Chapter 1

Introduction and preliminaries

1.1 Introduction and motivation

Many basic notions and results in the theory of functions have been obtained using Fourier or more generally trigonometric series. In 1981, while studying the convergence of Fourier series, C. Jordan [30] introduced the class $BV([0, 2\pi])$ of functions of bounded variation over $[0, 2\pi]$. The class $BV([0, 2\pi])$ distinguishes itself from other standard classes of functions in analysis because the Fourier analysis of the function of this class can be carried out in the most elegant way. If $f \in BV([0, 2\pi])$ then its m^{th} Fourier coefficient is of the order $O\left(\frac{1}{|m|}\right)$ [8, Vol.I, p.72], its Fourier series converges everywhere pointwise [8, Vol.I, p.114] and it converges absolutely if the modulus of continuity of the function tends to zero sufficiently rapidly [8, Vol.II, p.160]. Another important aspect of this class is that it is a Banach algebra with respect to the pointwise operations and the variation norm $\|f\| = \|f\|_\infty + V(f, [0, 2\pi])$, where $V(f, [0, 2\pi])$ is the total variation of the function $f \in BV([0, 2\pi])$.

A mathematician desires for more elegance and/or more generality in treating a particular problem leads further to interesting generalizations of the concept of bounded variation in many ways. Consequently, different classes of functions of generalized bounded variations are introduced.

In 1924, Wiener [73] introduced the class $BV^{(p)}([a, b])$ $(p \geq 1)$ of functions of $p-$bounded variation over $[a, b]$. The concept of $p-$bounded variation was subsequently generalized by L. C. Young [74] in 1937 to the class $\phi BV([a,b])$ of functions of $\phi-$bounded variation over $[a,b]$. Another class that was directly influenced by the study of the convergence problems in the theory of Fourier series namely $\Lambda BV([a,b])$ of functions of $\Lambda-$bounded variation over $[a,b]$ appeared in 1972 in Waterman's paper [72]. Subsequently in 1980, Shiba [54] introduced the class $\Lambda BV^{(p)}([a,b])$ of functions of $p-\Lambda-$bounded variation over $[a,b]$. The class $\phi\Lambda BV([a,b])$ of functions of $\phi-\Lambda-$bounded variation over $[a,b]$ was introduced by M. Schramm and D. Waterman [52] in 1982. In 1990, H. Kita and K. Yoneda [33] defined the class $BV(p(n)\uparrow p,[a,b])$ $(1\leq p\leq\infty)$ of functions of $p(n)-$bounded variation over $[a,b]$. It was generalized to the class $BV(p(n)\uparrow p,\varphi,[a,b])$ by T. Akhobadze [2] in 2000. Finally in 2011, the class $\Lambda BV(p(n)\uparrow p,\varphi,[a,b])$ of functions of $p(n)-\Lambda-$bounded variation over $[a,b]$ appeared in [58]. By considering the differences of order $r\geq 2$ the class $r-BV([a,b])$ of functions of bounded $r^{th}-$variation over $[a,b]$ is one of the important generalizations of the Jordan's class.

While investigating the convergence of Fourier series in the $L^1([0,2\pi])-$norm, in 1996 F. Móricz and A. H. Siddiqi [38] introduced the class $BVM([0,2\pi])$ of functions of bounded variation in the mean. The concept of bounded variation in the mean was subsequently generalized by R. E. Castillo [11] in 2005 and it was to the class $BV^{(p)}M([0,2\pi])$ of functions of $p-$bounded variation in the mean.

Similarly, the convergence problems in the theory of multiple Fourier series has motivated further to generalize the Jordan's class into higher dimensional space. The notion of bounded variation is generalized from a function of one variable to a function of several variables in different way. Several definitions are given under which function of two or more independent variables shall be said to be of bounded variation. Some of these definitions are associated with the names of Hardy, Vitali, Arzelà, Pierpont, Fréchet and Tonelli. For the first time, in 1906, G. H. Hardy [28] introduced the class $BV_H([a,b]\times[c,d])$ of two variables functions of bounded variation over $[a,b]\times[c,d]$. Then in 1908 G. Vitali [56] introduced the class $BV_V(\prod_{i=1}^{N}[a_i,b_i])$ of $N-$variables functions of bounded variation over $\prod_{i=1}^{N}[a_i,b_i]$. Considering the natural analogue of that of bounded variation for a function of one variable, Arzelà [4] introduced the class $BV_A(\prod_{i=1}^{N}[a_i,b_i])$. The inter-relations between these classes are studied by C. R. Adams and J. A. Clarkson [1]. Among all these classes, classes $BV_V([0,2\pi]^2)$ and $BV_H([0,2\pi]^2)$ are most convenient to consider in the study of double Fourier series. The Fourier analysis of these two classes are studied in detail [21, 37, 40]. These classes are also generalized in different ways. In 1958, J. Musielak [41] introduced the class $BV_V^{(p)}(\prod_{i=1}^N [a_i, b_i])$ of $N-$variables functions of $p-$bounded variation (in the sense of Vitali) over $\prod_{i=1}^N [a_i, b_i]$. In 1986, A. A. Saakyan [49] introduced the class $(\Lambda^1, \Lambda^2)^* BV([a, b] \times [c, d])$ of two variables functions of $(\Lambda^1, \Lambda^2)^*-$bounded variation (in the sense of Hardy) over $[a, b] \times [c, d]$ which was extended by A. I. Sablin [50] in 1987. It was to the class $(\Lambda^1, \cdots, \Lambda^N)^* BV(\prod_{i=1}^N [a_i, b_i])$ of $N-$variables functions of $(\Lambda^1, \cdots, \Lambda^N)^*-$bounded variation over $\prod_{i=1}^N [a_i, b_i]$. Finally, the class $(\Lambda^1, \cdots, \Lambda^N)^* BV^{(p)}(\prod_{i=1}^N [a_i, b_i])$ of $N-$variables functions of $p-(\Lambda^1, \cdots, \Lambda^N)^*-$bounded variation over $\prod_{i=1}^N [a_i, b_i]$ is generalized to the class $\phi(\Lambda^1, \cdots, \Lambda^N)^* BV(\prod_{i=1}^N [a_i, b_i])$ of $N-$variables functions of $\phi-(\Lambda^1, \cdots, \Lambda^N)^*-$bounded variation over $\prod_{i=1}^N [a_i, b_i]$. Also, the class $r - BV([a, b])$ is generalized to the class $r - BV(\prod_{i=1}^N [a_i, b_i])$ of $N-$variables functions of bounded $r^{th}-$variation over $\prod_{i=1}^N [a_i, b_i]$.

Similarly, the class $BV^{(p)}M([0, 2\pi])$ is generalized to the class $BV^{(p)}M([0, 2\pi]^2)$ of two variables functions of $p-$bounded variation in the mean.

The study of properties of one variable as well as several variables functions of generalized bounded variations is one of the interesting problems of recent researches. In fact, Waterman, Móricz, Dyachenko, Bakhvalov, Ashton, Doust and some other mathematicians have already studied properties of functions of these classes. In the present, we propose to carry the study further and prove many interesting properties of one variable as well as several variables functions of generalized bounded variations which generalize earlier results of Ashton, Doust, Castillo, Móricz, Fülöp, Kantrowitz, Veres, Schramm, Waterman [5, 12, 21, 31, 39, 52, 53] and of some others in Functional analysis and in Fourier analysis.

1.2 Notations and definitions

In the sequel $\mathbb{T} = [0, 2\pi)$; $\mathbb{I} = [0, 1)$; $\mathbb{L}$ is a class of non-decreasing sequences $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ of positive numbers such that $\sum_n \frac{1}{\lambda_n}$ diverges; ϕ is an increasing non-negative convex function defined on $[0, \infty)$ such that $\phi(0) = 0$, $\frac{\phi(x)}{x} \to 0$ as $x \to 0$ and $\frac{\phi(x)}{x} \to \infty$ as $x \to \infty$; $\varphi(n)$ is a real sequence such that $\varphi(1) \geq 2$ and $\varphi(n) \to \infty$ as $n \to \infty$; and $\mathbb{C}$ is a set of complex numbers.

1.2.1 Notations and definitions for functions of one variable

Definition 1.2.1.1. *Given sequence $\Lambda = \{\lambda_n\}_{n=1}^{\infty} \in \mathbb{L}$ and $p \geq 1$, a function f defined on an interval $I := [a,b]$ is said to be of $p-\Lambda-$bounded variation (that is, $f \in \Lambda BV^{(p)}(I)$) if*

$$V_{\Lambda_p}(f,I) = \sup_{\{I_i\}} \left\{ \left(\sum_i \frac{|f(I_i)|^p}{\lambda_i} \right)^{\frac{1}{p}} \right\} < \infty,$$

where $\{I_i\} = \{[a_i, b_i]\}$ is a finite collection of non-overlapping subintervals in I and $f(I_i) = f(b_i) - f(a_i)$.

In the Definition 1.2.1.1, for $\Lambda = \{1\}$ (that is, $\lambda_n = 1$, for all n) and $p = 1$ one gets the class $BV(I)$; for $p = 1$ one gets the class $\Lambda BV(I)$; and for $\Lambda = \{1\}$ one gets the class $BV^{(p)}(I)$.

For any $\Lambda \in \mathbb{L}$ and $p \geq 1$, we have

$$\left(\sum_i \frac{|f(I_i)|^p}{\lambda_i} \right)^{\frac{1}{p}} \leq \left(\frac{1}{\lambda_1} \right)^{\frac{1}{p}} \left(\sum_i |f(I_i)|^p \right)^{\frac{1}{p}} \leq \left(\frac{1}{\lambda_1} \right)^{\frac{1}{p}} \sum_i |f(I_i)|.$$

This implies

$$BV(I) \subset BV^{(p)}(I) \subset \Lambda BV^{(p)}(I).$$

It is observed that, if $f \in \Lambda BV^{(p)}(I)$ then it is a regulated function over I [61, Theorem 2, p.92] (that is, f has right hand and left hand limits at every point of the intervals $[a,b)$ and $(a,b]$ respectively). If f is a regulated function over I then $f \in \Lambda BV^{(p)}(I)$ for some sequence $\Lambda \in \mathbb{L}$ [61, Theorem 6, p.92]. Thus, the union of $\Lambda BV^{(p)}(I)$ functions over all sequences Λ is the class of regulated functions over I.

Moreover, if $f \in \Lambda BV^{(p)}(I)$ for every sequence Λ then $f \in BV^{(p)}(I)$ [61, Theorem 3, p.92]. Thus, the intersection of $\Lambda BV^{(p)}(I)$ functions over all sequences Λ is the class $BV^{(p)}(I)$ (that is, $\bigcap_\Lambda \Lambda BV^{(p)}(I) = BV^{(p)}(I)$).

Therefore, we can say that the class $\Lambda BV^{(p)}(I)$ lies between the class of regulated functions over I and the class $BV^{(p)}(I)$.

The following example shows that a continuous function need not be of $p-\Lambda-$bounded variation.

