Chapter 1

Introduction

1.1 Introduction

General Theory of Relativity is at present one of the most acclaimed theory of gravitation. Among the four fundamental forces of interaction categorized into gravitational interaction, electromagnetic interaction, strong interaction and weak interaction, gravitational force is the only effective long range force in a macroscopic level. General Theory of Relativity (GTR) is built upon certain fundamental assumptions or postulates. The first among them is the principle of covariance which states that in the neighbourhood of any point, one cannot distinguish between the gravitational field produced by the attraction of masses and the field produced by the accelerating frame of reference.

Another postulate is the principle of covariance which states that the laws of physics must take the same form in all coordinate systems. The introduction of tensor theory in general relativity is a direct consequence of this principle as tensor equations are invariant under coordinate transformations.

In the Newtonian theory of gravitation space and time are considered as separate absolute entities. That is why Newton described time in Principia as "*Absolute, true and mathematical time, of itself, and from its own nature, flows equably without relation to anything external*"[Johns, 2005]. The deviation of path, from straight line, of a particle in the 3-dimensional space is due to external force acting on it. In the GTR Einstein fused space and time together in a 4-dimensional entity called spacetime. Einstein suggested that the presence of gravitation curves up the geometry of the spacetime and an infinitesimal separation between any two events in the spacetime is given by the metric

$$ds^2 = g_{ij}dx^i dx^j, \tag{1.1}$$

where the metric coefficients g_{ij} are functions of the spacetime coordinates and the matrix has signature (+ - - -).

The next principle of general theory of relativity states that the spacetime curvature is created by the stress energy within the spacetime. In the presence of matter, this can be described by Einstein's field equations (EFEs)

$$R_{ij} - \frac{1}{2}Rg_{ij} = -\frac{8\pi G}{c^2}T_{ij}, \tag{1.2}$$

where R_{ij}, R and T_{ij} are components of Ricci tensor, Ricci scalar and energy momentum tensor describing the physical content of the spacetime, respectively. R_{ij} and R have the following expressions:

$$R_{ij} = \frac{\partial}{\partial x^j}\Gamma^k_{ik} + \Gamma^l_{ik}\Gamma^k_{lj} - \Gamma^l_{ij}\Gamma^k_{lk} - \frac{\partial}{\partial x^k}\Gamma^k_{ij} \tag{1.3}$$

where

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}\left(g_{lj,i} + g_{li,j} - g_{ij,l}\right) \tag{1.4}$$

are components of Christoffel symbol of second kind and

$$R = g^{ij} R_{ij}. \tag{1.5}$$

The most general spherically symmetric spacetime metric (line-element) in four-dimensions with signature -2 can be written in the canonical form

$$ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2 \left(d\theta^2 + sin^2\theta d\phi^2 \right), \tag{1.6}$$

where $\nu = \nu(r, t)$ and $\lambda = \lambda(r, t)$.

GTR is widely used at present to describe different physical phenomena in the universe related to the presence of strong gravitational field ever since the success of experimental observations generally known as the crucial tests of general relativity. Apart from this the recent detection of gravitational waves by researchers of LIGO [Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration), 2016] has corroborated the veracity of Einstein's theory of relativity.

GTR has played a dominant role in the areas of Cosmology, Astronomy and Astrophysics. Since the universe is filled with matter and radiation gravitational fields are present everywhere. Einstein examined cosmological problems using his theory. Different models of the universe, both static and non-static, were developed by Einstein, de Sitter, Robertson-Walker and Friedmann in which Robertson-Walker-Friedmann models are at present regarded as appropriate to describe the universe. Observations regarding the motion of galaxies indicate that the universe has emerged out from a big bang. The cosmic microwave background detected by scientists in the second half of the last century supported this finding.

Chandrasekhar [1931a,b] has shown that no stars with mass greater than 1.2 solar masses can be in a state of hydrostatic equilibrium. This mass limit is known as

Chandrasekhar limit [Misner et al., 1973].

When the thermonuclear energy in a star is exhausted the star loses its hydrostatic equilibrium and undergo a contraction that can be halted when it reaches a new state of equilibrium depending on its mass. The stellar remnant of a collapsing star under its own gravitation can be a White dwarf, Neutron star or Blackhole. White dwarfs keep its equilibrium by counterbalancing gravitational force by electron degeneracy pressure. Similarly in Neutron stars the gravitational collapse is avoided due to the repulsive neutron degeneracy pressure. When the mass of a star exceeds approximately three solar masses, no force is sufficient to keep the equilibrium of the star and the ultimate fate of such a massive star is a Blackhole.

1.2 Compact Stars

All the above three states described for a star during its evolution are known as compact stars [Shapiro and A., 1983]. There are two distinguishing properties for compact stars in comparison with normal stars. In normal stars the force against inward gravitational force is produced due to thermal pressure generated through burn of nuclear fuel, while in white dwarf stars the equilibrium is attained by the electron degeneracy pressure and in Neutron stars by the pressure due to degenerate neutrons.

The second property of compact stars is that they are extremely small compared to normal stars with very high densities. For example White dwarf stars are about 5000 kms in radius and about 1 ton/cm^3 in density [Misner et al., 1973], while a Neutron star is about 10 kms in radius and 10^{14} g/cm^3 in density same as that of atomic nucleus. Because of its large surface potential, GTR plays a dominant role in the determination of structure of compact stars.

