Chapter 3

Generalized q-Mittag-Leffler function

3.1 Introduction

The q-theory has remarkable property that a q-analogue of an "ordinary" expression is not unique; especially if the expression is a finite or infinite series. In such case the two q-forms are defined; one involving the factor of the form $q^{n(n-1)/2}, n$ being summation index, and the other without this factor. This is due to the definition of ${}_r\phi_s[z]$ (1.3.4) and the fact that the number 0 (zero) can be considered as a parameter in numerator or in denominator. This has given rise to two q-exponential functions, two q-cosine, q-sine functions and many other such cases. Following this tradition, the two q-extensions of the function (2.1.5) are defined in this chapter and derive the properties analogues to those obtained in Chapter-2.

Definition 3.1.1. *If $\alpha, \beta, \gamma, \lambda \in \mathbb{C}$ with $\Re(\alpha, \beta, \gamma, \lambda) > 0,\ \delta, \mu > 0,\ r \in \{-1, 0\} \cup \mathbb{N},\ s \in \mathbb{N} \cup \{0\}$ then*

$$E^{\gamma,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu}(z; s, r|q) = \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [\Gamma_q(\gamma+\delta n)]^s}{\Gamma_q(\beta+\alpha n)\ [\Gamma_q(\lambda+\mu n)]^r\ (q;q)_n} z^n, \quad (3.1.1)$$

where $p = \alpha^2 + r\mu^2 - s\delta^2 + 1$ with $\Re(p) > 0$.

Definition 3.1.2. *If $\alpha, \beta, \gamma, \lambda \in \mathbb{C}$ with $\Re(\alpha, \beta, \gamma, \lambda) > 0,\ \delta, \mu > 0,\ r \in \{-1, 0\} \cup \mathbb{N},\ s \in \mathbb{N} \cup \{0\}$ and $\alpha^2 + r\mu^2 + 1 = s\delta^2$ then*

$$e^{\gamma,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu}(z; s, r|q) = \sum_{n=0}^{\infty} \frac{[\Gamma_q(\gamma+\delta n)]^s}{\Gamma_q(\beta+\alpha n)\ [\Gamma_q(\lambda+\mu n)]^r\ (q;q)_n} z^n. \quad (3.1.2)$$

Alternatively in view of the definition of q-Gamma function (1.2.31) these q-forms can also be put in the form:

$$E^{\gamma,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu}(z;s,r|q) \;=\; \sum_{n=0}^{\infty} \frac{(-1)^{pn}\; q^{pn(n-1)/2}\; (q^{\alpha n+\beta};q)_\infty\; [(q^{\lambda+\mu n};q)_\infty]^r}{[(q^{\gamma+\delta n};q)_\infty]^s\; (q;q)_n} z^n, \quad (3.1.3)$$

and

$$e^{\gamma,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu}(z;s,r|q) \;=\; \sum_{n=0}^{\infty} \frac{(q^{\alpha n+\beta};q)_\infty\; [(q^{\lambda+\mu n};q)_\infty]^r}{[(q^{\gamma+\delta n};q)_\infty]^s\; (q;q)_n} z^n. \quad (3.1.4)$$

The q-analogues of the Shukla and Prajapati's function (2.1.4) and the other functions listed above in Table-1 of Chapter 2 are all yielded by (3.1.1) or (3.1.2). They are tabulated below together with the indicated substitutions.

Table-3

q-Function of	**r**	**s**	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\gamma}$	$\boldsymbol{\delta}$	$\boldsymbol{\lambda}$	$\boldsymbol{\mu}$	**Particular case of**
Mittag-Leffler	0	1	α	1	1	1	-	-	(3.1.1)
Wiman	0	1	α	β	1	1	-	-	(3.1.1)
Prabhakar	0	1	α	β	γ	1	-	-	(3.1.1)
Shukla and Prajapati	0	1	α	β	γ	q	-	-	(3.1.1)
Bessel-Maitland	0	0	μ	$\nu+1$	-	-	-	-	(3.1.1)
Dotsenko	-1	1	ω/ν	c	a	1	b	ω/ν	(3.1.2)
Saxena-Nishimoto	1	1	α_1	β_1	γ	K	β_2	α_2	(3.1.1)
Elliptic	-1	1	1	1	$\frac{1}{2}$	1	$\frac{1}{2}$	1	(3.1.2)

The explicit forms of the functions mentioned in this table are as stated below.

- q-Mittag-Leffler function:

$$E_\alpha(z|q) = \sum_{n=0}^{\infty} \left[(-1)^n\; q^{n(n-1)/2}\right]^{\alpha^2} (q^{\alpha n+1};q)_\infty\; z^n.$$

- q-Analogue of Wiman's function:

$$E_{\alpha,\beta}(z|q) = \sum_{n=0}^{\infty} \left[(-1)^n\; q^{n(n-1)/2}\right]^{\alpha^2} (q^{\alpha n+\beta};q)_\infty\; z^n.$$

- q-Analogue of Prabhakar's generalized ML-function:

$$E^{\gamma}_{\alpha,\beta}(z|q) = \sum_{n=0}^{\infty} \frac{\left[(-1)^n \; q^{n(n-1)/2}\right]^{\alpha^2} \; (q^{\alpha n+\beta}; q)_\infty}{(q^{\gamma+n}; q)_\infty \; (q; q)_n} z^n.$$

- q-ML-function of Shukla and Prajapati (q is replaced by δ):

$$E^{\gamma,\delta}_{\alpha,\beta}(z|q) = \sum_{n=0}^{\infty} \frac{\left[(-1)^n \; q^{n(n-1)/2}\right]^{(\alpha^2-\delta^2+1)} \; (q^{\alpha n+\beta}; q)_\infty}{(q^{\gamma+\delta n}; q)_\infty \; (q; q)_n} z^n.$$

- q-Bessel-Maitland function:

$$J^{\mu}_{\nu}(-z; q) = \sum_{n=0}^{\infty} \frac{\left[(-1)^n \; q^{n(n-1)/2}\right]^{(\mu^2+1)} \; (q^{\mu n+\nu+1}; q)_\infty}{(q; q)_n} z^n.$$

(Later on, this will be referred to this as q-BMF)

- q-Dotsenko function:

$${}_2R_1(a, b; c, \omega; \nu; z; q) = \sum_{n=0}^{\infty} \frac{(q^{c+\frac{\omega}{\nu}n}; q)_\infty}{(q^{b+\frac{\omega}{\nu}n}; q)_\infty \; (q^{n+a}; q)_\infty \; (q; q)_n} z^n.$$

- q-Form of the particular case $m = 2$ of the function due to Saxena and Nishimoto

$$\begin{aligned} E_{\gamma,K}[(\alpha_j, \beta_j)_{1,2}; z|q] \;\; &= \;\; \sum_{n=0}^{\infty} \frac{\left[(-1)^n \; q^{n(n-1)/2}\right]^{(\alpha_1^2+\alpha_2^2-K^2+1)} \; (q^{\alpha_1 n+\beta_1}; q)_\infty}{(q^{\gamma+Kn}; q)_\infty \; (q; q)_n} \\ &\quad \times (q^{\alpha_2 n+\beta_2}; q)_\infty \; z^n. \end{aligned}$$

(Later on, this will be referred to as q-SNF)

- q-Elliptic function:

$$K(\sqrt{z}|q) = \frac{\pi}{2} \; {}_2\phi_1 \left(\begin{matrix} \frac{1}{2}, & \frac{1}{2}; & z \\ 1; & & \end{matrix} \right).$$

3.2 Main results

3.2.1 Convergence

Theorem 3.2.1. *Let* $\Re(\alpha,\beta,\gamma,\lambda)>0,\ \Re(\alpha^2)+r\mu^2-s\delta^2+1>0,\ \delta,\mu>0, r\in\{-1,0\}\cup\mathbb{N},\ s\in\mathbb{N}\cup\{0\}$ *and* $0<q<1$. *Then* $E^{\gamma,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu}(z;s,r|q)$ *is an entire function of order zero.*

Proof. Put

$$V_n=\frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [\Gamma_q(\gamma+\delta n)]^s}{\Gamma_q(\beta+\alpha n)\ [\Gamma_q(\lambda+\mu n)]^r\ \Gamma_q(n+1)} \tag{3.2.1}$$

to get

$$E^{\gamma,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu}(z;s,r|q)=\sum_{n=0}^{\infty}V_n\ z^n.$$

Then in view of (1.2.32), one can get after some simplifications,

$$V_n \sim \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ (1+q)^{\frac{1}{2}(s-r-2)}\ (\Gamma_{q^2}(\frac{1}{2}))^{s-r-2}\ (1-q)^{n+\frac{1}{2}}}{(1-q)^{-s(\frac{1}{2}-\gamma-\delta n)}\ (1-q)^{\frac{1}{2}-\beta-\alpha n}\ (1-q)^{r(\frac{1}{2}-\lambda-\mu n)}}$$
$$\times e^{\frac{\theta q^{\gamma+\delta n}}{1-q-q^{\gamma+\delta n}}}\ e^{-\frac{\theta q^{\beta+\alpha n}}{1-q-q^{\beta+\alpha n}}}\ e^{-\frac{\theta q^{\lambda+\mu n}}{1-q-q^{\lambda+\mu n}}}\ e^{-\frac{\theta q^{1+n}}{1-q-q^{1+n}}}.$$

Hence,

$$\sqrt[n]{|V_n|} \sim \left|\frac{(1+q)^{\frac{1}{2}(s-r-2)}\ (\Gamma_{q^2}(\frac{1}{2}))^{(s-r-2)}\ (1-q)^{s(\frac{1}{2}-\gamma-\delta n)}\ (1-q)^{n+\frac{1}{2}}}{(1-q)^{\frac{1}{2}-\beta-\alpha n}\ (1-q)^{r(\frac{1}{2}-\lambda-\mu n)}}\right|^{\frac{1}{n}}$$
$$\times\left|e^{\frac{\theta q^{\gamma+\delta n}}{1-q-q^{\gamma+\delta n}}}\ e^{-\frac{\theta q^{\beta+\alpha n}}{1-q-q^{\beta+\alpha n}}}\ e^{-\frac{\theta q^{\lambda+\mu n}}{1-q-q^{\lambda+\mu n}}}\ e^{-\frac{\theta q^{1+n}}{1-q-q^{1+n}}}\right|^{\frac{1}{n}}$$
$$\times\left|(-1)^p\ q^{p(n-1)/2}\right|.$$

Making limit $n\to\infty$, this gives

$$\begin{aligned}\frac{1}{R} &= \lim_{n\to\infty}\sqrt[n]{|V_n|}\sim\left|(1-q)^{\alpha+r\mu-s\delta+1}\right|\ \lim_{n\to\infty}\left|q^{p(n-1)/2}\right| \\ &= 0\end{aligned}$$

when $\Re(\alpha^2)+r\mu^2-s\delta^2+1>0$. Thus, the function (3.1.1) is an *entire* function. Its order may be determined by Theorem 1.2.1.

