Chapter 4

Fourier coefficients properties of functions of generalized bounded variations

4.1 Order of magnitude of multiple Fourier coefficients of functions of generalized bounded variations

Riemann-Lebesgue lemma [8, Vol.I, p.67] says that for any function $f \in L^1(\overline{\mathbb{T}})$, where $\mathbb{T} = [0, 2\pi)$, its Fourier coefficients $\hat{f}(m) \to 0$ as $|m| \to \infty$. Often, this itself is an insufficient information for several purposes and it becomes necessary to estimate the rate at which $\hat{f}(m) \to 0$ as $|m| \to \infty$. But ingeneral, this rate cannot be determined. In fact, even for the subspace $\mathbf{C}(\overline{\mathbb{T}})$ of $L^1(\overline{\mathbb{T}})$, $\hat{f}(m) \to 0$ as slow as possible [8, Vol.I, p.229]. In this context, the subspace $BV(\overline{\mathbb{T}})$ of $L^1(\overline{\mathbb{T}})$ distinguishes itself from other subspaces of $L^1(\overline{\mathbb{T}})$. It is observed that if $f \in BV(\overline{\mathbb{T}})$ then its Fourier coefficients $\hat{f}(m) = O\left(\frac{1}{|m|}\right)$ as $|m| \to \infty$ [8, Vol.I, p.72]. Many mathematicians have generalized this result for functions of generalized bounded variations. In fact, for the class $BV^{(p)}(\overline{\mathbb{T}})$, Siddiqi [55] in 1972 proved the following:

$$f \in BV^{(p)}(\overline{\mathbb{T}}) \textit{ implies } \hat{f}(m) = O\left(\frac{1}{|m|^{\frac{1}{p}}}\right).$$

For the classes $\Lambda BV^{(p)}(\overline{\mathbb{T}})$ and $\phi\Lambda BV(\overline{\mathbb{T}})$, in 1982 M. Schramm and D. Waterman [53] proved the following:

$$f \in \Lambda BV^{(p)}(\overline{\mathbb{T}}) \textit{ implies } \hat{f}(m) = O\left(\frac{1}{\left(\sum_{j=1}^{|m|}\frac{1}{\lambda_j}\right)^{\frac{1}{p}}}\right)$$

and

$$f \in \phi\Lambda BV(\overline{\mathbb{T}}) \textit{ implies } \hat{f}(m) = O\left(\phi^{-1}\left(\frac{1}{\sum_{j=1}^{|m|}\frac{1}{\lambda_j}}\right)\right).$$

Also, for a function of $N-$variables, the Riemann-Lebesgue lemma holds: For any function $f \in L^1(\overline{\mathbb{T}}^N)$, its multiple Fourier coefficients $\hat{f}(\mathbf{k}) \to 0$ as $|\mathbf{k}| = |(k_1, \cdots, k_N)| = \sqrt{|k_1|^2 + \cdots + |k_N|^2} \to \infty$. Often, this itself is an insufficient information for several purposes and it becomes necessary to estimate the rate at which $\hat{f}(\mathbf{k}) \to 0$ as $|\mathbf{k}| \to \infty$. In 2004, V. Fülöp and F. Móricz [21] estimated the order of magnitude of multiple Fourier coefficients of $N-$variables functions of bounded variation in the sense of Vitali and Hardy (Theorem H and Corollary H, p.29). Here, we have generalized these results by estimating the order of magnitude of multiple Fourier coefficients of $N-$variables measurable functions of generalized bounded variations in the sense of Vitali and Hardy.

4.1.1 New results for functions of two variables

First we estimate the order of magnitude of double Fourier series coefficients of two variables measurable functions of generalized bounded variations in the sense of Vitali and Hardy as follow.

Theorem 4.1.1.1. *If $f \in \bigwedge BV^{(p)}(\overline{\mathbb{T}}^2) \cap L^p(\overline{\mathbb{T}}^2)$ $(p \geq 1)$ and $\boldsymbol{k} = (m, n) \in \mathbb{Z}^2$ is such that $mn \neq 0$, then*

$$\hat{f}(\boldsymbol{k}) = O\left(\frac{1}{\left(\sum_{j=1}^{|m|}\sum_{k=1}^{|n|}\frac{1}{\lambda_j^1\ \lambda_k^2}\right)^{\frac{1}{p}}}\right). \tag{4.1}$$

Proof of Theorem 4.1.1.1. For any $m, n \in \mathbb{Z} - \{0\}$,

$$\begin{aligned}
\hat{f}(m,n) &= \frac{1}{4\pi^2}\int\int_{\mathbb{T}^2} f(x,y)\, e^{-imx}\, e^{-iny}\, dx\, dy \\
&= \frac{1}{4\pi^2}\int\int_{\mathbb{T}^2} f\left(x+\frac{\pi}{m}, y+\frac{\pi}{n}\right)\; e^{-im\left(x+\frac{\pi}{m}\right)}\; e^{-in\left(y+\frac{\pi}{n}\right)}\, dx\, dy \\
&= \frac{1}{4\pi^2}\int\int_{\mathbb{T}^2} f\left(x+\frac{\pi}{m}, y+\frac{\pi}{n}\right)\; e^{-imx}\; e^{-i\pi}\; e^{-iny}\; e^{-i\pi}\; dx\; dy \\
&= \frac{1}{4\pi^2}\int\int_{\mathbb{T}^2} f\left(x+\frac{\pi}{m}, y+\frac{\pi}{n}\right)\; e^{-imx}\; e^{-iny}\; dx\; dy.
\end{aligned}$$

Also,

$$\begin{aligned}
\hat{f}(m,n) &= \frac{1}{4\pi^2}\int\int_{\mathbb{T}^2} f\left(x, y+\frac{\pi}{n}\right)\; e^{-imx}\; e^{-in\left(y+\frac{\pi}{n}\right)}\, dx\, dy \\
&= \frac{1}{4\pi^2}\int\int_{\mathbb{T}^2} f\left(x, y+\frac{\pi}{n}\right)\; e^{-imx}\; e^{-iny}\; e^{-i\pi}\; dx\; dy \\
&= -\frac{1}{4\pi^2}\int\int_{\mathbb{T}^2} f\left(x, y+\frac{\pi}{n}\right)\; e^{-imx}\;\; e^{-iny}\; dx\; dy.
\end{aligned}$$

Similarly, we get

$$\hat{f}(m,n) = -\frac{1}{4\pi^2}\int\int_{\mathbb{T}^2} f\left(x+\frac{\pi}{m}, y\right)\; e^{-imx}\; e^{-iny}\; dx\; dy.$$

Thus, we have

$$\begin{aligned}
4|\hat{f}(m,n)| = \frac{1}{4\pi^2}\bigg|\int\int_{\mathbb{T}^2}\bigg(& f\left(x+\frac{\pi}{m}, y+\frac{\pi}{n}\right) - f\left(x, y+\frac{\pi}{n}\right) \\
& - f\left(x+\frac{\pi}{m}, y\right) + f(x,y)\bigg)\; e^{-imx}\; e^{-iny}\; dx\; dy\bigg|. \quad (4.2)
\end{aligned}$$

Put

$$\begin{aligned}
\Delta f_{jk}(x,y) = f\left(x+\frac{j\pi}{m}, y+\frac{k\pi}{n}\right) &- f\left(x+\frac{(j-1)\pi}{m}, y+\frac{k\pi}{n}\right) \\
&- f\left(x+\frac{j\pi}{m}, y+\frac{(k-1)\pi}{n}\right) + f\left(x+\frac{(j-1)\pi}{m}, y+\frac{(k-1)\pi}{n}\right),
\end{aligned}$$

for any $j, k \in \mathbb{Z}$.

Then, because of the periodicity of f in each variable, we get

$$\int\int_{\overline{\mathbb{T}}^2} |\Delta f_{jk}(x,y)|\ dx\ dy$$

$$= \int\int_{\overline{\mathbb{T}}^2} \left| f\left(x+\frac{\pi}{m}, y+\frac{\pi}{n}\right) - f\left(x, y+\frac{\pi}{n}\right) - f\left(x+\frac{\pi}{m}, y\right) + f(x,y) \right| \ dx\ dy.$$

Therefore,

$$|\hat{f}(m,n)| \leq \frac{1}{16\pi^2} \int\int_{\overline{\mathbb{T}}^2} |\Delta f_{jk}(x,y)|\ dx\ dy. \tag{4.3}$$

Dividing both sides of the above inequality by $\lambda_j^1\ \lambda_k^2$ and then summing over $j=1$ to $|m|$ and $k=1$ to $|n|$, we have

$$|\hat{f}(m,n)| \left(\sum_{j=1}^{|m|} \sum_{k=1}^{|n|} \frac{1}{\lambda_j^1\ \lambda_k^2} \right) \leq \frac{1}{16\pi^2} \int\int_{\overline{\mathbb{T}}^2} \left(\sum_{j=1}^{|m|} \sum_{k=1}^{|n|} \frac{|\Delta f_{jk}(x,y)|}{(\lambda_j^1\ \lambda_k^2)^{\frac{1}{p}+\frac{1}{q}}} \right) \ dx\ dy,$$

where q is the index conjugate to p.

Applying Hölder's inequality on the right side of the above inequality, we get

$$|\hat{f}(m,n)| \left(\sum_{j=1}^{|m|} \sum_{k=1}^{|n|} \frac{1}{\lambda_j^1\ \lambda_k^2} \right)$$

$$\leq \frac{1}{16\pi^2} \int\int_{\overline{\mathbb{T}}^2} \left(\sum_{j=1}^{|m|} \sum_{k=1}^{|n|} \frac{|\Delta f_{jk}(x,y)|^p}{\lambda_j^1\ \lambda_k^2} \right)^{\frac{1}{p}} \left(\sum_{j=1}^{|m|} \sum_{k=1}^{|n|} \frac{1}{\lambda_j^1\ \lambda_k^2} \right)^{\frac{1}{q}} \ dx\ dy.$$

Hence,

$$|\hat{f}(m,n)| \left(\sum_{j=1}^{|m|} \sum_{k=1}^{|n|} \frac{1}{\lambda_j^1\ \lambda_k^2} \right)^{\frac{1}{p}} \leq \frac{1}{16\pi^2} \int\int_{\overline{\mathbb{T}}^2} \left(\sum_{j=1}^{|m|} \sum_{k=1}^{|n|} \frac{|\Delta f_{jk}(x,y)|^p}{\lambda_j^1\ \lambda_k^2} \right)^{\frac{1}{p}} \ dx\ dy$$

$$\leq \frac{1}{4} V_{\Lambda_p}(f, \overline{\mathbb{T}}^2).$$

This completes the proof of the theorem.

Remark 4.1.1.2. *The Theorem 4.1.1.1 is an extension of M. Schramm and D. Waterman Theorem F (p.29), for functions of two variables.*

Corollary 4.1.1.3. *If* $f \in \bigwedge^* BV^{(p)}(\overline{\mathbb{T}}^2)$ $(p \geq 1)$ *and* $\boldsymbol{k} = (m, n) \in \mathbb{Z}^2$ *is such that* $mn \neq 0$*, then* (4.1) *holds true.*

We need the following lemma to prove this corollary.

Lemma 4.1.1.4. *If* $f \in \bigwedge^* BV^{(p)}(R^2)$ $(p \geq 1)$ *then* f *is bounded on* R^2*.*

Proof of Lemma 4.1.1.4. For any $(x, y) \in R^2$,

$$
\begin{aligned}
&|f(x,y)| \\
&\leq |f(x,y)-f(a,y)-f(x,c)+f(a,c)|+|f(x,c)-f(a,c)|+|f(a,y)-f(a,c)| \\
&\qquad + |f(a,c)| \\
&= (\lambda_1^1\, \lambda_1^2)^{\frac{1}{p}} \left(\frac{|f(x,y) - f(a,y) - f(x,c) + f(a,c)|^p}{\lambda_1^1\, \lambda_1^2} \right)^{\frac{1}{p}} \\
&\qquad + (\lambda_1^1)^{\frac{1}{p}} \left(\frac{|f(x,c) - f(a,c)|^p}{\lambda_1^1} \right)^{\frac{1}{p}} + (\lambda_1^2)^{\frac{1}{p}} \left(\frac{|f(a,y) - f(a,c)|^p}{\lambda_1^2} \right)^{\frac{1}{p}} + |f(a,c)| \\
&\leq (\lambda_1^1\, \lambda_1^2)^{\frac{1}{p}}\, V_{\bigwedge_p}(f, R^2) + (\lambda_1^1)^{\frac{1}{p}}\, V_{\Lambda_p^1}(f(.,c), [a,b]) + (\lambda_1^2)^{\frac{1}{p}}\, V_{\Lambda_p^2}(f(a,.), [c,d]) \\
&\qquad + |f(a,c)| < \infty, \text{ as } f \in \textstyle\bigwedge^* BV^{(p)}(R^2).
\end{aligned}
$$

Hence, f is bounded on R^2.