1.3 Strange Stars

Strange stars (SS) are believed to have strange matter distribution in the form of quarks, in hydrostatic equilibrium. According to Bethe-Jonson model, the maximum allowable mass of such stellar configurations is estimated to be $1.64M\odot$, whereas Pandharipande-Smith suggested through their neutron star model, that such limit may reach up to $2.24M\odot$. It has been further pointed out that strange stars with masses exceeding the maximum mass of neutron stars will not be in hydrostatic equilibrium [Shapiro and A., 1983] and may start collapsing under its own gravitational effect.

Strange stars can be classified into two categories according to their compactification parameter $u = \left(\frac{M}{a}\right)$: Type I: SS with $\frac{M}{a} > 0.3$ and Type II: SS with $0.2 < \left(\frac{M}{a}\right) < 0.3$. In the later category, reliable information about the density profile, mass, radius will be essential to distinguish them from their neutron star counterparts and the lower limit may still be lower. Researchers believe that if the compact object is not a black hole, it may exist in the form of SS only [Jotania and Tikekar, 2006].

1.4 The Energy-Momentum Tensor

The EFEs (1.2) connect the geometry of the spacetime with the matter content producing curvatures in the spacetime. This according to John A. Wheeler can be stated as: "*Spacetime tells matter how to move, matter tells spacetime how to curve*".

1.4.1 Perfect Fluid

The type of matter extensively studied by researchers is a perfect fluid. It is characterized by a four-velocity vector u^i which may vary from point to point, energy density ρ and isotropic pressure p in the rest frame of each fluid element. Shear stress, anisotropic pressure and viscosity are absent in such fluid. The energy-momentum tensor for a perfect fluid is given by

$$T_{ij} = (\rho + p)\, u_i u_j - p g_{ij}. \tag{1.7}$$

A special case of equations (1.7) belongs to non-interacting incoherent matter or dust. Such a field is characterized by two quantities a four-velocity vector field of flow u^i and a proper density ρ measured by a co-moving observer. Such a matter content has energy-momentum tensor

$$T_{ij} = \rho u_i u_j. \tag{1.8}$$

1.4.2 Electromagnetic Field

The energy-momentum tensor associated with a distribution of charge is given by

$$E_i^j = \frac{1}{4\pi}\left(-F_{ik}F^{kj} + \frac{1}{4}F_{mn}F^{mn}\delta_i^j\right) \tag{1.9}$$

where F_{ij}'s are components of electromagnetic field tensor satisfying Maxwell's equations

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0 \tag{1.10}$$

and

$$\frac{\partial}{\partial x^k}\left(F^{ik}\sqrt{-g}\right) = 4\pi\sqrt{-g}J^i. \tag{1.11}$$

The four current J^i is defined as

$$J^i = \sigma u^i \tag{1.12}$$

where σ denotes the charge density of the distribution. For a static distribution

$$u^i = (e^{-\frac{\nu}{2}}, 0, 0, 0). \tag{1.13}$$

The spherical symmetry implies that electromagnetic field tensor F_{ij} has $F_{10} = -F_{01}$ as its only non-vanishing component. The Maxwell's equations (1.10) and (1.11) admit

$$F_{01} = -\frac{e^{\frac{\lambda+\nu}{2}}}{r^2} \int_0^r 4\pi r^2 \sigma e^{\frac{\lambda}{2}} dr \tag{1.14}$$

as their solution.

1.4.3 Anisotropic Fluid Distribution

A fluid distribution with radial pressure different from tangential (transverse) pressure is termed as anisotropic fluid distribution. Following Maharaj and Maartens [1989], we write the energy-momentum tensor as

$$T_{ij} = (\rho + p)u_i u_j - p g_{ij} + \pi_{ij} \tag{1.15}$$

where ρ is the proper density, p is the isotropic pressure, u^i the four-velocity field of the fluid. The anisotropic stress tensor is given by

$$\pi_{ij} = \sqrt{3} S \left[C_i C_j - \frac{1}{3}(u_i u_j - g_{ij}) \right] \tag{1.16}$$

where $c^i = (0, -e^{-\frac{\lambda}{2}}, 0, 0)$ is a radial vector. For a spherically symmetric anisotropic distribution $S = S(r)$ denotes the magnitude of the anisotropic stress. For the spacetime metric (1.6) the non-vanishing components of energy-momentum tensor

(1.15) are

$$T_0^0 = \rho,\; T_1^1 = -\left(p + \frac{2S}{\sqrt{3}}\right),\; T_2^2 = T_3^3 = -\left(p - \frac{S}{\sqrt{3}}\right). \tag{1.17}$$

Consequently the radial and transverse pressure have expressions

$$p_r = -T_1^1 = \left(p + \frac{2S}{\sqrt{3}}\right), \tag{1.18}$$

$$p_\perp = -T_2^2 = \left(p - \frac{S}{\sqrt{3}}\right). \tag{1.19}$$

From (1.18) and (1.19) we get

$$S = \frac{p_r - p_\perp}{\sqrt{3}} \tag{1.20}$$

as the magnitude of anisotropy. For a perfect fluid distribution $p_r = p_\perp$ and hence $S = 0$.

