Bor ([10],[11]) was the first who begun the study of $|\bar{N}, p_n|_k$ summability and obtained a relation between $|C,1|_k$ and $|\bar{N}, p_n|_k$ summability. Several authors like W.T.Sulaiman [50], M.A.Sarigöl [36], S.M.Mazhar [39], Ö.Cakar and C.Orhan [22] and many others have studied the $|\bar{N}, p_n|_k$ summability in the recent years and have improved upon the earlier results or obtained new results by considering weaker conditions and also considering more general summabilities.

We have studied $|C,1|_k$, $|C,1,\gamma|_k$, $|R,p_n|_k$, $|\overline{N},p_n|_k$, $|\overline{N},p_n,\phi_n|_k$ and $X-|\overline{N},p_n|_k$ summabilities and obtained several results, which in turn generalizes the earlier results obtained by various researchers. This thesis is the outcome of the researches carried out by the author mainly in this direction.

In order to state some of the main results proved by us in the present thesis, we first give the definitions and notations used in the present thesis.

1.2 DEFINITIONS AND NOTATIONS:

(i) ABSOLUTE CESÀRO SUMMABILITY:

Definition 1 :

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with (s_n) as the sequence of its partial sums. Let (σ_n) and (t_n) denote the n-th $(C,1)$ means of the sequences $(s_n)$and (na_n) respectively. Then the series $\sum_{n=0}^{\infty} a_n$ is said to be ***summable*** $|C,1|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |\sigma_n - \sigma_{n-1}|^k < \infty, \tag{1.2.1}$$

where

$$\sigma_n = \frac{1}{n+1}\sum_{\nu=0}^{n} s_\nu .$$

In view of the fact that $t_n = n(\sigma_n - \sigma_{n-1})$ by [33], (1.2.1) can be written as

$$\sum_{n=1}^{\infty} \frac{|t_n|^k}{n} < \infty . \tag{1.2.2}$$

The above definition of $|C,1|_k$ summability was first given by T.M.Flett [23] in 1957. Later on, in 1958 T.M.Flett [24] extended this definition to $|C,1,\gamma|_k$ summability by introducing $\gamma \geq 0$ as under :

Definition 2 :

The series $\sum_{n=0}^{\infty} a_n$ is said to be ***summable*** $|C,1,\gamma|_k$, $k \geq 1$ and $\gamma \geq 0$, if

$$\sum_{n=1}^{\infty} n^{\gamma k+k-1} |\sigma_n - \sigma_{n-1}|^k < \infty . \tag{1.2.3}$$

or

$$\sum_{n=1}^{\infty} \frac{|t_n|^{k\gamma+k}}{n} < \infty . \tag{1.2.4}$$

It is clear that when $\gamma = 0$, the summability $|C,1,\gamma|_k$ reduces to $|C,1|_k$ summability.

(ii) RIESZ SUMMABILITY :

In the year 1993, Mehmet Ali Sarigol [37] defined absolute Riesz summability as follows:

Definition 3:

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with (s_n) as the sequence of its partial sums, and let (p_n) be a sequence of positive real numbers such that

$$P_n = p_0 + p_1 + ---+ p_n \to \infty \text{ as } n \to \infty .$$

If

$$\sum_{n=1}^{\infty} n^{k-1} |t_n - t_{n-1}|^k < \infty , \tag{1.2.5}$$

where t_n denotes Riesz mean of $\sum_{n=0}^{\infty} a_n$, i.e.

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.2.6}$$

then the series $\sum_{n=0}^{\infty} a_n$ is said to be ***summable*** $|R, p_n|_k$, $k \geq 1$.

Note that for $k = 1$, the summability $|R, p_n|_k$ becomes absolute Riesz summability $|R, p_n|$ of order one, which is given by M.Riesz [49].