By choosing $f(z) = E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r|q)$ and $u_n = V_n$, Theorem 1.2.1 gets particularized to

$$\varrho(E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r|q)) = \lim_{n\to\infty}\sup\frac{n\ \log n}{\log(1/|V_n|)},$$

where

$$\begin{aligned}\log\left(\frac{1}{|V_n|}\right) &= \log\left(\left|\frac{\Gamma_q(\beta+\alpha n)[\Gamma_q(\lambda+\mu n)]^r\Gamma_q(n+1)}{q^{n(n-1)(\alpha^2+r\mu^2-s\delta^2+1)/2}[\Gamma_q(\gamma+\delta n)]^s}\right|\right)\\ &= \log|\Gamma_q(\alpha n+\beta) + r\log|\Gamma_q(\lambda+\mu n)|\\ &\quad +\log|\Gamma_q(n+1)| - \frac{1}{2}n(n-1)[\Re(\alpha^2+r\mu^2-s\delta^2+1)]\log q\\ &\quad -s\log|\Gamma_q(\gamma+\delta n)|. \end{aligned} \tag{3.2.2}$$

From the definition (1.2.31) of q-Gamma function, one finds

$$\begin{aligned}\log|\Gamma_q(\alpha n+\beta)| &= \log\left|\frac{(q;q)_\infty}{(q^{\alpha n+\beta};q)_\infty}(1-q)^{1-\alpha n-\beta}\right|\\ &= \log\left|\frac{(q;q)_\infty}{(q^{\alpha n+\beta};q)_\infty}\right|(1-q)^{1-n\Re(\alpha)-\Re(\beta)}\\ &= \log|(q;q)_\infty| + (1-n\Re(\alpha)-\Re(\beta))\log(1-q)\\ &\quad -\log\left|(q^{\alpha n+\beta};q)_\infty\right|; \end{aligned} \tag{3.2.3}$$

in which

$$\begin{aligned}\log\left|(q^{\alpha n+\beta};q)_\infty\right| &= \log\left(\prod_{k=0}^{\infty}\left|1-q^{\alpha n+\beta+k}\right|\right)\\ &= \log\left(\lim_{m\to\infty}\prod_{k=0}^{m}\left|1-q^{\alpha n+\beta+k}\right|\right)\\ &= \lim_{m\to\infty}\sum_{k=0}^{m}\log\left|1-q^{\alpha n+\beta+k}\right|\\ &= \sum_{k=0}^{\infty}\log\left|1-q^{\alpha n+\beta+k}\right|. \end{aligned}$$

Here it may be noted that [7, p.207]

$$\log\left|1-q^{\alpha n+\beta+k}\right| \le \log(1+\left|q^{\alpha n+\beta+k}\right|) \le \left|q^{\alpha n+\beta+k}\right| = q^{n\Re(\alpha+\beta)+k}$$

which leads to

$$\sum_{k=0}^{\infty}\log\left|1-q^{\alpha n+\beta+k}\right| \le \sum_{k=0}^{\infty}q^{n\Re(\alpha+\beta)+k} = \frac{q^{n\Re(\alpha+\beta)}}{1-q}.$$

This implies that

$$\lim_{n\to\infty} \frac{\log\left|(q^{\alpha n+\beta};q)_\infty\right|}{n\ \log n} = 0.$$

Consequently from (3.2.3) it follows that

$$\lim_{n\to\infty} \frac{\log|\Gamma_q(\alpha n+\beta)|}{n\ \log n} = 0.$$

This last limit and the trivial limit

$$\lim_{n\to\infty} \frac{n-1}{\log n} = \infty$$

when used in (3.2.2), yields

$$\lim_{n\to\infty} \frac{\log(1/|V_n|)}{n\log n} = \infty.$$

Thus,

$$\varrho(E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r|q)) = 0.$$

□

Theorem 3.2.2. *The function $e^{\gamma,\delta}_{\alpha,\ \beta,\ \lambda,\ \mu}(z;s,r|q)$ represents the series which converges absolutely for $|z| < \left|(1-q)^{(s\delta-\alpha-r\mu-1)}\right|$ and $|q|<1$.*

Proof. Take

$$U_n = \frac{[\Gamma_q(\gamma+\delta n)]^s}{\Gamma_q(\beta+\alpha n)\ [\Gamma_q(\lambda+\mu n)]^r\ \Gamma_q(n+1)} \tag{3.2.4}$$

then

$$e^{\gamma,\delta}_{\alpha,\ \beta,\ \lambda,\ \mu}(z;s,r|q) = \sum_{n=0}^{\infty} U_n\ z^n.$$

Now in view of the q- analogue of Stirling's asymptotic formula (1.2.32), one gets

$$U_n \quad\sim\quad \frac{(1+q)^{\frac{1}{2}(s-r-2)}\ (\Gamma_{q^2}(\frac{1}{2}))^{s-r-2}\ (1-q)^{n+\frac{1}{2}}}{(1-q)^{-s(\frac{1}{2}-\gamma-\delta n)}\ (1-q)^{\frac{1}{2}-\beta-\alpha n}\ (1-q)^{r(\frac{1}{2}-\lambda-\mu n)}}$$
$$\times e^{\frac{\theta q^{\gamma+\delta n}}{1-q-q^{\gamma+\delta n}}}\ e^{-\frac{\theta q^{\beta+\alpha n}}{1-q-q^{\beta+\alpha n}}}\ e^{-\frac{\theta q^{\lambda+\mu n}}{1-q-q^{\lambda+\mu n}}}\ e^{-\frac{\theta q^{1+n}}{1-q-q^{1+n}}}.$$

This gives

$$\sqrt[n]{|U_n|} \sim \left| \frac{(1+q)^{\frac{1}{2}(s-r-2)} \, (\Gamma_{q^2}(\frac{1}{2}))^{(s-r-2)} \, (1-q)^{s(\frac{1}{2}-\gamma-\delta n)} \, (1-q)^{n+\frac{1}{2}}}{(1-q)^{\frac{1}{2}-\beta-\alpha n} \, (1-q)^{r(\frac{1}{2}-\lambda-\mu n)}} \right|^{\frac{1}{n}}$$
$$\times \left| e^{\frac{\theta q^{\gamma+\delta n}}{1-q-q^{\gamma+\delta n}}} \, e^{-\frac{\theta q^{\beta+\alpha n}}{1-q-q^{\beta+\alpha n}}} \, e^{-\frac{\theta q^{\lambda+\mu n}}{1-q-q^{\lambda+\mu n}}} \, e^{-\frac{\theta q^{1+n}}{1-q-q^{1+n}}} \right|^{\frac{1}{n}}$$

whence

$$\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|U_n|} \sim \left|(1-q)^{\alpha+r\mu-s\delta+1}\right|.$$

Thus, the series in (3.1.2) converges absolutely if $|z| < R = (1-q)^{s\delta-\Re(\alpha)-r\mu-1}$. □

3.2.2 Contour integral

Theorem 3.2.3. *Let* $\alpha > 0; \beta, \gamma, \lambda \in \mathbb{C}$ *with* $\Re(\beta, \gamma, \lambda) > 0$ *and* $\delta, \mu > 0$. *Then the function* $E^{\gamma,\,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu}\,(z; s, r|q)$ *is expressible as the Mellin - Barnes q-integral given by*

$$E^{\gamma,\,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu}\,(z; s, r|q) = \frac{1}{2\pi i} \int\limits_L \frac{(-1)^{-pS} \; q^{-pS(-S-1)/2} \; \Gamma_q(S)[\Gamma_q(\gamma-\delta S)]^s \; (-z)^{-S}}{\Gamma_q(\beta-\alpha S) \; [\Gamma_q(\lambda-\mu S)]^r} \mathrm{d}_q S, \tag{3.2.5}$$

where $|argz| < \pi$. *The contour* L *of integration begins from* $-i\infty$ *and proceeds towards* $+i\infty$*, and is indented to keep the poles of integrand at* $S = -n$ *to the left; and the poles at* $S = (\gamma + n)/\delta$ *to the right of the path for all* $n \in \mathbb{N} \cup \{0\}$.