For compact objects with central density greater than nuclear density, matter distribution may not be isotropic with principal stresses equal. Theoretical investigations of Ruderman [1972] and Canuto [1974] have shown that matter distributions may show anisotropy in pressure when the density exceeds nuclear density where gravitational effects play a dominant role. Bowers and Liang [1974] have shown that anisotropy has non-negligible effect on the maximum equilibrium mass and surface red-shift. Herrera and Santos [1997] have given an exhaustive review of the works on anisotropic fluid distributions. Anisotropy can arise due to diverse reasons like the existence of a solid core or the presence of type 3A super fluid [Kippenhahn and Weigert, 1990], different kind of phase transitions [Sawyer, 1972].

A method was developed by Tolman [1939] to find exact solution of Einstein's field equations in terms of known functions for static fluid spheres. Pant and Sah [1979] obtained analytic solution for charged fluid on spherically symmetric spacetime. In their analysis, if charge is absent, the solution is Tolman's solution VI with $B = 0$. Cosenza et al. [1981] developed the procedure to obtain solution of Einstein's field equations for anisotropic matter from known solutions of isotropic matter. Bayin [1982] found the solution for anisotropic fluid sphere by generalizing equation of state $p = \alpha\rho$ and also studied radiating anisotropic fluid sphere. By generalizing Tolman's I, IV and V solutions and the de Sitter solution Pant and Sah [1982] have obtained class of new static solutions by assuming an equation of state. Durgapal [1982] has obtained a class of new exact solutions for spherically symmetric static fluid spheres with the ansatz $e^{\nu} \propto (1 + x)^{n}$, and found that for each integer value of n, one can have new exact solution. In 1984, Krori *et al* have obtained the exact solutions of Einstein's field equations for anisotropic matter with the modification in Tolman III, IV, V and VI Solutions [Krori et al., 1984]. Maartens and Maharaj [1985] have developed a new ansatz to find an exact solution of Einstein's field equations. Ram and Pandey [1986] have obtained static and spherically symmetric solutions of the field equations in the bimetric theory of gravitation with the consideration of both isotropic and anisotropic matter content when the physical metric admits a 1-parameter family of conformal motions. de León [1987] has presented two exact analytical solutions to Einstein's field equations for anisotropic matter distribution describing the maximum mass, causality condition and central and surface redshifts. The charged analog of Vaidya and Tikekar [1982] solution on spheroidal spacetime was obtained by Patel and Koppar [1987]. Maharaj and Maartens [1989] have developed new ansatz to obtain interior solution of the Einstein's field equations. Delgaty and Lake [1998] analysed physical plausibility conditions for 127 solutions of Einstein's field equations and found that only 16 of them satisfies all the conditions and only for 9 solutions sound speed is decreasing with radius. Tikekar [1990] obtained new exact solution for a static fluid sphere on spheroidal spacetime. Tikekar and Thomas [1999] found exact solution of Einstein's field equations for anisotropic fluid sphere on pseudo spheroidal spacetime. The key feature of their model is the high variation of density from centre to boundary of stellar configuration also radial and tangential pressure are equal at the centre and boundary of the star. Mak and Harko [2003] obtained classes of exact anisotropic solutions of Einstein's field equations on spherically symmetric spacetime metric. Komathiraj and Maharaj [2007a] studied analytical models of quark stars where they found a class of solutions of Einstein-Maxwell system by considering linear equation of state. Karmarkar et al. [2007] analysed the role of pressure anisotropy for Vaidya-Tikekar model [Vaidya and Tikekar, 1982]. The exact solutions for Einstein-Maxwell system were extensively studied by Komathiraj and Maharaj [2007b], Maharaj and Komathiraj [2007] & Thirukkanesh and Maharaj [2008]. Chattopadhyay and Paul [2010] obtained the solutions of static compact stars on higher dimensional spacetime. The space part of spacetime metric considered by them is $(D-1)$ pseudo spheroid immersed in D-dimensional Euclidean space. Numerous researchers have contributed in recent past on mathematical model of compact superdense stars such as pulsars and quark stars compatible with observational data [Maurya and Gupta, 2011a,b,c, Sharma and Ratanpal, 2013, Murad, 2013a,b, Murad and Fatema, 2013, Fatema and Murad, 2013, Pandya et al., 2015, Murad and Fatema, 2015a, Thomas and Pandya, 2015c,b, Ratanpal et al., 2015b,a, Murad and Fatema, 2015b, Maurya et al., 2017b, 2016a,b,c, Dayanandan et al., 2016].

1.5 Some Important Spacetime Metrics

1.5.1 Spheroidal Spacetime Metric

A three-spheroid immersed in a 4-dimensional Euclidean flat space with metric

$$d\sigma^2 = dx^2 + dy^2 + dz^2 + dw^2 \tag{1.21}$$

has the Cartesian equation

$$\frac{w^2}{b^2} + \frac{x^2 + y^2 + z^2}{R^2} = 1. \tag{1.22}$$

The sections $w = constant$ of equation (1.22) represent a family of concentric spheres for $w < b$ while sections $w = constant,\ y = constant$ or $z = constant$ represent a family of ellipsoids. Using the transformation

$$\begin{aligned} x &= Rsin\lambda sin\theta cos\phi, \\ y &= Rsin\lambda sin\theta sin\phi, \\ z &= Rsin\lambda cos\theta, \\ w &= bcos\lambda, \end{aligned} \tag{1.23}$$

metric (1.21) takes the form

$$d\sigma^2 = \left(R^2cos^2\lambda + b^2sin^2\lambda\right)d\lambda^2 + R^2sin^2\lambda(d\theta^2 + sin^2\theta d\phi^2). \tag{1.24}$$

Choosing $r = Rsin\lambda$, metric (1.24) reduces to

$$d\sigma^2 = \frac{1 - K\frac{r^2}{R^2}}{1 - \frac{r^2}{R^2}}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{1.25}$$

where

$$K = 1 - \frac{b^2}{R^2}. \tag{1.26}$$

The metric (1.25) is regular for all points for which $r^2 < R^2$ and $K < 1$. When $K = 1$, the metric (1.25) corresponds to the metric of flat space. When $K = 0$, the metric (1.25) represents metric on a 3-sphere.