(iii) NÖRLUND SUMMABILITY :

The general definition of Nörlund mean occurs first in Voronoi, Proc. of the eleventh Congress of Russian naturalists and scientists (in Russian), St. Petersburg in 1902,. Voronoi's article was short note in a rare publication, and was unnoticed until Tamarkin called attention to it. Nörlund [45] gave the following definition independently in 1919.

Definition 4 :

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with sequence of partial sums (s_n). Let (p_n) be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + ---- + p_n, (P_{-1} = p_{-1} = 0). \qquad (1.2.7)$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v \ (P_n \neq 0) \qquad (1.2.8)$$

defines the sequence (t_n) of Nörlund means of the sequence (s_n) generated by the sequence of coefficients (p_n). The series $\sum_{n=0}^{\infty} a_n$ is said to be ***summable*** (N, p_n) to the sum s, if

$$\lim_{n \to \infty} t_n = s. \qquad (1.2.9)$$

In the above definition, G.H.Hardy [27] replaced (1.2.8) by

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

and defined a new method of summability, which is denoted by $(\overline{N}, p_n)$ as follows:

Definition 5 :

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with sequence of partial sums (s_n). Let (p_n) be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + --- + p_n \ , \quad (P_{-1} = p_{-1} = 0).$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v \ \ (P_n \neq 0) \tag{1.2.10}$$

defines the sequence (t_n) of Nörlund means of the sequence (s_n) generated by the sequence of coefficients (p_n). The series $\sum_{n=0}^{\infty} a_n$ is said to be ***summable*** $(\overline{N}, p_n)$ to the sum s, if

$$\lim_{n\to\infty} t_n = s \tag{1.2.11}$$

and if the sequence (t_n) is of bounded variation , that is

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty \ ,$$

then the series $\sum_{n=0}^{\infty} a_n$ is said to be absolutely ***summable*** $(\overline{N}, p_n)$, or simply summable $|\overline{N}, p_n|$.

The conditions for regularity of the method of summability $(\overline{N}, p_n)$ defined by (1.2.10) are

$$\lim_{n\to\infty} |P_n| \to \infty \tag{1.2.12}$$

and

$$\sum_{v=0}^{n} |p_v| = O(|P_n|). \tag{1.2.13}$$

In 1983, Hüseyin Bor [8] introduced the following concept of absolute $(\overline{N}, p_n)$ summability.

Definition 6 :

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with (s_n) as the sequence of its partial sums. Let (p_n) be a sequence of positive real constants such that

$$P_n = p_0 + p_1 + - - - + p_n \to \infty \text{ as } n \to \infty .$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \ , (P_n \neq 0) \qquad (1.2.14)$$

defines the sequence (t_n) of the $(\overline{N}, p_n)$ means of the sequence (s_n), generated by the sequence of coefficients (p_n) . Then the series $\sum_{n=0}^{\infty} a_n$ is said to be ***summable*** $\left|\overline{N}, p_n\right|_k$, $k \geq 1$ if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} \left| t_n - t_{n-1} \right|^k < \infty. \qquad (1.2.15)$$

In the special case, when $p_n = 1$ for all values of n, the summability $\left|\overline{N}, p_n\right|_k$ reduces to $|C,1|_k$ summability and for $k = 1$, the summability $\left|\overline{N}, p_n\right|_k$ gives rise to $|R, p_n|$.

Further Hüseyin Bor [14] extended the $\left|\overline{N}, p_n\right|_k$ summability to $\left|\overline{N}, p_n, \gamma\right|_k$ summability by introducing $\gamma \geq 0$ in the year 1988 as follows :

Definition 7 :

The series $\sum_{n=0}^{\infty} a_n$ is said to be ***summable*** $\left|\overline{N}, p_n, \gamma\right|_k$, $\gamma \geq 0$ and $k \geq 1$ if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\gamma k+k-1} \left|t_n - t_{n-1}\right|^k < \infty, \qquad (1.2.16)$$

where t_n is given by (1.2.10).

