Chapter 5

Chain Recurrent Sets in Nonautonomous Discrete Dynamical Systems

In this chapter, we define, give examples and study chain recurrent sets and weak isolated sets in a nonautonomous discrete dynamical system induced by a sequence of homeomorphisms.

5.1 Properties of Chain Recurrent Sets in Nonautonomous Discrete Dynamical Systems

We first define chain recurrent point for a time varying homeomorphism.

Definition 5.1.1 *Let (X, d) be a metric space and $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on X. A point $x \in X$ is said to be a* **chain recurrent point for** *F if for any $\delta > 0$ and for any $n > 0$, there exist $m \geq n$ and a finite sequence $\{x_i\}_{i=0}^{k}$ of points of X with $x_0 = x_k = x$ such* *that*

$$d(f_{m+i}(x_i), x_{i+1}) < \delta,$$

for all $i = 0, 1, \ldots, k-1$ *or*

$$d(f^{-1}_{m+i}(x_i), x_{i+1}) < \delta,$$

for all $i = 0, 1, \ldots, k-1$. *The sequence* $\{x_i\}_{i=0}^{k}$ *is said to be* **a** δ**-chain for x with action starting at m**. *The set of all chain-recurrent points of F is denoted by* $CR(F)$.

Remark 5.1 *Let* $x_0 \in X$ *be a periodic point for F, then there exists* $k > 0$ *such that* $x_{i+k} = x_i$, *for all* $i \in \mathbb{Z}$, *where* $x_{n+1} = f_{n+1}(x_n)$, *for all* $n \geq 0$ *and* $x_n = f^{-1}_{-n}(x_{n+1})$, *for all* $n < 0$. *Now for any* $\delta > 0$ *and for any* $n \geq 0$, *there exist* $m = jk \geq n$ *and a finite sequence* $\{x_{jk+i}\}_{i=0}^{k}$ *of points of X with* $x_{jk} = x_{(j+1)k} = x_0$ *such that*

$$\begin{aligned} d(f_{m+i}(x_{m+i}), x_{m+i+1}) &= d(f_{jk+i}(x_{jk+i}), x_{jk+i+1}) \\ &= d(x_{jk+i+1}, x_{jk+i+1}) \\ &= 0 < \delta, \end{aligned}$$

for all $i = 0, 1, \ldots, k-1$. *Hence* x_0 *is a chain recurrent point. Thus* $Per(F) \subseteq CR(F)$. *Also note that the set of periodic points need not be closed when* $CR(F)$ *is closed.*

The following example shows that $Per(F)$ is a proper subset of $CR(F)$.

Example 5.1 *Let* $F = \{f_n\}_{n=0}^{\infty}$ *be a time varying homeomorphism on* $X = [0, 1]$, *where for* $x \in X$ *and* $n \geq 0$,

$$f_n(x) = \begin{cases} x^n, & if \;\; n \;\; is \;\; non-prime; \\ x & if \;\; n \;\; is \;\; prime. \end{cases}$$

Note that 0 *and* 1 *are the only periodic points. For any* $x \in [0, 1]$, $\delta > 0$ *and* $n > 0$, *there exist* $m \geq n$ *which is prime. Hence* $\{x, f_m(x) = x\}$ *is a* δ*-chain for x with action starting at m. Hence x is chain recurrent. Thus* $CR(F) = X$.

Following is an example of a chain recurrent set which is a proper subset of X.

Example 5.2

$$X = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\},$$

where $\mathbb{N}$ is the set of all positive integers, under the usual metric d given by $d(x, y) = |x - y|$. Consider the map σ on X defined as follows :

$$\sigma(x) = \begin{cases} \frac{1}{n-1}, & if \;\; x = \frac{1}{n},\ n > 2; \\ 1 - \frac{1}{n+1}, & if \;\; x = 1 - \frac{1}{n},\ n \geq 2, \\ x, & if \;\; x = 0 \;\; or \;\; x = 1. \end{cases}$$

Consider time varying homeomorphism $F = \{f_n\}_{n=0}^{\infty}$ on X where $f_n = \sigma^n,\ n \geq 0$.

