Nonwandering set in Nonautonomous Discrete Dynamical Systems

In this chapter, we define and study nonwandering set, α-limit set, ω-limit set and recurrent set for an invertible nonautonomous discrete dynamical systems. Our definition of nonwandering point is different form those given in [36] and [77].

4.1 Properties of Nonwandering set

We first define the notion of a nonwandering point for a time varying homeomorphism.

Definition 4.1.1 *Let (X, d) be a metric space and $f_k : X \to X$ be a sequence of homeomorphisms, $k = 0, 1, 2, \ldots$. A point $x \in X$ is said to be* **a nonwandering point** *for time varying homeomorphism $F = \{f_k\}_{k=0}^{\infty}$ if for any neighborhood U of x and for any $n > 0$ there exist $m \geq n$ and $r \geq 0$ such that*

$$F_{[m,m+r]}(U) \cap U \neq \emptyset \quad or \quad F^{-1}_{[m,m+r]}(U) \cap U \neq \emptyset.$$

The set of all nonwandering points for F is denoted by $\Omega(F)$.

The following three results show that the set of all nonwandering points for an invertible nonautonomous discrete dynamical system F is a nonempty closed set containing the set of all periodic points of F.

Theorem 4.1.1 *Let* (X, d) *be a compact metric space and* $F = \{f_n\}_{n=0}^{\infty}$ *be a time varying homeomorphism on* X. *Then* $\Omega(F)$ *is a nonempty set.*

Proof : Suppose $\Omega(F)$ is empty. Then for any $x \in X$, $x \notin \Omega(F)$. Therefore there exist a neighborhood U_x of x and an integer $n_x > 0$ such that for any $y \in U_x$,

$$F_{[m_x,m_x+r_x]}(y) \notin U_x \ \text{ and } \ F^{-1}_{[m_x,m_x+r_x]}(y) \notin U_x,$$

for every $m_x \geq n_x$ and $r_x \geq 0$. Note that $\{U_x \colon x \in X\}$ forms an open cover of X. Since X is compact, there exist $x_0, x_1, ..., x_m$ such that $\cup_{k=0}^{m} U_{x_k} = X$. Let

$$N = \max\{n_{x_i} \colon i = 0, 1, \cdots, m\}.$$

For $m_{x_0} = N \geq n_{x_0}$ and $r_{x_0} = 1$, for any $z \in U_{x_0}$,

$$F_{[N,N+1]}(z) = F_{[m_{x_0},m_{x_0}+r_{x_0}]}(z) \notin U_{x_0}.$$

Suppose $F_{[N,N+1]}(z) \in U_{x_1}$. Now for $m_{x_0} = N \geq n_{x_0}$, $r_{x_0} = 2 > 0$ and $z \in U_{x_0}$,

$$F_{[N,N+2]}(z) = F_{[m_{x_0},m_{x_0}+r_{x_0}]}(z) \notin U_{x_0}.$$

Then for $m_{x_1} = N + 1 \geq n_{x_1}$, $r_{x_1} = 1 > 0$ and $F_{[N,N+1]}(z) \in U_{x_1}$,

$$\begin{aligned} F_{[N,N+2]}(z) &= F_{[N+1,N+2]}(F_{[N,N+1]}(z)) \\ &= F_{[m_{x_1},m_{x_1}+r_{x_1}]}(F_{[N,N+1]}(z)) \\ &\notin U_{x_1}. \end{aligned}$$

Thus

$$F_{[N,N+2]}(z) = F_{[N+1,N+2]}(F_{[N,N+1]}(z)) \notin U_{x_0} \cup U_{x_1}$$

Continuing in this way, we have

$$F_{[N,N+m+1]}(z) \notin \cup_{i=0}^{m} U_{x_i} = X$$

which gives a contradiction. Hence $\Omega(F)$ is nonempty.