Proof. The integral on the right hand side of (3.2.5) may be evaluated as the sum of the residues at the poles $S = 0, -1, -2, \ldots$.
In fact, in view of the definition of residue,

$$\begin{aligned}
I &= \frac{1}{2\pi i} \int\limits_L \frac{(-1)^{-pS} \; q^{-pS(-S-1)/2} \; \Gamma_q(S) \; [\Gamma_q(\gamma-\delta S)]^s \; (-z)^{-S}}{\Gamma_q(\beta-\alpha S) \; [\Gamma_q(\lambda-\mu S)]^r} \mathrm{d}_q S \\
&= \sum_{n=0}^{\infty} \underset{S=-n}{Res} \left[\frac{(-1)^{-pS} \; q^{-pS(-S-1)/2} \; \Gamma_q(S) \; (-z)^{-S}}{\Gamma_q(\beta-\alpha S) \; [\Gamma_q(\lambda-\mu S)]^r \; [\Gamma_q(\gamma-\delta S)]^{-s}} \right] \\
&= \sum_{n=0}^{\infty} \lim_{S\to-n} \frac{\pi(S+n)}{sin\pi S} \frac{(-1)^{-pS} \; q^{-pS(-S-1)/2} \; [\Gamma_q(\gamma-\delta S)]^s \; (-z)^{-S}}{\Gamma_q(\beta-\alpha S) \; [\Gamma_q(\lambda-\mu S)]^r \; \Gamma_q(1-S)}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [\Gamma_q(\gamma+\delta n)]^s}{\Gamma_q(\beta+\alpha n)\ [\Gamma_q(\lambda+\mu n)]^r\ \Gamma_q(n+1)} z^n$$
$$= E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r|q).$$

□

By dropping the factor $q^{N(N-1)/2}$ in this proof, one gets

Theorem 3.2.4. *Let* $\alpha \in \mathbb{R}_+; \beta,\ \gamma,\ \lambda \in \mathbb{C}$, *with* $\Re(\beta,\gamma,\lambda) > 0$ *and* $\delta,\ \mu > 0$. *Then the function* $e^{\gamma,\ \delta}_{\alpha,\ \beta,\ \lambda,\ \mu}\ (z;s,r|q)$ *is expressible as the Mellin - Barnes q-integral given by*

$$e^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r|q) = \frac{1}{2\pi i}\int_L \frac{\Gamma_q(S)\ [\Gamma_q(\gamma-\delta S)]^s\ (-z)^{-S}}{\Gamma_q(\beta-\alpha S)\ [\Gamma_q(\lambda-\mu S)]^r} \mathrm{d}_q S, \qquad (3.2.6)$$

where $|argz| < \pi$; *the contour* L *of integration begins from* $-i\infty$ *and proceeds towards* $+i\infty$, *and is indented to keep the poles of integrand at* $S = -n$ *to the left; and the poles at* $S = (\gamma+n)/\delta$ *to the right of the path, for all* $n \in \mathbb{N} \cup \{0\}$.

3.2.3 Difference equation

With the aid of the following operators, the difference equations of both q-analogues will be derived. Put

$$\Lambda_q f(x) = f(x) - f(xq^{-1}), \quad \Theta f(x) = f(x) - f(xq), \qquad (3.2.7)$$

$$\mathcal{D}_q\ f(x) = (1-q)\ D_q f(x) := (1-q)\ \frac{f(x)-f(xq)}{x-xq} = \frac{f(x)-f(xq)}{x}, \qquad (3.2.8)$$

$$\frac{\left\{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[\Theta + c^{-u}q^{1-(b+v)/a} - 1]^m\right\}}{\left\{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[c^{-u}q^{1-(b+v)/a}]^m\right\}} = \Phi^{(a,b,c;m)}_{u,v} \qquad (3.2.9)$$

and

$$\frac{\left\{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[\Theta + c^{-u}q^{(b+v)/a} - 1]^m\right\}}{\left\{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[c^{-u}q^{-(b+v)/a}]^m\right\}} = \Psi^{(a,b,c;m)}_{u,v}. \qquad (3.2.10)$$

In these notations, the q-difference equation satisfied by (3.1.1) is derived in the following theorem.

Theorem 3.2.5. *Let $\alpha, \mu, \delta \in \mathbb{N}$, then $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z; s, r|q)$ satisfies the equation*

$$\begin{aligned}&\left[\Phi^{(\mu,\lambda,\eta;r)}_{\ell,k}\ \Phi^{(\alpha,\beta,\sigma;1)}_{h,m}\ \Theta\right] E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z; s, r|q)\\ &- \left[(-1)^p\ z\ \Psi^{(\delta,\gamma,\zeta;s)}_{j,i}\right] E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(zq^p; s, r|q) = 0 \end{aligned} \tag{3.2.11}$$

in which ζ is δ^{th} root of unity, η is μ^{th} root of unity, σ is α^{th} root of unity.

Proof. The first attempt is to express the coefficient of z^n in the series of $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z; s, r|q)$ in the q-factorial notations with the help of the set of the formulas [18, Appendix I]:

$$(a; q)_{kn} = (a, aq, \ldots, aq^{k-1}; q^k)_n,$$

$$(a^k; q^k)_n = (a, a\omega_k, \ldots, a\omega_k^{k-1}; q^k)_n\ \ ; \omega_k = e^{(2\pi i)/k},$$

$$(A; q^n)_{\nu k} = (A^{1/n}; q)_{\nu k}\ (A^{1/n}\omega; q)_{\nu k} \ldots (A^{1/n}\omega^{n-1}; q)_{\nu k},\ \ \omega^n = 1,$$

and

$$(q^\gamma; q^\delta)_n = (q^{\gamma/\delta}; q)_n\ (\varpi q^{\gamma/\delta}; q)_n \ldots (\varpi^{\delta-1} q^{\gamma/\delta}; q)_n = \prod_{i=0}^{\delta-1} (\varpi^i q^{\gamma/\delta}; q)_n,\ \ \varpi^\delta = 1.$$

Following the notation for this coefficient in (3.2.1), one gets

$$\begin{aligned} V_n &= \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^\gamma; q)_{\delta n}]^s}{[(q^\lambda; q)_{\mu n}]^r\ (q^\beta; q)_{\alpha n}\ (q; q)_n}\\ &= \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^\gamma; q)_{\delta n}]^s}{[(q^\lambda; q)_{\mu n}]^r\ (q^\beta; q)_{\alpha n}\ (q; q)_n}\\ &= \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^\gamma; q^\delta)_n]^s\ [(q^{\gamma+1}; q^\delta)_n]^s \cdots [(q^{\gamma+\delta-1}; q^\delta)_n]^s}{[(q^\lambda; q^\mu)_n]^r\ [(q^{\lambda+1}; q^\mu)_n]^r \cdots [(q^{\lambda+\mu-1}; q^\mu)_n]^r}\\ &\quad\times \frac{1}{(q^\beta; q^\alpha)_n\ (q^{\beta+1}; q^\alpha)_n \cdots (q^{\beta+\alpha-1}; q^\alpha)_n (q; q)_n}\\ &= \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ \left\{\prod_{j=0}^{\delta-1}\prod_{i=0}^{\delta-1} [(\zeta^j\ q^{(\gamma+i)/\delta}; q)_n]^s\right\}}{\left\{\prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1} [(\eta^\ell\ q^{(\lambda+k)/\mu}; q)_n]^r\right\}\ \left\{\prod_{h=0}^{\alpha-1}\prod_{m=0}^{\alpha-1} (\sigma^h\ q^{(\beta+m)/\alpha}; q)_n\right\} (q; q)_n}, \end{aligned} \tag{3.2.12}$$

where ζ is δ^{th} root of unity, η is μ^{th} root of unity, σ is α^{th} root of unity.
Now take

$$\prod_{j=0}^{\delta-1}\prod_{i=0}^{\delta-1}[(\zeta^j\ q^{(\gamma+i)/\delta};q)_n]^s=\mathcal{A}_n,\ \prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1}[(\eta^\ell\ q^{(\lambda+k)/\mu};q)_n]^r=\mathcal{B}_n, \tag{3.2.13}$$

and

$$\prod_{h=0}^{\alpha-1}\prod_{m=0}^{\alpha-1}(\sigma^h\ q^{(\beta+m)/\alpha};q)_n=\mathcal{C}_n,\ (-1)^{pn}\ q^{pn(n-1)/2}=D_n \tag{3.2.14}$$

then

$$\sum_{n=0}^{\infty}V_n\ z^n=\sum_{n=0}^{\infty}\frac{\mathcal{A}_n\ D_n}{\mathcal{B}_n\ \mathcal{C}_n\ (q;q)_n}z^n=W,\quad \text{say}.$$

Since the series in (3.1.2) has infinite radius of convergence, one can apply $'\Theta'$ to get

$$\Theta W\ =\ \sum_{n=0}^{\infty}\frac{\mathcal{A}_n\ D_n}{\mathcal{B}_n\ \mathcal{C}_n\ (q;q)_n}\Theta z^n=\sum_{n=0}^{\infty}\frac{\mathcal{A}_n\ D_n}{\mathcal{B}_n\ \mathcal{C}_n}\frac{1-q^n}{(q;q)_n}z^n=\sum_{n=1}^{\infty}\frac{\mathcal{A}_n\ D_n}{\mathcal{B}_n\ \mathcal{C}_n}\frac{z^n}{(q;q)_{n-1}}.$$