For any $x \in X - \{0, 1\}$, $0 < \delta < \frac{1}{6}$ there exists $n > 0$ such that for any $m \geq n$,

$$d(f_m(x), 1) < \delta \;\; and \;\; d(f_m^{-1}(x), 0) < \delta.$$

Thus there does not exist any δ-chain for x with action starting at $m \geq n$ and therefore $x \notin CR(F)$. Since 0 and 1 are fixed points, we have $CR(F) = \{0, 1\}$.

Next, we show that for an invertible nonautonomous discrete dynamical system on a compact metric space, if the family of homeomorphisms generating the time varying homeomorphism and its inverse maps is equicontinuous, then the set of all chain recurrent points is a closed set. Moreover, in this case, it contains the set of all nonwandering points .

Theorem 5.1.1 *Let $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on a compact metric space (X, d). If the family of homeomorphisms $\{f_n, f_n^{-1}\}_{n=0}^{\infty}$ is equicontinuous then $CR(F)$ is a closed set.*

Proof : Let $\{y_j\}_{j=0}^{\infty}$ be a sequence of points of $CR(F)$ converging to some $y \in X$. Let $\varepsilon > 0$ and $N > 0$ be given. Now X being compact, the family $\{f_n, f_n^{-1}\}_{n=0}^{\infty}$ is uniformly equicontinuous on X therefore there exists $0 < \delta < \frac{\varepsilon}{2}$ such that

$$d(f_i(x), f_i(y)) < \frac{\varepsilon}{2} \ \ and \ \ d(f_i^{-1}(x), f_i^{-1}(y)) < \frac{\varepsilon}{2},$$

for all $i \geq 0$, whenever $d(x, y) < \delta$. Since $y_j \to y$, there exists $n > 0$ such that $d(y_n, y) < \delta$. Now since $y_n \in CR(F)$ therefore there exist $m \geq N$ and a finite sequence $\{x_i\}_{i=0}^{k}$ with $x_0 = x_k = y_n$ such that

$$d(f_{m+i}(x_i), x_{i+1}) < \frac{\varepsilon}{2}, \ \ for \ \ all \ \ i = 0, 1, \ldots, k-1 \tag{5.1}$$

or

$$d(f_{m+i}^{-1}(x_i), x_{i+1}) < \frac{\varepsilon}{2}, \ \ for \ \ all \ \ i = 0, 1, \ldots, k-1. \tag{5.2}$$

Now

$$d(y, y_n) < \delta \Rightarrow d(f_m(y), f_m(y_n)) < \frac{\varepsilon}{2},$$

$$d(f_m(y_n), x_1) = d(f_m(x_0), x_1) < \frac{\varepsilon}{2}$$

and

$$d(f_{m+k-1}(x_{k-1}), y_n) = d(f_{m+k-1}(x_{k-1}), x_k) < \frac{\varepsilon}{2}.$$

In case 5.1 holds then we have

$$d(f_m(y), x_1) \leq d(f_m(y), f_m(y_n)) + d(f_m(y_n), x_1) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Also

$$d(f_{m+k-1}(x_{k-1}), y) \leq d(f_{m+k-1}(x_{k-1}), y_n) + d(y_n, y) < \frac{\varepsilon}{2} + \delta < \varepsilon.$$

Thus, taking $z_0 = z_k = y$ and $z_i = x_i$, for $i = 1, 2, \ldots, k-1$, we have a finite sequence $\{z_i\}_{i=0}^{k}$ such that

$$d(f_{m+i}(z_i), z_{i+1}) < \varepsilon,$$

for all $i = 0, 1, \ldots, k-1$ an therefore $y \in CR(F)$.
Similarly

$$d(y, y_n) < \delta \implies d(f_m^{-1}(y), f_m^{-1}(y_n)) < \frac{\varepsilon}{2},$$
$$d(f_m^{-1}(y_n), x_1) = d(f_m^{-1}(x_0), x_1) < \frac{\varepsilon}{2}$$

and

$$d(f_{m+k-1}^{-1}(x_{k-1}), y_n) = d(f_{m+k-1}^{-1}(x_{k-1}), x_k) < \frac{\varepsilon}{2}.$$

