Chapter 6

Spectral Decomposition Theorem in Equicontinuous Nonautonomous Discrete Dynamical Systems

In this chapter, we prove a decomposition theorem, similar to Smale's spectral decomposition theorem, in an equicontinuous nonautonomous discrete dynamical system induced by a sequence of homeomorphisms.

6.1 Weak Chain Recurrence

In this section, we define weak chain recurrence, give examples and study properties of weak chain recurrent sets in a nonautonomous discrete dynamical systems induced by a sequence of homeomorphisms.

Definition 6.1.1 *Let X be a metric space with metric d and $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on X. For any $\delta > 0$, sequence $\{(x_i, t_i)\}_{i=-\infty}^{\infty} \subset X \times \mathbb{Z}$ is called* **a weak δ- pseudo orbit for** *F if $\{t_i\}$ is an increasing sequence of integers, $\{t_{2i}\}, \{t_{2i+1}\}$ are strictly increasing sequences, $t_0 = 0$ and*

$$d(F_{[t_{2i},\, t_{2i+1}]}(x_i), x_{i+1}) < \delta, \quad for \;\; i \geq 0$$

and

$$d(F^{-1}_{[|t_{2i+2}|,\, |t_{2i+1}|]}(x_{i+1}), x_i) < \delta, \quad for \;\; i < 0.$$

Additionally, if $x_n = x$ for infinitely many $n \in \mathbb{Z}$ then the sequence $\{(x_i, t_i)\}_{i=-\infty}^{\infty}$ is said to be **a weak δ-chain for x**.

We call $\{(x_i, t_i)\}_{i=0}^{\infty}$ **a positive infinite weak δ-chain for x** *and $\{(x_i, t_i)\}_{i=-\infty}^{0}$* **a negative infinite weak δ-chain for x**. *Also $\{(x_i, t_i)\}_{i=m}^{n}$ is called* **a finite weak δ-pseudo orbit from $\mathbf{x_m}$ to $\mathbf{x_n}$** *if $0 \leq m \leq n$ or $n \leq m \leq 0$. In addition if $x_m = x_n = x$ then we call $\{(x_i, t_i)\}_{i=0}^{\infty}$* **a finite weak δ-chain for x** *with action starting at m.*

Remark 6.1 *Note that since $\{t_i\}$ is an increasing sequence therefore $t_{2i+2} \geq t_{2i+1}$ i.e. $-t_{2i+2} \leq -t_{2i+1}$. Hence using the fact that if $n < 0$ then $t_n \leq 0$, we get $|t_{2i+2}| \leq |t_{2i+1}|$, for $i < 0$. Thus*

$$F^{-1}_{[|t_{2i+2}|,\, |t_{2i+1}|]} = f_{\beta}^{-1} \circ \cdots \circ f_{\alpha+1}^{-1} \circ f_{\alpha}^{-1},$$

where $\alpha = |t_{2i+2}|$ and $\beta = |t_{2i+1}|$.

Definition 6.1.2 *Let X be a metric space with metric d and $F = \{f_n\}_{n=0}^{\infty}$ be a time-varying homeomorphism on X. A point $x \in X$ is said to be* **a weak chain recurrent point for F** *if for any $\delta > 0$ there exists a weak δ-chain $\{(x_i, t_i)\}_{i=-\infty}^{\infty}$ for x.*

The **set of all weak chain recurrent points for F** *is denoted by $WCR(F)$.*

In the following example $WCR(F) = X$.

Example 6.1 *Let $X = [0,1]$ with the usual metric d defined by $d(x,y) = |x-y|$ and $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on X such that f_0 is the identity function and*

$$f_n(x) = \begin{cases} x^{100} & for \;\; n = i^2, \;\; i = 0,1,\dots \\ x^{1/(k+1)} & for \;\; n = i^2 + k, \;\; , \;\; and \;\; k = 0,1,\dots,i \\ x^{k+2} & for \;\; n = (i+1)^2 - k, \;\; i = 0,1,2,\dots \;\; and \;\; k = 0,1,\dots,i. \end{cases}$$

Note that for

$$n = i^2, \; f_n(x) = x^{100}$$

and for

$$i^2 < n < (i+1)^2 \;\; if \;\; i^2 < n \leq i^2 + i, \;\; f_n(x) = x^{1/(k+1)},$$

where $n = i^2 + k$ and if

$$(i+1)^2 - i = i^2 + i \leq n < (i+1)^2, \;\; f_n(x) = x^{k+2},$$

where $n = (i+1)^2 - k$.