The spacetime metric with the above properties for its physical three space may be written as

$$ds^2 = e^{\nu(r)}dt^2 - \frac{1 - K\frac{r^2}{R^2}}{1 - \frac{r^2}{R^2}}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{1.27}$$

(i) When $K = 0$ and $e^{\nu(r)} = \left[A + B\sqrt{1 - \frac{r^2}{R^2}}\right]^2$ metric (1.27) gives Schwarzschild interior solution.

(ii) When $K = 0$ and $\nu = 0$ metric (1.27) gives the metric of Einstein's universe.

(iii) When $K = 0$, and $e^{\nu(r)} = 1 - \frac{r^2}{R^2}$ metric (1.27) gives the metric of the de Sitter's universe.

A number of researchers have used spacetime metric (1.27) to describe physically viable models of superdense stars with matter distribution in the form of perfect fluid, charged fluid distribution, anisotropic fluid distribution and fluid distributions accompanied by radial heat flux [Vaidya, 1951, Vaidya and Tikekar, 1982, Finch and Skea, 1989, Singh and Kotambkar, 2005, Paul et al., 2011, Thirukkanesh, 2013, Pandya et al., 2015].

1.5.2 Pseudo-Spheroidal Spacetime Metric

A 3-hyperboloid immerse in a 4-dimensional Euclidean flat space with metric (1.21) has the Cartesian equation

$$\frac{w^2}{b^2} - \frac{x^2 + y^2 + z^2}{R^2} = 1. \tag{1.28}$$

The sections $w = constant$ are spheres of real or imaginary radii according as $w^2 > b^2$ or $w^2 < b^2$, while the sections $x = constant, y = constant$ or $z = constant$ represent hyperboloids of two sheets.

Using the transformation

$$\begin{aligned} x &= Rsinh\lambda sin\theta cos\phi, \\ y &= Rsinh\lambda sin\theta sin\phi, \\ z &= Rsinh\lambda cos\theta, \\ w &= bcosh\lambda, \end{aligned} \tag{1.29}$$

the Euclidean metric (1.21) takes the form

$$d\sigma^2 = \left(R^2cosh^2\lambda + b^2sinh^2\lambda\right) d\lambda^2 + R^2sinh^2\lambda(d\theta^2 + sin^2\theta d\phi^2) \tag{1.30}$$

Taking $r = Rsinh\lambda$, metric (1.30) reduces to

$$d\sigma^2 = \frac{1 + K\frac{r^2}{R^2}}{1 + \frac{r^2}{R^2}} dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{1.31}$$

where

$$K = 1 + \frac{b^2}{R^2}. \tag{1.32}$$

The metric (1.31) which is regular at all points with $K > 1$ is called Pseudo-spheroidal metric [Tikekar and Thomas, 1998]. We have extensively used the spacetime metric

$$ds^2 = e^{\nu(r)}dt^2 - \frac{1 + K\frac{r^2}{R^2}}{1 + \frac{r^2}{R^2}}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{1.33}$$

with $K > 1$ to describe anisotropic fluid distributions. The spacetime metric (1.33), has been studied by many researchers [Tikekar and Thomas, 1998, 1999, 2005, Thomas et al., 2005, Thomas and Ratanpal, 2007, Paul et al., 2011, Chattopadhyay and Paul, 2010, Chattopadhyay et al., 2012].

1.5.3 Paraboloidal Spacetime Metric

A 3-paraboloid embedded in a 4-dimensional Euclidean flat space with metric (1.21) has the Cartesian equation

$$x^2 + y^2 + z^2 = 2wR. \tag{1.34}$$

The $w = constant$ sections are spheres, while the sections $x = constant, y = constant$ or $z = constant$ give 3 - paraboloids.

Using the transformation

$$\begin{aligned} x &= rsin\theta cos\phi, \\ y &= rsin\theta sin\phi, \\ z &= rcos\theta, \\ w &= \frac{r^2}{2R}, \end{aligned} \tag{1.35}$$

the Euclidean metric (1.21) takes the form

$$d\sigma^2 = \left(1 + \frac{r^2}{R^2}\right) dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{1.36}$$

where R is a geometric parameter. The spacetime metric whose physical 3-space has paraboloidal geometry has the form

$$ds^2 = e^{\nu(r)}dt^2 - \left(1 + \frac{r^2}{R^2}\right) dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{1.37}$$

is called paraboloidal spacetime. This spacetime metric has been extensively studied by Jotania and Tikekar [2006].

1.6 Karmarkar Condition

We noted earlier that, a purely gravitational field is represented by a 4-dimensional Riemannian metric. Sometimes a useful consideration would be the immersion of a field in a flat space of higher dimensions. If the lowest number of dimensions of a flat space in which a given Riemannian space of n dimensions can be immersed is $n + p$, the latter is said to be of class p. For example, Schwarzschild's exterior solution is a Riemannian metric of class 2 while Einstein's and de Sitter's cosmological models are Riemannian spaces of class 1. Karmarkar [1948] has stated that, the general spherically symmetric metric is of class 2.