In case 5.2 holds then we have

$$d(f_m^{-1}(y), x_1) \leq d(f_m^{-1}(y), f_m^{-1}(y_n)) + d(f_m^{-1}(y_n), x_1) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Also

$$d(f_{m+k-1}^{-1}(x_{k-1}), y) \leq d(f_{m+k-1}^{-1}(x_{k-1}), y_n) + d(y_n, y) < \frac{\varepsilon}{2} + \delta < \varepsilon.$$

Thus taking $z_0 = z_k = y$ and $z_i = x_i$, for $i = 1, 2, \ldots, k-1$, we have a finite sequence $\{z_i\}_{i=0}^{k}$ such that

$$d(f_{m+i}^{-1}(z_i), z_{i+1}) < \varepsilon,$$

for all $i = 0, 1, \ldots, k-1$. Thus $y \in CR(F)$. Hence $CR(F)$ is closed.

Theorem 5.1.2 *Let $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on a compact metric space (X, d). If the family $\{f_n, f_n^{-1}\}_{n=0}^{\infty}$ is equicontinuous on X then $\Omega(F) \subseteq CR(F)$.*

Proof : Let $x \in \Omega(F)$, $\varepsilon > 0$ and $n > 0$ be given. Now X being compact, the family $\{f_n, f_n^{-1}\}_{n=0}^{\infty}$ is uniformly equicontinuous on X therefore there exists $0 < \delta < \varepsilon$ such that

$$d(f_i(x), f_i(y)) < \varepsilon \;\; and \;\; d(f_i^{-1}(x), f_i^{-1}(y)) < \varepsilon,$$

for all $i \geq 0$, whenever $d(x, y) < \delta$. For $\delta > 0$ there exist $m \geq n$ and $r \geq 0$ such that

$$F_{[m,m+r]}(U_\delta(x)) \cap U_\delta(x) \neq \phi \;\; or \;\; F_{[m,m+r]}^{-1}(U_\delta(x)) \cap U_\delta(x) \neq \phi,$$

where $U_\delta(x) = \{y \in X \colon d(x,y) < \delta\}$. Equivalently, there exists $y \in X$ such that $d(x,y) < \delta$ and

$$d(F_{[m,m+r]}(y), x) < \delta \ \ \textit{or} \ \ d(F^{-1}_{[m,m+r]}(y), x) < \delta.$$

If

$$d(F_{[m,m+r]}(y), x) < \delta$$

then put $x_0 = x_{r+1} = x$ and $x_i = F_{[m,m+i-1]}(y)$, for $i = 1, 2, \ldots, r$. We get

$$d(f_m(x_0), x_1) = d(f_m(x), f_m(y)) < \varepsilon$$

(as $x_0 = x$ and $x_1 = F_{[m,m]}(y) = f_m(y)$) and

$$d(f_{m+r}(x_r), x_{r+1}) = d(F_{[m,m+r]}(y), x) < \delta < \varepsilon$$

(as $f_{m+r}(x_r) = f_{m+r}(F_{[m,m+r-1]}(y)) = F_{[m,m+r]}(y)$.) Thus

$$d(f_{m+i}(x_i), x_{i+1}) < \varepsilon,$$

for $i = 0, 1, \ldots, r$.

On the other hand, if

$$d(F^{-1}_{[m,m+r]}(y), x) < \delta$$

then put $x_0 = x_{r+1} = x$ and $x_i = F^{-1}_{[m,m+i-1]}(y)$, for $i = 1, 2, \ldots, r$. We get

$$d(f^{-1}_m(x_0), x_1) = d(f^{-1}_m(x), f^{-1}_m(y)) < \varepsilon$$

(using the facts $x_0 = x$ and $x_1 = F^{-1}_{[m,m]}(y) = f^{-1}_m(y)$) and

$$d(f^{-1}_{m+r}(x_r), x_{r+1}) = d(F^{-1}_{[m,m+r]}(y), x) < \delta < \varepsilon$$

(using the facts $f^{-1}_{m+r}(x_r) = f^{-1}_{m+r}(F^{-1}_{[m,m+r-1]}(y)) = F^{-1}_{[m,m+r]}(y)$). Hence

$$d(f^{-1}_{m+i}(x_i), x_{i+1}) < \varepsilon,$$

for $i = 0, 1, \ldots, r$. Thus, in any case $\{x_i\}_{i=0}^{r+1}$ is an ε-chain for x with action starting at m which proves that $x \in CR(F)$.