Thus we have sequence

$$\{x, x^{100}, \sqrt{x}, x^3, x^{100}, \sqrt{x}, \sqrt[3]{x}, x^4, x^3, x^{100}, \dots\}$$

of homeomorphisms. Define sequence $\{t_n\}_{n=0}^{\infty}$ as follows :

$$t_0 = t_1 = 0;$$

$$t_{2i} = t_{2i+1} = \begin{cases} (\frac{i-1}{2})^2 + 3(\frac{i-1}{2}) + 3 & if \;\; i \;\; is \;\; odd; \\ (\frac{i-2}{2})^2 + 5(\frac{i-2}{2}) + 6 & if \;\; i \;\; is \;\; even, \end{cases}$$

for $i = 0,1,2,\dots$ Thus

$$t_{4n} = t_{4n+1} = (n+1)^2 + (n+1)$$

and

$$t_{4n+2} = t_{4n+3} = (n+2)^2 - (n+1).$$

Note that

$$F_{[t_{4n},t_{4n+1}]}(x) = f_{[(n+1)^2+(n+1)]}(x) = x^{1/(n+2)}$$

and

$$F_{[t_{4n+2},t_{4n+3}]}(x) = f_{[(n+2)^2-(n+1)]}(x) = x^{n+3}$$

Now for any $x \in X$, put

$$x_0 = x,\ x_{2i} = x^{i+2}\ \ and\ \ x_{2i+1} = x,$$

for all $i = 1, 2, 3, \ldots$. Now

$$\begin{aligned} d(F_{[t_{4n},t_{4n+1}]}(x_{2n}), x_{2n+1}) &= d(f_{[(n+1)^2+(n+1)]}(x^{n+2}), x) \\ &= d(x, x) \\ &= 0 < \delta \end{aligned}$$

and

$$\begin{aligned} d(F_{[t_{4n+2},\, t_{4n+3}]}(x_{2n+1}), x_{2n+2}) &= d(f_{[(n+2)^2-(n+1)]}(x), x^{n+3}) \\ &= d(x^{n+3}, x^{n+3}) \\ &= 0 < \delta. \end{aligned}$$

Thus for any $\delta > 0$, $(t_n, x_n)_{n=0}^{\infty}$ is an infinite positive weak δ-chain for x that can be extended to an infinite weak δ-chain for x which shows that $x \in WCR(F)$ which implies that $WCR(F) = X$.

We define the notion of equicontinuous invertible nonautonomous discrete dynamical system.

Definition 6.1.3 *Let (X, d) be a metric space and $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on X. Then F is said to be* **equicontinuous time varying homeomorphism or equicontinuous invertible nonautonomous discrete dynamical system** *if*

$$\{F_{[m,n]}, F_{[m,n]}^{-1} : 0 \le m \le n\}$$

is an equicontinuous family of functions.

The following result shows that the set of all chain recurrent points is contained in the set of all weak chain recurrent points for an invertible nonautonomous discrete dynamical system.

Theorem 6.1.1 *Let $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on a compact metric space (X, d). Then the set of all chain recurrent points of F is contained in the set of all weak chain recurrent points of F i.e. $CR(F) \subseteq WCR(F)$.*

Proof: Let $x \in CR(F)$ and $\delta > 0$ be given. Then for any $n \geq 0$, there exist $m \geq n$ and a finite sequence $\{x_i\}_{i=0}^{k}$ of points of X with $x_0 = x_k = x$ such that either

$$d(f_{m+i}(x_i), x_{i+1}) < \delta, \ \ for \ \ all \ \ i \in \{0, 1, \ldots, k-1\}$$

or

$$d(f_{m+i}^{-1}(x_i), x_{i+1}) < \delta, \ \ for \ \ all \ \ i \in \{0, 1, \ldots, k-1\}.$$

For $n_0 = 0$, there exist $m_0 \geq n_0$ and a finite sequence $\{x_i\}_{i=0}^{k_0}$ of points of X with $x_0 = x = x_{k_0}$ such that either

$$d(f_{m_0+i}(x_i), x_{i+1}) < \delta, \quad for \quad all \quad i \in \{0, 1, \ldots, k_0-1\}$$

or

$$d(f_{m_0+i}^{-1}(x_i), x_{i+1}) < \delta, \quad for \quad all \quad i \in \{0, 1, \ldots, k_0-1\}$$

equivalently either

$$d(F_{[m_0+i, m_0+i]}(x_i), x_{i+1}) < \delta, \quad for \quad all \quad i \in \{0, 1, \ldots, k_0-1\}$$

or

$$d(F_{[m_0+i, m_0+i]}^{-1}(x_i), x_{i+1}) < \delta, \quad for \quad all \quad i \in \{0, 1, \ldots, k_0-1\}.$$