In 1948, Karmarkar (Karmarkar [1948]) established the necessary and sufficient condition for a spherically symmetric spacetime in 4-dimensions embedded into 5-dimensional flat spacetime. Later Pandey and Sharma [1982] have shown that the condition is only a necessary condition for the embedding. Such spacetimes are called embedding class one type spacetimes. The condition for the spacetime metric

(1.6) is of class one is that the Riemannian curvature tensor satisfy the relation

$$R_{1414}R_{2323} = R_{1212}R_{3434} + R_{1224}R_{1334} \tag{1.38}$$

with $R_{2323} \neq 0$. The components of R_{hijk} for metric (1.6) are given by

$$R_{2323} = r^2 sin^2\theta \left[1 - e^{-\lambda}\right], \tag{1.39}$$

$$R_{1212} = \frac{1}{2}\lambda' r, \tag{1.40}$$

$$R_{1334} = R_{1224}\ sin^2\theta = 0, \tag{1.41}$$

$$R_{1414} = -e^{\nu}\left[\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{1}{4}\lambda'\nu'\right], \tag{1.42}$$

$$R_{2424} = -\frac{1}{4}\nu' r e^{\nu-\lambda}, \tag{1.43}$$

$$R_{3434} = sin^2\theta R_{2424}. \tag{1.44}$$

Using (1.39) – (1.44) in (1.38), we obtain the differential equation

$$\frac{2\nu''}{\nu'} + \nu' = \frac{\lambda' e^{\lambda}}{e^{\lambda} - 1} \tag{1.45}$$

For a given expression for e^{λ}, the expression for e^{ν} can be determined by solving equation (1.45) for ν.

1.7 Einstein's Field Equations with Spherical Symmetry

A non-static spherically symmetric spacetime metric can be written in the form

$$ds^2 = e^{\nu(r,t)}dt^2 - e^{\lambda(r,t)}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2), \tag{1.46}$$

The EFEs (1.2) corresponding to metric (1.46) are equivalent to the following set of equations

$$
\begin{aligned}
8\pi T_1^1 &= -e^{-\lambda}\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)+\frac{1}{r^2}, &(1.47)\\
8\pi T_2^2 &= 8\pi T_3^3 = -e^{-\lambda}\left(\frac{\nu''}{2}-\frac{\nu'\lambda'}{4}+\frac{\nu'^2}{4}+\frac{\nu'-\lambda'}{2r}\right)\\
&+ e^{-\nu}\left(\frac{\ddot{\lambda}}{2}+\frac{\dot{\lambda}^2}{4}-\frac{\dot{\lambda}\dot{\nu}}{4}\right), &(1.48)\\
8\pi T_0^0 &= e^{-\lambda}\left(\frac{\lambda'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}, &(1.49)\\
8\pi T_0^1 &= -e^{-\lambda}\left(\frac{\dot{\lambda}}{r}\right), &(1.50)\\
8\pi T_1^0 &= e^{-\nu}\left(\frac{\dot{\lambda}}{r}\right), &(1.51)
\end{aligned}
$$

where the accents denote differentiation with respect to r and the dots differentiation with respect to t.

For a spherically symmetric static metric $\nu(r,t)=\nu(r)$ and $\lambda(r,t)=\lambda(r)$ and hence equations (1.47) – (1.51) reduces to the following set of three equations

$$
\begin{aligned}
8\pi T_0^0 &= e^{-\lambda}\left(\frac{\lambda'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}, &(1.52)\\
8\pi T_2^2 &= 8\pi T_3^3 = -e^{-\lambda}\left(\frac{\nu''}{2}-\frac{\nu'\lambda'}{4}+\frac{\nu'^2}{4}+\frac{\nu'-\lambda'}{2r}\right), &(1.53)\\
8\pi T_1^1 &= -e^{-\lambda}\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)+\frac{1}{r^2}. &(1.54)
\end{aligned}
$$

Equations (1.52) through (1.54) form a system of three highly non-linear second order ordinary differential equations.

1.7.1 Field Equations for Anisotropic Fluid Distribution

The energy-momentum tensor for anisotropic fluid distribution is given by Maharaj and Maartens [1989]

$$T_{ij} = (\rho + p)u_i u_j - p g_{ij} + \sqrt{3}S\left[C_i C_j - \frac{1}{3}(u_i u_j - g_{ij})\right] \tag{1.55}$$

where $C^i = \left(0, -e^{-\frac{\lambda}{2}}, 0, 0\right)$ is a radial vector and $S = S(r)$ is the magnitude of anisotropic stress. For the energy momentum tensor (1.55) the EFEs (1.52) – (1.54) is equivalent to the following system of three equations

$$\begin{aligned}
\frac{1-e^{-\lambda}}{r^2} + \frac{e^{-\lambda}\lambda'}{r} &= 8\pi T_0^0 = 8\pi\rho, && (1.56)\\
\frac{e^{-\lambda}-1}{r^2} + \frac{e^{-\lambda}\nu'}{r} &= -8\pi T_1^1 = 8\pi\left(p + \frac{2S}{\sqrt{3}}\right) = 8\pi p_r, && (1.57)\\
e^{-\lambda}\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu'\lambda'}{4} + \frac{\nu'-\lambda'}{2r}\right) &= -8\pi T_2^2 = 8\pi\left(p - \frac{S}{\sqrt{3}}\right) = 8\pi p_\perp, && (1.58)
\end{aligned}$$