It can be seen that:

(i) If $\gamma = 0$ and $p_n = 1$ for all values of n , then the summability $\left|\overline{N}, p_n, \gamma\right|_k$ reduces to $|C,1|_k$ summability.

(ii) For $\gamma = 0$, the summability $\left|\overline{N}, p_n, \gamma\right|_k$ becomes $\left|\overline{N}, p_n\right|_k$.

Later on in the year 1992, W.T.Sulaiman [50] extended the $\left|\overline{N}, p_n\right|_k$ summability to $\left|\overline{N}, p_n, \phi_n\right|_k$ summability by introducing a sequence (ϕ_n) as follows :

Definition 8 :

The series $\sum_{n=0}^{\infty} a_n$ is said to be ***summable*** $\left|\overline{N}, p_n, \phi_n\right|_k$, $k \geq 1$ if

$$\sum_{n=1}^{\infty} \phi_n^{\,k-1} |t_n - t_{n-1}|^k < \infty, \qquad (1.2.17)$$

where (ϕ_n) is a sequence of positive real constants and t_n is given by (1.2.10).

In particular we observe that :

(i) For $\phi_n = \frac{P_n}{p_n}$, the summability $\left|\overline{N}, p_n, \phi_n\right|_k$ reduces to $\left|\overline{N}, p_n\right|_k$ summability.

(ii) If $\phi_n = n$ for all n , then the summability $\left|\overline{N}, p_n, \phi_n\right|_k$ reduces to $\left|R, p_n\right|_k$ summability.

The above definition of $\left|\overline{N}, p_n, \phi_n\right|_k$ was reformed by S.M.Mazhar [39] as under :

Definition 9 :

The series $\sum_{n=0}^{\infty} a_n$ is said to be ***summable*** $X - \left|\overline{N}, p_n\right|_k$, $k \geq 1$ if

$$\sum_{n=1}^{\infty} X_n^{k-1} |t_n - t_{n-1}|^k < \infty, \qquad (1.2.18)$$

where (X_n) is a sequence of positive real constants and t_n is given by (1.2.10).

It is interesting to observe that:

(i) For $X_n = \frac{P_n}{p_n}$, the summability $X - \left|\overline{N}, p_n\right|_k$ reduces to $\left|\overline{N}, p_n\right|_k$ summability.

(ii) If $X_n = n$ for all n, then the summability $X - \left|\overline{N}, p_n\right|_k$ reduces to $|R, p_n|_k$ summability.

1.3 MAIN RESULTS:

In order to explain the significance of the results established by us, it is desirable to recall briefly the development that has taken place regarding the study of summability methods used in our thesis.

The study of $\left|\overline{N}, p_n\right|_k$ summability methods of infinite series was originally introduced by Hüseyin Bor ([10],[11]) and proved two well known results in this direction . These results are as under:

THEOREM IA :

Let (p_n) be a sequence of positive real constants such that as $n \to \infty$

$$np_n = O(P_n), \qquad (1.3.1)$$

$$P_n = O(np_n). \qquad (1.3.2)$$

If $\sum_{n=0}^{\infty} a_n$ is summable $|C,1|_k$, then it is also summable $\left|\overline{N}, p_n\right|_k$, $k \geq 1$.

THEOREM IB:

Let (p_n) be a sequence of positive real constants such that it satisfies the conditions (1.3.1) and (1.3.2). If $\sum_{n=0}^{\infty} a_n$ is summable $\left|\bar{N}, p_n\right|_k$, then it is also summable $|C,1|_k$, $k \geq 1$.

Further, Hüseyin Bor [12] generalized Theorem IA for more general summability as follows :

THEOREM IC:

Let (p_n) be a sequence of positive real constants such that it satisfies the conditions (1.3.1), (1.3.2) and

$$\sum_{n=v}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}}=O\left[\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{1}{P_v}\right]. \qquad (1.3.3)$$

If $\sum_{n=0}^{\infty} a_n$ is summable $|C,1;\delta|_k$, then it is also summable $\left|\bar{N}, p_n;\delta\right|_k$, $\delta \geq 0$ and $k \geq 1$.