Next operating by $\Phi_{h,m}^{(\alpha,\beta,\sigma;1)}$, one gets

$$\begin{aligned}
\Phi_{h,m}^{(\alpha,\beta,\sigma;1)}\ \Theta W\ &=\ \sum_{n=1}^{\infty}\frac{\mathcal{A}_n\ D_n}{\mathcal{B}_n\ (q;q)_{n-1}}\frac{\left\{\prod_{h=0}^{\alpha-1}\prod_{m=0}^{\alpha-1}(\Theta+\sigma^{-h}q^{1-(\beta+m)/\alpha}-1)\right\}}{\left\{\prod_{h=0}^{\alpha-1}\prod_{m=0}^{\alpha-1}(\sigma^{-h}q^{1-(\beta+m)/\alpha})\right\}}\\
&\quad\times\frac{z^n}{\left\{\prod_{h=0}^{\alpha-1}\prod_{m=0}^{\alpha-1}(\sigma^h q^{(\beta+m)/\alpha};q)_n\right\}}\\
&=\ \sum_{n=1}^{\infty}\frac{\mathcal{A}_n\ D_n}{\mathcal{B}_n\ (q;q)_{n-1}}\frac{\left\{\prod_{h=0}^{\alpha-1}\prod_{m=0}^{\alpha-1}(1-\sigma^h q^{n-1+(\beta+m)/\alpha})\right\}}{\left\{\prod_{h=0}^{\alpha-1}\prod_{m=0}^{\alpha-1}(\sigma^h q^{(\beta+m)/\alpha};q)_n\right\}}z^n\\
&=\ \sum_{n=1}^{\infty}\frac{\mathcal{A}_n\ D_n}{\mathcal{B}_n\ \mathcal{C}_{n-1}\ (q;q)_{n-1}}z^n.
\end{aligned}$$

Finally,

$$\begin{aligned}
\Phi_{\ell,k}^{(\mu,\lambda,\eta;r)}\ \Phi_{h,m}^{(\alpha,\beta,\sigma;1)}\ \Theta W &= \sum_{n=1}^{\infty}\frac{\mathcal{A}_n\ D_n}{\mathcal{C}_{n-1}\ (q;q)_{n-1}}\frac{\left\{\prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1}[(\Theta+\eta^{-\ell}q^{1-(\lambda+k)/\mu}-1)]^r\right\}}{\left\{\prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1}[(\eta^{-\ell}q^{1-(\lambda+k)/\mu})]^r\right\}} \\
&\quad\times\frac{z^n}{\left\{\prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1}[(\eta^{\ell}\ q^{(\lambda+k)/\mu};q)_n]^r\right\}} \\
&= \sum_{n=1}^{\infty}\frac{\mathcal{A}_n\ D_n}{\mathcal{C}_{n-1}\ (q;q)_{n-1}}\frac{\left\{\prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1}[(-q^n+\eta^{-\ell}q^{1-(\lambda+k)/\mu})]^r\right\}}{\left\{\prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1}[(\eta^{-\ell}q^{1-(\lambda+k)/\mu})]^r\right\}} \\
&\quad\times\frac{z^n}{\left\{\prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1}[(\eta^{\ell}\ q^{(\lambda+k)/\mu};q)_n]^r\right\}} \\
&= \sum_{n=1}^{\infty}\frac{\mathcal{A}_n\ D_n}{\mathcal{B}_{n-1}\ \mathcal{C}_{n-1}\ (q;q)_{n-1}}z^n.
\end{aligned}$$

Thus,

$$\Phi_{\ell,k}^{(\mu,\lambda,\eta;r)}\ \Phi_{h,m}^{(\alpha,\beta,\sigma;1)}\ \Theta W = \sum_{n=0}^{\infty}\frac{\mathcal{A}_{n+1}\ \mathcal{D}_{n+1}}{\mathcal{B}_n\ \mathcal{C}_n\ (q;q)_n}z^{n+1}. \tag{3.2.15}$$

On the other hand,

$$\begin{aligned}
&\Psi_{j,i}^{(\delta,\gamma,\zeta;s)}\ E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(zq^p;s,r|q) \\
&= \sum_{n=0}^{\infty}\frac{\mathcal{A}_n\ D_n\ q^{pn}}{\mathcal{B}_n\ \mathcal{C}_n\ (q;q)_n}\frac{\left\{\prod_{j=0}^{\delta-1}\prod_{i=0}^{\delta-1}[(\Theta+\zeta^{-j}q^{-(\gamma+i)/\delta}-1)]^s\right\}}{\left\{\prod_{j=0}^{\delta-1}\prod_{i=0}^{\delta-1}[(\zeta^{-j}q^{-(\gamma+i)/\delta})]^s\right\}}z^n \\
&= \sum_{n=0}^{\infty}\frac{D_n\ q^{pn}}{\mathcal{B}_n\ \mathcal{C}_n\ (q;q)_n}\left\{\prod_{j=0}^{\delta-1}\prod_{i=0}^{\delta-1}[(\zeta^j\ q^{(\gamma+i)/\delta};q)_n]^s\right\} \\
&\quad\times\left\{\prod_{j=0}^{\delta-1}\prod_{i=0}^{\delta-1}[(1-\zeta^j q^{n+(\gamma+i)/\delta})]^s\right\}z^n,
\end{aligned}$$

that is,

$$z\ (-1)^p\ \Psi_{j,i}^{(\delta,\gamma,\zeta;s)}\ \ E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(zq^p;s,r|q) = \sum_{n=0}^{\infty} \frac{\mathcal{A}_{n+1}\ D_{n+1}}{\mathcal{B}_n\ \mathcal{C}_n\ (q;q)_n} z^{n+1}. \tag{3.2.16}$$

On comparing (3.2.15) and (3.2.16), the equation (3.2.11) is obtained. □

The q-difference equation satisfied by the function (3.1.2) is given in following theorem whose proof follows line-to-line just dropping $q^{n(n-1)/2}$ that is, dropping D_n in (3.2.14).

Theorem 3.2.6. *Let $\alpha, \mu, \delta \in \mathbb{N}$, then $Y = e_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(z;s,r|q)$ satisfies the equation*

$$\left[\Phi_{\ell,k}^{(\mu,\lambda,\eta;r)}\ \Phi_{h,m}^{(\alpha,\beta,\sigma;1)}\ \Theta - z\ \Psi_{j,i}^{(\delta,\gamma,\zeta;s)}\right] Y = 0, \tag{3.2.17}$$

where ζ is δ^{th} root of unity, η is μ^{th} root of unity, σ is α^{th} root of unity.

3.2.4 Eigen function property

Take

$$\frac{\prod\limits_{u=0}^{a-1}\prod\limits_{v=0}^{a-1}[(\Lambda_q + c^{-u}q^{1-(b+v)/a} - 1)]^m}{\left\{\prod\limits_{u=0}^{a-1}\prod\limits_{v=0}^{a-1}[c^{-u}q^{1-(b+v)/a}]^m\right\}} = \Omega_{u,v}^{(a,b,c;m)}, \tag{3.2.18}$$

and

$$\Delta_q = \mathcal{D}_q\ \Omega_{j,i}^{(\delta,\gamma,\zeta;-s)}\ \Phi_{\ell,k}^{(\mu,\lambda,\eta;r)}\ \Phi_{h,m}^{(\alpha,\beta,\sigma;1)}. \tag{3.2.19}$$

Here the operators $\Omega_{j,i}^{(\delta,\gamma,\zeta;-s)}, \Phi_{\ell,k}^{(\mu,\lambda,\eta;r)}, \Phi_{h,m}^{(\alpha,\beta,\sigma;1)}$ in (3.2.19) are not commutative with the operator $\mathcal{D}_q$.
In making an attempt to examine this property for the functions $E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(z;s,r|q)$ and $e_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(z;s,r|q)$, it was found that $e_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(z;s,r|q)$ turns out to be an eigen function with respect to the operator Δ_q whereas this property fails for the function $E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(z;s,r|q)$.

Theorem 3.2.7. *Let $\alpha, \mu, \delta \in \mathbb{N}$ and q-difference operator Θ be defined by (3.2.7) then $e_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(z;s,r|q)$ is an eigen function with respect to the operator Δ_q defined* *by (3.2.19). That is,*

$$\Delta_q \; e^{\gamma,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu}(cz;s,r|q) = c\; e^{\gamma,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu}(cz;s,r|q). \tag{3.2.20}$$

Proof. With $\mathcal{A}_n$, $\mathcal{B}_n$ and $\mathcal{C}_n$ as in (3.2.13) and in (3.2.14),

$$e^{\gamma,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu}(cz;s,r|q) \;=\; \sum_{n=0}^{\infty} \frac{c^n\;\mathcal{A}_n}{\mathcal{B}_n\;\mathcal{C}_n\;(q;q)_n} z^n.$$

Now if $e^{\gamma,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu}(cz;s,r|q) = Y_c$ then in the notation (3.2.9),

$$\begin{aligned}
\Phi^{(\alpha,\beta,\sigma;1)}_{h,m}\; Y_c \;&=\; \sum_{n=0}^{\infty} \frac{c^n\;\mathcal{A}_n}{\mathcal{B}_n\;(q;q)_n} \frac{\left\{\prod_{h=0}^{\alpha-1}\prod_{m=0}^{\alpha-1}(\Theta+\sigma^{-h}q^{1-(\beta+m)/\alpha}-1)\right\}}{\left\{\prod_{h=0}^{\alpha-1}\prod_{m=0}^{\alpha-1}(\sigma^{-h}q^{1-(\beta+m)/\alpha})\right\}} \\
&\quad\times \frac{z^n}{\left\{\prod_{h=0}^{\alpha-1}\prod_{m=0}^{\alpha-1}(\sigma^h q^{(\beta+m)/\alpha};q)_n\right\}} \\
&=\; \sum_{n=0}^{\infty} \frac{c^n\;\mathcal{A}_n}{\mathcal{B}_n\;(q;q)_n} \frac{\left\{\prod_{h=0}^{\alpha-1}\prod_{m=0}^{\alpha-1}(1-\sigma^h q^{n-1+(\beta+m)/\alpha})\right\}}{\left\{\prod_{h=0}^{\alpha-1}\prod_{m=0}^{\alpha-1}(\sigma^h q^{(\beta+m)/\alpha};q)_n\right\}} z^n \\
&=\; \sum_{n=0}^{\infty} \frac{c^n\;\mathcal{A}_n}{\mathcal{B}_n\;\mathcal{C}_{n-1}\;(q;q)_n} z^n.
\end{aligned}$$