Example 1.2.1.2. *Given a $p \geq 1$. Let $f : [0,1] \to \mathbb{R}$ be defined as*

$$f(x) = \begin{cases} x^{\frac{1}{p}} \ cos\left(\frac{\pi}{2x}\right), & if\ x \in (0,1], \\ 0, & otherwise. \end{cases}$$

Obviously, $f \in \mathbf{C}([0,1])$. For any $m = 2k$, where $k \in \mathbb{N}$, if we consider the points $x_0 = 0$ and $x_i = \frac{1}{m+1-i}$, for $i = 1, 2, \cdots, m$, then we have $0 = x_0 \leq x_1 \leq \cdots \leq x_m = 1$ and

$$f(x_i) = \begin{cases} 0, & \text{if } i \text{ is even,} \\ \pm(x_i)^{\frac{1}{p}}, & \text{if } i \text{ is odd.} \end{cases}$$

Therefore,

$$\begin{aligned} \sum_{i=0}^{m-1} |f(x_{i+1}) - f(x_i)|^p &= \frac{1}{m} + \frac{1}{m} + \frac{1}{m-2} + \frac{1}{m-2} + \cdots + \frac{1}{2} + \frac{1}{2} \\ &= \sum_{i=1}^{k} \frac{1}{i} \to \infty \ as\ k \to \infty. \end{aligned}$$

Thus, $f \notin BV^{(p)}([0,1])$. Since $\bigcap_\Lambda \Lambda BV^{(p)}([0,1]) = BV^{(p)}([0,1])$, $f \notin \Lambda BV^{(p)}([0,1])$ for at lest one sequence $\Lambda \in \mathbb{L}$.

Since f is regulated function over $[0,1]$, $f \in \Lambda' BV^{(p)}([0,1])$, for some $\Lambda' \in \mathbb{L}$ (in view of the result [61, Theorem 6, p.92]).

Hence, $BV^{(p)}(I) \subsetneqq \Lambda' BV^{(p)}(I)$.

Definition 1.2.1.3. *Given a continuous function ϕ defined on $[0,\infty)$ and strictly increasing from 0 to ∞, a function f defined on an interval I is said to be of $\phi - \Lambda-$bounded variation (that is, $f \in \phi\Lambda BV(I)$) if*

$$V_{\Lambda_\phi}(f, I) = \sup_{\{I_i\}} \left\{ \sum_i \frac{\phi(|f(I_i)|)}{\lambda_i} \right\} < \infty,$$

where I, Λ, $\{I_i\}$ and $f(I_i)$ are as defined above in the Definition 1.2.1.1.

Here, function ϕ is said to have property Δ_2 if there is a constant $d \geq 2$ such that $\phi(2x) \leq d\phi(x)$, for all $x \geq 0$.

In the Definition 1.2.1.3, for $\phi(x) = x$ and $\Lambda = \{1\}$ one gets the class $BV(I)$; for $\phi(x) = x$ one gets the class $\Lambda BV(I)$; for $\phi(x) = x^p$ one gets the class $\Lambda BV^{(p)}(I)$; and for $\Lambda = \{1\}$ one gets the class $\phi BV(I)$.

Definition 1.2.1.4. *Given sequence $\varphi(n)$ and 1$\leq$ $p(n) \uparrow p$ as $n \to \infty$, where $1 \leq p \leq \infty$, a function f defined on an interval I is said to be of $p(n)-\Lambda-$bounded variation (that is, $f \in \Lambda BV(p(n) \uparrow p, \varphi, I)$) if*

$$V_{\Lambda_{p(n)}}(f, \varphi, I) = \sup_{n\geq 1} \sup_{\{I_i\}} \left\{ \left(\sum_i \frac{|f(I_i)|^{p(n)}}{\lambda_i} \right)^{\frac{1}{p(n)}} : \delta\{I_i\} \geq \frac{b-a}{\varphi(n)} \right\} < \infty,$$

where I, Λ, $\{I_i\}$ and $f(I_i)$ are as defined earlier in the Definition 1.2.1.1, and

$$\delta\{I_i\} = \inf_i \{|a_i - b_i|\}.$$

In the Definition 1.2.1.4, for $\varphi(n) = 2^n$, for all n, and $\Lambda = \{1\}$ one gets the class $BV(p(n) \uparrow p, I)$; for $\Lambda = \{1\}$ one gets the class $BV(p(n) \uparrow p, \varphi, I)$; for $p = \infty$ one gets the class $\Lambda BV(p(n) \uparrow \infty, \varphi, I)$; for $\Lambda = \{1\}$ and $p = \infty$ one gets the class $BV(p(n) \uparrow \infty, \varphi, I)$; and for $p(n) = p$, for all n, one gets the class $\Lambda BV^{(p)}(I)$.

It is observed that [62, Lemma 2.7, p.226], for $1 \leq p < \infty$,

$$BV^{(p)}(I) \subseteq BV(p(n) \uparrow \infty, \varphi, I)$$

and

$$\bigcup_{1\leq q<p} BV^{(q)}(I) \subseteq BV(p(n) \uparrow p, \varphi, I) \subseteq BV^{(p)}(I).$$

Definition 1.2.1.5. *Given a positive integer r, a function f defined on an interval I is said to be of bounded $r^{th}-$variation (that is, $f \in r - BV(I)$) if*

$$V_r(f, I) = \sup_P \left\{ \sum_{i=0}^{m-r} |\Delta^r f(x_i)| \right\} < \infty,$$

where I is as defined earlier in the Definition 1.2.1.1, $P : a = x_0 < x_1 < \cdots < x_m = b$,

$$\Delta f(x_i) = f(x_{i+1}) - f(x_i)$$

and

$$\Delta^k f(x_i) = \Delta^{k-1}(\Delta f(x_i)),\ k \geq 2,$$

so that

$$\Delta^r f(x_i) = \sum_{u=0}^{r} (-1)^u \binom{r}{u} f(x_{i+r-u}).$$

Obviously, $BV(I) \subset r - BV(I) \subset B(I)$, where $B(I)$ is a class of all bounded functions on I.

The following example shows that $BV(I) \neq r - BV(I)$.

Example 1.2.1.6. *Consider everywhere continuous but nowhere differentiable function of Weierstrass [29], defined as*

$$f(x) = \sum_{n=1}^{\infty} b^{-n}\ cos(b^n x), \quad b \text{ an integer} > 1,$$

satisfies the condition

$$|f(x+h) + f(x-h) - 2f(x)| = O(|h|) \text{ as } h \to 0$$

uniformly in x in $\overline{\mathbb{T}}$ and, therefore, it is of bounded second variation over $\overline{\mathbb{T}}$ (that is, $f \in 2 - BV(\overline{\mathbb{T}})$) [75]. However, f being a nowhere differentiable function, it is not of bounded variation over $\overline{\mathbb{T}}$ (that is, $f \notin BV(\overline{\mathbb{T}})$).

Definition 1.2.1.7. *Given a function $f \in L^p(\overline{\mathbb{T}})$, where $p \geq 1$, the $p-$integral modulus of continuity of f of higher differences of order $r \geq 1$ is defined as*

$$\omega_r^{(p)}(f;\delta) = \sup\left\{ \left(\frac{1}{2\pi} \int_{\overline{\mathbb{T}}} |\Delta^r f(x;h)|^p\, dx \right)^{\frac{1}{p}}\ :\ 0 < h \leq \delta \right\},$$

where

$$\Delta^r f(x;h) = \sum_{u=0}^{r} (-1)^u \binom{r}{u} f(x + (r-u)h).$$

In the Definition 1.2.1.7, for $r = 1$, we omit writing r, one gets $\omega^{(p)}(f;\delta)$, the $p-$integral modulus of continuity of f.

For $p \geq 1$ and $\alpha \in (0,1]$, we say that $f \in Lip(p;\alpha)(\overline{\mathbb{T}})$ if

$$\omega^{(p)}(f;\delta) = O(\delta^\alpha).$$

In the Definition 1.2.1.7, for $p = \infty$ and $r = 1$, we omit writing p and r, one gets $\omega(f;\delta)$, the modulus of continuity of f, and in that case the class $Lip(p;\alpha)(\overline{\mathbb{T}})$ reduces to the Lipschitz class $Lip(\alpha)(\overline{\mathbb{T}})$.

For a $2\pi-$periodic function, the notion of $p-$bounded variation in the sense of L^p-norm is defined as follows.

Definition 1.2.1.8. *Let $f \in L^p(\overline{\mathbb{T}})$ with $p \geq 1$. We say $f \in BV^{(p)}M(\overline{\mathbb{T}})$ (that is, f is a function of $p-$bounded variation in the mean over $\overline{\mathbb{T}}$) if*

$$V_p^m(f,\overline{\mathbb{T}}) = \sup_{\{I_i\}} \left\{ \sum_i \int_{\overline{\mathbb{T}}} \frac{|f(I_{ix})|^p}{|I_{ix}|^{p-1}} \, dx \right\} < \infty,$$

where $\{I_i\} = \{[x_i, x_{i+1}]\}$ is a finite collection of non-overlapping subintervals in $\overline{\mathbb{T}}$, $\{I_{ix}\} = \{[x+x_i, x+x_{i+1}]\}$, $f(I_{ix}) = f(x+x_{i+1})-f(x+x_i)$ and $|I_{ix}| = |x_{i+1}-x_i|$.

In the Definition 1.2.1.8, for $p = 1$ one gets the class $BVM(\overline{\mathbb{T}})$.

Definition 1.2.1.9. *A function f defined on an interval I is said to be absolutely continuous (that is, $f \in AC(I)$) if for a given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $\sum_i |f(I_i)| < \epsilon$ whenever $\{I_i\} = \{[a_i, b_i]\}$ is a finite collection of non-overlapping subintervals in I with $\sum_i (b_i - a_i) < \delta$, where I and $f(I_i)$ are as defined earlier in the Definition 1.2.1.1.*

The class of one variable functions of bounded variation as well as the classes of one variable functions of generalized bounded variations are of great interest because of their valuable properties like additivity, differentiability, measurability, integrability, etc. Because of all such properties, functions of these classes owe their important role in the study of Operator theory, Fourier series, Walsh$-$Fourier series, Fourier$-$Haar series, Fourier$-$Jacobi series and other orthogonal series, Stieltjes and other integrals, and the calculus of variations.

For a $2\pi-$periodic complex valued function $f \in L^1(\overline{\mathbb{T}})$, its Fourier series is defined as

$$f(x) \sim \sum_{m\in\mathbb{Z}} \hat{f}(m)\, e^{imx},$$

where

$$\hat{f}(m) = \frac{1}{2\pi} \int_{\overline{\mathbb{T}}} f(x)\, e^{-imx} \, dx$$

denotes the m^{th} Fourier coefficient of f.

The Fourier series of a function f is said to be $\beta-$absolute convergence if

$$\sum_{m\in\mathbb{Z}} |\hat{f}(m)|^\beta < \infty, \quad 0 < \beta \leq 2.$$

Let us denote

$$A_1(\beta) = \left\{ f \in L^1(\overline{\mathbb{T}}) : \sum_{m\in\mathbb{Z}} |\hat{f}(m)|^\beta < \infty \right\}.$$

For $\beta = 1$, we omit writing β, one gets the absolute convergence of the Fourier series of f and in that case the class $A_1(\beta)$ is denoted by A_1.

Wiener [48, §11.6, p.278] proved that the class A_1 is a commutative complex Banach algebra. Therefore, the class A_1 is also known as Wiener algebra.

Like trigonometric system, the Walsh system also forms a complete orthonormal system. It is observed that the Walsh system perform all the usual applications of trigonometric system. Like Fourier series, the Walsh$-$Fourier series is interesting also from theoretical as well as application points of view.

The Walsh orthonormal system $\{\psi_m(x) : m \in \mathbb{N}_0\}$, where $\mathbb{N}_0 = \{0\} \cup \mathbb{N} = \{0, 1, 2, \cdots\}$, on the unit interval $\mathbb{I} = [0, 1)$ in the Paley enumeration is defined as follows.

Let

$$r_0(x) = \begin{cases} 1, & \text{if } x \in \left[0, \frac{1}{2}\right), \\ -1, & \text{if } x \in \left[\frac{1}{2}, 1\right); \end{cases}$$

and extend $r_0(x)$ for the half-axis $[0, \infty)$ with period 1.