of three equations, where

$$p_r = \left(p + \frac{2S}{\sqrt{3}}\right), \tag{1.59}$$

$$p_\perp = \left(p - \frac{S}{\sqrt{3}}\right), \tag{1.60}$$

so that the anisotropy is given by

$$S = \frac{p_r - p_\perp}{\sqrt{3}}. \tag{1.61}$$

Equations (1.56) – (1.58) can be couched to an alternative form

$$\begin{aligned} e^{-\lambda} &= 1-\frac{2m}{r}, & (1.62)\\ \left(1-\frac{2m}{r}\right)\nu' &= 8\pi p_r r+\frac{2m}{r^2}, & (1.63)\\ -\frac{4}{r}(8\pi\sqrt{3}S) &= (8\pi\rho+8\pi p_r)\nu'+16\pi p_r', & (1.64) \end{aligned}$$

where m given by

$$m(r)=4\pi\int_0^r u^2\rho(u)du \tag{1.65}$$

is the mass enclosed within a radius r.

1.7.2 Field Equations for Anisotropic Charged Fluid Distributions

Einstein's field equations for a charged anisotropic fluid distribution is given by

$$R_i^j-\frac{1}{2}R\delta_i^j=8\pi\left(T_i^j+\pi_i^j+E_i^j\right), \tag{1.66}$$

where

$$T_i^j=(\rho+p)\,u_iu^j-p\delta_i^j, \tag{1.67}$$

$$\pi_i^j=\sqrt{3}S\left[c_ic^j-\frac{1}{2}\left(u_iu^j-\delta_i^j\right)\right], \tag{1.68}$$

and

$$E_i^j=\frac{1}{4\pi}\left(-F_{ik}F^{jk}+\frac{1}{4}F_{mn}F^{mn}\delta_i^j\right). \tag{1.69}$$

where ρ, p, u_i, S and c^i denote the proper density, fluid pressure, unit-four velocity vector, magnitude of anisotropic tensor, and a radial vector given by $\left(0,-e^{-\frac{\lambda}{2}},0,0\right)$. F_{ij} denote the anti-symmetric electromagnetic field strength tensor whose properties are given in subsection (1.4.2).

Equation (1.66) for the spacetime metric (1.6) is equivalent to the following system of three equations:

$$\frac{1-e^{-\lambda}}{r^2}+\frac{e^{-\lambda}\lambda'}{r}=8\pi\rho+E^2, \tag{1.70}$$

$$\frac{e^{-\lambda}-1}{r^2}+\frac{e^{-\lambda}\nu'}{r}=8\pi p_r-E^2, \tag{1.71}$$

$$e^{-\lambda}\left(\frac{\nu''}{2}+\frac{\nu'^2}{4}-\frac{\nu'\lambda'}{4}+\frac{\nu'-\lambda'}{2r}\right)=8\pi p_\perp+E^2, \tag{1.72}$$

where p_r and $p_\perp$ are given by (1.18) and (1.19).

1.8 Elementary Criteria for Physical Acceptability

We have already seen that the EFEs are highly non-linear and hence it is difficult to obtain exact solutions. However, several different exact solutions are now available [Kramer et al., 1980]. If the solution so obtained is a complicated expression, it is difficult to discuss the physical properties of the model and in this case one has to resort to numerical or graphical techniques. Delgaty and Lake [1998] examined 127 models for physical acceptability conditions and only 16 of them are found to satisfy the test and out of which only 9 of them have a sound speed monotonically decreasing with radius.

A physically acceptable solution must comply with the following requirements, [Knutsen, 1988a,b, 1989, Murad and Fatema, 2015b].

(a). Regularity Conditions:

(i) The solution should be free from the physical and geometric singularities. That is, $e^{\lambda(r)}>0$, $e^{\nu(r)}>0$ in the range $0\le r\le a$, where a is the radius of the spherical distribution.

(ii) The radial and transverse pressures and density of the distribution should be positive. That is $p_r(r) \geq 0, \quad p_\perp(r) \geq 0, \quad \rho(r) \geq 0$ in the range $0 \leq r \leq a$.

(iii) Radial pressure $p_r = 0$ at $r = a$. The energy density and transverse pressure may follow $\rho \geq 0$, and $p_\perp \geq 0$ for $r = a$.

(b). Stability Conditions:

(iv) In order to have an equilibrium configuration, the matter must be stable against collapse of local regions. This requires that the radial pressure p_r must be a non-decreasing function of density ρ. That is, $\frac{dp_r}{d\rho} \geq 0$.

(v) The relativistic adiabatic index is given by $\Gamma = \frac{p_r+\rho}{p_r}\frac{dp_r}{d\rho}$. A necessary condition for the exact solution to serve as a model for a relativistic star is that $\Gamma > \frac{4}{3}$.

(c). Causality Conditions:

(vi) For isotropic fluids $\sqrt{\frac{dp_r}{d\rho}}$ and $\sqrt{\frac{dp_\perp}{d\rho}}$ represent the speed of sound, which should be less than the speed of light. This requires that $0 \leq \sqrt{\frac{dp_r}{d\rho}} \leq 1, \ 0 \leq \sqrt{\frac{dp_\perp}{d\rho}} \leq 1$.