On the other hand, the above Theorem IA of Huseyin Bor was also generalized by Ö.Cakar and C.Orhan [22] replacing condition (1.3.2) by a weaker condition. Their result is given below:

THEOREM ID:

Let (p_n) be a sequence of positive real constants such that as $n \to \infty$

$$np_n = O(P_n)$$

$$\sum_{v=1}^{n}\left(\frac{P_v}{p_v}\right)=O(P_{n-1}). \tag{1.3.4}$$

If $\sum_{n=0}^{\infty} a_n$ is summable $|C,1|_k$, then it is also summable $|\overline{N},p_n|_k$, $k\geq 1$.

It is worth noting that, by dropping condition (1.3.2) from Theorem 1A and condition (1.3.1) from Theorem 1B, M.A.Sarigol [51] proved the same results in 1989.

Further, it was proved by G.Sunouchi and L.S.Bosenquent ([52],[4]) that the necessary and sufficient condition for a series (1.1.1) to be summable $|\overline{N},q_n|$, whenever it is summable $|\overline{N},p_n|$ is

$$q_n P_n = O(Q_n p_n) \quad \text{as} \quad n\to\infty.$$

On the other hand H.Bor and B.Thorpe [17] have shown that the summabilities given by G.Sunouchi and L.S.Bosenquent can be replaced by more general summabilities $|\overline{N},q_n|_k$ and $|\overline{N},p_n|_k$, $k\geq 1$.

In chapter II of the thesis, we have studied $|C,1,\gamma|_k$ and $|\overline{N},p_n,\gamma|_k$ summability of an infinite series and obtained relation between them, which generalizes Theorem IA to Theorem IC due to Huseyin Bor and Theorem ID of Ö.Caker and C.Orhan. Also in this chapter, we have discussed the $X-|\overline{N},p_n|_k$ summability and proved some results which generalizes the results of G.Sunouchi and L.S.Bosenquent and also of H.Bor and B.Thorpe. Moreover in this chapter, we have also obtained a relation between absolute Riesz summabilities with respect to sequences (p_n) and (q_n).

To state the result obtained by us in next chapter, we need following definition of almost increasing sequence given by S.Aljancic and D.Arandelovic [1].

Definition 10 :

A positive sequence (b_n) is said to ***almost increasing*** if there exists a positive sequence (c_n) and two positive constants A and B such that

$$Ac_n \leq b_n \leq Bc_n \text{ for all } n.$$

Obviously, every increasing sequence is almost increasing but the converse need not be true as can be seen from the example

$$b_n = ne^{(-1)^n} \qquad (\text{see [1]}).$$

S.M.Mazhar [40] proved a result on absolute Cesàro summability of an infinite series. This result was further extended by Hüseyin Bor [13] for $\left|\overline{N}, p_n\right|_k$ summability as under:

THEOREM IE:

Let (p_n) be a sequence of positive numbers such that

$$P_n = O(np_n) \text{ as } n \to \infty .$$

If (X_n) is a positive monotonic non-decreasing sequence such that

$$\lambda_m X_m = O(1) \text{ as } m \to \infty$$

$$\sum_{n=1}^{m} nX_n \left|\Delta^2 \lambda_n\right| = O(1) \ ,$$

and

$$\sum_{n=1}^{m}\left(\frac{p_n}{P_n}\right)|t_n|^k = O(X_m) \text{ as } m \to \infty \ ,$$

then the series $\sum_{n=0}^{\infty} a_n\lambda_n$ is summable $\left|\overline{N}, p_n\right|_k$, $k \geq 1$.