The Rademacher orthonormal system $\{r_k(x) : \ k \in \mathbb{N}_0\}$ is defined as

$$r_k(x) = r_0(2^k x), \quad k = 1, 2, \cdots ; \ x \in \mathbb{I}.$$

If

$$m = \sum_{k=0}^{\infty} m_k 2^k, \quad \textit{each } m_k = 0 \textit{ or } 1,$$

is the binary decomposition of $m \in \mathbb{N}_0$, then

$$\psi_m(x) = \prod_{k=0}^{\infty} r_k^{m_k}(x), \quad x \in \mathbb{I},$$

is called the m^{th} Walsh function in the Paley enumeration.

In particular, we have

$$\psi_0(x) = 1 \ \ and \ \ \psi_{2^m}(x) = r_m(x), \ m \in \mathbb{N}_0.$$

Any $x \in \mathbb{I}$ can be written as

$$x = \sum_{k=0}^{\infty} x_k \ 2^{-(k+1)}, \quad each \ x_k = 0 \ or \ 1.$$

For any $x \in \mathbb{I} \backslash Q$, there is only one expression of this form, where Q is a class of dyadic rationals in $\mathbb{I}$. When $x \in Q$ there are two expressions of this form, one which terminates in 0's and one which terminates in 1's.

For any $x, y \in \mathbb{I}$ their dyadic sum is defined as

$$x \dotplus y = \sum_{k=0}^{\infty} |x_k - y_k| \ 2^{-(k+1)}.$$

Observed that, for each $m \in \mathbb{N}_0$, we have

$$\psi_m(x \dotplus y) = \psi_m(x) \ \psi_m(y), \quad x, y \in \mathbb{I}, \quad x \dotplus y \notin Q.$$

For a $1-$periodic real valued function $f \in L^1(\bar{\mathbb{I}})$, its Walsh$-$Fourier series is defined as

$$f(x) \sim \sum_{m \in \mathbb{N}_0} \hat{f}(m) \ \psi_m(x),$$

where

$$\hat{f}(m) = \int_{\bar{\mathbb{I}}} f(x) \ \psi_m(x) \ dx$$

denotes the m^{th} Walsh$-$Fourier coefficient of f.

1.2.2 Notations and definitions for functions of two variables

Soon following the Jordan's research study, many mathematicians began to study notion of bounded variation for functions of several variables. Those who proposed definitions of bounded variation for functions of two variables (that is, $f(x, y)$) were actuated mainly by the desire to single out for attention a class of functions having properties analogous to some particular properties of a function $g(x)$ of bounded variation. To preserve properties of one sort, the definition of bounded variation of function of one variable is extended to the function of two variables in one way, whereas to preserve properties of another sort, a quite different extension may be needed. Some of such extensions are as follow.

Definition 1.2.2.1. *A function f defined on a rectangle $R^2 := [a, b] \times [c, d]$ is said to be of bounded variation in the sense of Arzelà (that is, $f \in BV_A(R^2)$) if*

$$V_A(f, R^2) = \sup_P \left\{ \sum_i |f(x_{i+1}, y_{i+1}) - f(x_i, y_i)| \right\} < \infty,$$

where

$$P : (a, c) = (x_0, y_0) \leq (x_1, y_1) \leq \cdots \leq (x_N, y_N) = (b, d),$$

in which $(x_k, y_k) \leq (x_{k+1}, y_{k+1})$ is defined as

$$(x_k, y_k) \leq (x_{k+1}, y_{k+1}) \Leftrightarrow x_k \leq x_{k+1} \text{ and } y_k \leq y_{k+1}, \text{ for all } k = 0, 1, \cdots, N - 1.$$

The class $BV_A(R^2, \mathbb{R})$, of real functions of bounded variation on R^2 in the sense of Arzelà, generalizes many properties of the class $BV([a, b])$. Some of them are as follow.

Remark 1.2.2.2. *Let $f_1 : [a, b] \to \mathbb{R}$ and $f_2 : [c, d] \to \mathbb{R}$ be functions of one variable. Define real functions f and g on $R^2 := [a, b] \times [c, d]$ as $f(x, y) = f_1(x) + f_2(y)$ and $g(x, y) = f_1(x)f_2(y)$. Then the following will hold [23, Exercise 43 (i) and (ii), p.40]:*

(i) $f \in BV_A(R^2, \mathbb{R})$ if and only if $f_1 \in BV([a, b])$ and $f_2 \in BV([c, d])$.

(ii) Let f_1 and f_2 be non-zero functions. $g \in BV_A(R^2, \mathbb{R})$ if and only if $f_1 \in BV([a, b])$ and $f_2 \in BV([c, d])$.

If a function $f : R^2 \to \mathbb{R}$ satisfies the condition

$$|f(x_2, y_2) - f(x_1, y_1)| \leq K \ (x_2 - x_1 + y_2 - y_1),$$

for all $(x_1, y_1) \leq (x_2, y_2)$ in R^2, where K is constant, then $f \in BV_A(R^2, \mathbb{R})$ and $V_A(f, R^2) \leq K(b - a + d - c)$ [23, Exercise 49 (i), p.41].

Definition 1.2.2.3. *A real function f defined on a rectangle R^2 is said to be monotonically increasing (respectively monotonically decreasing) if, for all (x_1, y_1), $(x_2, y_2) \in R^2$, we have*

$$(x_1, y_1) \leq (x_2, y_2) \Rightarrow f(x_1, y_1) \leq f(x_2, y_2) \ (\textit{respcetively} \ f(x_1, y_1) \geq f(x_2, y_2)),$$

where R^2 and $(x_1, y_1) \leq (x_2, y_2)$ are as defined earlier in the Definition 1.2.2.1.

If f is either monotonically increasing on R^2 or monotonically decreasing on R^2 then f is said to be monotonic on R^2.

A function of the class $BV_A(R^2, \mathbb{R})$ is characterized as the difference of two monotonic functions on R^2 [23, Proposition 1.12, p.19].

Consider function f on $\mathbb{R}^k$. For $k = 1$ and $I = [a, b]$, define $\Delta f_a^b = f(I) = f(b) - f(a)$. For $k = 2$, $I = [a, b]$ and $J = [c, d]$, define

$$\Delta f_{(a,c)}^{(b,d)} = f(I \times J) = f(I, d) - f(I, c) = f(b, d) - f(a, d) - f(b, c) + f(a, c).$$

Definition 1.2.2.4. *A function f defined on a rectangle R^2 is said to be of bounded variation in the sense of Vitali (that is, $f \in BV_V(R^2)$) if*

$$V(f, R^2) = \sup_{I_1, \ I_2} \left\{ \sum_i \sum_j |f(I_i \times I_j)| \right\} < \infty,$$

where R^2 is as defined earlier in the Definition 1.2.2.1; I_1 and I_2 are finite collections of non-overlapping subintervals $\{I_i\}$ and $\{I_j\}$ in $[a, b]$ and $[c, d]$ respectively.

The following example shows that a function $f \in BV_V(R^2)$ need not be bounded.

Example 1.2.2.5. ([*23, Example* 1.19 (*i*), *p.*23]) *Let $f : [0, 1]^2 \to \mathbb{R}$ be defined as*

$$f(x,y) = \begin{cases} \frac{1}{x}+\frac{1}{y}, & \textit{if } x \neq 0 \textit{ and } y \neq 0, \\ \frac{1}{x}, & \textit{if } x \neq 0 \textit{ and } y = 0, \\ \frac{1}{y}, & \textit{if } x = 0 \textit{ and } y \neq 0, \\ 0, & \textit{if } x = 0 \textit{ and } y = 0. \end{cases}$$

Then $V(f,[0,1]^2) = 0$. Thus, unbounded function $f \in BV_V([0,1]^2)$.

This class is further generalized to the class $BV_H(R^2)$ as follows.

If $f \in BV_V(R^2)$ is such that the marginal functions $f(.,c) \in BV([a,b])$ and $f(a,.) \in BV([c,d])$ then f is said to be of bounded variation in the sense of Hardy (that is, $f \in BV_H(R^2)$).

If $f \in BV_H(R^2)$ then f is bounded and each of the marginal functions $f(.,s) \in BV([a,b])$ and $f(t,.) \in BV([c,d])$, where $s \in [c,d]$ and $t \in [a,b]$ are fixed.

Definition 1.2.2.6. *A real function f defined on a rectangle R^2 is said to be $2-$fold monotonically increasing (respectively $2-$fold monotonically decreasing) if, for all $(x_1,y_1),(x_2,y_2) \in R^2$, we have*

$$(x_1,y_1) \leq (x_2,y_2) \Rightarrow \Delta f_{(x_1,y_1)}^{(x_2,y_2)} \geq 0 \ \left(respectively \ \Delta f_{(x_1,y_1)}^{(x_2,y_2)} \leq 0 \right),$$

where R^2 and $(x_1,y_1) \leq (x_2,y_2)$ are as defined earlier in the Definition 1.2.2.1.

If f is either $2-$fold monotonically increasing on R^2 or $2-$fold monotonically decreasing on R^2 then f is said to be $2-$fold monotonic on R^2.

A function of the class $BV_V(R^2,\mathbb{R})$, of real functions of bounded variation on R^2 in the sense of Vitali, is characterized as the difference of two $2-$fold monotonic functions on R^2 [23, Proposition 1.17, p.22].

Definition 1.2.2.7. *Given $\bigwedge = (\Lambda^1,\Lambda^2)$, where $\Lambda^k = \{\lambda_n^k\}_{n=1}^{\infty} \in \mathbb{L}$, for $k = 1,\ 2$, and $p \geq 1$, a measurable function f defined on a rectangle R^2 is said to be of $p-\bigwedge-$bounded variation (that is, $f \in \bigwedge BV^{(p)}(R^2)$) if*

$$V_{\bigwedge_p}(f,R^2) = \sup_{I_1,\ I_2} \left\{ \left(\sum_i \sum_j \frac{|f(I_i \times I_j)|^p}{\lambda_i^1\ \lambda_j^2} \right)^{\frac{1}{p}} \right\} < \infty,$$

where R^2 is as defined earlier in the Definition 1.2.2.1; I_1 and I_2 are as defined earlier in the Definition 1.2.2.4.

Let f be an unbounded function as defined earlier in the Example 1.2.2.5 (p.12). Then $V_{\bigwedge_p}(f,[0,1]^2)=0$. Thus, a function $f\in\bigwedge BV^{(p)}(R^2)$ need not be bounded.

This class is further generalized to the class $\bigwedge^* BV^{(p)}(R^2)$ in the sense of Hardy as follows.

If $f\in\bigwedge BV^{(p)}(R^2)$ is such that the marginal functions $f(.,c)\in\Lambda^1 BV^{(p)}([a,b])$ and $f(a,.)\in\Lambda^2 BV^{(p)}([c,d])$ then f is said to be of $p-\bigwedge^*-$bounded variation (that is, $f\in\bigwedge^* BV^{(p)}(R^2)$).

If $f\in\bigwedge^* BV^{(p)}(R^2)$ then f is bounded and each of the marginal functions $f(.,s)\in\Lambda^1 BV^{(p)}([a,b])$ and $f(t,.)\in\Lambda^2 BV^{(p)}([c,d])$, where $s\in[c,d]$ and $t\in[a,b]$ are fixed.