(d). Energy Condition:

(vii) A physically reasonable energy-momentum tensor has to obey the conditions: $\rho \geq p_r + 2p_\perp$ and $\rho + p_r + 2p_\perp \geq 0$.

(e). Monotone decrease of physical parameters:

(viii) The pressure and density should be maximum at the centre of the star and monotonically decrease towards the boundary. Mathematically, this means

$$\frac{dp_r}{dr} = 0, \ \frac{dp_\perp}{dr} = 0, \ \text{and} \ \frac{d\rho}{dr} = 0 \text{ at } r = 0$$

and

$$\frac{d^2p_r}{dr^2} < 0, \ \frac{d^2p_\perp}{dr^2} < 0, \ \text{and} \ \frac{d^2\rho}{dr^2} < 0 \text{ at } r = 0,$$

so that

$$\frac{dp_r}{dr} < 0, \ \frac{dp_\perp}{dr} < 0, \ \text{and} \ \frac{d\rho}{dr} < 0, \ \text{for } 0 < r < a.$$

(ix) Velocity of sound should decrease radially outward. That is, $\frac{d}{dr}\left(\frac{dp_r}{d\rho}\right) < 0$, $\frac{d}{dr}\left(\frac{dp_\perp}{d\rho}\right) < 0$ for $0 \leq r \leq a$.

(x) Further the ratio of pressure to density, $\frac{p_r}{\rho}$ and $\frac{p_\perp}{\rho}$, should be monotonically decreasing towards the boundary. That is, $\frac{d}{dr}\left(\frac{p_r}{\rho}\right) = 0$ and $\frac{d}{dr}\left(\frac{p_\perp}{\rho}\right) = 0$ at $r = 0$ and $\frac{d^2}{dr^2}\left(\frac{p_r}{\rho}\right) < 0$ and $\frac{d^2}{dr^2}\left(\frac{p_\perp}{\rho}\right) < 0$ at $r = 0$.

(f). Matching Conditions:

(xi) For uncharged matter distribution, the interior solution obtained should match continuously with Schwarzschild exterior metric

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 \left(dr^2 + sin^2\theta d\phi^2\right), \ r \geq a$$

across the boundary $r = a$. This gives

$$e^{\nu(a)} = e^{-\lambda(a)} = 1 - \frac{2M}{a}.$$

(xii) For a charged matter distribution, the interior metric should match with the exterior Riessner-Nordström metric

$$ds^2 = \left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right) dt^2 - \left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + sin^2\theta d\phi^2\right), \ r \geq a$$

across the boundary $r = a$. This gives

$$e^{\nu(a)} = e^{-\lambda(a)} = 1 - \frac{2M}{a} + \frac{q^2}{a^2}.$$

(g). Charge Distribution:

(xiii) The electric field intensity E should be such that $E(0) = 0$ and monotonically increasing towards the boundary. That is, $\frac{dE}{dr} > 0$ for $0 \leq r \leq a$.

(h). Pressure Anisotropy:

(xiv) Pressure anisotropy $\Delta = p_r - p_\perp$ should vanish at the centre. That is, $\Delta(0) = 0$.

(i). Mass to Radius Ratio:

(xv) For isotropic fluid spheres, the allowable mass to radius ratio is given by $\frac{M}{a} \leq \frac{4}{9}$ $(c = G = 1)$. [Buchdahl, 1979].

For a charged fluid sphere the ratio $\frac{M}{a}$ must be bounded between two limits as follows

$$\left(\frac{q^2}{2a^2}\frac{18a^2 + q^2}{12a^2 + q^2}\right) \leq \frac{M}{a} \leq \left(\frac{a + \sqrt{a^2 + 3q^2}}{3a}\right)^2.$$

(j). Gravitational Redshift:

(xvi) The gravitational redshift z should be monotonically decreasing towards the boundary of the star. The central redshift z_c and boundary redshift z_a must be positive and finite. That is,

$$z_c = e^{-\frac{\nu(r)}{2}} - 1 > 0, \text{ at } r = 0$$

and

$$z_a = e^{-\frac{\lambda(r)}{2}} - 1 > 0, \text{ at } r = a.$$

1.9 Layout of the thesis

The thesis is organized as follows:

Chapter 1 contains introduction to general theory of relativity and the theoretical background needed for the problems studied in the subsequent chapters. It also contains the summary of each chapter of the thesis .

In **Chapter 2** we study the solution of Einsteins field equations (EFEs) for a static spherically symmetric anisotropic distribution by generalizing the ansatz of Finch and Skea [Class. Quantum Grav. **6** 467, 1989] described by $g_{rr} = \left(1 + \frac{r^2}{R^2}\right)^n$. By using the physical acceptability and regularity conditions we have obtained the bounds on the model parameter p_0 in terms of the dimensionless parameter n which lies in the interval $\left(1, \frac{4}{\sqrt{3}}\right)$. The model so developed is in good agreement with the observational data of pulsars , viz., 4U 1820-30, PSR J1903+327, 4U 1608-52, Vela X-1, PSR J1614-2230, SAX J1808.4-3658 and Her X-1 (referred in Gangopadhyay *et al* Gangopadhyay et al. [2013]).