Further, for $n_1 \geq m_0 + k_0$ there exist $m_1 \geq n_1$ and a finite sequence $\{x_i\}_{i=k_0}^{k_0+k_1}$ of points of X with $x_{k_0} = x_{k_0+k_1} = x$ such that either

$$d(F_{[m_1+i, m_1+i]}(x_{k_0+i}), x_{k_0+i+1}) < \delta, \quad for \quad all \quad i \in \{0, 1, \ldots, k_1-1\}$$

or

$$d(F^{-1}_{[m_1+i,m_1+i]}(x_{k_0+i}), x_{k_0+i+1}) < \delta, \quad for \quad all \quad i \in \{0, 1, \ldots, k_1 - 1\}.$$

Repeating this process we get infinitely many positive or negative finite weak δ-chains for x. Without loss of generality we can assume that we get infinitely many positive finite weak δ-chains for x.

Collecting those finite weak δ-chains, we obtain a positive weak δ-chain $\{(y_n, t_n)\}$ for x, where $\{y_n\}_{n=2}^{\infty}$ is subsequence of $\{x_i\}_{i=0}^{\infty}$ with $y_0 = y_1 = x_0 = x$, $\{t_n\}_{n=0}^{\infty}$ is an increasing sequence with $t_0 = t_1 = 0$ and for $n \geq 2$, $t_{2n} = t_{2n+1} = m_j + i$ for some $j \geq 0$ and for some $i \in \{0, 1, \ldots, k_j - 1\}$. Clearly each of $\{t_{2n}\}$ and $\{t_{2n+1}\}$ is a strictly increasing sequence. This sequence can be extended to a weak δ-chain for x.

Thus $x \in WCR(F)$ and this proves $CR(F) \subseteq WCR(F)$.

Remark 6.2 *We know that a periodic point is a chain recurrent point and therefore any periodic point is a weak chain recurrent point equivalently* $Per(F) \subseteq WCR(F)$. *Also note that every* δ*-pseudo orbit* $\{(x_i)\}_{i=0}^{\infty}$ *is a weak* δ*-pseudo orbit* $\{(x_i, t_i)\}_{i=-\infty}^{\infty}$ *with* $t_{2i} = t_{2i+1} = i$*, for all* $i \in \mathbb{Z}$.

6.2 Weak Pseudo Orbit Extending Property

We define the weak pseudo orbit extending property for an invertible nonautonomous discrete dynamical system and show that for an invertible nonautonomous discrete dynamical system having this property, set of all chain recurrent points and the set of all weak chain recurrent points coincide.

Definition 6.2.1 *Let* $F = \{f_n\}_{n=0}^{\infty}$ *be a time varying homeomorphism on a compact metric space* (X, d). *Then* F *is said to have* **the weak pseudo orbit extending property (W.P.O.E.P.)** *if for every* $\varepsilon > 0$*, there exists a* $\delta > 0$ *such that for any weak* δ*-pseudo orbit* $\{(x_n, t_n)\}_{n=-\infty}^{\infty}$*, there exists an*

ε-pseudo orbit $\{y_m\}_{m=-\infty}^{\infty}$ such that $\{x_n\}_{n=-\infty}^{\infty}$ is a subsequence of $\{y_m\}_{m=-\infty}^{\infty}$.

Theorem 6.2.1 *Let $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on a compact metric space (X, d). If F has the weak pseudo orbit extending property (W.P.O.E.P.) then the set of all chain recurrent points of F and the set of all weak chain recurrent points of F are same equivalently $CR(F) = WCR(F)$.*

Proof: In view of Theorem 6.1.1, it remains to show that if F has the W.P.O.E.P. then $WCR(F) \subseteq CR(F)$. Let $x \in WCR(F)$, $\varepsilon > 0$ and $n \geq 0$ be given. By the W.P.O.E.P. of F, there exists a $\delta > 0$ such that any weak δ-pseudo orbit for F is a subsequence of some ε-pseudo orbit for F. Since $x \in WCR(F)$ therefore there exists a weak δ-chain $\{(x_i, t_i)\}_{i=-\infty}^{\infty}$ for x.