Note that, for $\Lambda^1=\Lambda$ and $\Lambda^2=\{1\}$ (that is, $\lambda_n^1=\lambda_n$ and $\lambda_n^2=1$, for all n), the classes $\bigwedge BV^{(p)}(R^2)$ and $\bigwedge^* BV^{(p)}(R^2)$ reduce to the classes $\Lambda BV^{(p)}(R^2)$ and $\Lambda^* BV^{(p)}(R^2)$ respectively; for $\Lambda^1=\Lambda^2=\{1\}$ and $p=1$, the classes $\bigwedge BV^{(p)}(R^2)$ and $\bigwedge^* BV^{(p)}(R^2)$ reduce to the classes $BV_V(R^2)$ and $BV_H(R^2)$ respectively; for $p=1$, the classes $\bigwedge BV^{(p)}(R^2)$, $\bigwedge^* BV^{(p)}(R^2)$, $\Lambda BV^{(p)}(R^2)$ and $\Lambda^* BV^{(p)}(R^2)$ reduce to the classes $\bigwedge BV(R^2)$, $\bigwedge^* BV(R^2)$, $\Lambda BV(R^2)$ and $\Lambda^* BV(R^2)$ respectively; and for $\Lambda^1=\Lambda^2=\{1\}$, the classes $\bigwedge BV^{(p)}(R^2)$ and $\bigwedge^* BV^{(p)}(R^2)$ reduce to the classes $BV_V^{(p)}(R^2)$ and $BV_H^{(p)}(R^2)$ respectively.

Dyachenko and Waterman [18, Proposition 1, p.401] proved that there exists a function $f\in\bigwedge^* BV(R^2)$ which is everywhere discontinuous.

For any $\Lambda^1,\Lambda^2\in\mathbb{L}$ and $p\geq 1$, we have

$$\begin{aligned}\left(\sum_i\sum_j\frac{|f(I_i\times I_j)|^p}{\lambda_i^1\,\lambda_j^2}\right)^{\frac{1}{p}} &\leq \left(\frac{1}{\lambda_1^1\,\lambda_1^2}\right)^{\frac{1}{p}}\left(\sum_i\sum_j|f(I_i\times I_j)|^p\right)^{\frac{1}{p}}\\ &\leq \left(\frac{1}{\lambda_1^1\,\lambda_1^2}\right)^{\frac{1}{p}}\sum_i\sum_j|f(I_i\times I_j)|.\end{aligned}$$

This implies

$$BV_V(R^2)\subset BV_V^{(p)}(R^2)\subset\bigwedge BV^{(p)}(R^2)$$

and hence

$$BV_H(R^2)\subset BV_H^{(p)}(R^2)\subset\bigwedge^* BV^{(p)}(R^2).$$

The class $\bigwedge BV^{(p)}(R^2, \mathbb{R})$, of real functions of $p - \bigwedge -$bounded variation on R^2, generalizes many properties of the class $BV([a, b])$. Some of them are as follow.

Let f_1, f_2, f and g be as in the Remark 1.2.2.2 (p.11) earlier. Then the following will hold:

(i) Let f_1 and f_2 be not constant functions. $g \in \bigwedge BV^{(p)}(R^2, \mathbb{R})$ if and only if $f_1 \in \Lambda^1 BV^{(p)}([a, b])$ and $f_2 \in \Lambda^2 BV^{(p)} BV([c, d])$.

(ii) $V_{\bigwedge_p}(f, R^2) = 0$ implies $f \in \bigwedge BV^{(p)}(R^2, \mathbb{R})$.

If a function $h : R^2 \to \mathbb{R}$ is such that $V_{\bigwedge_p}(h, R^2) = 0$ then $V(h, R^2) = 0$. Therefore, in view of [23, Exercise 44, p.40], there exist functions $h_1 : [a, b] \to \mathbb{R}$ and $h_2 : [c, d] \to \mathbb{R}$ such that $h(x, y) = h_1(x) + h_2(y)$, for all $(x, y) \in R^2$.

If a function $f : R^2 \to \mathbb{R}$ satisfies the condition

$$\left| \Delta f_{(x_1,y_1)}^{(x_2,y_2)} \right| \le M \; |x_2 - x_1| \; |y_2 - y_1|,$$

for all $(x_1, y_1) \le (x_2, y_2)$ in R^2, where M is constant, then $f \in \bigwedge BV^{(p)}(R^2, \mathbb{R})$ and

$$V_{\bigwedge_p}(f, R^2) \le \frac{M(b-a)(d-c)}{(\lambda_1^1 \; \lambda_1^2)^{\frac{1}{p}}}.$$

The following example shows that a continuous function need not be of $p-$bounded variation in the sense of Vitali.

Example 1.2.2.8. *Given a $p \ge 1$. Let $f : [0, 1]^2 \to \mathbb{R}$ be defined as*

$$f(x, y) = \begin{cases} (xy)^{\frac{1}{p}} \; cos\left(\frac{\pi}{2x}\right), & \text{if } x \in (0, 1] \text{ and } y \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, $f \in \mathbf{C}([0, 1]^2)$. For any $m = 2k$, where $k \in \mathbb{N}$, if we consider the points $x_0 = 0$ and $x_i = \frac{1}{m+1-i}$, for $i = 1, 2, \cdots, m$, then we have $0 = x_0 \le x_1 \le \cdots \le x_m = 1$. Take $n = 1$, $y_0 = 0$ and $y_1 = 1$ then we have $0 = y_0 \le y_1 = 1$.

Moreover, for any $i \ge 0$, $f(x_i, 0) = 0$ and

$$f(x_i, 1) = \begin{cases} 0, & \text{if } i \text{ is even,} \\ \pm(x_i)^{\frac{1}{p}}, & \text{if } i \text{ is odd.} \end{cases}$$

Therefore,

$$\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|f(x_{i+1},y_{j+1})-f(x_i,y_{j+1})-f(x_{i+1},y_j)+f(x_i,y_j)|^p$$

$$=\frac{1}{m}+\frac{1}{m}+\frac{1}{m-2}+\frac{1}{m-2}+\cdots+\frac{1}{2}+\frac{1}{2}$$

$$=\sum_{i=1}^{k}\frac{1}{i}\to\infty \; as \; k\to\infty.$$

Thus, $f\notin BV_V^{(p)}([0,1]^2)$.

Definition 1.2.2.9. *A measurable function f defined on a rectangle R^2 is said to be of $\phi-\bigwedge-$bounded variation (that is, $f\in\phi\bigwedge BV(R^2)$) if*

$$V_{\bigwedge_\phi}(f,R^2)=\sup_{I_1,\ I_2}\left\{\sum_i\sum_j\frac{\phi(|f(I_i\times I_j)|)}{\lambda_i^1\ \lambda_j^2}\right\}<\infty,$$

where R^2 is as defined earlier in the Definition 1.2.2.1; $\bigwedge$ is as defined in the Definition 1.2.2.7; I_1 and I_2 are as defined in the Definition 1.2.2.4; and ϕ is as defined in the Definition 1.2.1.3.

Let f be an unbounded function as defined earlier in the Example 1.2.2.5 (p.12). Then $V_{\bigwedge_\phi}(f,[0,1]^2)=0$. Thus, a function $f\in\phi\bigwedge BV(R^2)$ need not be bounded.

This class is further generalized to the class $\phi\bigwedge^* BV(R^2)$ in the sense of Hardy as follows.

If $f\in\phi\bigwedge BV(R^2)$ is such that the marginal functions $f(.,c)\in\phi\Lambda^1 BV([a,b])$ and $f(a,.)\in\phi\Lambda^2 BV([c,d])$ then f is said to be of $\phi-\bigwedge^*-$bounded variation (that is, $f\in\phi\bigwedge^* BV(R^2)$).

If $f\in\phi\bigwedge^* BV(R^2)$ then f is bounded and each of the marginal functions $f(.,s)\in\phi\Lambda^1 BV([a,b])$ and $f(t,.)\in\phi\Lambda^2 BV([c,d])$, where $s\in[c,d]$ and $t\in[a,b]$ are fixed.

Observe that for $\phi(x)=x$ the conditions $\frac{\phi(x)}{x}\to 0$ as $x\to 0$ and $\frac{\phi(x)}{x}\to\infty$ as $x\to\infty$ are not valid.

Note that, for $\phi(x)=x$ and $\Lambda^1=\Lambda^2=\{1\}$, the classes $\phi\bigwedge BV(R^2)$ and $\phi\bigwedge^* BV(R^2)$ reduce to the classes $BV_V(R^2)$ and $BV_H(R^2)$ respectively; for $\phi(x)=x$, the classes $\phi\bigwedge BV(R^2)$ and $\phi\bigwedge^* BV(R^2)$ reduce to the classes $\bigwedge BV(R^2)$

and $\bigwedge^* BV(R^2)$ respectively; and for $\phi(x) = x^p$, the classes $\phi \bigwedge BV(R^2)$ and $\phi \bigwedge^* BV(R^2)$ reduce to the classes $\bigwedge BV^{(p)}(R^2)$ and $\bigwedge^* BV^{(p)}(R^2)$ respectively.

Definition 1.2.2.10. *Given a positive integer r, a function f defined on a rectangle R^2 is said to be of bounded $r^{th}-$variation (that is, $f \in r - BV(R^2)$) if the following two conditions are satisfied:*

(i)

$$V_r(f, R^2) = \sup_{P=P_1\times P_2} \left\{ \sum_{i=0}^{m-r} \sum_{j=0}^{n-r} |\Delta^r f(x_i, y_j)| \right\} < \infty,$$

where R^2 is as defined earlier in the Definition 1.2.2.1, $P_1 : a = x_0 < x_1 < \cdots < x_m = b$, $P_2 : c = y_0 < y_1 < \cdots < y_n = d$,

$$\Delta f(x_i, y_j) = f([x_i, x_{i+1}] \times [y_j, y_{j+1}])$$

and

$$\Delta^k f(x_i, y_j) = \Delta^{k-1}(\Delta f(x_i, y_j)),\ k \geq 2,$$

so that

$$\Delta^r f(x_i, y_j) = \sum_{u=0}^{r} \sum_{v=0}^{r} (-1)^{u+v} \binom{r}{u} \binom{r}{v} f(x_{i+r-u}, y_{j+r-v}).$$

(ii) The marginal functions $f(., c) \in r - BV([a, b])$ and $f(a, .) \in r - BV([c, d])$.

If $f \in r - BV(R^2)$ then f is bounded and each of the marginal functions $f(., s) \in r - BV([a, b])$ and $f(t, .) \in r - BV([c, d])$, where $s \in [c, d]$ and $t \in [a, b]$ are fixed.

Obviously, $BV_H(R^2) \subset r - BV(R^2)$.

Definition 1.2.2.11. *Given a function $f \in L^p(\overline{\mathbb{T}}^2)$, where $p \geq 1$, the $p-$integral modulus of continuity of f of higher differences of order $r \geq 1$ is defined as*

$$\omega_r^{(p)}(f; \delta_1, \delta_2)$$

$$= \sup \left\{ \left(\frac{1}{4\pi^2} \int\int_{\overline{\mathbb{T}}^2} |\Delta^r f(x, y; h_1, h_2)|^p \ dx\ dy \right)^{\frac{1}{p}} :\ 0 < h_1 \leq \delta_1,\ 0 < h_2 \leq \delta_2 \right\},$$

where

$$\Delta^r f(x, y; h_1, h_2) = \sum_{u=0}^{r} \sum_{v=0}^{r} (-1)^{u+v} \binom{r}{u} \binom{r}{v} f(x + (r - u)h_1, y + (r - v)h_2).$$

In the Definition 1.2.2.11, for $r = 1$, we omit writing r, one gets $\omega^{(p)}(f; \delta_1, \delta_2)$, the $p-$integral modulus of continuity of f.