Proof of Corollary 4.1.1.3. In view of the Lemma 4.1.1.4, $f \in \bigwedge^* BV^{(p)}(\overline{\mathbb{T}}^2)$ implies f is bounded on $\overline{\mathbb{T}}^2$, and hence $f \in L^p(\overline{\mathbb{T}}^2)$, for all $p \geq 1$. Thus, $\bigwedge^* BV^{(p)}(\overline{\mathbb{T}}^2) \subset \bigwedge BV^{(p)}(\overline{\mathbb{T}}^2) \cap L^p(\overline{\mathbb{T}}^2)$. Therefore, the corollary follows from the Theorem 4.1.1.1.

Corollary 4.1.1.5. *If* $f \in \bigwedge^* BV^{(p)}(\overline{\mathbb{T}}^2)$ $(p \geq 1)$ *and* $\boldsymbol{k} = (m, 0) \in \mathbb{Z}^2$ *is such that* $m \neq 0$*, then*

$$
\hat{f}(\boldsymbol{k}) = O\left(\frac{1}{\left(\sum_{j=1}^{|m|} \frac{1}{\lambda_j^1} \right)^{\frac{1}{p}}} \right).
$$

We need the following lemma to prove this corollary.

Lemma 4.1.1.6. *If* $f \in \bigwedge^* BV^{(p)}(\overline{\mathbb{T}}^2)$ $(p \geq 1)$ *then*

$$\|V_{\Lambda_p^1}(f(.,y),\overline{\mathbb{T}})\|_\infty \leq (\lambda_1^2)^{\frac{1}{p}}\ V_{\bigwedge_p}(f,\overline{\mathbb{T}}^2) + V_{\Lambda_p^1}(f(.,0),\overline{\mathbb{T}}),$$

where

$$\|V_{\Lambda_p^1}(f(.,y),\overline{\mathbb{T}})\|_\infty = \sup_{y\in\overline{\mathbb{T}}} V_{\Lambda_p^1}(f(.,y),\overline{\mathbb{T}}).$$

Proof of Lemma 4.1.1.6. For any $y \in \overline{\mathbb{T}}$ and for any finite collection of non-overlapping subintervals $\{[x_j, x_{j+1}]\}$ in $\overline{\mathbb{T}}$, we have

$$\left(\sum_j \frac{|f(x_{j+1},y)-f(x_j,y)|^p}{\lambda_j^1}\right)^{\frac{1}{p}}$$

$$= \left(\sum_j \frac{|f(x_{j+1},y)-f(x_j,y)-f(x_{j+1},0)+f(x_j,0)+f(x_{j+1},0)-f(x_j,0)|^p}{\lambda_j^1}\right)^{\frac{1}{p}}$$

$$= \left(\sum_j \left(\frac{|f(x_{j+1},y)-f(x_j,y)-f(x_{j+1},0)+f(x_j,0)+f(x_{j+1},0)-f(x_j,0)|}{(\lambda_j^1)^{\frac{1}{p}}}\right)^p\right)^{\frac{1}{p}}$$

$$\leq \left(\sum_j \left(\frac{|f(x_{j+1},y)-f(x_j,y)-f(x_{j+1},0)+f(x_j,0)|+|f(x_{j+1},0)-f(x_j,0)|}{(\lambda_j^1)^{\frac{1}{p}}}\right)^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_j \left(\frac{|f(x_{j+1},y)-f(x_j,y)-f(x_{j+1},0)+f(x_j,0)|}{(\lambda_j^1)^{\frac{1}{p}}} + \frac{|f(x_{j+1},0)-f(x_j,0)|}{(\lambda_j^1)^{\frac{1}{p}}}\right)^p\right)^{\frac{1}{p}}$$

$$\leq \left(\sum_j \frac{|f(x_{j+1},y)-f(x_j,y)-f(x_{j+1},0)+f(x_j,0)|^p}{\lambda_j^1\ \lambda_1^2}\right)^{\frac{1}{p}} (\lambda_1^2)^{\frac{1}{p}}$$

$$+ \left(\sum_j \frac{|f(x_{j+1},0)-f(x_j,0)|^p}{\lambda_j^1}\right)^{\frac{1}{p}},$$

by Minkowski's inequality.

Thus,

$$V_{\Lambda_p^1}(f(.,y),\overline{\mathbb{T}}) \leq (\lambda_1^2)^{\frac{1}{p}}\ V_{\bigwedge_p}(f,\overline{\mathbb{T}}^2) + V_{\Lambda_p^1}(f(.,0),\overline{\mathbb{T}}),\ for\ all\ y \in \overline{\mathbb{T}}.$$

Hence, the lemma follows.

Proof of Corollary 4.1.1.5. For any $m \in \mathbb{Z} - \{0\}$,

$$\begin{aligned}
\hat{f}(m,0) &= \frac{1}{4\pi^2} \int\int_{\mathbb{T}^2} f(x,y)\ e^{-imx}\ dx\ dy \\
&= \frac{1}{4\pi^2} \int\int_{\mathbb{T}^2} f\left(x+\frac{\pi}{m},y\right)\ e^{-im\left(x+\frac{\pi}{m}\right)}\ dx\ dy \\
&= \frac{1}{4\pi^2} \int\int_{\mathbb{T}^2} f\left(x+\frac{\pi}{m},y\right)\ e^{-imx}\ \ e^{-i\pi}\ dx\ dy \\
&= -\frac{1}{4\pi^2} \int\int_{\mathbb{T}^2} f\left(x+\frac{\pi}{m},y\right)\ e^{-imx}\ dx\ dy.
\end{aligned}$$

Thus, we have

$$2|\hat{f}(m,0)| = \frac{1}{4\pi^2} \left| \int\int_{\mathbb{T}^2} \left(f(x,y) - f\left(x+\frac{\pi}{m},y\right)\right)\ e^{-imx}\ dx\ dy \right|. \tag{4.4}$$

Put

$$\Delta f_j(x,y) = f\left(x+\frac{j\pi}{m},y\right) - f\left(x+\frac{(j-1)\pi}{m},y\right),\ \ for\ any\ j \in \mathbb{Z}.$$

Then, because of the periodicity of f in each variable, we get

$$\int\int_{\mathbb{T}^2} |\Delta f_j(x,y)|\ dx\ dy = \int\int_{\mathbb{T}^2} \left| f(x,y) - f\left(x+\frac{\pi}{m},y\right)\right|\ dx\ dy.$$

Therefore,

$$|\hat{f}(m,0)| \leq \frac{1}{8\pi^2} \int\int_{\mathbb{T}^2} |\Delta f_j(x,y)|\ dx\ dy. \tag{4.5}$$

Dividing both sides of the above inequality by λ_j^1 and then summing over $j = 1$ to $|m|$, we have

$$|\hat{f}(m,0)| \left(\sum_{j=1}^{|m|} \frac{1}{\lambda_j^1}\right) \leq \frac{1}{8\pi^2} \int\int_{\mathbb{T}^2} \left(\sum_{j=1}^{|m|} \frac{|\Delta f_j(x,y)|}{(\lambda_j^1)^{\frac{1}{p}+\frac{1}{q}}}\right)\ dx\ dy,$$

where q is the index conjugate to p.

Applying Hölder's inequality on the right side of the above inequality, we get

$$|\hat{f}(m,0)| \left(\sum_{j=1}^{|m|} \frac{1}{\lambda_j^1}\right) \leq \frac{1}{8\pi^2} \int\int_{\mathbb{T}^2} \left(\sum_{j=1}^{|m|} \frac{|\Delta f_j(x,y)|^p}{\lambda_j^1}\right)^{\frac{1}{p}} \left(\sum_{j=1}^{|m|} \frac{1}{\lambda_j^1}\right)^{\frac{1}{q}}\ dx\ dy.$$

Thus,

$$|\hat{f}(m,0)|\left(\sum_{j=1}^{|m|}\frac{1}{\lambda_j^1}\right)^{\frac{1}{p}} \le \frac{1}{8\pi^2}\int\int_{\overline{\mathbb{T}}^2}\left(\sum_{j=1}^{|m|}\frac{|\Delta f_j(x,y)|^p}{\lambda_j^1}\right)^{\frac{1}{p}} dx\ dy$$
$$\le \frac{1}{2}V_{\Lambda_p^1}(f(.,y),\overline{\mathbb{T}})$$
$$\le \frac{1}{2}\|V_{\Lambda_p^1}(f(.,y),\overline{\mathbb{T}})\|_\infty$$
$$\le \frac{1}{2}\left((\lambda_1^2)^{\frac{1}{p}}\ V_{\bigwedge_p}(f,\overline{\mathbb{T}}^2)+V_{\Lambda_p^1}(f(.,0),\overline{\mathbb{T}})\right),$$

in view of the above Lemma 4.1.1.6.

Hence, the corollary follows.

Theorem 4.1.1.7. *If ϕ satisfies Δ_2 condition, $f \in \phi\bigwedge BV(\overline{\mathbb{T}}^2)\cap L^1(\overline{\mathbb{T}}^2)$ and $\boldsymbol{k}=(m,n)\in\mathbb{Z}^2$ is such that $mn\neq 0$, then*

$$\hat{f}(\boldsymbol{k}) = O\left(\phi^{-1}\left(\frac{1}{\sum_{j=1}^{|m|}\sum_{k=1}^{|n|}\frac{1}{\lambda_j^1\ \lambda_k^2}}\right)\right). \tag{4.6}$$

Proof of Theorem 4.1.1.7. Put

$$\Delta f_{jk}(x,y) = f\left(x+\frac{j\pi}{m},y+\frac{k\pi}{n}\right)-f\left(x+\frac{(j-1)\pi}{m},y+\frac{k\pi}{n}\right)$$
$$-f\left(x+\frac{j\pi}{m},y+\frac{(k-1)\pi}{n}\right)+f\left(x+\frac{(j-1)\pi}{m},y+\frac{(k-1)\pi}{n}\right),$$

for any $j,k\in\mathbb{Z}$.

Then, proceeding as in the proof of the Theorem 4.1.1.1, we get (4.3)

$$|\hat{f}(m,n)| \le \frac{1}{16\pi^2}\int\int_{\overline{\mathbb{T}}^2}|\Delta f_{jk}(x,y)|\ dx\ dy$$
$$\le \frac{1}{4\pi^2}\int\int_{\overline{\mathbb{T}}^2}|\Delta f_{jk}(x,y)|\ dx\ dy.$$

For $c>0$, by Jensen's inequality for integrals, we have

$$\phi(c|\hat{f}(m,n)|) \le \frac{1}{4\pi^2}\int\int_{\overline{\mathbb{T}}^2}\phi(c|\Delta f_{jk}(x,y)|)\ dx\ dy.$$

Dividing both sides of the above inequality by $\lambda_j^1\ \lambda_k^2$ and then summing over $j=1$ to $|m|$ and $k=1$ to $|n|$, we get

$$\phi(c|\hat{f}(m,n)|)\left(\sum_{j=1}^{|m|}\sum_{k=1}^{|n|}\frac{1}{\lambda_j^1\ \lambda_k^2}\right) \leq \frac{1}{4\pi^2}\int\int_{\overline{\mathbb{T}}^2}\left(\sum_{j=1}^{|m|}\sum_{k=1}^{|n|}\frac{\phi(c|\Delta f_{jk}(x,y)|)}{\lambda_j^1\ \lambda_k^2}\right)\, dx\, dy$$

$$\leq V_{\bigwedge_\phi}(cf,\overline{\mathbb{T}}^2), \tag{4.7}$$

as ϕ satisfies Δ_2 condition implies $cf \in \phi \bigwedge BV(\overline{\mathbb{T}}^2)$.