Theorem 4.1.2 *Let (X, d) be a compact metric space and $F = \{f_n\}_{n=0}^{\infty}$ be the time varying homeomorphism on X. Then $\Omega(F)$ is a closed set.*

Proof Let $\{x_n\}_{n=0}^{\infty}$ be a sequence in $\Omega(F)$ which converges to $x \in X$. Let U be an open neighbourhood of x and $n > 0$. Since $x_k \to x$, $\exists\, l \geq 0$ such that $x_l \in U$. Since U is open, there exists $\epsilon > 0$ such that $U_\epsilon(x_l) = \{y \in X : d(y, x_l) < \epsilon\} \subset U$. Since x_l is a nonwandering point, there exist $m \geq n$ and $r \geq 0$ such that

$$F_{[m,m+r]}(U_\epsilon(x_l)) \cap U_\epsilon(x_l) \neq \phi \;\; or \;\; F^{-1}_{[m,m+r]}(U_\epsilon(x_l)) \cap U_\epsilon(x_l) \neq \phi.$$

Since $U_\epsilon(x_l) \subset U$, we have

$$F_{[m,m+r]}(U_\epsilon(x_l)) \subset F_{[m,m+r]}(U) \;\; or \;\; F^{-1}_{[m,m+r]}(U_\epsilon(x_l)) \subset F^{-1}_{[m,m+r]}(U).$$

Thus

$$F_{[m,m+r]}(U) \cap U \supset F_{[m,m+r]}(U_\epsilon(x_l)) \cap U_\epsilon(x_l) \neq \phi$$

or

$$F^{-1}_{[m,m+r]}(U) \cap U \supset F^{-1}_{[m,m+r]}(U_\epsilon(x_l)) \cap U_\epsilon(x_l) \neq \phi.$$

Thus $x \in \Omega(F)$ and hence $\Omega(F)$ is closed.

Theorem 4.1.3 *Let (X, d) be a compact metric space and $F = \{f_n\}_{n=0}^{\infty}$ be the time varying homeomorphism on X. Then $Per(F) \subset \Omega(F)$.*

Poof Let $x \in Per(F)$ then there exists $k > 0$ such that $F_{ik+j}(x) = F_j(x)$, for every $i \in \mathbb{Z}$ and $0 \leq j < k$. Now for any $n > 0$, there exists $i > 0$ such that $ik \geq n$. Now $F_{ik}(x) = x$ and $F_{(i+1)k}(x) = x$. Thus

$$F_{[ik+1,(i+1)k]}(x) = F_{[ik+1,(i+1)k]}(F_{ik}(x)) = F_{(i+1)k}(x) = x.$$

Thus for any neighborhood U of x and any $n > 0$ there exist $m = ik + 1 > n$ and $r = k - 1 \geq 0$ such that

$$F_{[m,m+r]}(U) \cap U = F_{[ik+1,ik+1+(k-1)]}(U) \cap U \neq \phi.$$

Hence $x \in \Omega(F)$ which proves that $Per(F) \subset \Omega(F)$.

We show that for a time varying homeomorphism F, nonwandering set for its k^{th} iterate F^k is a subset of nonwandering set of F.

Theorem 4.1.4 *Let $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on a metric space (X, d). Then for any $k \geq 1$, $\Omega(F^k) \subseteq \Omega(F)$.*

Proof : Fix $k \geq 1$. Given $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism then $F^k = G = \{g_n\}_{n=0}^{\infty}$ where

$$g_n = F_{[(n-1)k+1,nk]}.$$

Let $x \in \Omega(F^k)$ then for any neighborhood U of x and for any $n > 0$ there exists $m \geq n$ and $r \geq 0$ such that

$$G_{[m,m+r]}(U) \cap U \neq \emptyset \ \ or \ \ G^{-1}_{[m,m+r]}(U) \cap U \neq \emptyset.$$

Now here

$$\begin{aligned}
G_{[m,m+r]} &= g_{m+r} \circ \cdots \circ g_{m+1} \circ g_m \\
&= F_{[(m+r-1)k+1,(m+r)k]} \circ \cdots \circ F_{[mk+1,(m+1)k]} \circ F_{[(m-1)k+1,mk]} \\
&= f_{(m+r)k} \circ \cdots \circ f_{(m+r-1)k+1} \circ \cdots \circ f_{(m+1)k} \circ \cdots \\
&\qquad\qquad\qquad \cdots \circ f_{mk+1} \circ f_{mk} \circ \cdots \circ f_{(m-1)k+1} \\
&= F_{[(m-1)k+1,(m+r)k]} \\
&= F_{[m_1,m_1+r_1]}
\end{aligned}$$

where $m_1 = (m-1)k + 1 \geq n$ and $r_1 = rk + k - 1 \geq 0$. Note that since $m \geq n$, $m - 1 \geq n - 1$ and $(m-1)k \geq (n-1)k \geq n - 1$ (*as* $k \geq 1$), we have $m_1 = (m-1)k + 1 \geq (n-1) + 1 = n$.