For $p \geq 1$ and $\alpha_1, \alpha_2 \in (0, 1]$, we say that $f \in Lip(p; \alpha_1, \alpha_2)(\overline{\mathbb{T}}^2)$ if

$$\omega^{(p)}(f; \delta_1, \delta_2) = O(\delta_1^{\alpha_1} \delta_2^{\alpha_2}).$$

In the Definition 1.2.2.11, for $p = \infty$ and $r = 1$, we omit writing p and r, one gets $\omega(f; \delta_1, \delta_2)$, the modulus of continuity of f, and in that case the class $Lip(p; \alpha_1, \alpha_2)(\overline{\mathbb{T}}^2)$ reduces to the Lipschitz class $Lip(\alpha_1, \alpha_2)(\overline{\mathbb{T}}^2)$.

For a function f of two variables, where f is $2\pi-$periodic in each variable, the notion of $p-$bounded variation in the sense of L^p-norm is defined as follows.

Definition 1.2.2.12. *Let $f \in L^p(\overline{\mathbb{T}}^2)$ with $p \geq 1$. We say $f \in BV^{(p)}M(\overline{\mathbb{T}}^2)$ (that is, f is a function of $p-$bounded variation in the mean over $\overline{\mathbb{T}}^2$) if each of*

$$V_p^m(f, \overline{\mathbb{T}}^2) = \sup_{I_1,\ I_2} \left\{ \sum_i \sum_j \int\int_{\overline{\mathbb{T}}^2} \frac{|f(I_{ix} \times I_{jy})|^p}{|I_{ix}|^{p-1}\ |I_{jy}|^{p-1}}\ dx\ dy \right\},$$

$$V_p^m(f(., s), \overline{\mathbb{T}}) = \sup_{I_1} \left\{ \sum_i \int_{\overline{\mathbb{T}}} \frac{|f(I_{ix} \times s)|^p}{|I_{ix}|^{p-1}}\ dx \right\},\ s \in \overline{\mathbb{T}},$$

and

$$V_p^m(f(t, .), \overline{\mathbb{T}}) = \sup_{I_2} \left\{ \sum_j \int_{\overline{\mathbb{T}}} \frac{|f(t \times I_{jy})|^p}{|I_{jy}|^{p-1}}\ dy \right\},\ t \in \overline{\mathbb{T}},\ are\ finite,$$

where I_1 and I_2 are finite collections of non-overlapping subintervals $\{[x_i, x_{i+1}]\}$ and $\{[y_j, y_{j+1}]\}$ respectively in $\overline{\mathbb{T}}$, $\{I_{ix}\} = \{[x+x_i, x+x_{i+1}]\}$, $\{I_{jy}\} = \{[y+y_j, y+y_{j+1}]\}$, $|I_{ix}| = |x_{i+1} - x_i|$ and $|I_{jy}| = |y_{j+1} - y_j|$.

Definition 1.2.2.13. *A function f defined on a rectangle R^2 is said to be absolutely continuous (that is, $f \in AC(R^2)$) if the following two conditions are satisfied:*

(i) Given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$\sum_{R_k^2 \in R} |f([a_k, b_k] \times [c_k, d_k])| < \epsilon$$

whenever $R = \{R_k^2 := [a_k, b_k] \times [c_k, d_k]\}$ *is a finite collection of pairwise non-overlapping sub-rectangles of* R^2 *with*

$$\sum_{R_k^2 \in R} (b_k - a_k)(d_k - c_k) < \delta,$$

where R^2 *is as defined earlier in the Definition 1.2.2.1.*

(ii) *The marginal functions* $f(.,c) \in AC([a,b])$ *and* $f(a,.) \in AC([c,d])$.

An absolutely continuous function f on R^2 is uniformly continuous and each of the marginal functions $f(.,s) \in AC([a,b])$ and $f(t,.) \in AC([c,d])$, where $s \in [c,d]$ and $t \in [a,b]$ are fixed.

E. Berkson and T. Gillespie [9] observed that, the class $AC(R^2)$ is a closed linear subspace of the class $BV_H(R^2)$ and functions in $AC(R^2)$ can be characterized in terms of indefinite integrals of Lebesgue integrable functions.

It is observed that these classes of two variables functions of generalized bounded variations possess many interesting properties in Fourier analysis.

For any $\mathbf{x} = (x,y) \in \overline{\mathbb{T}}^2$ and $\mathbf{k} = (m,n) \in \mathbb{Z}^2$, denote their scalar product by $\mathbf{k} \cdot \mathbf{x} = mx + ny$.

For a complex valued function $f \in L^1(\overline{\mathbb{T}}^2)$, where f is $2\pi-$periodic in each variable, its double Fourier series is defined as

$$f(\mathbf{x}) \sim \sum_{\mathbf{k} \in \mathbb{Z}^2} \hat{f}(\mathbf{k})\ e^{i(\mathbf{k}\cdot\mathbf{x})} = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \hat{f}(m,n)\ e^{i(mx+ny)},$$

where

$$\hat{f}(\mathbf{k}) = \frac{1}{(2\pi)^2} \int\int_{\overline{\mathbb{T}}^2} f(\mathbf{x})\ e^{-i(\mathbf{k}\cdot\mathbf{x})}\ \mathbf{dx} = \frac{1}{(2\pi)^2} \int\int_{\overline{\mathbb{T}}^2} f(x,y)\ e^{-i(mx+ny)}\ dx\ dy$$

denotes the $\mathbf{k}^{th}$ Fourier coefficient of f.

The double Fourier series of a function f is said to be $\beta-$absolute convergence if

$$\sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} |\hat{f}(m,n)|^{\beta} < \infty, \quad 0 < \beta \leq 2.$$

Let us denote

$$A_2(\beta) = \left\{ f \in L^1(\overline{\mathbb{T}}^2) : \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} |\hat{f}(m,n)|^\beta < \infty \right\}.$$

For $\beta = 1$, we omit writing β, one gets the absolute convergence of the double Fourier series of f and in that case the class $A_2(\beta)$ is denoted by A_2.

For a real valued function $f \in L^1(\overline{\mathbb{I}}^2)$, where f is $1-$periodic in each variable, its double Walsh$-$Fourier series is defined as

$$f(\mathbf{x}) \sim \sum_{\mathbf{k}\in\mathbb{N}_0^2} \hat{f}(\mathbf{k})\ \psi_m(x)\ \psi_n(y) = \sum_{m\in\mathbb{N}_0} \sum_{n\in\mathbb{N}_0} \hat{f}(m,n)\ \psi_m(x)\ \psi_n(y),$$

where

$$\hat{f}(\mathbf{k}) = \hat{f}(m,n) = \int\int_{\overline{\mathbb{I}}^2} f(x,y)\ \psi_m(x)\ \psi_n(y)\ dx\ dy$$

denotes the $\mathbf{k}^{th}$ Walsh$-$Fourier coefficient of f.

1.2.3 Notations and definitions for functions of N$-$variables

Let $I^k = [a_k, b_k] \subset \mathbb{R}$, for $k = 1, 2, \cdots, N$. In the Subsection 1.2.2 above, we defined $f(I^1)$ for a function f of one variable and $f(I^1 \times I^2)$ for a function f of two variables (p.12). Similarly, for a function f on $\mathbb{R}^N$, by induction, defining the expression $f(I^1 \times \cdots \times I^{N-1})$ for a function of $N-1$ variables, one gets

$$f(I^1 \times \cdots \times I^N) = f(I^1 \times \cdots \times I^{N-1}, b_N) - f(I^1 \times \cdots \times I^{N-1}, a_N).$$

Observe that, $f(I^1 \times \cdots \times I^N)$ can also be expressed as

$$f(I^1 \times \cdots \times I^N) = \Delta f_{\mathbf{a}}^{\mathbf{b}} = \sum_{\mathbf{c}} k(\mathbf{c}) f(\mathbf{c}),$$

where $\mathbf{a} = (a_1, a_2, \cdots, a_N),\ \mathbf{b} = (b_1, b_2, \cdots, b_N) \in \mathbb{R}^N$, the summation is over all $\mathbf{c} = (c_1, c_2, \cdots, c_N) \in \mathbb{R}^N$ such that $c_i \in \{a_i, b_i\}$, for $i = 1, \cdots, N$, and for any such $\mathbf{c}$, $k(\mathbf{c}) = k_1 \cdots k_N,\ in\ which,\ for\ 1 \leq i \leq N,$

$$k_i = \begin{cases} 1, & \text{if } c_i = b_i, \\ -1, & \text{if } c_i = a_i. \end{cases}$$

Then, for $N = 1$ we get

$$f(I^1) = \Delta f_{\mathbf{a}}^{\mathbf{b}} = \Delta f_{a_1}^{b_1} = \sum_{c_1} k(\mathbf{c}) f(\mathbf{c}) = f(b_1) - f(a_1).$$

For $N = 2$ we get

$$f(I^1 \times I^2) = \Delta f_{\mathbf{a}}^{\mathbf{b}} = \Delta f_{(a_1,a_2)}^{(b_1,b_2)} = \sum_{(c_1,c_2)} k(\mathbf{c}) f(\mathbf{c})$$

$$= f(b_1, b_2) + f(a_1, a_2) - f(b_1, a_2) - f(a_1, b_2).$$

Similarly, for $N = 3$ we get

$$f(I^1 \times I^2 \times I^3) = \Delta f_{\mathbf{a}}^{\mathbf{b}} = \Delta f_{(a_1,a_2,a_3)}^{(b_1,b_2,b_3)} = \sum_{(c_1,c_2,c_3)} k(\mathbf{c}) f(\mathbf{c})$$

$$= f(b_1, b_2, b_3) + f(b_1, a_2, a_3) + f(a_1, b_2, a_3) + f(a_1, a_2, b_3)$$

$$- f(b_1, b_2, a_3) - f(a_1, b_2, b_3) - f(b_1, a_2, b_3) - f(a_1, a_2, a_3).$$

Definition 1.2.3.1. *Given $\bigwedge = (\Lambda^1, \cdots, \Lambda^N)$, where $\Lambda^k = \{\lambda_n^k\}_{n=1}^{\infty} \in \mathbb{L}$, for $k = 1, 2, \cdots, N$, and $p \geq 1$, a measurable function f defined on $R^N := \prod_{k=1}^{N} [a_k, b_k]$ is said to be of $p - \bigwedge -$bounded variation (that is, $f \in \bigwedge BV^{(p)}(R^N)$) if*

$$V_{\bigwedge_p}(f, R^N) = \sup_{J^1, \cdots, J^N} \left\{ \left(\sum_{k_1} \cdots \sum_{k_N} \frac{|f(I_{k_1}^1 \times \cdots \times I_{k_N}^N)|^p}{\lambda_{k_1}^1 \cdots \lambda_{k_N}^N} \right)^{\frac{1}{p}} \right\} < \infty,$$

where $J^1, \cdots, J^{N-1}$ and J^N are finite collections of non-overlapping subintervals $\{I_{k_1}^1\}, \cdots, \{I_{k_{N-1}}^{N-1}\}$ and $\{I_{k_N}^N\}$ in $[a_1, b_1], \cdots, [a_{N-1}, b_{N-1}]$ and $[a_N, b_N]$ respectively.

This class is further generalized to the class $\bigwedge^* BV^{(p)}(R^N)$ in the sense of Hardy as follows.

A function $f \in \bigwedge BV^{(p)}(R^N)$ is said to be of $p - \bigwedge^* -$bounded variation (that is, $f \in \bigwedge^* BV^{(p)}(R^N)$) if for each of its marginal functions

$$f(x_1, \cdots, x_{i-1}, a_i, x_{i+1}, \cdots, x_N) \in (\Lambda^1, \cdots, \Lambda^{i-1}, \Lambda^{i+1}, \cdots, \Lambda^N)^* BV^{(p)}(R^N(a_i)),$$

for all $i = 1, 2, \cdots, N$, where

$R^N(a_i)$

$= \{(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N) \in \mathbb{R}^{N-1} : x_k \in [a_k, b_k] \, for \; k = 1, \cdots, i-1, i+1, \cdots, N\}.$

If $f \in \bigwedge^* BV^{(p)}(R^N)$ then f is bounded on R^N.