Chapter 3 deals with two models obtained as the solutions of Einstein's field equations representing a static spherically symmetric anisotropic matter distribution on the background of pseudo-spheroidal spacetime characterized by the metric potential $g_{rr} = \frac{1+K\frac{r^2}{R^2}}{1+\frac{r^2}{R^2}}$, where K and R represent geometric parameters. In the first model, the field equations are integrated by assuming a particular, physically acceptable form for the radial pressure p_r given by the expression $p_r = \frac{p_0}{R^2}\frac{\left(1-\frac{r^2}{R^2}\right)\left(1+\frac{r^2}{R^2}\right)}{\left(1+K\frac{r^2}{R^2}\right)^2}$. We have obtained suitable bounds of model parameters K and p_0 on the basis of the physical acceptability conditions viz., regularity, stability and energy conditions. It is found that the model is compatible with the wide range of compact stars viz., 4U 1820-30, PSR J1903+327, 4U 1608-52, Vela X-1, PSR J1614-2230, SMC X-4 and Cen X-3.

In order to validate the model for physical acceptability , we have studied in detail the regularity, energy and stability conditions using numerical and graphical methods for the pulsar 4U 1820-30 by taking the mass of the pulsar as $1.58M_{\odot}$ and radius $9.1km$ The values of the parameters in this case are $p_0 = 1.08$ and $K = 3.1$.

In the second model, we have taken a different form for radial pressure, viz., $p_r = \frac{K-1}{R^2}\frac{\left(1-\frac{r^2}{R^2}\right)}{\left(1+K\frac{r^2}{R^2}\right)^2}$. The bound for the geometric parameter K is obtained as $2.4641 \leq K \leq 4.1231$ using the physical acceptability conditions. For validating the present model , we have studied in detail the regularity, stability and energy conditions for the pulsar candidate PSR J1614-2230 having mass equal to $1.97M_{\odot}$ and radius $9.69km$ corresponding to $K = 3.997$.

In **Chapter 4**, we have studied anisotropic charged fluid distributions on pseudo-spheroidal spacetime. By choosing suitable expressions for radial pressure $p_r = \frac{p_0}{R^2}\frac{\left(1-\frac{r^2}{R^2}\right)\left(1+\frac{r^2}{R^2}\right)}{\left(1+K\frac{r^2}{R^2}\right)^2}$ and electric field intensity $E = \sqrt{\alpha}\frac{r}{R^2}$, where $\alpha \geq 0$ is a constant, the field equations are integrated. The parameters K, R and α are determined by imposing the physical acceptability conditions. The present model is in good agreement with the observational data of various compact stars like 4U 1820-30, PSR J1903+327, 4U 1608-52, Vela X-1, SMC X-4, Cen X-3 given by Gangopadhyay *et al.* (Gangopadhyay et al. [2013]). When $\alpha = 0$, the model reduces to the uncharged anisotropic distribution described as first model in chapter 3. In order to examine the nature of physical quantities throughout the distribution, we have considered a particular pulsar 4U 1820-30, whose tabulated mass and radius are $M = 1.58M_{\odot}$, and $R = 9.1(km)$, respectively, for $K = 2.718$ and $\alpha = 0.05$. It is found that all physical variables behave well for this particular pulsar.

We have studied a second model in this chapter by assuming a different form for radial pressure p_r and electric field intensity E, namely, $p_r = \frac{K-1}{R^2}\frac{\left(1-\frac{r^2}{R^2}\right)}{\left(1+K\frac{r^2}{R^2}\right)^2}$ and $E^2 = \frac{\alpha(K-1)}{R^2}\frac{\frac{r^2}{R^2}}{\left(1+K\frac{r^2}{R^2}\right)^2}$. The bounds of geometric parameter K and the parameter α appearing in the expression for E^2 are obtained by imposing the requirements for a physically acceptable model. It is found that the model is in good agreement with the observational data of number of compact stars like 4U 1820-30, PSR J1903+327, 4U 1608-52, Vela X-1, PSR J1614-2230, Cen X-3 given by Gangopadhyay *et al.* (Gangopadhyay et al. [2013]). When $\alpha = 0$, the model reduces to the uncharged anisotropic distribution discussed as a second model in chapter 3.

Chapter 5 provides new exact solutions of Einstein's field equations (EFEs) by assuming a linear equation of state, $p_r = \alpha(\rho - \rho_R)$ where p_r is the radial pressure and ρ_R is the surface density. The background spacetime metric is a paraboloidal spacetime metric characterized by the metric potential $g_{rr} = 1 + \frac{r^2}{R^2}$. By assuming estimated mass and radius of strange star candidate 4U 1820-30, various physical and energy conditions are used for estimating the range of parameter α. The suitability of the model for describing pulsars like PSR J1903+327, Vela X-1, Her X-1 and SAX J1808.4-3658 has been explored and respective ranges of α, for which all physical and energy conditions are satisfied throughout the distribution, are obtained.

In **Chapter 6** we have obtained an exact solutions of Einstein's field equations on the background of paraboloidal spacetime using Karmarkar condition, namely, $R_{1414}R_{2323} = R_{1212}R_{3434} + R_{1224}R_{1334}$. For a spherically symmetric static paraboloidal spacetime this condition is equivalent to $\frac{2\nu''}{\nu'} + \nu' = \frac{2}{r}$, where the metric potential $g_{tt} = e^{\nu}$. The physical acceptability conditions of the model are investigated and found that the model is compatible with a number of compact star candidates like Her X-1, LMC X-4, EXO 1785-248, PSR J1903+327, Vela X-1 and PSR J1614-2230. A noteworthy feature of the model is that it is geometrically significant and simple in form.

Chapter 6 is followed by the appendix of units used in throughout the course of research and then the Bibliography is provided.