Since ϕ is convex and $\phi(0) = 0$, for $c \in (0,1]$ we have $\phi(cx) \leq c\phi(x)$ and hence we can choose sufficiently small $c \in (0,1]$ such that $V_{\bigwedge_\phi}(cf,\overline{\mathbb{T}}^2) \leq 1$. Thus, from (4.7), we have

$$|\hat{f}(m,n)| \leq \frac{1}{c}\phi^{-1}\left(\frac{1}{\sum_{j=1}^{|m|}\sum_{k=1}^{|n|}\frac{1}{\lambda_j^1\ \lambda_k^2}}\right).$$

This completes the proof of the theorem.

Remark 4.1.1.8. *The Theorem 4.1.1.7 is an extension of M. Schramm and D. Waterman Theorem G (p.29), for functions of two variables.*

Corollary 4.1.1.9. *If ϕ satisfies Δ_2 condition, $f \in \phi \bigwedge^* BV(\overline{\mathbb{T}}^2)$ and $\boldsymbol{k} = (m,n) \in \mathbb{Z}^2$ is such that $mn \neq 0$, then (4.6) holds true.*

We need the following lemma to prove this corollary.

Lemma 4.1.1.10. *If $f \in \phi \bigwedge^* BV(R^2)$ then f is bounded on R^2.*

Proof of Lemma 4.1.1.10. For any $(x,y) \in R^2$,

$$\begin{aligned}
&|f(x,y)| \\
&\leq |f(x,y)-f(a,y)-f(x,c)+f(a,c)|+|f(x,c)-f(a,c)|+|f(a,y)-f(a,c)| \\
&\qquad\qquad\qquad\qquad + |f(a,c)| \\
&= (\lambda_1^1\ \lambda_1^2)\left(\frac{|f(x,y)-f(a,y)-f(x,c)+f(a,c)|}{\lambda_1^1\ \lambda_1^2}\right) \\
&\qquad + (\lambda_1^1)\left(\frac{|f(x,c)-f(a,c)|}{\lambda_1^1}\right) + (\lambda_1^2)\left(\frac{|f(a,y)-f(a,c)|}{\lambda_1^2}\right) + |f(a,c)| \\
&\leq (\lambda_1^1\ \lambda_1^2)\,\phi^{-1}(V_{\bigwedge_\phi}(f,R^2))+(\lambda_1^1)\,\phi^{-1}(V_{\Lambda_\phi^1}(f(.,c),[a,b]))+(\lambda_1^2)\,\phi^{-1}(V_{\Lambda_\phi^2}(f(a,.),[c,d])) \\
&\qquad + |f(a,c)| < \infty, \quad \text{as } f \in \phi \textstyle\bigwedge^* BV(R^2).
\end{aligned}$$

Hence, f is bounded on R^2.

Proof of Corollary 4.1.1.9. In view of the Lemma 4.1.1.10, $f \in \phi \bigwedge^* BV(\overline{\mathbb{T}}^2)$ implies f is bounded on $\overline{\mathbb{T}}^2$, and hence $f \in L^1(\overline{\mathbb{T}}^2)$. Thus, $\phi \bigwedge^* BV(\overline{\mathbb{T}}^2) \subset \phi \bigwedge BV(\overline{\mathbb{T}}^2) \cap L^1(\overline{\mathbb{T}}^2)$. Therefore, the corollary follows from the Theorem 4.1.1.7.

Corollary 4.1.1.11. *If ϕ satisfies Δ_2 condition, $f \in \phi \bigwedge^* BV(\overline{\mathbb{T}}^2)$ and $\boldsymbol{k} = (m, 0) \in \mathbb{Z}^2$ is such that $m \neq 0$, then*

$$\hat{f}(\boldsymbol{k}) = O\left(\phi^{-1}\left(\frac{1}{\sum_{j=1}^{|m|} \frac{1}{\lambda_j^1}}\right)\right).$$

We need the following lemma to prove this corollary.

Lemma 4.1.1.12. *If ϕ satisfies Δ_2 condition and $f \in \phi \bigwedge^* BV(\overline{\mathbb{T}}^2)$, then*

$$\|V_{\Lambda^1_\phi}(f(.,y), \overline{\mathbb{T}})\|_\infty \leq d\left(\lambda_1^2 \, V_{\bigwedge_\phi}(f, \overline{\mathbb{T}}^2) + V_{\Lambda^1_\phi}(f(.,0), \overline{\mathbb{T}})\right),$$

where

$$\|V_{\Lambda^1_\phi}(f(.,y), \overline{\mathbb{T}})\|_\infty = \sup_{y \in \overline{\mathbb{T}}} V_{\Lambda^1_\phi}(f(.,y), \overline{\mathbb{T}}).$$

Proof of Lemma 4.1.1.12. As ϕ is satisfying Δ_2 condition and is increasing implies

$$\phi(u+v) \leq \phi(2max\{u,v\}) \leq d\phi(max\{u,v\}) \leq d(\phi(u) + \phi(v)), \;\; for \; any \; u, v \geq 0.$$

For any $y \in \overline{\mathbb{T}}$ and for any finite collection of non-overlapping subintervals $\{[x_j, x_{j+1}]\}$ in $\overline{\mathbb{T}}$, we have

$$\sum_j \frac{\phi(|f(x_{j+1}, y) - f(x_j, y)|)}{\lambda_j^1}$$

$$= \sum_j \frac{\phi(|f(x_{j+1}, y) - f(x_j, y) - f(x_{j+1}, 0) + f(x_j, 0) + f(x_{j+1}, 0) - f(x_j, 0)|)}{\lambda_j^1}$$

$$\leq \sum_j \frac{\phi(|f(x_{j+1}, y) - f(x_j, y) - f(x_{j+1}, 0) + f(x_j, 0)| + |f(x_{j+1}, 0) - f(x_j, 0)|)}{\lambda_j^1}$$

$$\leq \sum_j \frac{d(\phi(|f(x_{j+1}, y) - f(x_j, y) - f(x_{j+1}, 0) + f(x_j, 0)|) + \phi(|f(x_{j+1}, 0) - f(x_j, 0)|))}{\lambda_j^1}$$

$$= d\Bigg(\lambda_1^2 \sum_j \frac{\phi(|f(x_{j+1},y) - f(x_j,y) - f(x_{j+1},0) + f(x_j,0)|)}{\lambda_j^1\ \lambda_1^2}$$

$$+ \sum_j \frac{\phi(|f(x_{j+1},0) - f(x_j,0)|)}{\lambda_j^1}\Bigg).$$

Thus,

$$V_{\Lambda_\phi^1}(f(.,y),\overline{\mathbb{T}}) \leq d\left(\lambda_1^2\ V_{\bigwedge_\phi}(f,\overline{\mathbb{T}}^2) + V_{\Lambda_\phi^1}(f(.,0),\overline{\mathbb{T}})\right),\ \textit{for all } y \in \overline{\mathbb{T}}.$$

Hence, the lemma follows.

Proof of Corollary 4.1.1.11. Put

$$\Delta f_j(x,y) = f\left(x + \frac{j\pi}{m}, y\right) - f\left(x + \frac{(j-1)\pi}{m}, y\right),\ \textit{for any } j \in \mathbb{Z}.$$

Then, proceeding as in the proof of the Corollary 4.1.1.5, we get (4.5)

$$\begin{aligned} |\hat{f}(m,0)| &\leq \frac{1}{8\pi^2}\int\int_{\overline{\mathbb{T}}^2} |\Delta f_j(x,y)|\ dx\ dy \\ &\leq \frac{1}{4\pi^2}\int\int_{\overline{\mathbb{T}}^2} |\Delta f_j(x,y)|\ dx\ dy. \end{aligned}$$

For $c > 0$, by Jensen's inequality for integrals, we have

$$\phi(c|\hat{f}(m,0)|) \leq \frac{1}{4\pi^2}\int\int_{\overline{\mathbb{T}}^2} \phi(c|\Delta f_j(x,y)|)\ dx\ dy.$$

Dividing both sides of the above inequality by λ_j^1 and then summing over $j = 1$ to $|m|$, we have

$$\begin{aligned} \phi(c|\hat{f}(m,0)|)\left(\sum_{j=1}^{|m|}\frac{1}{\lambda_j^1}\right) &\leq \frac{1}{4\pi^2}\int\int_{\overline{\mathbb{T}}^2}\left(\sum_{j=1}^{|m|}\frac{\phi(c|\Delta f_j(x,y)|)}{\lambda_j^1}\right)\ dx\ dy \\ &\leq V_{\Lambda_\phi^1}(cf(.,y),\overline{\mathbb{T}}) \\ &\leq \|V_{\Lambda_\phi^1}(cf(.,y),\overline{\mathbb{T}})\|_\infty \\ &\leq d\left(\lambda_1^2\ V_{\bigwedge_\phi}(cf,\overline{\mathbb{T}}^2) + V_{\Lambda_\phi^1}(cf(.,0),\overline{\mathbb{T}})\right), \qquad (4.8) \end{aligned}$$

in view of the above Lemma 4.1.1.12.

Since ϕ is convex and $\phi(0) = 0$, so we can choose sufficiently small $c \in (0, 1]$ such that $V_{\Lambda_\phi}(cf, \overline{\mathbb{T}}^2) \leq \frac{1}{2d\lambda_1^2}$ and $V_{\Lambda_\phi^1}(cf(.,0), \overline{\mathbb{T}}) \leq \frac{1}{2d}$. Hence, from (4.8), we have

$$|\hat{f}(m,0)| \leq \frac{1}{c}\phi^{-1}\left(\frac{1}{\sum_{j=1}^{|m|}\frac{1}{\lambda_j^1}}\right).$$

This completes the proof of the corollary.

Theorem 4.1.1.13. *If $f \in r-BV(\overline{\mathbb{T}}^2)$ $(r \geq 1)$ and $\boldsymbol{k} = (m,n) \in \mathbb{Z}^2$ is such that $mn \neq 0$, then*

$$\hat{f}(\boldsymbol{k}) = O\left(\frac{1}{|mn|}\right).$$

Proof of Theorem 4.1.1.13. Put

$$\Delta f_{jk}(x,y) = f\left(x + \frac{j\pi}{m}, y + \frac{k\pi}{n}\right) - f\left(x + \frac{(j-1)\pi}{m}, y + \frac{k\pi}{n}\right)$$
$$- f\left(x + \frac{j\pi}{m}, y + \frac{(k-1)\pi}{n}\right) + f\left(x + \frac{(j-1)\pi}{m}, y + \frac{(k-1)\pi}{n}\right),$$

for any $j, k \in \mathbb{Z}$.

Then, proceeding as in the proof of the Theorem 4.1.1.1, we get (4.3)

$$|\hat{f}(m,n)| \leq \frac{1}{16\pi^2}\int\int_{\overline{\mathbb{T}}^2} |\Delta f_{jk}(x,y)|\ dx\ dy.$$

Similarly, we get

$$|\hat{f}(m,n)| \leq \left(\frac{1}{4^{r+1}\pi^2}\right)\int\int_{\overline{\mathbb{T}}^2} |\Delta^r f_{jk}(x,y)|\ dx\ dy.$$

Summing both sides of the above inequality over $j = 1$ to $|m| - r$ and $k = 1$ to $|n| - r$, we get

$$(|m|-r)(|n|-r)|\hat{f}(m,n)| \leq \left(\frac{1}{4^{r+1}\pi^2}\right)\int\int_{\overline{\mathbb{T}}^2}\left(\sum_{j=1}^{|m|-r}\sum_{k=1}^{|n|-r}|\Delta^r f_{jk}(x,y)|\right) dx\ dy.$$

This together with

$$\sum_{j=1}^{|m|-r}\sum_{k=1}^{|n|-r}|\Delta^r f_{jk}(x,y)| \leq V_r(f, \overline{\mathbb{T}}^2)\ ,\ |m| \approx |m| - r \text{ and } |n| \approx |n| - r$$

imply that

$$|\hat{f}(m,n)| = O\left(\frac{1}{|mn|}\right).$$

This completes the proof of the theorem.