Note that, for $\Lambda^1 = \cdots = \Lambda^N = \{1\}$ and $p = 1$, the classes $\bigwedge BV^{(p)}(R^N)$ and $\bigwedge^* BV^{(p)}(R^N)$ reduce to the classes $BV_V(R^N)$ and $BV_H(R^N)$ respectively; for $p = 1$, the classes $\bigwedge BV^{(p)}(R^N)$ and $\bigwedge^* BV^{(p)}(R^N)$ reduce to the classes $\bigwedge BV(R^N)$ and $\bigwedge^* BV(R^N)$ respectively; and for $\Lambda^1 = \cdots = \Lambda^N = \{1\}$, the classes $\bigwedge BV^{(p)}(R^N)$ and $\bigwedge^* BV^{(p)}(R^N)$ reduce to the classes $BV_V^{(p)}(R^N)$ and $BV_H^{(p)}(R^N)$ respectively.

For any $\Lambda^k \in \mathbb{L}$, for $k = 1, 2, \cdots, N$, and $p \geq 1$, we have

$$\left(\sum_{k_1} \cdots \sum_{k_N} \frac{|f(I^1_{k_1} \times \cdots \times I^N_{k_N})|^p}{\lambda^1_{k_1} \cdots \lambda^N_{k_N}}\right)^{\frac{1}{p}}$$

$$\leq \left(\frac{1}{\lambda^1_1 \cdots \lambda^N_1}\right)^{\frac{1}{p}} \left(\sum_{k_1} \cdots \sum_{k_N} |f(I^1_{k_1} \times \cdots \times I^N_{k_N})|^p\right)^{\frac{1}{p}}$$

$$\leq \left(\frac{1}{\lambda^1_1 \cdots \lambda^N_1}\right)^{\frac{1}{p}} \sum_{k_1} \cdots \sum_{k_N} |f(I^1_{k_1} \times \cdots \times I^N_{k_N})|.$$

This implies

$$BV_V(R^N) \subset BV_V^{(p)}(R^N) \subset \bigwedge BV^{(p)}(R^N)$$

and hence

$$BV_H(R^N) \subset BV_H^{(p)}(R^N) \subset \bigwedge^* BV^{(p)}(R^N).$$

Definition 1.2.3.2. *A measurable function f defined on R^N is said to be of $\phi - \bigwedge -$bounded variation (that is, $f \in \phi \bigwedge BV(R^N)$) if*

$$V_{\Lambda_\phi}(f, R^N) = \sup_{J^1, \cdots, J^N} \left\{\sum_{k_1} \cdots \sum_{k_N} \frac{\phi(|f(I^1_{k_1} \times \cdots \times I^N_{k_N})|)}{\lambda^1_{k_1} \cdots \lambda^N_{k_N}}\right\} < \infty,$$

where R^N, $\bigwedge$ and $J^1, \cdots, J^N$ are as defined earlier in the Definition 1.2.3.1, and ϕ is as defined earlier in the Definition 1.2.1.3.

This class is further generalized to the class $\phi \bigwedge^* BV(R^N)$ in the sense of Hardy as follows.

A function $f \in \phi \bigwedge BV(R^N)$ is said to be of $\phi - \bigwedge^* -$bounded variation (that is, $f \in \phi \bigwedge^* BV(R^N)$) if for each of its marginal functions

$$f(x_1, \cdots, x_{i-1}, a_i, x_{i+1}, \cdots, x_N) \in \phi(\Lambda^1, \cdots, \Lambda^{i-1}, \Lambda^{i+1}, \cdots, \Lambda^N)^* BV(R^N(a_i)),$$

for all $i = 1, 2, \cdots, N$.

If $f \in \phi \bigwedge^* BV(R^N)$ then f is bounded on R^N.

Note that, for $\phi(x) = x$ and $\Lambda^1 = \cdots = \Lambda^N = \{1\}$, the classes $\phi \bigwedge BV(R^N)$ and $\phi \bigwedge^* BV(R^N)$ reduce to the classes $BV_V(R^N)$ and $BV_H(R^N)$ respectively; for $\phi(x) = x$, the classes $\phi \bigwedge BV(R^N)$ and $\phi \bigwedge^* BV(R^N)$ reduce to the classes $\bigwedge BV(R^N)$ and $\bigwedge^* BV(R^N)$ respectively; and for $\phi(x) = x^p$, the classes $\phi \bigwedge BV(R^N)$ and $\phi \bigwedge^* BV(R^N)$ reduce to the classes $\bigwedge BV^{(p)}(R^N)$ and $\bigwedge^* BV^{(p)}(R^N)$ respectively.

Definition 1.2.3.3. *Given a positive integer r, a function f defined on R^N is said to be of bounded $r^{th}-$variation (that is, $f \in r - BV(R^N)$) if the following two conditions are satisfied:*

(i)

$$V_r(f, R^N) = \sup_{P = P_1 \times \cdots \times P_N} \left\{ \sum_{k_1=0}^{s_1 - r} \cdots \sum_{k_N=0}^{s_N - r} |\Delta^r f(x_1^{k_1}, \cdots, x_N^{k_N})| \right\} < \infty,$$

where R^N is as defined earlier in the Definition 1.2.3.1,

$$P_i : a_i = x_i^0 < x_i^1 < \cdots < x_i^{s_i} = b_i, \ \textit{for all } i = 1, 2, \cdots, N,$$

$$\Delta f(x_1^{k_1}, \cdots, x_N^{k_N}) = f([x_1^{k_1}, x_1^{k_1+1}] \times \cdots \times [x_N^{k_N}, x_N^{k_N+1}])$$

and

$$\Delta^k f(x_1^{k_1}, \cdots, x_N^{k_N}) = \Delta^{k-1}(\Delta f(x_1^{k_1}, \cdots, x_N^{k_N})), \ k \geq 2,$$

so that

$$\Delta^r f(x_1^{k_1}, \cdots, x_N^{k_N}) = \sum_{u_1=0}^{r} \cdots \sum_{u_N=0}^{r} (-1)^{u_1 + \cdots + u_N} \binom{r}{u_1} \cdots \binom{r}{u_N} f(x_1^{k_1+r-u_1}, \cdots, x_N^{k_N+r-u_N}).$$

(ii) *Each of its marginal functions*

$$f(x_1, \cdots, x_{i-1}, a_i, x_{i+1}, \cdots, x_N) \in r - BV(R^N(a_i)), \ for\ all\ i = 1, 2, \cdots, N.$$

Obviously, $BV_H(R^N) \subset r - BV(R^N) \subset B(R^N)$, where $B(R^N)$ is a class of all bounded functions on R^N.

Definition 1.2.3.4. *Given* $\mathbf{x} = (x_1, \cdots, x_N) \in \overline{\mathbb{T}}^N$ *and* $f \in L^p(\overline{\mathbb{T}}^N)$, *where* $p \geq 1$, *the* $p-$*integral modulus of continuity of* f *of higher differences of order* $r \geq 1$ *is defined as*

$$\omega_r^{(p)}(f; \delta_1, \cdots, \delta_N)$$

$$= \sup \Bigg\{ \left(\frac{1}{(2\pi)^N} \int \cdots \int_{\overline{\mathbb{T}}^N} |\Delta^r f(x_1, \cdots, x_N; h_1, \cdots, h_N)|^p \ d\mathbf{x} \right)^{\frac{1}{p}}$$

$$: 0 < h_i \leq \delta_i \ for\ all\ i = 1, 2, \cdots, N \Bigg\}, \ where$$

$$\Delta^r f(x_1, \cdots, x_N; h_1, \cdots, h_N)$$
$$= \sum_{u_1=0}^{r} \cdots \sum_{u_N=0}^{r} (-1)^{u_1+\cdots+u_N} \binom{r}{u_1} \cdots \binom{r}{u_N} f(x_1+(r-u_1)h_1, \cdots, x_N+(r-u_N)h_N).$$

In the Definition 1.2.3.4, for $r = 1$, we omit writing r, one gets $\omega^{(p)}(f; \delta_1, \cdots, \delta_N)$, the $p-$integral modulus of continuity of f.

For $p \geq 1$ and $\alpha_i \in (0, 1]$, for all $i = 1, 2, \cdots, N$, we say that $f \in Lip(p; \alpha_1, \cdots, \alpha_N)(\overline{\mathbb{T}}^N)$ if

$$\omega^{(p)}(f; \delta_1, \cdots, \delta_N) = O(\delta_1^{\alpha_1} \cdots \delta_N^{\alpha_N}).$$

In the Definition 1.2.3.4, for $p = \infty$ and $r = 1$, we omit writing p and r, one gets $\omega(f; \delta_1, \cdots, \delta_N)$, the modulus of continuity of f, and in that case the class $Lip(p; \alpha_1, \cdots, \alpha_N)(\overline{\mathbb{T}}^N)$ reduces to the Lipschitz class $Lip(\alpha_1, \cdots, \alpha_N)(\overline{\mathbb{T}}^N)$.

Definition 1.2.3.5. *A function* f *defined on* R^N *is said to be absolutely continuous (that is,* $f \in AC(R^N)$*) if the following two conditions are satisfied:*

(i) *Given* $\epsilon > 0$, *there exists* $\delta = \delta(\epsilon) > 0$ *such that*

$$\sum_{R_k^N \in R} |f([a_1^k, b_1^k] \times \cdots \times [a_N^k, b_N^k])| < \epsilon$$

whenever

$$R = \{R_k^N = [a_1^k, b_1^k] \times \cdots \times [a_N^k, b_N^k]\}$$

is a finite collection of pairwise non-overlapping sub-rectangles of R^N *with*

$$\sum_{R_k^N \in R} \prod_{i=1}^{N} (b_i^k - a_i^k) < \delta,$$

where R^N *is as defined earlier in the Definition 1.2.3.1.*

(ii) *Each of its marginal functions*

$$f(x_1, \cdots, x_{i-1}, a_i, x_{i+1}, \cdots, x_N) \in AC(R^N(a_i)), \ for \ all \ i = 1, 2, \cdots, N.$$

For any $\mathbf{x} = (x_1, \cdots, x_N) \in \overline{\mathbb{T}}^N$ and $\mathbf{k} = (k_1, \cdots, k_N) \in \mathbb{Z}^N$, denote their scalar product by $\mathbf{k} \cdot \mathbf{x} = k_1 x_1 + \cdots + k_N x_N$.

For a complex valued function $f \in L^1(\overline{\mathbb{T}}^N)$, where f is $2\pi-$periodic in each variable, its multiple Fourier series is defined as

$$f(\mathbf{x}) \sim \sum_{\mathbf{k} \in \mathbb{Z}^N} \hat{f}(\mathbf{k}) \ e^{i(\mathbf{k} \cdot \mathbf{x})} = \sum_{k_1 \in \mathbb{Z}} \cdots \sum_{k_N \in \mathbb{Z}} \hat{f}(k_1, \cdots, k_N) \ e^{i(k_1 x_1 + \cdots + k_N x_N)},$$

where

$$\begin{aligned} \hat{f}(\mathbf{k}) &= \frac{1}{(2\pi)^N} \int \cdots \int_{\overline{\mathbb{T}}^N} f(\mathbf{x}) \ e^{-i(\mathbf{k} \cdot \mathbf{x})} \ d\mathbf{x} \\ &= \frac{1}{(2\pi)^N} \int \cdots \int_{\overline{\mathbb{T}}^N} f(x_1, \cdots, x_N) \ e^{-i(k_1 x_1 + \cdots + k_N x_N)} \ dx_1 \cdots dx_N \end{aligned}$$

denotes the $\mathbf{k}^{th}$ Fourier coefficient of f.