Without loss of generality we may assume that $x_i = x$ for infinitely many $i \geq 0$. Then there exist $m \geq n$ and $r \geq 0$ such that $x_m = x_{m+r} = x$. By W.P.O.E.P. of F, there exists an ε-pseudo orbit $\{y_m\}_{m=-\infty}^{\infty}$ such that $\{x_n\}_{n=-\infty}^{\infty}$ is a subsequence of $\{y_m\}_{m=-\infty}^{\infty}$.

Hence we get a finite ε-pseudo orbit from $y_k = x_m = x$ to $y_{k+j} = x_{m+r} = x$ with action starting at k. Hence $x \in CR(F)$. This proves that $WCR(F) \subseteq CR(F)$.

We show that the converse of above theorem is not true. We construct a time varying homeomorphism F on a compact metric space such that $CR(F) = WCR(F)$ although F does not have W.P.O.E.P.

Example 6.2 *Let*

$$Y = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{1 - \frac{1}{n} : n \in \mathbb{N}\},$$

where $\mathbb{N}$ is the set of all positive integers, under the usual metric d given *by $d(x, y) = |x - y|$. Define a map $f\colon Y \to [0, 1]$ by*

$$f(y) = \begin{cases} 0 & if \;\; y \in \{0, 1\}; \\ y & if \;\; y \in Y - \{0, 1\}. \end{cases}$$

Consider the quotient space $X = Y/f$ and the shift map σ on X defined by

$$\sigma(x) = \begin{cases} \{\frac{1}{n-1}\} & if \;\; x = \{\frac{1}{n}\}, \; n > 2; \\ \{1 - \frac{1}{n+1}\} & if \;\; x = \{1 - \frac{1}{n}\}, \; n \geq 2; \\ x & if \;\; x = \{0, 1\}. \end{cases}$$

Consider time varying homeomorphism $F = \{f_n\}_{n=0}^{\infty}$ on X, where f_0 is the identity map and

$$f_n(x) = \begin{cases} \sigma(x) & if \;\; n \;\; is \;\; evem; \\ \sigma^{-1}(x) & if \;\; n \;\; is \;\; odd. \end{cases}.$$

Define the sequence of integers $\{t_i\}$ as $t_{2i} = t_{2i+1} = 2i$, for all $i \in \mathbb{Z}$. Then for any $x \in X$, $F_{[t_{2i}, t_{2i+1}]}(x) = \sigma(x)$. Let $\delta > 0$. There exist $m > 0$ such that $\sigma^m(x) \in U_{\frac{\delta}{2}}(A)$ and $\sigma^{-m}(x) \in U_{\frac{\delta}{2}}(A)$, where $A = \{0, 1\}$. Put

$$\{x_0 = x, x_1 = \sigma(x), \ldots, x_m = \sigma^m x, x_{m+1} = \sigma^{-m}(x), x_{m+2} = \sigma^{-m+1}(x), \ldots$$
$$\ldots, x_{2m} = \sigma^{-1}(x), x_{2m+1} = x\}.$$

Now

$$d(F_{[t_{2i}, t_{2i+1}]}(x_i), x_{i+1}) < \delta,$$

for all $i = 0, 1, \ldots, 2m$.

Hence $\{x_m\}_{i=0}^{2m+1}$ is a positive finite weak δ-pseudo orbit for x. Repeating this process for infinitely many times, we get an infinite positive weak δ-pseudo orbit for x which can be extended to an infinite weak δ-pseudo orbit for x. Thus $x \in WCR(F)$. Since $x \in X$ is arbitrary, we have $WCR(F) = X$.

Let $0 < \varepsilon < \frac{1}{6}$. Then ε-pseudo orbit for $x = \{\frac{1}{2}\}$ is $\{\{\frac{1}{2}\}, \{\frac{1}{3}\}\}$. If possible, suppose F has W.P.O.E.P. Then we get a $\delta > 0$ such that any weak δ-pseudo *orbit of $x \in X$ can be extended to an ε-pseudo orbit of x. Analogously as above, we get an infinite weak δ-pseudo orbit for $x = \{\frac{1}{2}\}$ which can never be extended to an ε-pseudo orbit for x. Thus F does not have the W.P.O.E.P.*

Moreover, for any $x \in X, \delta > 0$ and $n \geq 0$,

$$\{x, f_n(x), f_{n+1}(x) = x\}$$

is a finite δ-chain for x with action starting at n. Thus $x \in CR(F)$ and hence $CR(F) = X$.