Theorem 4.1.1.14. *If* $f \in Lip(p;\alpha_1,\alpha_2)(\overline{\mathbb{T}}^2)$ $(p \geq 1,\ \alpha_1,\alpha_2 \in (0,1])$ *and* $\boldsymbol{k} = (m,n) \in \mathbb{Z}^2$ *is such that* $mn \neq 0$*, then*

$$\hat{f}(\boldsymbol{k}) = O\left(\frac{1}{|m|^{\alpha_1}|n|^{\alpha_2}}\right).$$

Proof of Theorem 4.1.1.14. Proceeding as in the proof of the Theorem 4.1.1.1, we get (4.2)

$$4|\hat{f}(m,n)| = \frac{1}{4\pi^2}\left|\int\int_{\overline{\mathbb{T}}^2}\left(f\left(x+\frac{\pi}{m}, y+\frac{\pi}{n}\right) - f\left(x, y+\frac{\pi}{n}\right)\right.\right.$$
$$\left.\left. - f\left(x+\frac{\pi}{m}, y\right) + f(x,y)\right) e^{-imx}\, e^{-iny}\, dx\, dy\right|.$$

Therefore,

$$|\hat{f}(m,n)| \leq \frac{1}{16\pi^2}\int\int_{\overline{\mathbb{T}}^2} |\Delta f_{11}(x,y)|\, dx\, dy,$$

where

$$\Delta f_{11}(x,y) = f\left(x+\frac{\pi}{m}, y+\frac{\pi}{n}\right) - f\left(x, y+\frac{\pi}{n}\right) - f\left(x+\frac{\pi}{m}, y\right) + f(x,y).$$

Applying Hölder's inequality on the right side of the above inequality, we have

$$|\hat{f}(m,n)| = O(1)\ \left(\int\int_{\overline{\mathbb{T}}^2} |\Delta f_{11}(x,y)|^p\, dx\, dy\right)^{\frac{1}{p}}$$
$$= O\left(\frac{1}{|m|^{\alpha_1}|n|^{\alpha_2}}\right),$$

as $f \in Lip(p;\alpha_1,\alpha_2)(\overline{\mathbb{T}}^2)$.

Theorem 4.1.1.15. *If* $f \in AC(\overline{\mathbb{T}}^2)$ *and* $\boldsymbol{k} = (m,n) \in \mathbb{Z}^2$ *is such that* $mn \neq 0$*, then*

$$\hat{f}(\boldsymbol{k}) = o\left(\frac{1}{|mn|}\right).$$

Proof of Theorem 4.1.1.15. The Theorem 4.1.1.15 can be proved in a similar way to the proof of the Theorem 4.1.1.1.

Corollary 4.1.1.16. *If* $f \in AC(\overline{\mathbb{T}}^2)$ *and* $\boldsymbol{k} = (m, 0) \in \mathbb{Z}^2$ *is such that* $m \neq 0$, *then*

$$\hat{f}(\boldsymbol{k}) = o\left(\frac{1}{|m|}\right).$$

Proof of Corollary 4.1.1.16. The Corollary 4.1.1.16 can be proved in a similar way to the proof of the Corollary 4.1.1.5.

4.1.2 New results for functions of N−variables

Now, we extend the results of the Subsection 4.1.1 for functions of $N-$variables in the following way.

Theorem 4.1.2.1. *If* $f \in \bigwedge BV^{(p)}(\overline{\mathbb{T}}^N) \cap L^p(\overline{\mathbb{T}}^N)$ $(p \geq 1)$ *and* $\boldsymbol{k} = (k_1, \cdots, k_N) \in \mathbb{Z}^N$ *is such that* $k_1 \cdots k_N \neq 0$, *then*

$$\hat{f}(\boldsymbol{k}) = O\left(\frac{1}{\left(\sum_{r_1=1}^{|k_1|} \cdots \sum_{r_N=1}^{|k_N|} \frac{1}{\lambda^1_{r_1} \cdots \lambda^N_{r_N}}\right)^{\frac{1}{p}}}\right). \tag{4.9}$$

Remark 4.1.2.2. *The Theorem 4.1.2.1, with* $\Lambda^1 = \cdots = \Lambda^N = \{1\}$ *and* $p = 1$, *reduces to V. Fülöp and F. Móricz Theorem H (p.29) as a particular case.*

Corollary 4.1.2.3. *If* $f \in \bigwedge^* BV^{(p)}(\overline{\mathbb{T}}^N)$ $(p \geq 1)$ *and* $\boldsymbol{k} = (k_1, \cdots, k_N) \in \mathbb{Z}^N$ *is such that* $k_1 \cdots k_N \neq 0$, *then* (4.9) *holds true.*

Corollary 4.1.2.4. *If* $f \in \bigwedge^* BV^{(p)}(\overline{\mathbb{T}}^N)$ $(p \geq 1)$ *and* $\boldsymbol{k} = (k_1, \cdots, k_N) \in \mathbb{Z}^N$ *is such that* $k_j \neq 0$ *for* $(1 \leq) j_1 < \cdots < j_M (\leq N)$ *and* $k_j = 0$ *for* $(1 \leq) l_1 < \cdots < l_{N-M} (\leq N)$, *where* $\{l_1, \cdots, l_{N-M}\}$ *is the complementary set of* $\{j_1, \cdots, j_M\}$ *with respect to* $\{1, \cdots, N\}$, *then*

$$\hat{f}(\boldsymbol{k}) = O\left(\frac{1}{\left(\sum_{r_1=1}^{|k_{j_1}|} \cdots \sum_{r_M=1}^{|k_{j_M}|} \frac{1}{\lambda^{j_1}_{r_1} \cdots \lambda^{j_M}_{r_M}}\right)^{\frac{1}{p}}}\right).$$

Theorem 4.1.2.5. *If ϕ satisfies Δ_2 condition, $f \in \phi \bigwedge BV(\overline{\mathbb{T}}^N) \cap L^1(\overline{\mathbb{T}}^N)$ and $\boldsymbol{k} = (k_1, \cdots, k_N) \in \mathbb{Z}^N$ is such that $k_1 \cdots k_N \neq 0$, then*

$$\hat{f}(\boldsymbol{k}) = O\left(\phi^{-1}\left(\frac{1}{\sum_{r_1=1}^{|k_1|} \cdots \sum_{r_N=1}^{|k_N|} \frac{1}{\lambda^1_{r_1} \cdots \lambda^N_{r_N}}}\right)\right). \tag{4.10}$$

Corollary 4.1.2.6. *If ϕ satisfies Δ_2 condition, $f \in \phi \bigwedge^* BV(\overline{\mathbb{T}}^N)$ and $\boldsymbol{k} = (k_1, \cdots, k_N) \in \mathbb{Z}^N$ is such that $k_1 \cdots k_N \neq 0$, then (4.10) holds true.*

Corollary 4.1.2.7. *If ϕ satisfies Δ_2 condition, $f \in \phi \bigwedge^* BV(\overline{\mathbb{T}}^N)$ and $\boldsymbol{k} = (k_1, \cdots, k_N) \in \mathbb{Z}^N$ is such that $k_j \neq 0$ for $(1 \leq) j_1 < \cdots < j_M (\leq N)$ and $k_j = 0$ for $(1 \leq) l_1 < \cdots < l_{N-M} (\leq N)$, where $\{l_1, \cdots, l_{N-M}\}$ is the complementary set of $\{j_1, \cdots, j_M\}$ with respect to $\{1, \cdots, N\}$, then*

$$\hat{f}(\boldsymbol{k}) = O\left(\phi^{-1}\left(\frac{1}{\sum_{r_1=1}^{|k_{j_1}|} \cdots \sum_{r_M=1}^{|k_{j_M}|} \frac{1}{\lambda^{j_1}_{r_1} \cdots \lambda^{j_M}_{r_M}}}\right)\right).$$

Theorem 4.1.2.8. *If $f \in r - BV(\overline{\mathbb{T}}^N)$ $(r \geq 1)$ and $\boldsymbol{k} = (k_1, \cdots, k_N) \in \mathbb{Z}^N$ is such that $k_1 \cdots k_N \neq 0$, then*

$$\hat{f}(\boldsymbol{k}) = O\left(\frac{1}{|\prod_{j=1}^N k_j|}\right).$$

Theorem 4.1.2.9. *If $f \in Lip(p; \alpha_1, \cdots, \alpha_N)(\overline{\mathbb{T}}^N)$ $(p \geq 1,\ \alpha_1, \cdots, \alpha_N \in (0, 1])$ and $\boldsymbol{k} = (k_1, \cdots, k_N) \in \mathbb{Z}^N$ is such that $k_1 \cdots k_N \neq 0$, then*

$$\hat{f}(\boldsymbol{k}) = O\left(\frac{1}{\prod_{j=1}^N |k_j|^{\alpha_j}}\right).$$

Theorem 4.1.2.10. *If $f \in AC(\overline{\mathbb{T}}^N)$ and $\boldsymbol{k} = (k_1, \cdots, k_N) \in \mathbb{Z}^N$ is such that $k_1 \cdots k_N \neq 0$, then*

$$\hat{f}(\boldsymbol{k}) = o\left(\frac{1}{|\prod_{j=1}^N k_j|}\right).$$

Corollary 4.1.2.11. *If $f \in AC(\overline{\mathbb{T}}^N)$ and $\boldsymbol{k} = (k_1, \cdots, k_N) \in \mathbb{Z}^N$ is such that $k_j \neq 0$ for $(1 \leq) j_1 < \cdots < j_M (\leq N)$ and $k_j = 0$ for $(1 \leq) l_1 < \cdots < l_{N-M} (\leq N)$,* *where* $\{l_1, \cdots, l_{N-M}\}$ *is the complementary set of* $\{j_1, \cdots, j_M\}$ *with respect to* $\{1, \cdots, N\}$*, then*

$$\hat{f}(\boldsymbol{k}) = o\left(\frac{1}{|\prod_{u=1}^{M} k_{j_u}|}\right).$$

All extended results of this subsection can be proved in the same way as the results in the Subsection 4.1.1.

4.2 Absolute convergence of multiple Fourier series of functions of generalized bounded variations

In the rest of this chapter we study the $\beta-$absolute convergence ($0 < \beta \leq 2$) of multiple Fourier series of $N-$variables functions of generalized bounded variations. Generalizing sufficient conditions for the absolute convergence of Fourier series of one variable functions of generalized bounded variations, many mathematicians have obtained sufficient conditions for the $\beta-$absolute convergence of Fourier series of functions of such classes. In fact, the conditions (1.16), (1.18), (1.19) and (1.20) (p.31−32) are sufficient conditions for the $\beta-$absolute convergence of Fourier series of functions of the classes $L^2(\overline{\mathbb{T}})$, $\Lambda BV(\overline{\mathbb{T}})$, $\Lambda BV^{(p)}(\overline{\mathbb{T}})$ and $r - BV(\overline{\mathbb{T}})$ respectively. In 1947, Minakshisundaram and Szász [35] obtained sufficient condition for the $\beta-$absolute convergence of multiple Fourier series of a function f satisfies the condition (1.21) (Theorem Q, p.33). F. Móricz and A. Veres [39] in 2008 obtained sufficient conditions for the $\beta-$absolute convergence of multiple Fourier series of functions of the classes $L^2(\overline{\mathbb{T}}^N)$ and $BV_V^{(p)}(\overline{\mathbb{T}})$ (Theorem R and S, p.33). Here, we have obtained sufficient condition, in terms of quadratic modulus of continuity of higher differences of order $r \geq 1$, for the $\beta-$absolute convergence of multiple Fourier series of a function of the class $L^2(\overline{\mathbb{T}}^N)$. Also, we have obtained sufficient conditions, in terms of integral modulus of continuity or modulus of continuity, for the $\beta-$absolute convergence of multiple Fourier series of $N-$variables functions of generalized bounded variations.