Next

$$\begin{aligned}
\Phi^{(\mu,\lambda,\eta;r)}_{\ell,k}\;\Phi^{(\alpha,\beta,\sigma;1)}_{h,m}\; Y_c \;&=\; \sum_{n=0}^{\infty} \frac{c^n\;\mathcal{A}_n}{\mathcal{C}_{n-1}\;(q;q)_n} \frac{\left\{\prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1}[(\Theta+\eta^{-\ell}q^{1-(\lambda+k)/\mu}-1)]^r\right\}}{\left\{\prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1}[(\eta^{\ell}\;q^{(\lambda+k)/\mu};q)_n]^r\right\}} \\
&\quad\times \frac{z^n}{\left\{\prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1}[(\eta^{-\ell}q^{1-(\lambda+k)/\mu})]^r\right\}} \\
&=\; \sum_{n=0}^{\infty} \frac{c^n\;\mathcal{A}_n}{\mathcal{C}_{n-1}\;(q;q)_n} \frac{\left\{\prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1}[(-q^n+\eta^{-\ell}q^{1-(\lambda+k)/\mu})]^r\right\}}{\left\{\prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1}[(\eta^{-\ell}q^{1-(\lambda+k)/\mu})]^r\right\}}
\end{aligned}$$

$$\times \frac{z^n}{\left\{\prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1}[(\eta^\ell\, q^{(\lambda+k)/\mu};q)_n]^r\right\}}$$

$$= \sum_{n=0}^{\infty}\frac{c^n\,\mathcal{A}_n}{\mathcal{B}_{n-1}\,\mathcal{C}_{n-1}\,(q;q)_n}z^n.$$

Further using (3.2.18),

$$\begin{aligned}
&\Omega_{j,i}^{(\delta,\gamma,\zeta;-s)}\ \Phi_{\ell,k}^{(\mu,\lambda,\eta;r)}\ \Phi_{h,m}^{(\alpha,\beta,\sigma;1)}\ Y_c\\
&= \sum_{n=0}^{\infty}\frac{c^n}{\mathcal{B}_{n-1}\,\mathcal{C}_{n-1}\,(q;q)_n}\\
&\quad\times\frac{\left\{\prod_{j=0}^{\delta-1}\prod_{i=0}^{\delta-1}[(\zeta^{-j}q^{1-(\gamma+i)/\delta})]^s\right\}\prod_{j=0}^{\delta-1}\prod_{i=0}^{\delta-1}[(\zeta^j\,q^{(\gamma+i)/\delta};q)_n]^s}{\prod_{j=0}^{\delta-1}\prod_{i=0}^{\delta-1}[(\Delta_q+\zeta^{-j}q^{1-(\gamma+i)/\delta}-1)]^s\ z^{-n}}\\
&= \sum_{n=0}^{\infty}\frac{c^n}{\mathcal{B}_{n-1}\,\mathcal{C}_{n-1}\,(q;q)_n}\\
&\quad\times\frac{\left\{\prod_{j=0}^{\delta-1}\prod_{i=0}^{\delta-1}[(\zeta^{-j}q^{1-(\gamma+i)/\delta})]^s\right\}\left\{\prod_{j=0}^{\delta-1}\prod_{i=0}^{\delta-1}[(\zeta^j\,q^{(\gamma+i)/\delta};q)_n]^s\right\}}{\left\{\prod_{j=0}^{\delta-1}\prod_{i=0}^{\delta-1}[(-q^n+\zeta^{-j}q^{1-(\gamma+i)/\delta})]^s\right\}}z^n\\
&= \sum_{n=0}^{\infty}\frac{c^n\,\mathcal{A}_{n-1}}{\mathcal{B}_{n-1}\,\mathcal{C}_{n-1}\,(q;q)_n}z^n.
\end{aligned}$$

Finally,

$$\begin{aligned}
\Delta_q\,Y_c &= \mathcal{D}_q\,\Omega_{j,i}^{(\delta,\gamma,\zeta;-s)}\ \Phi_{\ell,k}^{(\mu,\lambda,\eta;r)}\ \Phi_{h,m}^{(\alpha,\beta,\sigma;1)}\ Y_c\\
&= \sum_{n=1}^{\infty}\frac{c^n\,\mathcal{A}_{n-1}\,z^{n-1}}{\mathcal{B}_{n-1}\,\mathcal{C}_{n-1}(q;q)_{n-1}}\\
&= \sum_{n=0}^{\infty}\frac{c^{n+1}\,\mathcal{A}_n}{\mathcal{B}_n\,\mathcal{C}_n\,(q;q)_n}z^n\\
&= c\ e_{\alpha,\,\beta,\,\lambda,\,\mu}^{\gamma,\delta}(cz;s,r|q).
\end{aligned}$$

□

3.2.5 Mixed relations

Theorem 3.2.8. *The mixed relation involving* $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z^\alpha; s, r|q)$ *is given by*

$$\begin{aligned} & D_q\big(z^\beta E^{\gamma,\delta}_{\alpha,\beta+k,\lambda,\mu}(z^\alpha; s, r|q)\big) \\ = & (1-q^{k-1})\ D_q\big(z^\beta E^{\gamma,\delta}_{\alpha,\beta+k+1,\lambda,\mu}(z^\alpha; s, r|q)\big) \\ & + q^{k-1}\,(1-q)\ D_q^2\big(z^{\beta+1} E^{\gamma,\delta}_{\alpha,\beta+k+1,\lambda,\mu}(z^\alpha; s, r|q)\big), \end{aligned} \tag{3.2.21}$$

where $k = 2, 3, 4, \ldots$.

Proof. The proofs for the cases $k = 2, 3$ and for $k \geq 4$ differ, hence they are given separately.

As the change is taking place only in the parameter β, hence for the sake of brevity, the following notations will be adopted here.

$$E^{\gamma,\delta}_{\alpha,\beta+j,\lambda,\mu}(z^\alpha; s, r|q) = {}_qE_{\beta+j}, \quad \alpha n + \beta = \sigma(n),$$

and

$$\frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\lambda+\mu n}; q)_\infty]^r\ z^{\alpha n}}{[(q^{\gamma+\delta n}; q)_\infty]^s\ (q;q)_n} = d_{n;q}.$$

CASE I: $k = 2$

Here

$$\begin{aligned} {}_qE_{\beta+2} &= \sum_{n=0}^{\infty} (q^{\sigma(n)+2}; q)_\infty\ d_{n;q} \\ &= \sum_{n=0}^{\infty} \frac{(q^{\sigma(n)}; q)_\infty}{(1-q^{\sigma(n)})\ (1-q^{\sigma(n)+1})}\ d_{n;q} \\ &= \sum_{n=0}^{\infty} \frac{(q^{\sigma(n)}; q)_\infty}{1-q} \left(\frac{1}{1-q^{\sigma(n)}} - \frac{q}{1-q^{\sigma(n)+1}}\right)\ d_{n;q} \\ &= \sum_{n=0}^{\infty} \frac{(q^{\sigma(n)}; q)_\infty}{(1-q)\ (1-q^{\sigma(n)})}\ d_{n;q} - \sum_{n=0}^{\infty} \frac{q\ \ (q^{\sigma(n)}; q)_\infty}{(1-q)\ (1-q^{\sigma(n)+1})}\ d_{n;q}. \end{aligned}$$

By rearranging the series, this becomes

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{q\ \ (q^{\sigma(n)}; q)_\infty}{(1-q)\ (1-q^{\sigma(n)+1})}\ d_{n;q} \\ = & \sum_{n=0}^{\infty} \frac{(q^{\sigma(n)}; q)_\infty}{(1-q)\ (1-q^{\sigma(n)})}\ d_{n;q} - {}_qE_{\beta+2}, \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(q^{\sigma(n)};q)_\infty}{(1-q)\,(1-q^{\sigma(n)})}\; d_{n;q} - \sum_{n=0}^{\infty} \,(q^{\sigma(n)+2};q)_\infty \; d_{n;q} \\
&= \sum_{n=0}^{\infty} \frac{(q^{\sigma(n)};q)_\infty}{(1-q)\,(1-q^{\sigma(n)})}\; d_{n;q} - \sum_{n=0}^{\infty} \frac{(q^{\sigma(n)};q)_\infty}{(1-q^{\sigma(n)})\,(1-q^{\sigma(n)+1})}\; d_{n;q} \\
&= \sum_{n=0}^{\infty} \frac{(q^{\sigma(n)};q)_\infty}{(1-q^{\sigma(n)})} \left(\frac{1}{1-q} - \frac{1}{(1-q^{\sigma(n)+1})}\right)\; d_{n;q} \\
&= \frac{q}{1-q} \sum_{n=0}^{\infty} \frac{(q^{\sigma(n)};q)_\infty}{(1-q^{\sigma(n)+1})}\; d_{n;q} \\
&= q \sum_{n=0}^{\infty} (q^{\sigma(n)};q)_\infty \left(\frac{1}{(1-q^{\sigma(n)+2})\,(1-q^{\sigma(n)+1})} + \frac{q}{(1-q^{\sigma(n)+2})\,(1-q)}\right)\; d_{n;q} \\
&= q \sum_{n=0}^{\infty} (q^{\sigma(n)+3};q)_\infty \,(1-q^{\sigma(n)})\; d_{n;q} \\
&\quad + \frac{q^2}{1-q} \sum_{n=0}^{\infty} \,(q^{\sigma(n)+3};q)_\infty \,(1-q^{\sigma(n)+1})\,(1-q^{\sigma(n)})\; d_{n;q}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{(q^{\sigma(n)};q)_\infty}{(1-q^{\sigma(n)+1})}\; d_{n;q} &= (1-q) \sum_{n=0}^{\infty} (q^{\sigma(n)+3};q)_\infty \,(1-q^{\sigma(n)})\; d_{n;q} \\
&\quad + q \sum_{n=0}^{\infty} \,(q^{\sigma(n)+3};q)_\infty \,(1-q^{\sigma(n)})(1-q^{\sigma(n)+1})\; d_{n;q}.
\end{aligned} \tag{3.2.22}
$$