Now we define an equivalence relation on the set of all weak chain recurrent points for an invertible nonautonomous discrete dynamical system and show that equivalence classes are closed as well as open set.

Definition 6.2.2 *Let $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on a compact metric space (X, d). For $x, y \in WCR(F)$, we say that* **x is** δ **related to y**, *denoted by $x \overset{\delta}{\sim} y$, if the following conditions are satisfied :*

(a) *for any weak δ-chain for x, infinitely many finite weak δ-chains for x contain y and*

(b) *for any weak δ-chain for y, infinitely many finite weak δ-chains for y contain x.*

We say $x \sim y$ if for every $\delta > 0$, $x \overset{\delta}{\sim} y$.

Lemma 6.2.1 *Let $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on a compact metric space (X, d). The relation $\overset{\delta}{\sim}$ defined on $WCR(F)$ is an equivalence relation.*

Proof : Clearly from the definition, $\sim$ is reflexive and symmetric. We prove transitivity.
Let $x \sim y$ and $y \sim z$. Then for any $\delta > 0$, $x \overset{\delta}{\sim} y$ and $y \overset{\delta}{\sim} z$.
Thus

(a_1) for any weak δ-chain for x, there are infinitely many finite weak δ-chains for x containing y and

(b_1) for any weak δ-chain for y, there are infinitely many finite weak δ-chains for y containing x.

Also

(a_2) for any weak δ-chain for y, there are infinitely many finite weak δ-chains for y containing z and

(b_2) for any weak δ-chain for z, there are infinitely many finite weak δ-chains for z containing y.

Let $\{(x_i, t_i)\}_{i=0}^{\infty}$ be a weak δ-chain for x. Then using $(a_1),\ (a_2),\ (b_1),\ (b_2)$, we get monotonically increasing subsequences $\{n_i\}_{i=0}^{\infty}$, $\{m_i\}_{i=0}^{\infty}$, $\{p_i\}_{i=0}^{\infty}$ and $\{k_i\}_{i=0}^{\infty}$ of positive integers such that for all $i \geq 0$,

$$n_i \leq m_i \leq n_{i+1} \leq m_{i+1}; \quad m_{k_i} \leq p_i \leq m_{k_i+1}$$

and $x_{n_i} = x,\ x_{m_i} = y,\ x_{p_i} = z$. Thus for $i \geq 0$,

$$n_{k_i} \leq m_{k_i} \leq p_i \leq m_{k_i+1} \leq n_{k_{i+1}+1}.$$

Hence we get

$$n_{k_i} \leq p_i \leq n_{k_{i+1}+1};\ \ x_{n_{k_i}} = x,\ x_{p_i} = z,\ x_{n_{k_{i+1}+1}} = x.$$

This implies that

(a) for any weak δ-chain for x, there are infinitely many finite weak δ-chains for x containing z.

Similarly one can show that

(b) for any weak δ-chain for z, there are infinitely many finite weak δ-chains for z containing y.

Thus $x \overset{\delta}{\sim} z$. Since δ is arbitrary therefore $x \sim z$ which implies that $\sim$ is transitive.
Hence $\sim$ is an equivalence relation on $WCR(F)$.

This equivalence relation gives a partition of $WCR(F)$ consisting of equivalence classes

$$[x] = \{y \in WCR(F) : y \sim x\}.$$

Using the above lemma we prove the following result :

Theorem 6.2.2 *Let $F = \{f_n\}_{n=0}^{\infty}$ be an equicontinuous time varying homeomorphism on a compact metric space (X, d). Then each equivalence class of $WCR(F)$ is a closed as well as an open set.*

Proof : Let $\{y_k\}_{k=0}^{\infty}$ be a sequence of points of $WCR(F)$ converging to some $y \in X$. Let $\varepsilon > 0$ be given. Then there exists an integer $k > 0$ such that $d(y_k, y) < \frac{\varepsilon}{2}$. Since X is compact therefore F is uniformly equicontinuous, i.e.