4.2.1 New results for functions of two variables

First we generalize the Theorem R (p.33) for a function of two variables as follows.

Theorem 4.2.1.1. *If* $f \in L^2(\overline{\mathbb{T}}^2)$ *and*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{\omega_r^{(2)}\left(f;\frac{\pi}{m},\frac{\pi}{n}\right)}{(mn)^{\frac{1}{2}}}\right)^{\beta}<\infty,$$

then

$$\sum_{|m|=1}^{\infty}\sum_{|n|=1}^{\infty}|\hat{f}(m,n)|^{\beta}<\infty, \quad 0<\beta\le 2. \tag{4.11}$$

Proof of Theorem 4.2.1.1. Let $h_1, h_2 > 0$ be given. For any $r \geq 1$ and for any $(x,y) \in \overline{\mathbb{T}}^2$, put $g_r(x,y) = \Delta^r f(x,y;h_1,h_2)$, where

$$\Delta^r f(x,y;h_1,h_2)=\sum_{u=0}^{r}\sum_{v=0}^{r}\ (-1)^{u+v}\binom{r}{u}\binom{r}{v}\ f(x+(r-u)h_1,y+(r-v)h_2).$$

Then, for any $m,n \in \mathbb{Z}$, we have

$$\begin{aligned}
\hat{g}_1(m,n) &= \frac{1}{4\pi^2}\int\int_{\overline{\mathbb{T}}^2} g_1(x,y)\ e^{-imx}\ e^{-iny}\ dx\ dy\\
&= \frac{1}{4\pi^2}\int\int_{\overline{\mathbb{T}}^2} \Delta^1 f(x,y;h_1,h_2)\ e^{-imx}\ e^{-iny}\ dx\ dy\\
&= \frac{1}{4\pi^2}\int\int_{\overline{\mathbb{T}}^2}\Big(f(x+h_1,y+h_2)-f(x,y+h_2)-f(x+h_1,y)\\
&\qquad\qquad + f(x,y)\Big)\ e^{-imx}\ e^{-iny}\ dx\ dy\\
&= \frac{1}{4\pi^2}\int\int_{\overline{\mathbb{T}}^2} f(x+h_1,y+h_2)\ e^{-imx}\ e^{-iny}\ dx\ dy\\
&\qquad -\frac{1}{4\pi^2}\int\int_{\overline{\mathbb{T}}^2} f(x,y+h_2)\ e^{-imx}\ e^{-iny}\ dx\ dy\\
&\qquad -\frac{1}{4\pi^2}\int\int_{\overline{\mathbb{T}}^2} f(x+h_1,y)\ e^{-imx}\ e^{-iny}\ dx\ dy\\
&\qquad +\frac{1}{4\pi^2}\int\int_{\overline{\mathbb{T}}^2} f(x,y)\ e^{-imx}\ e^{-iny}\ dx\ dy\\
&= e^{imh_1}\ e^{inh_2}\ \hat{f}(m,n) - e^{inh_2}\ \hat{f}(m,n) - e^{imh_1}\ \hat{f}(m,n) + \hat{f}(m,n)
\end{aligned}$$

$$
\begin{aligned}
&= \hat{f}(m,n)\ (e^{imh_1}\ e^{inh_2} - e^{inh_2} - e^{imh_1} + 1) \\
&= \hat{f}(m,n)\ (e^{imh_1} - 1)\ (e^{inh_2} - 1) \\
&= \hat{f}(m,n) \left(e^{i\frac{mh_1}{2}} e^{i\frac{mh_1}{2}} - e^{-i\frac{mh_1}{2}} e^{i\frac{mh_1}{2}} \right) \left(e^{i\frac{nh_2}{2}} e^{i\frac{nh_2}{2}} - e^{-i\frac{nh_2}{2}} e^{i\frac{nh_2}{2}} \right) \\
&= \hat{f}(m,n)\ e^{i\frac{mh_1}{2}} e^{i\frac{nh_2}{2}} \left(e^{i\frac{mh_1}{2}} - e^{-i\frac{mh_1}{2}} \right) \left(e^{i\frac{nh_2}{2}} - e^{-i\frac{nh_2}{2}} \right) \\
&= -4\hat{f}(m,n)\ e^{i\frac{mh_1}{2}} e^{i\frac{nh_2}{2}} \sin\frac{mh_1}{2}\ \sin\frac{nh_2}{2}
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{g}_2(m,n) &= \frac{1}{4\pi^2} \int\int_{\mathbb{T}^2} g_2(x,y)\ e^{-imx}\ e^{-iny}\ dx\ dy \\
&= \frac{1}{4\pi^2} \int\int_{\mathbb{T}^2} \Delta^2 f(x,y;h_1,h_2)\, e^{-imx}\ e^{-iny}\ dx\ dy \\
&= \frac{1}{4\pi^2} \int\int_{\mathbb{T}^2} \Big(f(x+2h_1, y+2h_2) - 2f(x+2h_1, y+h_2) + f(x+2h_1, y) \\
&\qquad - 2f(x+h_1, y+2h_2) + 4f(x+h_1, y+h_2) - 2f(x+h_1, y) \\
&\qquad + f(x, y+2h_2) - 2f(x, y+h_2) + f(x,y) \Big)\ e^{-imx}\ e^{-iny}\ dx\ dy \\
&= \hat{f}(m,n) \Big(e^{i2mh_1}\ e^{i2nh_2} - 2e^{i2mh_1}\ e^{inh_2} + e^{i2mh_1} - 2e^{imh_1}\ e^{i2nh_2} \\
&\qquad + 4e^{imh_1}\ e^{inh_2} - 2e^{imh_1} + e^{i2nh_2} - 2e^{inh_2} + 1 \Big) \\
&= \hat{f}(m,n)\ (e^{i2mh_1} - 2e^{imh_1} + 1)\ (e^{i2nh_2} - 2e^{inh_2} + 1) \\
&= \hat{f}(m,n)\ (e^{imh_1} - 1)^2\ (e^{inh_2} - 1)^2 \\
&= \hat{f}(m,n) \left(e^{i\frac{mh_1}{2}} e^{i\frac{mh_1}{2}} - e^{-i\frac{mh_1}{2}} e^{i\frac{mh_1}{2}} \right)^2 \left(e^{i\frac{nh_2}{2}} e^{i\frac{nh_2}{2}} - e^{-i\frac{nh_2}{2}} e^{i\frac{nh_2}{2}} \right)^2 \\
&= \hat{f}(m,n)\ e^{i2\frac{mh_1}{2}} e^{i2\frac{nh_2}{2}} \left(e^{i\frac{mh_1}{2}} - e^{-i\frac{mh_1}{2}} \right)^2 \left(e^{i\frac{nh_2}{2}} - e^{-i\frac{nh_2}{2}} \right)^2 \\
&= (-4)^2 \hat{f}(m,n)\ e^{i2\frac{mh_1}{2}} e^{i2\frac{nh_2}{2}} \sin^2\frac{mh_1}{2}\ \sin^2\frac{nh_2}{2}.
\end{aligned}
$$

Repeating this process r times, we see that

$$
\hat{g}_r(m,n) = (-4)^r \hat{f}(m,n)\ e^{ir\frac{mh_1}{2}} e^{ir\frac{nh_2}{2}} \sin^r\frac{mh_1}{2}\ \sin^r\frac{nh_2}{2}.
$$

Since $f \in L^2(\overline{\mathbb{T}}^2)$, it follows that $g_r \in L^2(\overline{\mathbb{T}}^2)$. Thus, Parseval's formula gives

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\left|\hat{f}(m,n)\ \sin^r \frac{mh_1}{2}\ \sin^r \frac{nh_2}{2}\right|^2 = O\left(\|g_r\|_2^2\right). \tag{4.12}$$

Putting $h_1 = \frac{\pi}{2^{t_1}}$ and $h_2 = \frac{\pi}{2^{t_2}}$, where $t_1, t_2 \in \mathbb{N}$, taking into the account that

$$\frac{\pi}{4} \le \frac{|m|\pi}{2^{t_1+1}} < \frac{\pi}{2}, \quad 2^{t_1-1} \le |m| < 2^{t_1}, \tag{4.13}$$

and

$$\frac{\pi}{4} \le \frac{|n|\pi}{2^{t_2+1}} < \frac{\pi}{2}, \quad 2^{t_2-1} \le |n| < 2^{t_2}, \tag{4.14}$$

we get

$$S_{t_1t_2} \equiv \sum_{2^{t_1-1}\le|m|<2^{t_1}}\ \sum_{2^{t_2-1}\le|n|<2^{t_2}} |\hat{f}(m,n)|^2 = O\left(\|g_r\|_2^2\right).$$

Thus,

$$S_{t_1t_2} = O\left(\left(\omega_r^{(2)}\left(f; \frac{\pi}{2^{t_1}}, \frac{\pi}{2^{t_2}}\right)\right)^2\right). \tag{4.15}$$

Now, for $0 < \beta \le 2$, in view of Hölder's inequality, we have

$$\begin{aligned}
&\sum_{2^{t_1-1}\le|m|<2^{t_1}}\ \sum_{2^{t_2-1}\le|n|<2^{t_2}} |\hat{f}(m,n)|^\beta \\
&= O\left((2^{t_1}2^{t_2})^{1-\frac{\beta}{2}} \left(\sum_{2^{t_1-1}\le|m|<2^{t_1}}\ \sum_{2^{t_2-1}\le|n|<2^{t_2}} |\hat{f}(m,n)|^2 \right)^{\frac{\beta}{2}} \right) \\
&= O\left(2^{t_1}2^{t_2} \left(\frac{\omega_r^{(2)}\left(f; \frac{\pi}{2^{t_1}}, \frac{\pi}{2^{t_2}}\right)}{(2^{t_1}2^{t_2})^{\frac{1}{2}}} \right)^\beta \right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{|m|=1}^{\infty}\sum_{|n|=1}^{\infty} |\hat{f}(m,n)|^\beta &= \sum_{t_1=1}^{\infty}\sum_{t_2=1}^{\infty}\ \sum_{2^{t_1-1}\le|m|<2^{t_1}}\ \sum_{2^{t_2-1}\le|n|<2^{t_2}} |\hat{f}(m,n)|^\beta \\
&= O\left(\sum_{t_1=1}^{\infty}\sum_{t_2=1}^{\infty} 2^{t_1}2^{t_2} \left(\frac{\omega_r^{(2)}\left(f; \frac{\pi}{2^{t_1}}, \frac{\pi}{2^{t_2}}\right)}{(2^{t_1}2^{t_2})^{\frac{1}{2}}} \right)^\beta \right).
\end{aligned}$$

Hence, the theorem follows.