Now

$$
\begin{aligned}
(1-q)\, z^{1-\beta} D_q\big(z^{\beta}\, {}_qE_{\beta+2}\big) &= \sum_{n=0}^{\infty} \frac{(q^{\sigma(n)};q)_\infty}{(1-q^{\sigma(n)+1})}\; d_{n;q}, \\
(1-q)\, z^{1-\beta} D_q\big(z^{\beta}\, {}_qE_{\beta+3}\big) &= \sum_{n=0}^{\infty} (q^{\sigma(n)+3};q)_\infty \,(1-q^{\sigma(n)})\; d_{n;q},
\end{aligned}
$$

and

$$
(1-q)^2\, z^{1-\beta} D_q^2\big(z^{\beta+1}\, {}_qE_{\beta+3}\big) = \sum_{n=0}^{\infty} (q^{\sigma(n)+3};q)_\infty \,(1-q^{\sigma(n)})\,(1-q^{\sigma(n)+1})\; d_{n;q}.
$$

Using these in (3.2.22), yields the relation (3.2.21) for $k = 2$.

CASE II: $k = 3$

In this case,

$$
\begin{aligned}
{}_qE_{\beta+3} &= \sum_{n=0}^{\infty}(q^{\sigma(n)+3};q)_\infty \; d_{n;q} \\
&= \sum_{n=0}^{\infty}\frac{(q^{\sigma(n)};q)_\infty}{(1-q^{\sigma(n)})\;(1-q^{\sigma(n)+1})\;(1-q^{\sigma(n)+2})}\; d_{n;q} \\
&= \sum_{n=0}^{\infty}\frac{(q^{\sigma(n)};q)_\infty}{1-q}\Bigg(\frac{1}{(1-q^{\sigma(n)})\;(1-q^{\sigma(n)+2})} \\
&\quad -\frac{q}{(1-q^{\sigma(n)+1})\;(1-q^{\sigma(n)+2})}\Bigg)\; d_{n;q} \\
&= \frac{1}{1-q}\sum_{n=0}^{\infty}\frac{(q^{\sigma(n)};q)_\infty}{(1-q^{\sigma(n)})\;(1-q^{\sigma(n)+2})}\; d_{n;q} \\
&\quad -\frac{q}{(1-q)}\;\sum_{n=0}^{\infty}\frac{(q^{\sigma(n)};q)_\infty}{(1-q^{\sigma(n)+1})\;(1-q^{\sigma(n)+2})}\; d_{n;q}.
\end{aligned}
$$

This is considered in the following form.

$$
\begin{aligned}
&\frac{q}{1-q}\;\sum_{n=0}^{\infty}\frac{(q^{\sigma(n)};q)_\infty}{(1-q^{\sigma(n)+1})\;(1-q^{\sigma(n)+2})}\; d_{n;q} \\
=\;& \frac{1}{1-q}\sum_{n=0}^{\infty}\frac{(q^{\sigma(n)};q)_\infty}{(1-q^{\sigma(n)})\;(1-q^{\sigma(n)+2})}\; d_{n;q} -\; {}_qE_{\beta+3}, \\
=\;& \frac{1}{1-q}\sum_{n=0}^{\infty}\frac{(q^{\sigma(n)};q)_\infty}{(1-q^{\sigma(n)})(1-q^{\sigma(n)+2})}\; d_{n;q} \\
&-\sum_{n=0}^{\infty}\frac{(q^{\sigma(n)};q)_\infty}{(1-q^{\sigma(n)})\;(1-q^{\sigma(n)+1})\;(1-q^{\sigma(n)+2})}\; d_{n;q} \\
=\;& \sum_{n=0}^{\infty}\frac{(q^{\sigma(n)};q)_\infty}{(1-q^{\sigma(n)})\;(1-q^{(\sigma)n+2})}\left(\frac{1}{1-q}-\frac{1}{1-q^{\sigma(n)+1}}\right) d_{n;q} \\
=\;& \frac{q}{1-q}\sum_{n=0}^{\infty}\frac{(q^{\sigma(n)};q)_\infty}{(1-q^{\sigma(n)+1})\;(1-q^{\sigma(n)+2})}\; d_{n;q} \\
=\;& q\sum_{n=0}^{\infty}(q^{\sigma(n)};q)_\infty\Bigg(\frac{1+q}{(1-q^{\sigma(n)+1})\;(1-q^{\sigma(n)+2})\;(1-q^{\sigma(n)+3})} \\
&+\frac{q^2}{(1-q)\;(1-q^{\sigma(n)+2})\;(1-q^{\sigma(n)+3})}\Bigg)\; d_{n;q} \\
=\;& q\sum_{n=0}^{\infty}\;(q^{\sigma(n)};q)_\infty\;\frac{(1+q)}{(1-q^{\sigma(n)+1})\;(1-q^{\sigma(n)+2})\;(1-q^{\sigma(n)+3})}\; d_{n;q} \\
&+\frac{q^3}{1-q}\sum_{n=0}^{\infty}\frac{(q^{\sigma(n)};q)_\infty}{(1-q^{\sigma(n)+2})\;(1-q^{\sigma(n)+3})}\; d_{n;q}.
\end{aligned}
$$

This gives

$$\begin{aligned}
&\frac{q}{1-q}\sum_{n=0}^{\infty}(q^{\sigma(n)+3};q)_\infty\,(1-q^{\sigma(n)})\;d_{n;q}\\
=\;& q(1+q)\sum_{n=0}^{\infty}(q^{\sigma(n)+4};q)_\infty\,(1-q^{\sigma(n)})\;d_{n;q}\\
&+\frac{q^3}{1-q}\sum_{n=0}^{\infty}(q^{\sigma(n)+4};q)_\infty\,(1-q^{\sigma(n)})\,(1-q^{\sigma(n)+1})\;d_{n;q}.
\end{aligned}$$

That is,

$$\begin{aligned}
&\sum_{n=0}^{\infty}(q^{\sigma(n)+3};q)_\infty\,(1-q^{\sigma(n)})\;d_{n;q}\\
=\;&(1-q^2)\sum_{n=0}^{\infty}(q^{\sigma(n)+4};q)_\infty\,(1-q^{\sigma(n)})\;d_{n;q}\\
&+q^2\sum_{n=0}^{\infty}(q^{\sigma(n)+4};q)_\infty\,(1-q^{\sigma(n)})\,(1-q^{\sigma(n)+1})\;d_{n;q}.
\end{aligned}$$

But

$$(1-q)\,z^{1-\beta}D_q\big(z^\beta\;{}_qE_{\beta+3}\big)=\sum_{n=0}^{\infty}(q^{\sigma(n)+3};q)_\infty(1-q^{\sigma(n)})\;d_{n;q},$$

$$(1+q)(1-q)^2\,z^{1-\beta}D_q\big(z^\beta\;{}_qE_{\beta+4}\big)=(1-q^2)\sum_{n=0}^{\infty}(q^{\sigma(n)+4};q)_\infty\,(1-q^{\sigma(n)})\;d_{n;q}$$

and

$$q^2\,(1-q)^2\,z^{1-\beta}D_q^2\big(z^{\beta+1}\;{}_qE_{\beta+4}\big)=q^2\sum_{n=0}^{\infty}(q^{\sigma(n)+4};q)_\infty\,(1-q^{\sigma(n)})\,(1-q^{\sigma(n)+1})\;d_{n;q},$$

hence one finds

$$\begin{aligned}
D_q\big(z^\beta E^{\gamma,\delta}_{\alpha,\,\beta+3,\,\lambda,\,\mu}(z^\alpha;s,r|q)\big)\;=\;&(1-q^2)\,D_q\big(z^\beta E^{\gamma,\delta}_{\alpha,\,\beta+4,\,\lambda,\,\mu}(z^\alpha;s,r|q)\big)\\
&+q^2\,(1-q)\;D_q^2\big(z^{\beta+1}E^{\gamma,\delta}_{\alpha,\,\beta+4,\,\lambda,\,\mu}(z^\alpha;s,r|q)\big).
\end{aligned}$$

CASE III: $k=4,5,\ldots$

Begining with

$${}_qE_{\beta+k}\;=\;\sum_{n=0}^{\infty}(q^{\sigma(n)+k};q)_\infty\;d_{n;q}$$

$$= \sum_{n=0}^{\infty} \frac{(q^{\sigma(n)};q)_\infty}{(1-q^{\sigma(n)})\ (1-q^{\sigma(n)+1})\ (1-q^{\sigma(n)+2})\ ...\ (1-q^{\sigma(n)+k-1})}\ d_{n;q}.$$

and putting $(1-q^{\sigma(n)+i}) = B_i$, one finds

$$\begin{aligned} {}_qE_{\beta+k} &= \sum_{n=0}^{\infty}(q^{\sigma(n)};q)_\infty \left[\left\{\prod_{i=0}^{k-2} B_i\right\}^{-1} + \frac{q^{k-1}}{(1-q^{k-2})\ B_{k-1}}\left\{\prod_{i=0}^{k-3} B_i\right\}^{-1} \right. \\ & \left. -\frac{q^{k-1}}{(1-q^{k-2})}\left\{\prod_{i=1}^{k-1} B_i\right\}^{-1}\right] d_{n;q} \\ &= \sum_{n=0}^{\infty}(q^{\sigma(n)};q)_\infty \left\{\prod_{i=0}^{k-2} B_i\right\}^{-1} d_{n;q} + \sum_{n=0}^{\infty} \frac{q^{k-1}\ (q^{\sigma(n)};q)_\infty}{(1-q^{k-2})\ B_{k-1}}\left\{\prod_{i=0}^{k-3} B_i\right\}^{-1} d_{n;q} \\ & -\sum_{n=0}^{\infty} \frac{q^{k-1}\ (q^{\sigma(n)};q)_\infty}{(1-q^{k-2})}\left\{\prod_{i=1}^{k-1} B_i\right\}^{-1} d_{n;q}. \end{aligned}$$