Also $r_1 = rk + k - 1 > 0$ as $r \geq 0$ and $k \geq k - 1 \geq 0$.

Thus we have $m_1 \geq n$ and $r_1 \geq 0$ such that

$$F_{[m_1, m_1+r_1]}(U) \cap U \neq \emptyset \quad or \quad F^{-1}_{[m_1, m_1+r_1]}(U) \cap U \neq \emptyset.$$

This implies that $x \in \Omega(F)$. Since $x \in \Omega(F^k)$ is arbitrary, we have $\Omega(F^k) \subseteq \Omega(F)$.

4.2 Transitivity in Nonautonomous Discrete Dynamical Systems

We now define the notions of transitivity and strong transitivity for an invertible nonautonomous discrete dynamical system and show that an invertible nonautonomous discrete dynamical system is topologically transitive then each point is a nonwandering point.

Definition 4.2.1 *Let $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on a metric space (X, d). Then F is said to be* **transitive on** *X if for any two nonempty open sets U and V of X and for any $k \geq 0$, there exists $n > m \geq k$ such that*

$$F_{[m+1,n]}(U) \cap V \neq \emptyset$$

or

$$F^{-1}_{[m+1,n]}(U) \cap V \neq \emptyset.$$

Definition 4.2.2 *Let $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on a metric space (X, d). Then F is said to be* **strongly transitive on** *X if there exists $x \in X$ such that for any $k \geq 0$,*

$$O_k(x) = \{F_n(x) \colon n \in \mathbb{Z} - \{-k, -k+1, \cdots, k\}\}$$

is dense in X i.e. $\overline{O_k(x)} = X$.

We first show that strongly transitivity implies transitivity for an invertible nonautonomous discrete dynamical system.

Theorem 4.2.1 *Let $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on a metric space (X, d). If F is strongly transitive on X then it is transitive on X.*

Proof : Given that F is strongly transitive on X. Then for any $k \geq 0$, $\overline{O_k(x)} = X$. Then for any two nonempty open sets U and V of X and for any $k \geq 0$, there exist $n > m \geq k$ such that

$$F_m(x) \in U \ \ and \ \ F_n(x) \in V \quad or \quad F_m^{-1}(x) \in U \ \ and \ \ F_n^{-1}(x) \in V.$$

Hence

$$F_m(x) \in U \ \ and \ \ F_{[m+1,n]}(F_m(x)) \in V$$

or

$$F_m^{-1}(x) \in U \ \ and \ \ F_{[m+1,n]}^{-1}(F_m^{-1}(x)) \in V.$$

Thus

$$F_m(x) \in F_{[m+1,n]}(U) \cap V \ \ or \ \ F_m^{-1}(x) \in F_{[m+1,n]}(U) \cap V$$

i.e.

$$F_{[m+1,n]}(U) \cap V \neq \emptyset \ \ or \ \ F_{[m+1,n]}^{-1}(U) \cap V \neq \emptyset.$$

Following example show that the converse is not true.

Example 4.1 *Let*

$$Y = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{1 - \frac{1}{n} : n \in \mathbb{N}\},$$

where $\mathbb{N}$ is the set of all positive integers, under the usual metric d given by $d(x, y) = |x - y|$. Define the map $f : Y \to [0, 1]$ by

$$f(y) = \begin{cases} 0 & if \ \ y = 0 \ or \ y = 1 \\ y & otherwise. \end{cases}$$

Consider the quotient space $X = Y/f$ and the shift map σ on X defined as follows :

$$\sigma(x) = \begin{cases} \{\frac{1}{n-1}\} & if \ \ x = \{\frac{1}{n}\},\ n > 2; \\ \{1 - \frac{1}{n+1}\} & if \ \ x = \{1 - \frac{1}{n}\},\ n \geq 2, \\ x & x = \{0, 1\}. \end{cases}$$

Consider time varying homeomorphism $F = \{f_n\}_{n=0}^{\infty}$ on X where $f_n = \sigma,\ n \geq 0$. It is easy to see that F is transitive but not strongly transitive.