In the following example $\Omega(F)$ is a proper subset of $CR(F)$.

Example 5.3 *Let*

$$Y = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\},$$

where $\mathbb{N}$ *is the set of all positive integers under the usual metric* d_0 *given by* $d_0(x, y) = |x - y|$. *Define a map* $f : Y \to Y$ *by*

$$f(y) = \begin{cases} 0 & if \;\; y = 0 \text{ or } y = 1; \\ y & y \in Y - \{0, 1\}. \end{cases}$$

Consider the quotient space $X = Y/f$ *with metric* d *defined on* X *as follows. For any* $\{a\}, \{b\} \in X$,

$$d(\{a\}, \{b\}) = \min\{d_0(a, b), 1 - d_0(a, b)\}$$

Define shift map σ *on* X *as follows :*

$$\sigma(x) = \begin{cases} \{\frac{1}{n-1}\} & if \;\; x = \{\frac{1}{n}\},\; n > 2; \\ \{1 - \frac{1}{n+1}\} & if \;\; x = \{1 - \frac{1}{n}\},\; n \geq 2, \\ x & if \;\; x = \{0, 1\}. \end{cases}$$

Consider the time varying homeomorphism $F = \{f_n\}_{n=0}^{\infty}$ *on* X, *where* $f_n = \sigma^n$, $n \geq 0$. *Let* $x \in X - \{A\}$, *where* $A = \{0, 1\}$ *be given. Put* $\varepsilon = \frac{d(x,A)}{4}$. *Then there exists* $n > 0$ *such that for any* $m \geq n$, *any* $r \geq 0$ *and for any* $y \in U_\varepsilon(x)$,

$$d(F_{[m,m+r]}(y), A) < \varepsilon.$$

Therefore

$$F_{[m,m+r]}(U_\varepsilon(x)) \cap U_\varepsilon(x) = \phi$$

which implies x *is not a nonwandering point. Let* $x \in X$ *be fixed. Now for any* $\varepsilon > 0$ *and* $n \geq 0$, *there exists* $m \geq n + 1$ *such that*

$$F_{[n,m-1]}(x) \in U_{\frac{\varepsilon}{2}}(A).$$

We can choose $y \in U_{\frac{\varepsilon}{2}}(A)$ such that $F_{[m,m+r]}(y) = x$ for some $r \geq 0$. Note that

$$\begin{aligned} d(f_{m-1}(F_{[n,m-2]}(x)), y) &= d(F_{[n,m-1]}(x), y) \\ &\leq d(F_{[n,m-1]}(x), A) + d(A, y) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus

$$\{x, f_n(x), F_{[n,n+1]}(x), \cdots, F_{[n,m-2]}(x), y, f_m(y), F_{[m,m+1]}(y), \cdots \\ \ldots, F_{[m,m+r]}(y) = x\}$$

is an ε-chain for x with action starting at n. Hence $x \in CR(F)$. Thus $CR(F) = X$ and $\Omega(F) = \{A\}$.

In the above example, if $f_n = \sigma,\ n \geq 0$ then for time varying homeomorphism $F = \{f_n\}_{n=0}^{\infty}$, we have $\Omega(F)$ is a proper subset of $CR(F)$ and the family $\{f_n, f_n^{-1}\}_{n=0}^{\infty}$ is equicontinuous on X being a finite family of homeomorphisms $\{\sigma, \sigma^{-1}\}$.