$$\{F_{[m,n]}, F_{[m,n]}^{-1} : 0 \leq m \leq n\}$$

is equicontinuous. Hence there exists δ, $0 < \delta < \frac{\varepsilon}{2}$ such that for any $0 \leq m \leq n$,

$$d(F_{[m,n]}(x), F_{[m,n]}(y)) < \frac{\varepsilon}{2} \;\; and \;\; d(F_{[m,n]}^{-1}(x), F_{[m,n]}^{-1}(y)) < \frac{\varepsilon}{2}$$

whenever $d(x, y) < \delta$. Since $y_k \in WCR(F)$ therefore there exists a weak δ-chain $\{(x_i, t_i)\}_{i=0}^{\infty}$ for y_k such that

$$d(F_{[t_{2i}, t_{2i+1}]}(x_i), x_{i+1}) < \delta, \;\; for \;\; i \geq 0,$$

$$d(F_{[|t_{2i+2}|, |t_{2i+1}|]}^{-1}(x_{i+1}), x_i) < \delta, \;\; for \;\; i < 0$$

and $x_n = y_k$ for infinitely many $n \in \mathbb{Z}$. Thus for some fixed $y_k = x_n$,

$$d(F_{[t_{2n}, t_{2n+1}]}(y_k), x_{n+1}) < \delta, \;\; if \;\; n \geq 0$$

and

$$d(F^{-1}_{[|t_{2n+2}|,\,|t_{2n+1}|]}(x_{n+1}), y_k) < \delta, \quad if \quad n < 0.$$

Now

$$\begin{aligned} d(F_{[t_{2n},\,t_{2n+1}]}(y), x_{n+1}) &= d(F_{[t_{2n},\,t_{2n+1}]}(y), F_{[t_{2n},\,t_{2n+1}]}(y_k)) + d(F_{[t_{2n},\,t_{2n+1}]}(y_k), x_{n+1}) \\ &< \frac{\varepsilon}{2} + \delta \\ &< \varepsilon \end{aligned}$$

and

$$\begin{aligned} d(F^{-1}_{[|t_{2n+2}|,\,|t_{2n+1}|]}(x_{n+1}), y) &= d(F^{-1}_{[|t_{2n+2}|,\,|t_{2n+1}|]}(x_{n+1}), y_k) + d(y_k, y) \\ &< \delta + \frac{\varepsilon}{2} \\ &< \varepsilon. \end{aligned}$$

Thus we get a weak ε-chain for y. Since $\delta > 0$ is arbitrary therefore $y \in WCR(F)$. This proves that $WCR(F)$ is closed.

Next we show that $WCR(F)$ is open. Let $\varepsilon > 0$ be given. Since X is compact therefore F is uniformly equicontinuous. Hence there exists $0 < \delta < \frac{\varepsilon}{2}$ such that for any $0 \leq m \leq n$,

$$d(F_{[m,n]}(x), F_{[m,n]}(y)) < \frac{\varepsilon}{2}$$

whenever $d(x, y) < \delta$. Let $x \in WCR(F)$, then there is a weak δ-chain $\{(x_i, t_i)\}_{i=0}^{\infty}$ for x. We have

$$d(F_{[t_{2i},\,t_{2i+1}]}(x_i), x_{i+1}) < \delta, \quad for \quad i \geq 0$$

and

$$d(F^{-1}_{[|t_{2i+2}|,\,|t_{2i+1}|]}(x_{i+1}), x_i) < \delta, \quad for \quad i < 0.$$

Also $x_i = x$ for infinitely many $i \in \mathbb{Z}$. Let $y \in U_\delta(x)$, where

$$U_\delta(x) = \{z \in X \colon d(z, x) < \delta\}.$$

For any such $x_n = x$, if $n \geq 0$,

$$d(F_{[t_{2n},\, t_{2n+1}]}(x), x_{n+1}) < \delta$$

and if $n < 0$,

$$d(F^{-1}_{[|t_{2n+2}|,\, |t_{2n+1}|]}(x_{n+1}), x) < \delta.$$

Note that

$$\begin{aligned} d(F_{[t_{2n},\, t_{2n+1}]}(y), x_{n+1}) &= d(F_{[t_{2n},\, t_{2n+1}]}(y), F_{[t_{2n},\, t_{2n+1}]}(x)) + d(F_{[t_{2n},\, t_{2n+1}]}(x), x_{n+1}) \\ &< \frac{\varepsilon}{2} + \delta < \varepsilon \end{aligned}$$

and

$$\begin{aligned} d(F^{-1}_{[|t_{2n+2}|,\, |t_{2n+1}|]}(x_{n+1}), y) &= d(F^{-1}_{[|t_{2n+2}|,\, |t_{2n+1}|]}(x_{n+1}), x) + d(x, y) \\ &< \delta + \delta < \varepsilon. \end{aligned}$$

Replacing $x_n = y$, for all n for which $x_n = x$, we get a weak ε-chain for y. Since $\varepsilon > 0$ is arbitrary therefore $y \in WCR(F)$. Since $y \in U_\delta(x)$ is arbitrary therefore $U_\delta(x) \subset WCR(F)$. This proves that $WCR(F)$ is open.