The multiple Fourier series of a function f is said to be $\beta-$absolute convergence if

$$\sum_{k_1 \in \mathbb{Z}} \cdots \sum_{k_N \in \mathbb{Z}} |\hat{f}(k_1, \cdots, k_N)|^\beta < \infty, \quad 0 < \beta \leq 2.$$

Let us denote

$$A_N(\beta) = \left\{ f \in L^1(\overline{\mathbb{T}}^N) : \sum_{k_1 \in \mathbb{Z}} \cdots \sum_{k_N \in \mathbb{Z}} |\hat{f}(k_1, \cdots, k_N)|^\beta < \infty \right\}.$$

For $\beta = 1$, we omit writing β, one gets the absolute convergence of the multiple Fourier series of f and in that case the class $A_N(\beta)$ is denoted by A_N.

For a real valued function $f \in L^1(\overline{\mathbb{I}}^N)$, where f is $1-$periodic in each variable, its multiple Walsh$-$Fourier series is defined as

$$f(\mathbf{x}) \sim \sum_{\mathbf{k}\in\mathbb{N}_0^N} \hat{f}(\mathbf{k})\, \psi_{k_1}(x_1)\cdots\psi_{k_N}(x_N) = \sum_{k_1\in\mathbb{N}_0} \cdots \sum_{k_N\in\mathbb{N}_0} \hat{f}(k_1, \cdots, k_N)\, \psi_{k_1}(x_1)\cdots\psi_{k_N}(x_N),$$

where

$$\begin{aligned} \hat{f}(\mathbf{k}) &= \int \cdots \int_{\overline{\mathbb{I}}^N} f(\mathbf{x})\ \psi_{k_1}(x_1) \cdots \psi_{k_N}(x_N)\ d\mathbf{x} \\ &= \int \cdots \int_{\overline{\mathbb{I}}^N} f(x_1, \cdots, x_N)\ \psi_{k_1}(x_1) \cdots \psi_{k_N}(x_N)\ dx_1 \cdots dx_N \end{aligned}$$

denotes the $\mathbf{k}^{th}$ Walsh$-$Fourier coefficient of f.

1.3 Recent development and layout of the thesis

This section aims at providing introduction to the subject area of the thesis through laying down the recent development regarding concerned aspects of these problems. It is well known that one variable as well as several variables functions of generalized bounded variations share many properties of functions of bounded variation. Several mathematicians have studied basic properties of classes of both one variable functions of generalized bounded variations and two variables functions of bounded variation. Some of them are as followed.

In 1976, D. Waterman [71, p.41] proved that the class $\Lambda BV(I)$ is a Banach space with respect to the pointwise operations and the $\Lambda-$variation norm

$$\|f\|_{\Lambda} = \|f\|_{\infty} + V_{\Lambda}(f, I), \quad f \in \Lambda BV(I). \tag{1.1}$$

This result was extended for the class $\Lambda BV^{(p)}(I)$ [61, Theorem 1, p.92] as follows.

Theorem A: The class $\Lambda BV^{(p)}(I)$ is a Banach space with respect to the pointwise operations and the Λ_p-variation norm

$$\|f\|_{\Lambda_p} = \|f\|_{\infty} + V_{\Lambda_p}(f, I), \quad f \in \Lambda BV^{(p)}(I). \tag{1.2}$$

In 2010, R. Kantrowitz [31, Theorem 1, p.171] observed that

$$\|fg\|_{\Lambda_p} \leq \|f\|_{\Lambda_p} \|g\|_{\Lambda_p}, \quad where\ f, g \in \Lambda BV^{(p)}(I). \tag{1.3}$$

Thus, the class $\Lambda BV^{(p)}(I)$ is closed under pointwise multiplication.

This inequality (1.3) together with the Theorem A give the following result.

Theorem B: The class $\Lambda BV^{(p)}(I)$ is a Banach algebra with respect to the pointwise operations and the Λ_p-variation norm, as defined in (1.2).

Motivated by the problems in the spectral theory of linear operators, in 2005 B. Ashton and I. Doust [5] introduced the class $BV(\sigma)$ as follows.

Definition 1.3.1. *Given a non-empty compact subset σ of $\mathbb{R}$, a function f defined on σ is said to be of bounded variation (that is, $f \in BV(\sigma)$) if*

$$V(f, \sigma) = \sup_{P \in \Pi(\sigma)} \left\{ \sum_i |\Delta f(x_i)| \right\} < \infty,$$

where $\Pi(\sigma) = \{P\ :\ P = \{x_i\}_{i=1}^m$ is an increasing finite sequence in $\sigma\}$ and $\Delta f(x_i) = f(x_{i+1}) - f(x_i)$.

Note that, for $\sigma = [a, b]$ one gets the class $BV([a, b])$.

B. Ashton and I. Doust [5, Theorem 2.7] proved the following.

Theorem C: The class $BV(\sigma)$ is a Banach algebra with respect to the pointwise operations and the variation norm

$$\|f\|_\sigma = \|f\|_\infty + V(f, \sigma), \quad f \in BV(\sigma). \tag{1.4}$$

For functions of two variables, in 1984 E. Berkson and T. Gillespie [9, Theorem 3, p.310] observed that the class $BV_H(R^2)$ is a commutative unital Banach algebra with respect to the pointwise operations and the variation norm

$$\|f\| = \|f\|_\infty + V(f, R^2) + V(f(., c), I) + V(f(a, .), J), \quad f \in BV_H(R^2). \tag{1.5}$$

In the Chapter 2, generalizing the classes $BV(\sigma)$ and $\Lambda BV^{(p)}(I)$ to the class $\Lambda BV^{(p)}(\sigma, \mathbb{B})$, of one variable functions of $p-\Lambda-$bounded variation from σ into a commutative unital Banach algebra $\mathbb{B}$, and generalizing the class $\bigwedge^* BV^{(p)}(R^N)$

to the class $\bigwedge^* BV^{(p)}(\Pi_{i=1}^N \sigma_i, \mathbb{B})$, of $N-$variables functions of $p - \bigwedge^* -$bounded variation from $\Pi_{i=1}^N \sigma_i$ into $\mathbb{B}$, where σ_i are non-empty compact subsets of $\mathbb{R}$, for all $i = 1, 2, \cdots, N$, some of their basic properties will be studied.

The classical Dirichlet-Jordan test [8, V.I, p.114] asserts that the Fourier series of a $2\pi-$periodic function $f \in BV(\overline{\mathbb{T}})$ converges at each point. It was observed that the Dirichlet-Jordan test may be generalized by weakening the requirement that f is of bounded variation [10]. Earlier in 1996, considering a function f of class $BVM(\overline{\mathbb{T}})$, F. Móricz and A. H. Siddiqi [38] showed that the n^{th} partial sum, S_n, of the Fourier series of f converges to f in $L^1(\overline{\mathbb{T}})-$norm. In 2000, P. B. Pierce and D. Waterman [45, p.2593] observed that the class $BVM(\overline{\mathbb{T}})$ is a Banach space with respect to the pointwise operations and the variation norm in mean

$$\|f\| = \|f\|_{L^1(\overline{\mathbb{T}})} + V^m(f, \overline{\mathbb{T}}), \quad f \in BVM(\overline{\mathbb{T}}). \tag{1.6}$$

R. E. Castillo [11] extended the class $BVM(\overline{\mathbb{T}})$ to the class $BV^{(p)}M(\overline{\mathbb{T}})$ in 2005. Further in 2011, Castillo [12] proved the following results.

Theorem D: The class $BV^{(p)}M(\overline{\mathbb{T}})$ is a Banach space with respect to the pointwise operations and the variation norm in mean

$$\|f\| = \|f\|_{L^p(\overline{\mathbb{T}})} + (V_p^m(f, \overline{\mathbb{T}}))^{\frac{1}{p}}, \quad f \in BV^{(p)}M(\overline{\mathbb{T}}).$$

Theorem E: Let $f \in BV^{(p)}M(\overline{\mathbb{T}})$ be such that its derivative f' is continuous on $\overline{\mathbb{T}}$. Then $f' \in L^p(\overline{\mathbb{T}})$ and

$$V_p^m(f, \overline{\mathbb{T}}) = 2\pi \|f'\|_{L^p(\overline{\mathbb{T}})}^p.$$

Above Theorem E is an analogous Riesz type result [47] for the class $BV^{(p)}M(\overline{\mathbb{T}})$.

Further in Chapter 3, these two results (Theorem D and Theorem E) will be extended for two variables functions of $p-$bounded variation in the mean.

The Fourier analysis of one variable as well as several variables functions of generalized bounded variations is one of the interesting topics explored in researches currently. Here in this chapter, we briefly summarize the known Fourier coefficients properties of one variable as well as several variables functions of bounded variation and of generalized bounded variations.

It is very well known that a rate at which the Fourier coefficients of a function $f \in L^1(\overline{\mathbb{T}})$ tend to zero depends on smoothness of a function. One may recall the well known classical result (1.7) below [8, Vol.I, p.72] concerning the order of magnitude of Fourier coefficients.

$$If\ f \in BV(\overline{\mathbb{T}})\ then\ \hat{f}(m) = O\left(\frac{1}{|m|}\right). \tag{1.7}$$

This result was extended for a function of the class $BV^{(p)}(\overline{\mathbb{T}})$ by R. N. Siddiqi [55] in 1972 in the following way.

$$If\ f \in BV^{(p)}(\overline{\mathbb{T}})\ then\ \hat{f}(m) = O\left(\frac{1}{|m|^{\frac{1}{p}}}\right). \tag{1.8}$$

Generalizing this result, M. Schramm and D. Waterman [53] proved in 1982 the following:

Theorem F: If $f \in \Lambda BV^{(p)}(\overline{\mathbb{T}})$ then

$$\hat{f}(m) = O\left(\frac{1}{\left(\sum_{j=1}^{|m|}\frac{1}{\lambda_j}\right)^{\frac{1}{p}}}\right). \tag{1.9}$$

Theorem G: If $f \in \phi\Lambda BV(\overline{\mathbb{T}})$ then

$$\hat{f}(m) = O\left(\phi^{-1}\left(\frac{1}{\sum_{j=1}^{|m|}\frac{1}{\lambda_j}}\right)\right). \tag{1.10}$$

In 2004, V. Fülöp and F. Móricz [21, Theorem, p.99] generalized the above classical result (1.7) for a function of $N-$variables as follows.

Theorem H: If $f \in BV_V(\overline{\mathbb{T}}^N) \cap L^1(\overline{\mathbb{T}}^N)$ and $\mathbf{k} = (k_1, \cdots, k_N) \in \mathbb{Z}^N$ is such that $k_1 \cdots k_N \neq 0$, then

$$\hat{f}(\mathbf{k}) = O\left(\frac{1}{|\prod_{j=1}^{N} k_j|}\right). \tag{1.11}$$

Corollary H: If $f \in BV_H(\overline{\mathbb{T}}^N)$ and $\mathbf{k} = (k_1, \cdots, k_N) \in \mathbb{Z}^N$ is such that $k_j \neq 0$ for $(1 \leq)j_1 < \cdots < j_M(\leq N)$ and $k_j = 0$ for $(1 \leq)l_1 < \cdots < l_{N-M}(\leq N)$, where $\{l_1, \cdots, l_{N-M}\}$ is the complementary set of $\{j_1, \cdots, j_M\}$ with respect to $\{1, \cdots, N\}$, then

$$\hat{f}(\mathbf{k}) = O\left(\frac{1}{|\prod_{\substack{j=1\\k_j\neq 0}}^{N} k_j|}\right). \tag{1.12}$$

The behavior of the multiple Fourier series of functions of generalized bounded variations regarding the $\beta-$absolute convergence is also typical. Some of the results on absolute convergence and $\beta-$absolute convergence of Fourier series of a function of one variable are as followed.