5.2 Chain Recurrence and Shadowing Property

In this section, we study chain recurrent sets in an invertible nonautonomous discrete dynamical system having shadowing property.

Theorem 5.2.1 *Let $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on a metric space (X, d). If F has P.O.T.P. on X then $CR(F) \subseteq \Omega(F)$.*

Proof : Let $x \in CR(F)$, $\varepsilon > 0$ and $n_0 > 0$ be given. Since F has P.O.T.P., there exists a $\delta > 0$ such that any δ-pseudo orbit can be ε-traced by an orbit of F.

Since $x \in CR(F)$, there exists m with $m + 1 \geq n_0$ and a finite sequence $\{x_i\}_{i=m}^{m+r}$ with $x_m = x_{m+r} = x$ such that

$$d(f_{m+i+1}(x_{m+i}), x_{m+i+1}) < \delta \ \ for \ \ i = 0, 1, \ldots, r-1,$$

or

$$d(f^{-1}_{m+i+1}(x_{m+i}), x_{m+i+1}) < \delta \ \ for \ \ i = 0, 1, \ldots, r-1.$$

In the first case, put $z = (F_m)^{-1}(x)$, $x_i = F_i(z)$, for all $i < m$ and $x_i = F_{[m+r+1,i]}(x)$, for all $i > m + r$. Hence

$$d(f_{n+1}(x_n), x_{n+1})$$

$$= \begin{cases} d(f_{n+1}(F_n(z)), F_{n+1}(z)) & if \ \ n < m; \\ d(f_{m+i+1}(x_{m+i}), x_{m+i+1}) & if \ \ n = m + i, \ \ o \leq i \leq r-1; \\ d(f_{m+r+1}(x_{m+r}), x_{m+r+1}) & if \ \ n = m + r; \\ d(f_{m+r+2+i}(F_{[m+r+1,m+r+1+i]}(x)), & \\ \quad F_{[m+r+1,m+r+2+i]}(x)) & if \ \ n = m + r + 1 + i, \ \ i \geq 0; \end{cases}$$

$$= \begin{cases} d(F_{n+1}(z), F_{n+1}(z)) & if \ \ n < m; \\ d(f_{m+i+1}(x_{m+i}), x_{m+i+1}) & if \ \ n = m + i, \ \ o \leq i \leq r-1; \\ d(f_{m+r+1}(x), f_{m+r+1}(x)) & if \ \ n = m + r; \\ d(F_{[m+r+1,m+r+2+i]}(x), & \\ \quad F_{[m+r+1,m+r+2+i]}(x)) & if \ \ n = m + r + 1 + i, \ \ i \geq 0; \end{cases}$$

$$< \delta \quad (in \ \ any \ \ case)$$

Thus we get a δ-pseudo orbit $\{x_i\}_{i=-\infty}^{\infty}$ for F. By the P.O.T.P. of F, there exists a $y \in X$ such that

$$d(F_i(y), x_i) < \varepsilon,$$

for every $i \in \mathbb{Z}$. Note that $x_m = x_{m+r} = x$,

$$d(F_m(y), x_m) < \varepsilon \quad and \quad d(F_{m+r}(y), x_{m+r}) < \varepsilon$$

i.e.

$$d(F_m(y), x) < \varepsilon \quad and \quad d(F_{[m+1,m+r]}(F_m(y)), x) < \varepsilon.$$

Thus

$$F_{[m+1,m+r]}(U_\varepsilon(x)) \cap U_\varepsilon(x) \neq \emptyset,$$

where $U_\varepsilon(x) = \{y \in X \colon d(x, y) < \varepsilon\}$. Similarly in the latter case, put $z = (F_{(-m)})^{-1}(x)$, $x_i = F_{(-i)}(z)$, for all $i < m$ and $x_i = F^{-1}_{[m+r+1,i]}(x)$, for all $i > m + r$. Hence