Corollary 4.2.1.2. *If* $f \in Lip(\alpha_1, \alpha_2)(\overline{\mathbb{T}}^2)$, $0 < \alpha_1, \alpha_2 \le 1$, *then, for*

$$\beta > max\left\{\frac{2}{2\alpha_1 + 1}, \frac{2}{2\alpha_2 + 1}\right\},$$

(4.11) *holds true.*

Proof of Corollary 4.2.1.2. For any $t_1, t_2 \in \mathbb{N}$, put

$$\Delta f\left(x, y; \frac{\pi}{2^{t_1}}, \frac{\pi}{2^{t_2}}\right)$$

$$= f\left(x + \frac{\pi}{2^{t_1}}, y + \frac{\pi}{2^{t_2}}\right) - f\left(x, y + \frac{\pi}{2^{t_2}}\right) - f\left(x + \frac{\pi}{2^{t_1}}, y\right) + f(x, y).$$

Then, for $r = 1$, $h_1 = \frac{\pi}{2^{t_1}}$ and $h_2 = \frac{\pi}{2^{t_2}}$, proceeding as in the proof of the Theorem 4.2.1.1, from (4.12), we get

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\left|\hat{f}(m,n)\ \sin\frac{m\pi}{2^{t_1+1}}\ \sin\frac{n\pi}{2^{t_2+1}}\right|^2 = O\left(\|\Delta f\|_2^2\right).$$

In view of (4.13) and (4.14), we have

$$S_{t_1 t_2} \equiv \sum_{2^{t_1-1}\le|m|<2^{t_1}}\ \sum_{2^{t_2-1}\le|n|<2^{t_2}} |\hat{f}(m,n)|^2$$

$$= O\left(\int\int_{\mathbb{T}^2}\left|\Delta f\left(x, y; \frac{\pi}{2^{t_1}}, \frac{\pi}{2^{t_2}}\right)\right|^2 dx\ dy\right).$$

Since

$$\left|\Delta f\left(x, y; \frac{\pi}{2^{t_1}}, \frac{\pi}{2^{t_2}}\right)\right| = O\left(\omega\left(f; \frac{\pi}{2^{t_1}}, \frac{\pi}{2^{t_2}}\right)\right),$$

we have

$$S_{t_1 t_2} = O\left(\left(\omega\left(f; \frac{\pi}{2^{t_1}}, \frac{\pi}{2^{t_2}}\right)\right)^2\right)$$

$$= O\left(\left(\frac{1}{2^{t_1}}\right)^{2\alpha_1}\left(\frac{1}{2^{t_2}}\right)^{2\alpha_2}\right),$$

as $f \in Lip(\alpha_1, \alpha_2)(\overline{\mathbb{T}}^2)$.

Now, applying Hölder's inequality and proceeding as in the proof of the Theorem 4.2.1.1 (from (4.15) onward), we obtain the corollary.

Similarly, we obtain following results for the functions of generalized bounded variations like $\bigwedge BV(\overline{\mathbb{T}}^2)$, $\bigwedge BV^{(p)}(\overline{\mathbb{T}}^2)$ and $r - BV(\overline{\mathbb{T}}^2)$.

Theorem 4.2.1.3. *If* $f \in \bigwedge BV(\overline{\mathbb{T}}^2) \cap \mathbf{C}(\overline{\mathbb{T}}^2)$ *and*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{\omega\left(f;\frac{\pi}{m},\frac{\pi}{n}\right)}{mn\left(\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{1}{\lambda_j^1\ \lambda_k^2}\right)}\right)^{\frac{\beta}{2}} < \infty, \tag{4.16}$$

then (4.11) *holds true.*

Proof of Theorem 4.2.1.3. Put

$$\begin{aligned}\Delta f_{jk}\left(x,y;\frac{\pi}{2^{t_1}},\frac{\pi}{2^{t_2}}\right) = {} & f\left(x+j\frac{\pi}{2^{t_1}},y+k\frac{\pi}{2^{t_2}}\right)-f\left(x+(j-1)\frac{\pi}{2^{t_1}},y+k\frac{\pi}{2^{t_2}}\right)\\ & - f\left(x+j\frac{\pi}{2^{t_1}},y+(k-1)\frac{\pi}{2^{t_2}}\right)\\ & + f\left(x+(j-1)\frac{\pi}{2^{t_1}},y+(k-1)\frac{\pi}{2^{t_2}}\right),\end{aligned}$$

for any $t_1, t_2 \in \mathbb{N}$.

Then, for any $m, n \in \mathbb{Z}$, we have

$$\begin{aligned}\widehat{\Delta f}_{jk}(m,n) &= \hat{f}(m,n)\left(e^{imj\frac{\pi}{2^{t_1}}}\ e^{ink\frac{\pi}{2^{t_2}}} - e^{im(j-1)\frac{\pi}{2^{t_1}}}\ e^{ink\frac{\pi}{2^{t_2}}}\right.\\ &\qquad \left. - e^{imj\frac{\pi}{2^{t_1}}}\ e^{in(k-1)\frac{\pi}{2^{t_2}}} + e^{im(j-1)\frac{\pi}{2^{t_1}}}\ e^{in(k-1)\frac{\pi}{2^{t_2}}}\right)\\ &= \hat{f}(m,n)\left(e^{imj\frac{\pi}{2^{t_1}}} - e^{im(j-1)\frac{\pi}{2^{t_1}}}\right)\left(e^{ink\frac{\pi}{2^{t_2}}} - e^{in(k-1)\frac{\pi}{2^{t_2}}}\right)\\ &= \hat{f}(m,n)\ e^{im(j-\frac{1}{2})\frac{\pi}{2^{t_1}}}\ e^{in(k-\frac{1}{2})\frac{\pi}{2^{t_2}}}\left(e^{im\frac{\pi}{2^{t_1+1}}} - e^{-im\frac{\pi}{2^{t_1+1}}}\right)\\ &\qquad \left(e^{in\frac{\pi}{2^{t_2+1}}} - e^{-in\frac{\pi}{2^{t_2+1}}}\right)\\ &= -4\hat{f}(m,n)\ e^{im(j-\frac{1}{2})\frac{\pi}{2^{t_1}}}\ e^{in(k-\frac{1}{2})\frac{\pi}{2^{t_2}}}\ \sin\frac{m\pi}{2^{t_1+1}}\ \sin\frac{n\pi}{2^{t_2+1}}.\end{aligned} \tag{4.17}$$

Since $f \in L^2(\overline{\mathbb{T}}^2)$, it follows from Parseval's formula that

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\left|\hat{f}(m,n)\ \sin\frac{m\pi}{2^{t_1+1}}\ \sin\frac{n\pi}{2^{t_2+1}}\right|^2 = O\left(\|\Delta f_{jk}\|_2^2\right).$$

In view of (4.13) and (4.14), we have

$$S_{t_1t_2} \equiv \sum_{2^{t_1-1}\leq|m|<2^{t_1}} \sum_{2^{t_2-1}\leq|n|<2^{t_2}} |\hat{f}(m,n)|^2$$

$$= O\left(\int\int_{\overline{\mathbb{T}}^2} \left|\Delta f_{jk}\left(x, y; \frac{\pi}{2^{t_1}}, \frac{\pi}{2^{t_2}}\right)\right|^2 \, dx\, dy\right), \tag{4.18}$$

for all $j = 1, \cdots, 2^{t_1}$ and for all $k = 1, \cdots, 2^{t_2}$.

Dividing both sides of the above equation by $\lambda_j^1\ \lambda_k^2$ and then summing over $j = 1\ to\ 2^{t_1}$ and $k = 1\ to\ 2^{t_2}$, we have

$$S_{t_1t_2} = O\left(\left(\frac{1}{\sum_{j=1}^{2^{t_1}}\sum_{k=1}^{2^{t_2}}\frac{1}{\lambda_j^1\ \lambda_k^2}}\right)\left(\int\int_{\overline{\mathbb{T}}^2}\sum_{j=1}^{2^{t_1}}\sum_{k=1}^{2^{t_2}}\frac{|\Delta f_{jk}\left(x, y; \frac{\pi}{2^{t_1}}, \frac{\pi}{2^{t_2}}\right)|^2}{\lambda_j^1\ \lambda_k^2}\, dx\ dy\right)\right).$$

Since $\left|\Delta f_{jk}\left(x, y; \frac{\pi}{2^{t_1}}, \frac{\pi}{2^{t_2}}\right)\right| = O\left(\omega\left(f; \frac{\pi}{2^{t_1}}, \frac{\pi}{2^{t_2}}\right)\right)$ and $f \in \bigwedge BV(\overline{\mathbb{T}}^2)$, it follows that

$$\sum_{j=1}^{2^{t_1}}\sum_{k=1}^{2^{t_2}}\frac{|\Delta f_{jk}\left(x, y; \frac{\pi}{2^{t_1}}, \frac{\pi}{2^{t_2}}\right)|}{\lambda_j^1\ \lambda_k^2} = O(1).$$

Thus, we have

$$S_{t_1t_2} = O\left(\frac{\omega\left(f; \frac{\pi}{2^{t_1}}, \frac{\pi}{2^{t_2}}\right)}{\sum_{j=1}^{2^{t_1}}\sum_{k=1}^{2^{t_2}}\frac{1}{\lambda_j^1\ \lambda_k^2}}\right).$$

Now, applying Hölder's inequality and proceeding as in the proof of the Theorem 4.2.1.1 (from (4.15) onward), we obtain the theorem.

Corollary 4.2.1.4. *(i) If a measurable function $f \in \bigwedge^* BV(\overline{\mathbb{T}}^2)$ satisfies the condition* (4.16) *then* (4.11) *holds true.*

(ii) If $f \in BV_V(\overline{\mathbb{T}}^2) \cap \mathbf{C}(\overline{\mathbb{T}}^2)$ and

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{\left(\omega\left(f; \frac{\pi}{m}, \frac{\pi}{n}\right)\right)^{\frac{1}{2}}}{mn}\right)^{\beta} < \infty, \tag{4.19}$$

then (4.11) *holds true.*

(iii) *If* $f \in BV_V(\overline{\mathbb{T}}^2) \cap Lip(\alpha_1, \alpha_2)(\overline{\mathbb{T}}^2)$, $0 < \alpha_1, \alpha_2 \leq 1$, *then, for*

$$\beta > max\left\{\frac{2}{2+\alpha_1}, \frac{2}{2+\alpha_2}\right\},$$

(4.11) *holds true.*

Proof of Corollary 4.2.1.4. Proof of Corollary 4.2.1.4 (i). In view of the earlier Lemma 4.1.1.4 with $p = 1$ (p.62), $f \in \bigwedge^* BV(\overline{\mathbb{T}}^2)$ implies f is bounded on $\overline{\mathbb{T}}^2$, and hence $f \in L^2(\overline{\mathbb{T}}^2)$. Thus, $\bigwedge^* BV(\overline{\mathbb{T}}^2) \subset \bigwedge BV(\overline{\mathbb{T}}^2) \cap L^2(\overline{\mathbb{T}}^2)$. Therefore, the Corollary 4.2.1.4 (i) follows from the Theorem 4.2.1.3.

Proof of Corollary 4.2.1.4 (ii). Take $\Lambda^1 = \Lambda^2 = \{1\}$ (that is, $\lambda_n^1 = \lambda_n^2 = 1$, for all n) in the Theorem 4.2.1.3.

Proof of Corollary 4.2.1.4 (iii). Put

$$\begin{aligned}\Delta f_{jk}\left(x, y; \frac{\pi}{2^{t_1}}, \frac{\pi}{2^{t_2}}\right) = f\left(x + j\frac{\pi}{2^{t_1}}, y + k\frac{\pi}{2^{t_2}}\right) - f\left(x + (j-1)\frac{\pi}{2^{t_1}}, y + k\frac{\pi}{2^{t_2}}\right) \\ - f\left(x + j\frac{\pi}{2^{t_1}}, y + (k-1)\frac{\pi}{2^{t_2}}\right) \\ + f\left(x + (j-1)\frac{\pi}{2^{t_1}}, y + (k-1)\frac{\pi}{2^{t_2}}\right),\end{aligned}$$

for any $t_1, t_2 \in \mathbb{N}$.

Then, proceeding as in the proof of the Theorem 4.2.1.3, we get (4.18)

$$\begin{aligned}S_{t_1 t_2} &\equiv \sum_{2^{t_1-1} \leq |m| < 2^{t_1}} \sum_{2^{t_2-1} \leq |n| < 2^{t_2}} |\hat{f}(m,n)|^2 \\ &= O\left(\int\int_{\overline{\mathbb{T}}^2} \left|\Delta f_{jk}\left(x, y; \frac{\pi}{2^{t_1}}, \frac{\pi}{2^{t_2}}\right)\right|^2 \, dx \, dy\right),\end{aligned}$$

for all $j = 1, \cdots, 2^{t_1}$ and for all $k = 1, \cdots, 2^{t_2}$.