6.3 Spectral Decomposition Theorem in Equicontinuous Nonautonomous Discrete Dynamical Systems

We recall the following result which is Smale's spectral decomposition theorem extended to homeomorphisms on compact metric spaces (refer [2]).

Theorem : Let X be a compact metric space and $f \colon X \to X$ be an expansive homeomorphism with shadowing property.Then the non-wandering set $\Omega(f)$ of f can be written as a finite union of disjoint closed invariant sets on which f is topologically transitive.

Using Theorem 6.2.1 and Theorem 6.2.2, we prove the above celebrated result in nonautonomous case.

We shall also use the following result :

Corollary 5.2.1 Let $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on a compact metric space X with metric d. If the family $\{f_n, f_n^{-1}\}_{n=0}^{\infty}$ is equicontinuous on X and if F has the P.O.T.P. on X then $CR(F) = \Omega(F)$.

Recall the definition of transitivity for an invertible nonautonomous discrete dynamical system.

Definition : Let $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on a metric space (X, d). Then F is said to be **transitive on** X if for any two nonempty open sets U and V of X and for any $k \geq 0$, there exists $n > m \geq k$ such that

$$F_{[m+1,n]}(U) \cap V \neq \emptyset \quad or \quad F^{-1}_{[m+1,n]}(U) \cap V \neq \emptyset.$$

Finally we prove Spectral Decomposition Theorem similar to Smale's decomposition theorem in an equicontinuous nonautonomous discrete dynamical system.

Theorem 6.3.1 *Let $F = \{f_n\}_{n=0}^{\infty}$ be an equicontinuous time varying homeomorphism on a compact metric space X with metric d. If F has the P.O.T.P. and the W.P.O.E.P on X and F restricted to $\Omega(F)$ has the P.O.T.P. and the W.P.O.E.P then $\Omega(F)$ is a union of pairwise disjoint finitely many closed and open sets, say $\{B_i\}$ such that F restricted to each B_i is transitive.*

Proof: Since F is an equicontinuous time-varying map, the family

$$\{F_{[m,n]}, F^{-1}_{[m,n]} : 0 \leq m \leq n\}$$

is equicontinuous. For $m = n$, $F_{[m,n]} = f_n$ and $F^{-1}_{[m,n]} = f_n^{-1}$ and hence

$$\{f_n, f_n^{-1} : n \geq 0\} \subseteq \{F_{[m,n]}, F^{-1}_{[m,n]} : 0 \leq m \leq n\}.$$

Thus $\{f_n, f_n^{-1} \colon n \geq 0\}$ is also equicontinuous family. By Corollary 5.2.1(on Page 82), since F has the P.O.T.P. and $\{f_n, f_n^{-1}\}_{n=0}^{\infty}$ is equicontinuous therefore $\Omega(F) = CR(F)$. Also from the Theorem 6.2.1 (on Page 91) since F has the W.P.O.E.P. and F is an equicontinuous time varying homeomorphism therefore $CR(F) = WCR(F)$. Hence we have

$$WCR(F) = CR(F) = \Omega(F).$$

Note that we can decompose $WCR(F) = \Omega(F)$ into a disjoint union of closed and open sets, say $\{B_i\}$, which are equivalence classes of $WCR(F)$(from the Theorem 6.2.2 on Page 95). Since X is compact therefore we can take the family $\{B_i\}$ to be finite.

It only remains to show that F restricted to each B_i is transitive. Let

$$B_i = [x] = \{y \in \Omega(F) \colon y \sim x\}$$

be fixed. Let U and V be two nonempty open sets in B_i, $y \in U$, $z \in V$ and $\varepsilon > 0$ be such that $U_\varepsilon(y) \subseteq U$ and $U_\varepsilon(z) \subseteq V$. By the P.O.T.P. of F we have $\delta_1 > 0$ such that any δ_1-pseudo orbit can be ε-traced. By the W.P.O.E.P., there exists $\delta > 0$ such that for any weak δ_1-chain $\{x_i\}$, there exists a δ-pseudo orbit $\{y_j\}$ such that $\{x_i\}$ is a subsequence of $\{y_j\}$. Now since $y, z \in B_i$ therefore $y \sim x$ and $z \sim x$ and by transitivity of $\sim$, we have $y \sim z$. Hence $y \overset{\delta_1}{\sim} z$. We have

(a) for any weak δ_1-chain for y, there are infinitely many finite weak δ_1-chains for y containing z and

(b) for any weak δ_1-chain for z, there are infinitely many finite weak δ_1-chains for z containing y.