$$d(f^{-1}_{n+1}(x_n), x_{n+1})$$

$$= \begin{cases} d(f^{-1}_{n+1}(F_{-n}(z)), F_{-(n+1)}(z)) & if \;\; n < m; \\ d(f^{-1}_{m+i+1}(x_{m+i}), x_{m+i+1}) & if \;\; n = m + i, \;\; o \leq i \leq r - 1; \\ d(f^{-1}_{m+r+1}(x_{m+r}), x_{m+r+1}) & if \;\; n = m + r; \\ d(f^{-1}_{m+r+2+i}(F^{-1}_{[m+r+1,m+r+1+i]}(x)), & \\ \quad F^{-1}_{[m+r+1,m+r+2+i]}(x)) & if \;\; n = m + r + 1 + i, \;\; i \geq 0; \end{cases}$$

$$= \begin{cases} d(F_{-(n+1)}(z)), F_{-(n+1)}(z))) & if \;\; n < m; \\ d(f^{-1}_{m+i+1}(x_{m+i}), x_{m+i+1}) & if \;\; n = m + i, \;\; o \leq i \leq r - 1; \\ d(f^{-1}_{m+r+1}(x), f^{-1}_{m+r+1}(x)) & if \;\; n = m + r; \\ d(F^{-1}_{[m+r+1,m+r+2+i]}(x), & \\ \quad F^{-1}_{[m+r+1,m+r+2+i]}(x)) & if \;\; n = m + r + 1 + i, \;\; i \geq 0; \end{cases}$$

$$< \delta \quad (in \;\; any \;\; case)$$

Thus we get a δ-pseudo orbit $\{x_i\}_{i=-\infty}^{\infty}$ for F. By P.O.T.P. of F, there is a $y \in X$ such that

$$d(F_i(y), x_i) < \varepsilon,$$

for every $i \in \mathbb{Z}$. Note that $x_m = x_{m+r} = x$,

$$d(F_m(y), x_m) < \varepsilon \quad and \quad d(F_{m+r}(y), x_{m+r}) < \varepsilon,$$

i.e.

$$d(F_m(y), x) < \varepsilon \quad and \quad d(F_{[m+1,m+r]}(F_m(y)), x) < \varepsilon.$$

Thus

$$F_{[m+1,m+r]}(U_\varepsilon(x)) \cap U_\varepsilon(x) \neq \emptyset,$$

where $U_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}$. Similarly we can show that

$$F^{-1}_{[m+1,m+r]}(U_\varepsilon(x)) \cap U_\varepsilon(x) \neq \emptyset.$$

Hence $x \in \Omega(F)$ which proves $CR(F) \subseteq \Omega(F)$.

From Theorem 5.1.2 and Theorem 5.2.1, we have the following result.

Corollary 5.2.1 *Let $F = \{f_n\}_{n=0}^{\infty}$ be a time-varying homeomorphism on a compact metric space (X, d). If F has P.O.T.P. on X and the family $\{f_n, f_n^{-1}\}_{n=0}^{\infty}$ is equicontinuous on X then $CR(F) = \Omega(F)$.*

Proof : Since $\{f_n, f_n^{-1}\}_{n=0}^{\infty}$ is equicontinuous therefore by Theorem 5.1.2 (76), $\Omega(F) \subseteq CR(F)$. Further since F has P.O.T.P. on X, by Theorem 5.2.1 (on Page 79) we have $CR(F) \subseteq \Omega(F)$ also. Hence $CR(F) = \Omega(F)$.

5.3 Weak Isolated set for an Invertible Nonautonomous Discrete Dynamical Systems

We define and study the notion of weak isolated set for an invertible nonautonomous discrete dynamical system.

Definition 5.3.1 *Let (X, d) be a compact metric space and $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on X. A subset E of X is said to be* **weak isolated** *for F if it is compact and there exist a neighborhood U of E such that for any $y \in X$, $F_n(y) \in cl(U)$, for every $n \in \mathbb{Z}$, where $cl(U)$ is the closure of set U in X, implies that $y \in E$ i.e.*

$$\cap_{n=-\infty}^{\infty}(F_n)^{-1}(cl(U)) \subseteq E.$$

The following result gives a sufficient condition under which the set of all chain recurrent points is weak isolated.