Since

$$\left|\Delta f_{jk}\left(x, y; \frac{\pi}{2^{t_1}}, \frac{\pi}{2^{t_2}}\right)\right| = O\left(\omega\left(f; \frac{\pi}{2^{t_1}}, \frac{\pi}{2^{t_2}}\right)\right)$$

and $f \in BV_V(\overline{\mathbb{T}}^2)$, it follows that

$$\sum_{j=1}^{2^{t_1}} \sum_{k=1}^{2^{t_2}} \left|\Delta f_{jk}\left(x, y; \frac{\pi}{2^{t_1}}, \frac{\pi}{2^{t_2}}\right)\right| = O(1).$$

Thus, we have

$$S_{t_1t_2} = O\left(\omega\left(f; \frac{\pi}{2^{t_1}}, \frac{\pi}{2^{t_2}}\right)\right)$$
$$= O\left(\left(\frac{1}{2^{t_1}}\right)^{\alpha_1}\left(\frac{1}{2^{t_2}}\right)^{\alpha_2}\right),$$

as $f \in Lip(\alpha_1, \alpha_2)(\overline{\mathbb{T}}^2)$.

Now, applying Hölder's inequality and proceeding as in the proof of the Theorem 4.2.1.1 (from (4.15) onward), we obtain the Corollary 4.2.1.4 (*iii*).

Theorem 4.2.1.5. *If* $f \in \bigwedge BV^{(p)}(\overline{\mathbb{T}}^2) \cap \mathbf{C}(\overline{\mathbb{T}}^2)$, $1 \leq p < 2r$, $1 < r < \infty$, *and*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{\left(\omega^{((2-p)s+p)}\left(f; \frac{\pi}{m}, \frac{\pi}{n}\right)\right)^{2-\frac{p}{r}}}{mn\left(\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{1}{\lambda_j^1\ \lambda_k^2}\right)^{\frac{1}{r}}}\right)^{\frac{\beta}{2}} < \infty, \tag{4.20}$$

where $\frac{1}{r} + \frac{1}{s} = 1$, *then* (4.11) *holds true.*

Proof of Theorem 4.2.1.5. Put

$$\Delta f_{jk}\left(x, y; \frac{\pi}{2^{t_1}}, \frac{\pi}{2^{t_2}}\right) = f\left(x + j\frac{\pi}{2^{t_1}}, y + k\frac{\pi}{2^{t_2}}\right) - f\left(x + (j-1)\frac{\pi}{2^{t_1}}, y + k\frac{\pi}{2^{t_2}}\right)$$
$$- f\left(x + j\frac{\pi}{2^{t_1}}, y + (k-1)\frac{\pi}{2^{t_2}}\right)$$
$$+ f\left(x + (j-1)\frac{\pi}{2^{t_1}}, y + (k-1)\frac{\pi}{2^{t_2}}\right),$$

for any $t_1, t_2 \in \mathbb{N}$.

Then, proceeding as in the proof of the Theorem 4.2.1.3, we get (4.18)

$$S_{t_1t_2} \equiv \sum_{2^{t_1-1}\leq|m|<2^{t_1}}\ \sum_{2^{t_2-1}\leq|n|<2^{t_2}} |\hat{f}(m,n)|^2$$
$$= O\left(\int\int_{\mathbb{T}^2}\left|\Delta f_{jk}\left(x, y; \frac{\pi}{2^{t_1}}, \frac{\pi}{2^{t_2}}\right)\right|^2\ dx\ dy\right),$$

for all $j = 1, \cdots, 2^{t_1}$ and for all $k = 1, \cdots, 2^{t_2}$.

Since

$$2 = \frac{(2-p)s + p}{s} + \frac{p}{r},$$

by using Hölder's inequality, we get

$$\int\int_{\mathbb{T}^2} \left|\Delta f_{jk}\left(x,y;\frac{\pi}{2^{t_1}},\frac{\pi}{2^{t_2}}\right)\right|^2\ dx\ dy$$
$$\leq \left(\int\int_{\mathbb{T}^2} \left|\Delta f_{jk}\left(x,y;\frac{\pi}{2^{t_1}},\frac{\pi}{2^{t_2}}\right)\right|^{(2-p)s+p}\ dx\ dy\right)^{\frac{1}{s}}$$
$$\times\left(\int\int_{\mathbb{T}^2} \left|\Delta f_{jk}\left(x,y;\frac{\pi}{2^{t_1}},\frac{\pi}{2^{t_2}}\right)\right|^{p}\ dx\ dy\right)^{\frac{1}{r}}$$
$$\leq \Omega^{\frac{1}{r}}_{\frac{1}{2^{t_1}2^{t_2}}} \left(\int\int_{\mathbb{T}^2} \left|\Delta f_{jk}\left(x,y;\frac{\pi}{2^{t_1}},\frac{\pi}{2^{t_2}}\right)\right|^{p}\ dx\ dy\right)^{\frac{1}{r}},$$
where
$$\Omega_{\frac{1}{2^{t_1}2^{t_2}}} = \left(\omega^{((2-p)s+p)}\left(f;\frac{\pi}{2^{t_1}},\frac{\pi}{2^{t_2}}\right)\right)^{2r-p}.$$

This, together with (4.18), implies

$$S_{t_1t_2} = O\left(\Omega^{\frac{1}{r}}_{\frac{1}{2^{t_1}2^{t_2}}} \left(\int\int_{\mathbb{T}^2} \left|\Delta f_{jk}\left(x,y;\frac{\pi}{2^{t_1}},\frac{\pi}{2^{t_2}}\right)\right|^{p}\ dx\ dy\right)^{\frac{1}{r}}\right),$$

for all $j = 1,\cdots,2^{t_1}$ and for all $k = 1,\cdots,2^{t_2}$.

Thus,
$$(S_{t_1t_2})^r = O\left(\Omega_{\frac{1}{2^{t_1}2^{t_2}}} \int\int_{\mathbb{T}^2} \left|\Delta f_{jk}\left(x,y;\frac{\pi}{2^{t_1}},\frac{\pi}{2^{t_2}}\right)\right|^{p}\ dx\ dy\right).$$

Dividing both sides of the above equation by $\lambda_j^1\ \lambda_k^2$ and then summing over $j = 1\ to\ 2^{t_1}$ and $k = 1\ to\ 2^{t_2}$, we have

$$(S_{t_1t_2})^r \left(\sum_{j=1}^{2^{t_1}}\sum_{k=1}^{2^{t_2}} \frac{1}{\lambda_j^1\ \lambda_k^2}\right)$$
$$= O\left(\left(\Omega_{\frac{1}{2^{t_1}2^{t_2}}}\right)\left(\int\int_{\mathbb{T}^2} \sum_{j=1}^{2^{t_1}}\sum_{k=1}^{2^{t_2}} \frac{|\Delta f_{jk}(x,y;\frac{\pi}{2^{t_1}},\frac{\pi}{2^{t_2}})|^p}{\lambda_j^1\ \lambda_k^2}\ dx\ dy\right)\right).$$

Therefore,

$$S_{t_1t_2} = O\left(\left(\frac{\Omega_{\frac{1}{2^{t_1}2^{t_2}}}}{\sum_{j=1}^{2^{t_1}}\sum_{k=1}^{2^{t_2}}\frac{1}{\lambda_j^1\ \lambda_k^2}}\right)^{\frac{1}{r}} \left(\int\int_{\mathbb{T}^2} \sum_{j=1}^{2^{t_1}}\sum_{k=1}^{2^{t_2}} \frac{|\Delta f_{jk}(x,y;\frac{\pi}{2^{t_1}},\frac{\pi}{2^{t_2}})|^p}{\lambda_j^1\ \lambda_k^2}\ dx\ dy\right)^{\frac{1}{r}}\right).$$

Since $f \in \bigwedge BV^{(p)}(\overline{\mathbb{T}}^2)$, it follows that

$$\sum_{j=1}^{2^{t_1}} \sum_{k=1}^{2^{t_2}} \frac{|\Delta f_{jk}\left(x, y; \frac{\pi}{2^{t_1}}, \frac{\pi}{2^{t_2}}\right)|^p}{\lambda_j^1 \ \lambda_k^2} = O(1).$$

Thus, we have

$$S_{t_1 t_2} = O\left(\left(\frac{\Omega_{\frac{1}{2^{t_1} 2^{t_2}}}}{\sum_{j=1}^{2^{t_1}} \sum_{k=1}^{2^{t_2}} \frac{1}{\lambda_j^1 \ \lambda_k^2}}\right)^{\frac{1}{r}}\right).$$

Now, applying Hölder's inequality and proceeding as in the proof of the Theorem 4.2.1.1 (from (4.15) onward), we obtain the theorem.

Corollary 4.2.1.6. *If a measurable function* $f \in \bigwedge^* BV^{(p)}(\overline{\mathbb{T}}^2)$, $1 \leq p < 2r$, $1 < r < \infty$, *satisfies the condition* (4.20) *then* (4.11) *holds true.*

Proof of Corollary 4.2.1.6. In view of the earlier Lemma 4.1.1.4 (p.62), $f \in \bigwedge^* BV^{(p)}(\overline{\mathbb{T}}^2)$ implies f is bounded on $\overline{\mathbb{T}}^2$, and hence $f \in L^2(\overline{\mathbb{T}}^2)$. Thus, $\bigwedge^* BV^{(p)}(\overline{\mathbb{T}}^2) \subset \bigwedge BV^{(p)}(\overline{\mathbb{T}}^2) \cap L^2(\overline{\mathbb{T}}^2)$. Therefore, the corollary follows from the Theorem 4.2.1.5.

Theorem 4.2.1.7. *If* $f \in r - BV(\overline{\mathbb{T}}^2)$ $(r \geq 1)$ *satisfies the condition* (4.19) *then* (4.11) *holds true.*

Proof of Theorem 4.2.1.7. Put

$$\begin{aligned}\Delta f_{jk}\left(x, y; \frac{\pi}{2^{t_1}}, \frac{\pi}{2^{t_2}}\right) = f\left(x + j\frac{\pi}{2^{t_1}}, y + k\frac{\pi}{2^{t_2}}\right) - f\left(x + (j-1)\frac{\pi}{2^{t_1}}, y + k\frac{\pi}{2^{t_2}}\right) \\ - f\left(x + j\frac{\pi}{2^{t_1}}, y + (k-1)\frac{\pi}{2^{t_2}}\right) \\ + f\left(x + (j-1)\frac{\pi}{2^{t_1}}, y + (k-1)\frac{\pi}{2^{t_2}}\right),\end{aligned}$$

for any $t_1, t_2 \in \mathbb{N}$.

Then, proceeding as in the proof of the Theorem 4.2.1.3, we get (4.17)

$$\widehat{\Delta f}_{jk}(m, n) = -4\hat{f}(m, n) \ e^{im(j-\frac{1}{2})\frac{\pi}{2^{t_1}}} \ e^{in(k-\frac{1}{2})\frac{\pi}{2^{t_2}}} \ \sin\frac{m\pi}{2^{t_1+1}} \ \sin\frac{n\pi}{2^{t_2+1}}.$$

Similarly, we have

$$\widehat{\Delta^2 f}_{jk}(m, n) = (-4)^2 \hat{f}(m, n) \ e^{i2m\left(j-\frac{1}{2}\right)\frac{\pi}{2^{t_1}}} \ e^{i2n\left(k-\frac{1}{2}\right)\frac{\pi}{2^{t_2}}} \sin^2\frac{m\pi}{2^{t_1+1}} \sin^2\frac{n\pi}{2^{t_2+1}}.$$

Repeating this process r times, we see that

$$\widehat{\Delta^r f}_{jk}(m,n) = (-4)^r \hat{f}(m,n)\ e^{irm\left(j-\frac{1}{2}\right)\frac{\pi}{2^{t_1}}}\ e^{irn\left(k-\frac{1}{2}\right)\frac{\pi}{2^{t_2}}} \sin^r \frac{m\pi}{2^{t_1+1}} \sin^r \frac{n\pi}{2^{t_2+1}}.$$

Since $f \in L^2(\overline{\mathbb{T}}^2)$, it follows from Parseval's formula that

$$\sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \left| \hat{f}(m,n) \sin^r \frac{m\pi}{2^{t_1+1}} \sin^r \frac{n\pi}{2^{t_2+1}} \right|^2 = O(\|\Delta^r f_{jk}\|_2^2).$$

In view of (4.13) and (4.14), we have

$$\begin{aligned} S_{t_1 t_2} &\equiv \sum_{2^{t_1-1}\le|m|<2^{t_1}} \sum_{2^{t_2-1}\le|n|<2^{t_2}} |\hat{f}(m,n)|^2 \\ &= O\left(\int\int_{\overline{\mathbb{T}}^2} \left| \Delta^r f_{jk}\left(x,y;\frac{\pi}{2^{t_1}},\frac{\pi}{2^{t_2}}\right)\right|^2\ dx\ dy \right). \end{aligned}$$

Summing both sides of the above equation over $j = 1$ to $2^{t_1} - r$ and $k = 1$ to $2^{t_2} - r$, we have

$$2^{t_1} 2^{t_2}\left(S_{t_1 t_2}\right) = O\left(\int\int_{\overline{\mathbb{T}}^2} \sum_{j=1}^{2^{t_1}-r} \sum_{k=1}^{2^{t_2}-r} \left| \Delta^r f_{jk}\left(x,y;\frac{\pi}{2^{t_1}},\frac{\pi}{2^{t_2}}\right)\right|^2\ dx\ dy \right),$$

as $2^{t_1} \approx 2^{t_1} - r$ and $2^{t_2} \approx 2^{t_2} - r$.