THEOREM IF:

Let (p_n) be a sequence of positive numbers such that

$$P_n = O(np_n) \text{ as } n \to \infty \ .$$

Let (X_n) be a positive non-decreasing sequence and suppose that there exist sequences (λ_n) and (β_n) such that

$$|\Delta\lambda_n| \leq \beta_n$$

$$\beta_n \to 0 \text{ as } n \to \infty,$$

$$\sum_{n=1}^{m} nX_n|\Delta\beta_n| < \infty \ ,$$

$$\lambda_m X_m = O(1) \text{ as } m \to \infty \ ,$$

and

$$\sum_{n=1}^{m}\left(\frac{p_n}{P_n}\right)|t_n|^k = O(X_m) \text{ as } m \to \infty \ ,$$

where

$$t_n = \frac{1}{n+1}\sum_{v=1}^{n} va_v$$

then the series $\sum_{n=0}^{\infty} a_n\lambda_n$ is summable $\left|\overline{N}, p_n\right|_k$, $k \geq 1$.

Now looking at the hypothesis of Theorem IE and Theorem IF, we would like to examine whether the condition on sequence (X_n) can be replaced by a weaker condition and the summability by some what more general summability. We investigated this aspect in chapter III and established two theorems obtaining more general summability under weaker condition on sequence (X_n).

Further in chapter IV, we have extended the summability $\left|\overline{N}, p_n; \gamma\right|_k$ to $\left|\overline{N}_p, \gamma, \alpha\right|_k$ by introducing $\alpha \geq 0$ and proved a theorem on it. Incidentally our theorem generalizes the results of F.M.Khan [32] and Hüseyin Bor [15].

Chapter V is devoted to the study of $Y - \left|\overline{N}, p_n\right|_k$ summability of a Fourier series. We have seen that $Y - \left|\overline{N}, p_n\right|_k$ summability of a Fourier series at a point can be ensured by a local property. The result proved by us in this chapter generalizes the result due to Hüseyin Bor [16].

Finally to state the result obtained by us in chapter-VI, we need the following definition due to J.R.Nurcome [46].

Definition 11:

A sequence (a_n) of positive numbers is said to be ***quasi-monotone*** if for some $\beta \geq 0$, the sequence $\left(\frac{a_n}{n^\beta}\right)$ is non-increasing.

Remark:

We observe that, the condition of quasimonotone sequence is weaker than that of positive decreasing sequence. i.e.

Every positive decreasing sequence is quasimonotone but the converse is not true.

Proof :

We will first show that, if (a_n) is positive decreasing then it is also quasimonotone.

Suppose (a_n) is decreasing sequence of positive numbers. Than

$$a_n \geq a_{n+1} \ \forall n.$$

$$\therefore a_n - a_{n+1} \geq 0. \qquad (1.3.5)$$

Now,

$$\frac{a_n}{n^\beta} - \frac{a_{n+1}}{(n+1)^\beta} > \frac{a_n}{(n+1)^\beta} - \frac{a_{n+1}}{(n+1)^\beta}$$

$$= \frac{1}{(n+1)^\beta}[a_n - a_{n+1}]$$

$\geq$ 0, by (1.3.5).

Therefore the sequence $\left(\frac{a_n}{n^\beta}\right)$ is nonincreasing for some $\beta \geq 0$.Hence (a_n) is quasimonotone. That the converse is not true can be seen by the following example.

Take $a_n = \log n$. Then $\frac{a_n}{n^\beta}$ = $\frac{\log n}{n^\beta}$ is non-increasing for some $\beta > 0$ and therefore (a_n) is quasimonotone but $(\log n)$ is increasing sequence.

In the year 1948, Lorentz [35] proved a simple but elegant result as follows:

Theorem IG:

If

$$f(x) = \sum a_n Cosnx,$$

where $a_n \downarrow 0$, then for $f(x) \in Lip\alpha$, $0 < \alpha < 1$, it is necessary and sufficient that

$$a_n = O\left(\frac{1}{n^{1+\alpha}}\right).$$

This is also valid for

$$g(x) = \sum a_n Sinnx.$$

In the last chapter of the thesis i.e. chapter-VI, we have extended the part of the Theorem IG due to Lorentz under a weaker condition.