Thus using (a),(b) we get a weak δ_1-chain $\{(x_i, t_i)\}$ which contains finite weak δ_1-pseudo orbits from y to z infinitely many times. By the W.P.O.E.P., there exists a δ-pseudo orbit $\{y_j\}$ such that $\{x_i\}$ is a subsequence of $\{y_j\}$. By the P.O.T.P. of F restricted to $\Omega(F)$, there exists $p \in \Omega(F)$ such that $x_i \in U_\varepsilon(F_i(p))$, for every $i \in \mathbb{Z}$. Thus for any $k > 0$, there exist $n > m \geq k$ such that

$$y = x_m \in U_\varepsilon(F_m(p)) \;\; and \;\; z = x_n \in U_\varepsilon(F_n(p))$$

or

$$y = x_{-m} \in U_\varepsilon(F_{-m}(p)) \ \ and \ \ z = x_{-n} \in U_\varepsilon(F_{-n}(p)).$$

This implies that

$$F_m(p) \in U \ \ and \ \ F_{[m+1,n]}(F_m(p)) = F_n(p) \in V$$

or

$$F_{-m}(p) \in U \ \ and \ \ F^{-1}_{[m+1,n]}(F_{-m}(p)) = F_{-n}(p) \in V.$$

Therefore

$$F_{[m+1,n]}(U) \cap V \neq \emptyset \ \ or \ \ F^{-1}_{[m+1,n]}(U) \cap V \neq \emptyset$$

proving that F restricted to each B_i is transitive.

Remark 6.3 *For proving the decomposition theorem in autonomous case, the following result is used.*

Theorem *[1] If a homeomorphism f on a compact metric space X has P.O.T.P., then its restriction on the nonwandering set, that is, $f|_{\Omega(f)}$ also has P.O.T.P.*

The above result can not be proved for a nonautonomous discrete dynamical system by similar arguments due to the following difference in the definition of pseudo-orbit for the autonomous case and for the nonautonomous case.

Let (X, d) be a metric space, $f : X \to X$ be a continuous map and $\{y_i\}_{i=0}^{\infty}$ is a δ-pseudo orbit for f then by definition, we have

$$d(y_{i+1}, f(y_i)) < \delta, \ \ for \ \ ecah \ \ i \geq 0.$$

Now for time varying map $F = \{f_n\}_{n=0}^{\infty}$ on X if $\{y_i\}_{i=0}^{\infty}$ is a δ-pseudo orbit for F then by definition, we have

$$d(y_{i+1}, f_n(y_i)) < \delta, \ \ if \ \ n = i+1$$

but not known for $n \neq i+1$. Even for equicontinuous nonautonomous discrete dynamical system we may not be able to replace f_n by f_{i+1} if $n \neq i+1$. For example in autonomous case, we have $y_1 = f(y_0)$ where as in nonautonomous case, we have $y_1 = f_1(y_0)$ but we may not have $y_1 = f_n(y_0)$, for any $n \neq 1$.

Due to this reason basic sets B_i in the Theorem 6.3.1 need not be invariant under F which is required to prove that if F has P.O.T.P. then $F|_{\Omega(F)}$ has P.O.T.P.

6.4 Scope of Further Research

- We recall that there exists no positively expansive map or homeomorphism or expansive $\mathbb{Z}^d$ action on an interval, circle. Also there exists no positively expansive homeomorphism on an infinite compact metric space, see [50]. It will be interesting to study problems related to existence/non-existence of time varying homeomorphisms on different spaces.

- Different types of shadowing properties like limit shadowing property, average shadowing property, asymptotic average shadowing property etc. can be defined and their interrelation can be studied in nonautonomous discrete dynamical systems.

- Notions of distal and proximal time varying homeomorphisms can be defined and studied. It may be useful to study their relations with expansiveness in invertible nonautonomous discrete dynamical systems.

- Using the notion of topological mixing of time varying map, one can try to prove Bowen's decomposition theorem in nonautonomous discrete dynamical system.

- It is proved by Aoki that for an expansive homeomorphism on a compact metric space having P.O.T.P, *Per*(f) is dense in

$\Omega(f)$. It will be interesting to see whether this result holds in nonautonomous case.

- One can try to define and study above notions in multidimensional nonautonomous discrete dynamical systems given by a sequence of continuous maps/homeomorphisms.