Now we show that topological transitivity implies every point is nonwandering point.

Theorem 4.2.2 *Let (X, d) be a metric space and $F = \{f_n\}_{=0}^{\infty}$ be a time varying homeomorphism on X. If F is topologically transitive then $\Omega(F) = X$.*

Proof : Let $x \in X$, U_x be a neighborhood of x and $k \geq 0$ be given. Since F is topologically transitive, for any two nonempty open sets $U, V \subset X$, there exist $n > m \geq k$ such that

$$F_{[m+1,n]}(U) \cap V \neq \phi \ \ or \ \ F^{-1}_{[m+1,n]}(U) \cap V \neq \phi.$$

In particular for $V = U = U_x$ and for $m_1 = m + 1 > k$ and $r_1 = n - m - 1 \geq 0$ (as $n > m$, $n - m > 0$ and therefore $n - m - 1 \geq 0$; also $n = m_1 + r_1$) we have

$$F_{[m_1,m_1+r_1]}(U_x) \cap U_x \neq \phi \ \ or \ \ F^{-1}_{[m_1,m_1+r_1]}(U_x) \cap U_x \neq \phi$$

Thus $x \in \Omega(F)$. Hence $\Omega(F) = X$.

4.3 Limit Sets and Nonwandering Set

We define α-limit set, ω-limit set and recurrent set for an invertible nonautonomous discrete dynamical system and show that they are contained in the set of all nonwandering points.

Definition 4.3.1 *Let (X, d) be a metric space and $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on X. By α-* **limit set of a point** *$x \in X$, we mean the set*

$$\alpha(x) = \{y \in X | \lim_{i\to\infty} d(F_{n_i}(x), y) = 0\},$$

where $\{n_i\}$ is some strictly decreasing sequence of negative integers.

Definition 4.3.2 *Let (X, d) be a metric space and $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on X. By ω-* **limit set of a point** *$x \in X$, we mean the set*

$$\omega(x) = \{y \in X | \lim_{i\to\infty} d(F_{m_i}(x), y) = 0\},$$

where $\{m_i\}$ is some strictly increasing sequence of positive integers.

Definition 4.3.3 *Let (X, d) be a metric space and $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on X. A point $x \in X$ is said to be* **recurrent** *if $x \in \alpha(x) \cap \omega(x)$. We denote the set of all recurrent points of F by $\mathcal{R}(F)$ and the closure of it by $C(F)$.*

Remark 4.1 *Now from the definition of $Per(F)$, if $x \in Per(F)$, then there is a $k > 0$ such that $F_{i+k}(x) = F_i(x)$; $i = 1, 2, 3, \ldots$ then there exist a strictly increasing sequence $n_i = ik$; $i = 0, 1, 2, \ldots$ and a strictly decreasing sequence $m_i = -ik$, $i = 0, 1, 2, \ldots$ such that $F_{n_i}(x) = x = F_{m_i}(x)$, $i = 0, 1, 2, \ldots$ and therefore*

$$\lim_{i\to\infty} d(F_{n_i}(x), x) = 0 = \lim_{i\to\infty} d(F_{m_i}(x), x).$$

Thus $x \in \alpha(x) \cap \omega(x)$ i.e. $x \in \mathcal{R}(x)$. Hence $Per(F) \subseteq \mathcal{R}(F)$.

Theorem 4.3.1 *Let (X, d) be a compact metric space and $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on X. Then for any $x \in X$, $\omega(x) \subseteq \Omega(F)$ and $\alpha(x) \subseteq \Omega(F)$.*

Proof : Let $x \in X$ and $y \in \omega(x)$ be given. Let $n > 0$ and U be a neighborhood of y. Then there exists a strictly increasing sequence $\{n_k\}_{k=0}^{\infty}$ of positive integers such that $F_{n_k}(x) \to y$.