Once again writing this as

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{q^{k-1}\ (q^{\sigma(n)};q)_\infty}{(1-q^{k-2})}\left\{\prod_{i=1}^{k-1} B_i\right\}^{-1} d_{n;q} \\ = & \sum_{n=0}^{\infty}(q^{\sigma(n)};q)_\infty \left\{\prod_{i=0}^{k-2} B_i\right\}^{-1} d_{n;q} \\ & +\sum_{n=0}^{\infty} \frac{q^{k-1}\ (q^{\sigma(n)};q)_\infty}{(1-q^{k-2})\ B_{k-1}}\left\{\prod_{i=0}^{k-3} B_i\right\}^{-1} d_{n;q} - {}_qE_{\beta+k}, \end{aligned}$$

one gets

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{q^{k-1}\ (q^{\sigma(n)};q)_\infty}{(1-q^{k-2})}\left\{\prod_{i=1}^{k-1} B_i\right\}^{-1} d_{n;q} \\ = & \sum_{n=0}^{\infty}(q^{\sigma(n)};q)_\infty \left\{\prod_{i=0}^{k-2} B_i\right\}^{-1} d_{n;q} + \sum_{n=0}^{\infty} \frac{q^{k-1}\ (q^{\sigma(n)};q)_\infty}{(1-q^{k-2})\ B_{k-1}}\left\{\prod_{i=0}^{k-3} B_i\right\}^{-1} d_{n;q} \\ & -\sum_{n=0}^{\infty}(q^{\sigma(n)};q)_\infty \left\{\prod_{i=0}^{k-1} B_i\right\}^{-1} d_{n;q} \\ = & \sum_{n=0}^{\infty}(q^{\sigma(n)};q)_\infty \left\{\prod_{i=1}^{k-3} B_i\right\}^{-1} \left(\frac{1}{B_{k-2}} + \frac{q^{k-1}}{(1-q^{k-2})}\ \frac{1}{B_{k-1}} - \frac{1}{B_{k-2}\ B_{k-1}}\right) d_{n;q} \\ = & \sum_{n=0}^{\infty} \frac{q^{k-1}(q^{\sigma(n)};q)_\infty}{(1-q^{k-2})}\left\{\prod_{i=1}^{k-1} B_i\right\}^{-1} d_{n;q} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{q^{k-1}\ (q^{\sigma(n)};q)_\infty}{(1-q^{k-2})} \left(\frac{1-q^{k-1}}{(1-q^{\sigma(n)+1})} + q^{k-1} \right) \left\{ \prod_{i=2}^{k} B_i \right\}^{-1} d_{n;q} \\
&= \sum_{n=0}^{\infty} \frac{(q^{\sigma(n)+k+1};q)_\infty\ (1-q^{\sigma(n)})\ q^{k-1}\ (1-q^{k-1})}{(1-q^{k-2})}\ d_{n;q} \\
&\quad + \sum_{n=0}^{\infty} \frac{(q^{\sigma(n)+k+1};q)_\infty\ (1-q^{\sigma(n)})\ (1-q^{\sigma(n)+1})\ q^{2k-2}}{(1-q^{k-2})}\ d_{n;q}.
\end{aligned}
$$

But since

$$
\begin{aligned}
&\frac{(1-q)\ q^{k-1}}{(1-q^{k-2})} z^{1-\beta} D_q\big(z^\beta E^{\gamma,\delta}_{\alpha,\,\beta+k,\,\lambda,\,\mu}(z^\alpha;s,r|q)\big) \\
&= \sum_{n=0}^{\infty} \frac{q^{k-1}\ (q^{\sigma(n)};q)_\infty}{(1-q^{k-2})} \left\{ \prod_{i=1}^{k-1} B_i \right\}^{-1} d_{n;q}, \\
&\frac{(1-q)^2\ q^{k-1}}{(1-q^{k-2})}(1+q+q^2+\ldots+q^{k-2})\ z^{1-\beta}\ D_q\big(z^\beta E^{\gamma,\delta}_{\alpha,\,\beta+k+1,\,\lambda,\,\mu}(z^\alpha;s,r|q)\big) \\
&= \sum_{n=0}^{\infty} \frac{q^{k-1}\ (1-q^{k-1})\ (q^{\sigma(n)+k+1};q)_\infty\ (1-q^{\sigma(n)})}{(1-q^{k-2})}\ d_{n;q},
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{(1-q)^2\ q^{2k-2}}{(1-q^{k-2})} z^{1-\beta}\ D_q^2\big(z^{\beta+1} E^{\gamma,\delta}_{\alpha,\,\beta+k+1,\,\lambda,\,\mu}(z^\alpha;s,r|q)\big) \\
&= \sum_{n=0}^{\infty} \frac{(q^{\sigma(n)+k+1};q)_\infty(1-q^{\sigma(n)})(1-q^{\sigma(n)+1})\ q^{2k-2}}{(1-q^{k-2})}\ d_{n;q},
\end{aligned}
$$

the recurrence relation (3.2.21) follows. □

Yet another such identity is obtained in straight forward manner as follows.

Theorem 3.2.9. *For $\alpha,\beta,\gamma,\lambda \in \mathbb{C}; \Re(\alpha,\beta,\gamma,\lambda) > 0,\ \ \delta,\mu > 0$ there holds the mixed recurrence relation:*

$$
\begin{aligned}
&(1-q^\beta)\ E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu}(z^\alpha;s,r|q) + (1-q)\ q^\beta\ z\ D_q E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu}(z^\alpha;s,r|q) \\
= \ &E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z^\alpha;s,r|q).
\end{aligned}
$$

Proof. With the above notations of $d_{n;q}$ and $\sigma(n)$,

$$
\begin{aligned}
l.h.s. &= (1-q^\beta)\sum_{n=0}^{\infty}(q^{\sigma(n)+1};q)_\infty\ d_{n;q} + (1-q)\ q^\beta\ z\ D_q \sum_{n=0}^{\infty}(q^{\sigma(n)+1};q)_\infty\ d_{n;q} \\
&= (1-q^\beta)\sum_{n=0}^{\infty}(q^{\sigma(n)+1};q)_\infty\ d_{n;q} + q^\beta \sum_{n=0}^{\infty}(q^{\sigma(n)+1};q)_\infty(1-q^{\alpha n})\ d_{n;q}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty}(q^{\sigma(n)};q)_\infty\ d_{n;q} \\
&= r.h.s.
\end{aligned}
$$

□

The function $e^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r|q)$ satisfies the following mixed recurrence relations, whose proofs are omitted as it differs only in the factor $q^{pn(n-1)/2}$.

Theorem 3.2.10. *The mixed relation for $k \in \mathbb{N}$ with $k \geq 2$, is given by*

$$
\begin{aligned}
& D_q\big(z^{\beta} e^{\gamma,\delta}_{\alpha,\,\beta+k,\,\lambda,\,\mu}(z^{\alpha};s,r|q)\big) \\
= \;& (1-q^{k-1})\ D_q\big(z^{\beta} e^{\gamma,\delta}_{\alpha,\,\beta+k+1,\,\lambda,\,\mu}(z^{\alpha};s,r|q)\big) \\
& +q^{k-1}\ (1-q)\ D_q^2\big(z^{\beta+1} e^{\gamma,\delta}_{\alpha,\,\beta+k+1,\,\lambda,\,\mu}(z^{\alpha};s,r|q)\big).
\end{aligned} \tag{3.2.23}
$$

Theorem 3.2.11. *For $\alpha,\beta,\gamma,\lambda \in \mathbb{C}; \Re(\alpha,\beta,\gamma,\lambda) > 0,\ \ \delta,\mu > 0$ there holds the mixed recurrence relation:*

$$
\begin{aligned}
& (1-q^{\beta})\ e^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu}(z^{\alpha};s,r|q) + (1-q)\ q^{\beta}\ z\ D_q E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu}(z^{\alpha};s,r|q) \\
= \;& e^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z^{\alpha};s,r|q).
\end{aligned}
$$

3.2.6 Finite summation formulas

By repeatedly applying q-derivatives to the function $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r|q)$ two finite series formulas are obtained here. For the sake of simplicity, this function will be symbolized by $E(z)$, then in view of the definition (1.2.38) of q-derivative, one gets

$$
\begin{aligned}
D_qE(z) &= \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}[(q^{\lambda+\mu n};q)_\infty]^r (q^{\beta+\alpha n};q)_\infty}{[(q^{\gamma+\delta n};q)_\infty]^s (q;q)_n} D_q z^n \\
&= \frac{1}{1-q} \sum_{n=0}^{\infty} \frac{(-1)^{p(n+1)}\ q^{pn(n+1)/2}[(q^{\lambda+\mu+\mu n};q)_\infty]^r}{[(q^{\gamma+\delta+\delta n};q)_\infty]^s (q;q)_n} (q^{\alpha+\alpha n+\beta};q)_\infty\ z^n.
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
D_q^2E(z) = \;& \frac{1}{(1-q)^2} \sum_{n=0}^{\infty} \frac{(-1)^{p(n+2)}\ q^{p(n+1)(n+2)/2}[(q^{\lambda+2\mu+n\mu};q)_\infty]^r}{[(q^{\gamma+2\delta+\delta n};q)_\infty]^s (q;q)_n} \\
& \times (q^{2\alpha+\alpha n+\beta};q)_\infty\ z^n.
\end{aligned}
$$