Since

$$\left| \Delta^r f_{jk}\left(x,y;\frac{\pi}{2^{t_1}},\frac{\pi}{2^{t_2}}\right)\right| = O\left(\omega\left(f;\frac{\pi}{2^{t_1}},\frac{\pi}{2^{t_2}}\right)\right)$$

and $f \in r - BV(\overline{\mathbb{T}}^2)$, it follows that

$$\sum_{j=1}^{2^{t_1}-r} \sum_{k=1}^{2^{t_2}-r} \left| \Delta^r f_{jk}\left(x,y;\frac{\pi}{2^{t_1}},\frac{\pi}{2^{t_2}}\right)\right| = O(1).$$

Thus, we have

$$S_{t_1 t_2} = O\left(\frac{\omega\left(f;\frac{\pi}{2^{t_1}},\frac{\pi}{2^{t_2}}\right)}{2^{t_1}2^{t_2}} \right).$$

Now, applying Hölder's inequality and proceeding as in the proof of the Theorem 4.2.1.1 (from (4.15) onward), we obtain the theorem.

Remark 4.2.1.8. *The double Fourier series of a function f is said to be $\beta-$absolute convergence* $(0 < \beta \leq 2)$ *if*

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |\hat{f}(m,n)|^\beta < \infty,$$

where

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |\hat{f}(m,n)|^\beta = \sum_{|m|\geq 1}\sum_{|n|\geq 1} |\hat{f}(m,n)|^\beta + \sum_{m\in\mathbb{Z}} |\hat{f}(m,0)|^\beta + \sum_{n\in\mathbb{Z}} |\hat{f}(0,n)|^\beta - |\hat{f}(0,0)|^\beta.$$

If a function $f \in L^1(\overline{\mathbb{T}}^2)$ is such that

$$\hat{f}(m,n) = 0 \quad for\ m = 0 \quad or \quad n = 0, \tag{4.21}$$

then condition (4.11) is sufficient for the $\beta-$absolute convergence of the double Fourier series of f.

If condition (4.21) is not satisfied, we may proceed as follows.

In the special case when $m = 0$ or $n = 0$, we write

$$\hat{f}(m,0) = \hat{f}_1(m), \quad where\ f_1(x) = \frac{1}{2\pi}\int_{\mathbb{T}} f(x,y)\ dy, \quad x \in \mathbb{T}; \tag{4.22}$$

and

$$\hat{f}(0,n) = \hat{f}_2(n), \quad where\ f_2(y) = \frac{1}{2\pi}\int_{\mathbb{T}} f(x,y)\ dx, \quad y \in \mathbb{T}. \tag{4.23}$$

Combining Corollary L (i) and (ii) (p.31), for the $\beta-$absolute convergence of a Fourier series, of a function of one variable, with Corollary 4.2.1.2 (p.77) and Corollary 4.2.1.4 (iii) (p.80) respectively, we obtain the following corollaries.

Corollary 4.2.1.9. *If $f \in Lip(\alpha_1,\alpha_2)(\overline{\mathbb{T}}^2)$, $f_1 \in Lip(\alpha_3)(\overline{\mathbb{T}})$ and $f_2 \in Lip(\alpha_4)(\overline{\mathbb{T}})$ for $\alpha_j \in (0,1]$, where $j = 1,2,3,4$ and the functions f_1 and f_2 are as defined in* (4.22) *and* (4.23)*, then, for*

$$\beta > max\left\{\frac{2}{2\alpha_1+1}, \frac{2}{2\alpha_2+1}, \frac{2}{2\alpha_3+1}, \frac{2}{2\alpha_4+1}\right\},$$

the double Fourier series of f is $\beta-$absolute convergence.

In view of the Corollary 4.2.1.9, if $f \in Lip(\alpha_1,\alpha_2)(\overline{\mathbb{T}}^2)$, $f_1 \in Lip(\alpha_3)(\overline{\mathbb{T}})$ and $f_2 \in Lip(\alpha_4)(\overline{\mathbb{T}})$ for $\alpha_j \in (\frac{1}{2},1]$, $j = 1,2,3,4$, then the double Fourier series of f is absolute convergence.

Corollary 4.2.1.10. *If $f \in BV_V(\overline{\mathbb{T}}^2) \cap Lip(\alpha_1,\alpha_2)(\overline{\mathbb{T}}^2)$, $f_1 \in BV(\overline{\mathbb{T}}) \cap Lip(\alpha_3)(\overline{\mathbb{T}})$ and $f_2 \in BV(\overline{\mathbb{T}}) \cap Lip(\alpha_4)(\overline{\mathbb{T}})$ for $\alpha_j \in (0,1]$, where $j = 1,2,3,4$ and the functions f_1 and f_2 are as defined in (4.22) and (4.23), then, for*

$$\beta > max\left\{\frac{2}{2+\alpha_1}, \frac{2}{2+\alpha_2}, \frac{2}{2+\alpha_3}, \frac{2}{2+\alpha_4}\right\},$$

the double Fourier series of f is $\beta-$absolute convergence.

In view of the Corollary 4.2.1.10, if $f \in BV_V(\overline{\mathbb{T}}^2) \cap Lip(\alpha_1,\alpha_2)(\overline{\mathbb{T}}^2)$, $f_1 \in BV(\overline{\mathbb{T}}) \cap Lip(\alpha_3)(\overline{\mathbb{T}})$ and $f_2 \in BV(\overline{\mathbb{T}}) \cap Lip(\alpha_4)(\overline{\mathbb{T}})$ for $\alpha_j \in (0,1]$, $j = 1,2,3,4$, then the double Fourier series of f is absolute convergence.

Similarly, combining Theorem 4.2.1.1 (p.74) and Theorem L (p.31) or Theorem 4.2.1.3 (p.78) and Theorem N (p.32), we can easily find sufficient conditions imposed on f, f_1 and f_2 for the $\beta-$absolute convergence of the double Fourier series of f. Finally, combining Theorem 4.2.1.5 (p.81) and Theorem O (p.32) or Theorem 4.2.1.7 (p.83) and Theorem P (p.32), we can easily find sufficient conditions imposed on f, f_1 and f_2 for the $\beta-$absolute convergence of the double Fourier series of f.

4.2.2 New results for functions of N−variables

Now, we extend the results of the Subsection 4.2.1 for functions of $N-$variables in the following way.

Theorem 4.2.2.1. *If $f \in L^2(\overline{\mathbb{T}}^N)$ and*

$$\sum_{k_1=1}^{\infty}\cdots\sum_{k_N=1}^{\infty}\left(\frac{\left(\omega_r^{(2)}\left(f;\frac{\pi}{k_1},\cdots,\frac{\pi}{k_N}\right)\right)}{(k_1\cdots k_N)^{\frac{1}{2}}}\right)^{\beta} < \infty,$$

then

$$\sum_{|k_1|=1}^{\infty}\cdots\sum_{|k_N|=1}^{\infty}|\hat{f}(k_1,\cdots,k_N)|^{\beta} < \infty, \quad 0<\beta\leq 2. \tag{4.24}$$

Corollary 4.2.2.2. *If* $f \in Lip(\alpha_1, \cdots, \alpha_N)(\overline{\mathbb{T}}^N),\ 0 < \alpha_1, \cdots, \alpha_N \le 1$, *then, for*

$$\beta > max\left\{\frac{2}{2\alpha_1 + 1}, \cdots, \frac{2}{2\alpha_N + 1}\right\},$$

(4.24) *holds true.*

Theorem 4.2.2.3. *If* $f \in \bigwedge BV(\overline{\mathbb{T}}^N) \cap \mathbf{C}(\overline{\mathbb{T}}^N)$ *and*

$$\sum_{k_1=1}^{\infty} \cdots \sum_{k_N=1}^{\infty} \left(\frac{\omega\left(f; \frac{\pi}{k_1}, \cdots, \frac{\pi}{k_N}\right)}{k_1 \cdots k_N \left(\sum_{r_1=1}^{k_1} \cdots \sum_{r_N=1}^{k_N} \frac{1}{\lambda^1_{r_1} \cdots \lambda^N_{r_N}}\right)}\right)^{\frac{\beta}{2}} < \infty, \tag{4.25}$$

then (4.24) *holds true.*

Corollary 4.2.2.4. (*i*) *If a measurable function* $f \in \bigwedge^* BV(\overline{\mathbb{T}}^N)$ *satisfies the condition* (4.25) *then* (4.24) *holds true.*

(*ii*) *If* $f \in BV_V(\overline{\mathbb{T}}^N) \cap \mathbf{C}(\overline{\mathbb{T}}^N)$ *and*

$$\sum_{k_1=1}^{\infty} \cdots \sum_{k_N=1}^{\infty} \left(\frac{\left(\omega\left(f; \frac{\pi}{k_1}, \cdots, \frac{\pi}{k_N}\right)\right)^{\frac{1}{2}}}{k_1 \cdots k_N}\right)^{\beta} < \infty, \tag{4.26}$$

then (4.24) *holds true.*

(*iii*) *If* $f \in BV_V(\overline{\mathbb{T}}^N) \cap Lip(\alpha_1, \cdots, \alpha_N)(\overline{\mathbb{T}}^N),\ 0 < \alpha_1, \cdots, \alpha_N \le 1$, *then, for*

$$\beta > max\left\{\frac{2}{2 + \alpha_1}, \cdots, \frac{2}{2 + \alpha_N}\right\},$$

(4.24) *holds true.*

Theorem 4.2.2.5. *If* $f \in \bigwedge BV^{(p)}(\overline{\mathbb{T}}^N) \cap \mathbf{C}(\overline{\mathbb{T}}^N),\ 1 \le p < 2r,\ 1 < r < \infty$, *and*

$$\sum_{k_1=1}^{\infty} \cdots \sum_{k_N=1}^{\infty} \left(\frac{\left(\omega^{((2-p)s+p)}\left(f; \frac{\pi}{k_1}, \cdots, \frac{\pi}{k_N}\right)\right)^{2-\frac{p}{r}}}{k_1 \cdots k_N \left(\sum_{r_1=1}^{k_1} \cdots \sum_{r_N=1}^{k_N} \frac{1}{\lambda^1_{r_1} \cdots \lambda^N_{r_N}}\right)^{\frac{1}{r}}}\right)^{\frac{\beta}{2}} < \infty, \tag{4.27}$$

where $\frac{1}{r} + \frac{1}{s} = 1$, *then* (4.24) *holds true.*

Corollary 4.2.2.6. *If a measurable function* $f \in \bigwedge^* BV^{(p)}(\overline{\mathbb{T}}^N)$, $1 \leq p < 2r$, $1 < r < \infty$, *satisfies the condition* (4.27) *then* (4.24) *holds true.*

Theorem 4.2.2.7. *If* $f \in r - BV(\overline{\mathbb{T}}^N)$ $(r \geq 1)$ *satisfies the condition* (4.26) *then* (4.24) *holds true.*

All extended results of this subsection can be proved in the same way as the results in the Subsection 4.2.1.