Thus there exist $n_j > n_i > n$ such that $F_{n_i}(x) \in U$ and $F_{n_j}(x) \in U$,i.e. $F_{[n_i+1,n_j]}(F_{n_i(x)}) \in U$. Hence

$$F_{[n_i+1,n_i+1+(n_j-n_i-1)]}(U) \cap U \neq \phi$$

$$i.e. \quad F_{[m,m+r]}(U) \cap U \neq \phi,$$

where $m = n_i + 1 \geq n$ and $r = (n_j - n_i - 1) \geq 0$. Thus $y \in \Omega(F)$. Therefore $\omega(x) \subseteq \Omega(F)$, for any $x \in X$.

Similarly let $x \in X$ and $y \in \alpha(x)$ be given. Let $n > 0$ and U be a neighborhood of y. Then there exists a strictly decreasing sequence $\{m_k\}_{k=0}^{\infty}$ of negative integers such that $F_{m_k}(x) \to y$. Thus there exist $m_j < m_i < -n$ such that

$$F_{m_i}(x) \in U \;\; and \;\; F_{m_j}(x) \in U,$$

$$i.e. \quad F^{-1}_{[-m_i+1,-m_j]}(F_{m_i(x)}) \in U.$$

Thus

$$F^{-1}_{[-m_i+1,-m_i+1+(m_i-m_j-1)]}(U) \cap U \neq \phi$$

$$i.e. \;\; F^{-1}_{[m,m+r]}(U) \cap U \neq \phi,$$

where $m = -m_i + 1 \geq n$ and $n = (m_i - m_j - 1) \geq 0$. Thus $y \in \Omega(F)$. Therefore $\alpha(x) \subseteq \Omega(F)$, for any $x \in X$.

Remark 4.2 *From the definition of $\mathcal{R}(F)$, for any $x \in \mathcal{R}(F)$, $x \in \alpha(x) \cap \omega(x)$ and from the above theorem $\alpha(x) \cap \omega(x) \subseteq \Omega(F)$. Thus $\mathcal{R}(F) \subseteq \Omega(F)$. For a compact metric space $\Omega(F)$ is closed and we have $C(F) \subseteq \Omega(F)$.*

Example 4.2 *We construct a time varying homeomorphism for which every point is a recurrent point and hence nonwandering point but not all points are periodic. Hence the set of all periodic points is a proper subset of the set of all nonwandering points.*

Let $X = [0, 1]$ and $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism such that

$$f_n(x) = \begin{cases} x & for \;\; n = i^2, \;\; i = 1, 2, 3, \dots; \\ x^{1/(k+1)} & for \;\; n = i^2 + k, \;\; k = 1, 2, \dots i \;\; where \;\; i = 1, 2, 3, \dots; \\ x^{k+1} & for \;\; n = (i+1)^2 - k, \;\; k = 1, 2, \dots i \;\; where \;\; i = 1, 2, 3, \dots. \end{cases}$$

Now for

- $n = 1 = (1)^2, \; f_1(x) = x;$
- $n = 2 = (1)^2 + 1, \; f_2(x) = x^{1/(1+1)} = \sqrt{x};$
- $n = 3 = (1+1)^2 - 1, \; f_3(x) = x^{1+1} = x^2;$
- $n = 4 = (2)^2, \; f_4(x) = x;$
- $n = 5 = (2)^2 + 1, \; f_5(x) = x^{1/(1+1)} = \sqrt{x};$
- $n = 6 = (2)^2 + 2, \; f_6(x) = x^{1/(1+2)} = \sqrt[3]{x};$
- $n = 7 = (3)^2 - 2, \; f_7(x) = x^{(1+2)} = x^3;$
- $n = 8 = (3)^2 - 1, \; f_8(x) = x^{(1+1)} = x^2;$
- $n = 9 = (3)^2, \; f_9(x) = x;$

and so on...

Thus we have sequence

$$\{x, \sqrt{x}, x^2, x, \sqrt{x}, \sqrt[3]{x}, x^3, x^2, x, \dots\}$$

of homeomorphisms. Clearly, for each $x \in [0, 1]$,

$$F_{i^2}(x) = x, \; i = 1, 2, 3, \dots.$$

Thus x is a recurrent point as it infinitely many times re-appears in its orbit and period of appearance is getting larger and larger , except for 0 and 1 which are fixed points. So no point except 0 and 1 is periodic. Thus $Per(F)$ is proper subset of $\mathcal{R}(F)$.