The well known result of Bernstein [8, Vol.II, p.154] concerning the absolute convergence of the Fourier series of f is as follows.

Theorem I: If

$$\sum_{m=1}^{\infty} \frac{\omega\left(f; \frac{1}{m}\right)}{m^{\frac{1}{2}}} < \infty, \tag{1.13}$$

then $f \in A_1$.

The following corollary follows from Theorem I.

Corollary I: If $f \in Lip(\alpha)(\overline{\mathbb{T}})$, $\alpha > \frac{1}{2}$, then $f \in A_1$.

Zygmund [8, Vol.II, p.160] shows that if $f \in BV(\overline{\mathbb{T}})$ then the condition posed on its modulus of continuity for $f \in A_1$ can be significantly weakened. This theorem of Zygmund is as follows.

Theorem J: If $f \in BV(\overline{\mathbb{T}})$ and

$$\sum_{m=1}^{\infty} \frac{\left(\omega\left(f; \frac{1}{m}\right)\right)^{\frac{1}{2}}}{m} < \infty, \tag{1.14}$$

then $f \in A_1$.

The following corollary follows from Theorem J.

Corollary J: If $f \in BV(\overline{\mathbb{T}}) \cap Lip(\alpha)(\overline{\mathbb{T}})$, $\alpha > 0$, then $f \in A_1$.

From M_n-Weierstrass test, it is easy to observe that the class A_1 is a subspace of the space $\mathbf{C}(\overline{\mathbb{T}})$. A_1 is a proper subspace of $\mathbf{C}(\overline{\mathbb{T}})$ and it is seen from the following example [19, V.I, p.173].

Let f be the sum of the sine series $\sum_{n=2}^{\infty} \frac{sinnx}{nlogn}$. Then it is observed that $f \in AC(\overline{\mathbb{T}})$, but $f \notin A_1$.

Thus, $f \in BV(\overline{\mathbb{T}})$ and $f \notin A_1$. In view of the Corollary J, $f \notin Lip(\alpha)(\overline{\mathbb{T}})$ for any $\alpha > 0$. Thus, f is an example of absolutely continuous function not satisfying Lipschitz condition of order $\alpha > 0$.

The above theorems of Bernstein (Theorem I) and Zygmund (Theorem J) follow from the following theorem of Szász [8, Vol.II, p.155].

Theorem K: If $f \in L^2(\overline{\mathbb{T}})$ and

$$\sum_{m=1}^{\infty} \frac{\omega^{(2)}\left(f; \frac{1}{m}\right)}{m^{\frac{1}{2}}} < \infty, \tag{1.15}$$

then $f \in A_1$.

This theorem was generalized as follows [43, Theorem 2, for $n_k = k$, for all k, p.116].

Theorem L: If $f \in L^2(\overline{\mathbb{T}})$ and

$$\sum_{m=1}^{\infty} \left(\frac{\omega_r^{(2)}\left(f; \frac{1}{m}\right)}{m^{\frac{1}{2}}}\right)^{\beta} < \infty, \quad 0 < \beta \leq 2, \tag{1.16}$$

then $f \in A_1(\beta)$.

Corollary L: (i) If $f \in Lip(\alpha)(\overline{\mathbb{T}})$, $0 < \alpha \leq 1$, then, for $\beta > \frac{2}{2\alpha+1}$, $f \in A_1(\beta)$ [76, V.I, Theorem 3.10, p.243].

(ii) If $f \in BV(\overline{\mathbb{T}}) \cap Lip(\alpha)(\overline{\mathbb{T}})$, $0 < \alpha \leq 1$, then, for $\beta > \frac{2}{2+\alpha}$, $f \in A_1(\beta)$ [76, V.I, Theorem 3.13, p.243].

In 1982, M. Schramm and D. Waterman [52] extended the above Theorem J of Zygmund for a function of the class $\Lambda BV^{(p)}(\overline{\mathbb{T}})$ as it is shown below.

Theorem M: If $f \in \Lambda BV^{(p)}(\overline{\mathbb{T}})$, $1 \leq p < 2r$, $1 < r < \infty$, and

$$\sum_{m=1}^{\infty} \frac{\left(\omega^{((2-p)s+p)}\left(f; \frac{\pi}{m}\right)\right)^{1-\frac{p}{2r}}}{m^{\frac{1}{2}}\left(\sum_{j=1}^{m} \frac{1}{\lambda_j}\right)^{\frac{1}{2r}}} < \infty, \tag{1.17}$$

where $\frac{1}{r}+\frac{1}{s}=1$, then $f\in A_1$.

The sufficiency condition for the $\beta-$absolute convergence of Fourier series of a function $f\in \Lambda BV(\overline{\mathbb{T}})$ is obtained as it is follows [68, Theorem 1, for $n_k=k$, for all k, p.131].

Theorem N: If $f\in \Lambda BV(\overline{\mathbb{T}})$ and

$$\sum_{m=1}^{\infty}\left(\frac{\omega\left(f;\frac{1}{m}\right)}{m\left(\sum_{j=1}^{m}\frac{1}{\lambda_j}\right)}\right)^{\frac{\beta}{2}}<\infty, \tag{1.18}$$

then $f\in A_1(\beta)$.

This theorem was extended for a function of the class $\Lambda BV^{(p)}(\overline{\mathbb{T}})$ as shown below [59, Theorem 1, for $n_k=k$, for all k, p.770].

Theorem O: If $f\in \Lambda BV^{(p)}(\overline{\mathbb{T}})$, $1\leq p<2r$, $1<r<\infty$, and

$$\sum_{m=1}^{\infty}\left(\frac{\left(\omega^{((2-p)s+p)}\left(f;\frac{1}{m}\right)\right)^{2-\frac{p}{r}}}{m\left(\sum_{j=1}^{m}\frac{1}{\lambda_j}\right)^{\frac{1}{r}}}\right)^{\frac{\beta}{2}}<\infty, \tag{1.19}$$

where $\frac{1}{r}+\frac{1}{s}=1$, then $f\in A_1(\beta)$.

The sufficiency condition for the $\beta-$absolute convergence of Fourier series of a function $f\in r-BV(\overline{\mathbb{T}})$ is obtained as shown below [68, Theorem 3, for $n_k=k$, for all k, p.132].

Theorem P: If $f\in r-BV(\overline{\mathbb{T}})$ and

$$\sum_{m=1}^{\infty}\left(\frac{\left(\omega\left(f;\frac{1}{m}\right)\right)^{\frac{1}{2}}}{m}\right)^{\beta}<\infty, \tag{1.20}$$

then $f\in A_1(\beta)$.

Thus, there are interesting relationships established between the modulus of continuity or the integral modulus of continuity and the $\beta-$absolute convergence of Fourier series of one variable functions of generalized bounded variations. Some analogue results for the $\beta-$absolute convergence of multiple Fourier series are as indicated below.

The $\beta-$absolute convergence of multiple Fourier series was studied for the first time by Minakshisundaram and Szász [35]. They proved the following.

Theorem Q: If a function f satisfies the condition

$$|f(x_1, \cdots, x_N) - f(y_1, \cdots, y_N)| \leq M \ |(x_1, \cdots, x_N) - (y_1, \cdots, y_N)|^{\alpha}, \tag{1.21}$$

for all $(x_1, \cdots, x_N),\ (y_1, \cdots, y_N) \in \overline{\mathbb{T}}^N$, where $0 < \alpha \leq 1$ and $|(x_1, \cdots, x_N) - (y_1, \cdots, y_N)|$ is the usual distance between the two given points, then, for $\beta > \frac{2N}{2\alpha+N}$, $f \in A_N(\beta)$. Moreover, the result is not true for $\beta = \frac{2N}{2\alpha+N}$.

Extending the above theorems of Szász (Theorem K) and Zygmund (Theorem J), F. Móricz and A. Veres [39] in 2008 obtained sufficient conditions for the $\beta-$absolute convergence of multiple Fourier series of functions of the classes $L^2(\overline{\mathbb{T}}^N)$ and $BV_V^{(p)}(\overline{\mathbb{T}}^N)$ as they are shown below:

Theorem R: If $f \in L^2(\overline{\mathbb{T}}^N)$ and

$$\sum_{k_1=1}^{\infty} \cdots \sum_{k_N=1}^{\infty} \left(\frac{\omega^{(2)}\left(f; \frac{\pi}{k_1}, \cdots, \frac{\pi}{k_N}\right)}{(k_1 \cdots k_N)^{\frac{1}{2}}} \right)^{\beta} < \infty, \tag{1.22}$$

then

$$\sum_{|k_1|=1}^{\infty} \cdots \sum_{|k_N|=1}^{\infty} |\hat{f}(k_1, \cdots, k_N)|^{\beta} < \infty. \tag{1.23}$$

Theorem S: If $f \in BV_V^{(p)}(\overline{\mathbb{T}}^N) \cap \mathbf{C}(\overline{\mathbb{T}}^N)$, $0 < p < 2$, and

$$\sum_{k_1=1}^{\infty} \cdots \sum_{k_N=1}^{\infty} \left(\frac{\left(\omega\left(f; \frac{\pi}{k_1}, \cdots, \frac{\pi}{k_N}\right)\right)^{\frac{(2-p)}{2}}}{k_1 \cdots k_N} \right)^{\beta} < \infty, \tag{1.24}$$

then (1.23) holds true.

The order of magnitude of multiple Fourier coefficients of $N-$variables functions of generalized bounded variations will be estimated in the first section of Chapter 4. In the second section of Chapter 4, sufficiency conditions will be obtained, in terms of modulus of continuity or integral modulus of continuity, for the $\beta-$absolute convergence of multiple Fourier series of $N-$variables functions of generalized bounded variations.

Walsh [69] points out great similarities between the trigonometric system and the Walsh system. He establishes continuity of $f(x)$ at a point x_0 as a sufficient condition for the Walsh$-$Fourier series of $f(x)$ to be $(C, 1)$ summable to $f(x_0)$ at x_0. He observes that, like trigonometric system, the Walsh system is also interesting from theoretical point of view. Therefore, naturally the study of Walsh$-$Fourier coefficients properties of functions of these generalized bounded variations also makes a equally interesting problem for researchers. Walsh, Fine, Móricz, Goginava [20, 24, 36, 69] and many others have obtained several interesting results in this direction. In 1949, N. J. Fine [20, Theorem VI, p.383] estimated the order of magnitude of Walsh$-$Fourier coefficients as follows.

$$If\ f \in BV(\bar{\mathbb{I}})\ then\ \hat{f}(m) = O\left(\frac{1}{m}\right). \tag{1.25}$$

This result was generalized as follow [22].

Theorem T: If $f \in \Lambda BV^{(p)}(\bar{\mathbb{I}})$ then

$$\hat{f}(m) = O\left(\frac{1}{\left(\sum_{j=1}^{m} \frac{1}{\lambda_j}\right)^{\frac{1}{p}}}\right). \tag{1.26}$$

Theorem U: If $f \in \phi\Lambda BV(\bar{\mathbb{I}})$ then

$$\hat{f}(m) = O\left(\phi^{-1}\left(\frac{1}{\sum_{j=1}^{m} \frac{1}{\lambda_j}}\right)\right). \tag{1.27}$$

Later on, in Chapter 5, we are going to estimate the order of magnitude of Walsh$-$Fourier coefficients of one variable as well as $N-$variables functions of generalized bounded variations.