Theorem 5.3.1 *Let (X, d) be a compact metric space and $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on X with the family $\{f_n, f_n^{-1}\}_{n=0}^{\infty}$ being equicontinuous. If F is expansive on X and $F|_{CR(F)}$ has P.O.T.P. then $CR(F)$ is weak isolated.*

Proof : Let $\varepsilon > 0$ be an expansive constant for F. Since $F|_{CR(F)}$ has P.O.T.P., for $0 < \beta < \frac{\varepsilon}{2}$, there exists $\alpha > 0$ such that any α-pseudo orbit is β-traced by F.

Since X is compact, therefore $\{f_n, f_n^{-1}\}_{n=0}^{\infty}$ is uniformly continuous and therefore there exists $0 < \gamma < \min\{\frac{\alpha}{2}, \frac{\varepsilon}{2}\}$, such that for any $x, y \in X$,

$$d(x, y) < \gamma \Rightarrow d(f_n(x), f_n(y)) < \frac{\alpha}{2} \ \ and \ \ d(f_n^{-1}(x), f_n^{-1}(y)) < \frac{\alpha}{2},$$

for any $n \geq 0$. Let $0 < \delta < \gamma$ and $U = \{y \in X \colon d(y, CR(F)) < \delta\}$. Choose $y \in X$ such that $F_n(y) \in cl(U)$, for every $n \in \mathbb{Z}$. It remains to show that $y \in CR(F)$. Note that

$$cl(U) = \{y \in X \colon d(y, CR(F)) \leq \delta\}.$$

Hence there is $x_n \in CR(F)$ with

$$d(F_n(y), x_n) \leq \delta, \ \ for \ \ every \ \ n \in \mathbb{Z}.$$

Now

$$d(x_n, F_n(y)) \leq \delta < \gamma$$

and therefore

$$d(f_{n+1}(x_n), f_{n+1}(F_n(y))) < \frac{\alpha}{2}$$

and

$$d(f_{-n+1}^{-1}(x_n), f_{-n+1}^{-1}(F_n(y))) < \frac{\alpha}{2}.$$

Thus for $n \geq 0$

$$d(f_{n+1}(x_n), x_{n+1}) \leq d(f_{n+1}(x_n), f_{n+1}(F_n(y))) + d(F_{n+1}(y), x_{n+1}) < \frac{\alpha}{2} + \gamma < \alpha,$$

and for $n < 0$ we have,

$$d(f^{-1}_{-n+1}(x_n), x_{n-1}) \leq d(f^{-1}_{-n+1}(x_n), f^{-1}_{-n+1}(F_n(y))) + d(F_{n-1}(y), x_{n-1}) < \frac{\alpha}{2} + \gamma < \alpha$$

which implies $\{x_n\}_{n=-\infty}^{\infty}$ is an α-pseudo orbit for F. Hence there is a β-tracing point $x \in CR(F)$ satisfying

$$d(F_n(x), x_n) < \beta,$$

for every $n \in \mathbb{Z}$. Thus for any $n \in \mathbb{Z}$,

$$d(F_n(y), F_n(x)) \leq d(F_n(y), x_n) + d(x_n, F_n(x)) < \gamma + \beta < \varepsilon.$$

Since F is expansive, we have $x = y$ and therefore $y \in CR(F)$. So $CR(F)$ is weak isolated.

Remark 5.2 *Let (X, d) be a compact metric space and $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on X, where $\{f_n, f_n^{-1}\}_{n=0}^{\infty}$ is an equicontinuous family. Then for any $x \in X$, $\omega(x) \subseteq \Omega(F) \subseteq CR(F)$ and $\alpha(x) \subseteq \Omega(F) \subseteq CR(F)$. From the definition of $\mathcal{R}(F)$ for any $x \in \mathcal{R}(F)$, $x \in \alpha(x) \cap \omega(x)$ and thus $\mathcal{R}(F) \subseteq \Omega(F) \subseteq CR(F)$. From Theorem 4.1.1 (63) $\Omega(F)$ is nonempty, so $CR(F)$ is also a nonempty set.*