In general,

$$\begin{aligned} D_q^m E(z) &= \frac{1}{(1-q)^m} \sum_{n=0}^{\infty} \frac{(-1)^{p(n+m)} \; q^{p(n+m)(n+m-1)/2} [(q^{\lambda+m\mu+n\mu};q)_\infty]^r}{[(q^{\gamma+m\delta+n\delta};q)_\infty]^s (q;q)_n} \\ &\quad \times (q^{m\alpha+\alpha n+\beta};q)_\infty \; z^n \\ &= \frac{(-1)^{pm} \; q^{pm(m-1)/2}}{(1-q)^m} E^{\gamma+m\delta,\delta}_{\alpha,\beta+m\alpha,\lambda+m\mu,\mu}(zq^{pm};s,r|q). \end{aligned}$$

Here using the formula [51, Eq.6.9, p.33]:

$$x^m(1-q)^m D_q^m f(x) = (-1)^m q^{-m(m-1)/2} \sum_{k=0}^{m} (-1)^k q^{k(k-1)/2} \begin{bmatrix} m \\ k \end{bmatrix} f(xq^{m-k}) \quad (3.2.24)$$

with $f(x) = E(z) = E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r|q)$, one obtains

$$\begin{aligned} z^m E^{\gamma+m\delta,\delta}_{\alpha,\beta+m\alpha,\lambda+m\mu,\mu}(zq^{pm};s,r|q) &= (-1)^{pm} \; q^{-pm(m-1)/2} \sum_{k=0}^{m} (-1)^k q^{pk(k-2m+1)/2} \\ &\quad \times \begin{bmatrix} m \\ k \end{bmatrix}_{q^p} E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(zq^{pk};s,r|q). \qquad (3.2.25) \end{aligned}$$

This last identity enables one to derive one more such sum; the method of obtaining this uses inverse series relations [10] which is taken up below.

Lemma 3.2.1. *For the finite sequences $\{G_r\}$ and $\{g_r\}$ of complex numbers, the following series relations hold.*

$$G_m = \sum_{k=0}^{m} (-1)^k q^{k(k-2m+1)/2} \begin{bmatrix} m \\ k \end{bmatrix} g_k \qquad (3.2.26)$$

if and only if

$$g_m = \sum_{k=0}^{m} (-1)^k q^{k(k-1)/2} \begin{bmatrix} m \\ k \end{bmatrix} G_k. \qquad (3.2.27)$$

Proof. Part-I. Suppose (3.2.26) holds true. Then denoting the right hand side of (3.2.27) by Ω and substituting the series for G_k from (3.2.26),

$$\Omega = \sum_{k=0}^{m} (-1)^k q^{k(k-1)/2} \begin{bmatrix} m \\ k \end{bmatrix} \sum_{j=0}^{k} (-1)^j q^{j(j-2k+1)/2} \begin{bmatrix} k \\ j \end{bmatrix} g_j.$$

This in view of the double series relation:

$$\sum_{k=0}^{n}\sum_{j=0}^{k} A(k,j) = \sum_{j=0}^{n}\sum_{k=0}^{n-j} A(k+j,j), \tag{3.2.28}$$

and finite series-product identity:

$$\sum_{k=0}^{n} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} x^k = \prod_{k=1}^{n}(1+xq^{k-1}), \tag{3.2.29}$$

simplifies to

$$\begin{aligned}
\Omega &= \sum_{j=0}^{m}\sum_{k=j}^{m}(-1)^{k+j}\begin{bmatrix} m \\ k \end{bmatrix}\begin{bmatrix} k \\ j \end{bmatrix} q^{k(k-1)/2} q^{j(j-2k+1)/2} g_j \\
&= \sum_{j=0}^{m}\sum_{k=0}^{m-j}(-1)^k q^{k(k-1)/2}\frac{(q;q)_m}{(q;q)_{m-j-k}(q;q)_k(q;q)_j} g_j \\
&= \sum_{j=0}^{m}\begin{bmatrix} m \\ j \end{bmatrix} g_j \sum_{k=0}^{m-j} q^{k(k-1)/2}\begin{bmatrix} m-j \\ k \end{bmatrix}(-1)^k \\
&= g_m + \sum_{j=0}^{m-1}\begin{bmatrix} m \\ j \end{bmatrix} g_j \prod_{k=1}^{m-j}(1-q^{k-1}) \\
&= g_m,
\end{aligned}$$

completing the first part.

Part-II. Assuming now that the series (3.2.27) holds true then proceeding as above,

$$\begin{aligned}
\zeta &= \sum_{k=0}^{m}(-1)^k q^{k(k-2m+1)/2}\begin{bmatrix} m \\ k \end{bmatrix} g_k \\
&= \sum_{k=0}^{m}(-1)^k q^{k(k-2m+1)/2}\begin{bmatrix} m \\ k \end{bmatrix}\sum_{j=0}^{k}(-1)^j q^{j(j-1)/2}\begin{bmatrix} k \\ j \end{bmatrix} G_j \\
&= \sum_{j=0}^{m} q^{j^2-mj}\begin{bmatrix} m \\ j \end{bmatrix} G_j \sum_{k=0}^{m-j}(-1)^k q^{k(k+2j-2m+1)/2}\begin{bmatrix} m-j \\ k \end{bmatrix} \\
&= G_m + \sum_{j=0}^{m-1} q^{j^2-mj}\begin{bmatrix} m \\ j \end{bmatrix} G_j \sum_{k=0}^{m-j} q^{k(k-1)/2}\begin{bmatrix} m-j \\ k \end{bmatrix}\left(-q^{j-m+1}\right)^k \\
&= G_m + \sum_{j=0}^{m-1} q^{j^2-mj}\begin{bmatrix} m \\ j \end{bmatrix} G_j \prod_{k=1}^{m-j}\left(1-q^{j-m+k}\right) \\
&= G_m.
\end{aligned}$$

Hence the converse part and the proof of the lemma. □

The comparison of (3.2.26) with (3.2.25) suggests that

$$g_k = E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(zq^{pk}; s, r|q)$$

and

$$G_m = \frac{(-1)^{pm}}{q^{-pm(m-1)/2}}\ z^m E^{\gamma+m\delta,\delta}_{\alpha,\beta+m\alpha,\lambda+m\mu}(zq^{pm}; s, r|q).$$

With this choice, the inverse series (3.2.27) provides the identity:

$$\begin{aligned} E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(zq^{pm}; s, r|q) &= \sum_{k=0}^{m}(-1)^k q^{pk(k-1)} \begin{bmatrix} m \\ k \end{bmatrix}_{q^p} \\ &\quad \times(-1)^{pk}\ z^k E^{\gamma+k\delta,\delta}_{\alpha,\beta+k\alpha,\lambda+k\mu,\mu}(zq^{pk}; s, r|q). \qquad (3.2.30) \end{aligned}$$

Similarly, by repeatedly applying q-derivatives to the function $e^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z; s, r|q)$ two finite series formulas may be obtained. Here the m^{th} order q-derivative occurs in the form:

$$\begin{aligned} D^m_q e^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z; s, r|q) &= \frac{1}{(1-q)^m} \sum_{n=0}^{\infty} \frac{[(q^{\lambda+m\mu+n\mu}; q)_\infty]^r}{[(q^{\gamma+m\delta+n\delta}; q)_\infty]^s (q; q)_n} (q^{m\alpha+\alpha n+\beta}; q)_\infty\ z^n \\ &= \frac{1}{(1-q)^m}\ e^{\gamma+m\delta,\delta}_{\alpha,\beta+m\alpha,\lambda+m\mu,\mu}(z; s, r|q). \end{aligned}$$

Once again use of the formula (3.2.24) with $f(x) = e^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z; s, r|q)$, one finds

$$z^m e^{\gamma+m\delta,\delta}_{\alpha,\beta+m\alpha,\lambda+m\mu,\mu}(z; s, r|q) = \sum_{k=0}^{m}(-1)^k q^{k(k-2m+1)/2} \begin{bmatrix} m \\ k \end{bmatrix}\ e^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z; s, r|q). \quad (3.2.31)$$

This last identity enables one to derive one more such sum by using the inverse series relations of Lemma 3.2.1. The comparison of (3.2.26) with (3.2.31) suggests that

$$g_k = e^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(zq^k; s, r|q)$$

and

$$G_m = z^m e^{\gamma+m\delta,\delta}_{\alpha,\beta+m\alpha,\lambda+m\mu}(z; s, r|q).$$

With this choice, the inverse series (3.2.27) provides the identity:

$$e^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z; s, r|q) = \sum_{k=0}^{m}(-1)^k q^{k(k-1)/2} \begin{bmatrix} m \\ k \end{bmatrix}\ z^k e^{\gamma+k\delta,\delta}_{\alpha,\beta+k\alpha,\lambda+k\mu,\mu}(zq^k; s, r|q). \ (3.2.32)$$

3.2.7 Double series representation

Next, the double series representation of the function $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r|q)$ is derived as follows.

Theorem 3.2.12. *With the restrictions to the parameters as in earlier properties,*

$$^*E^{\gamma,\delta,\rho}_{\alpha,\beta,\lambda,\mu}(z;s,r|q)=\sum_{i,j=0}^{\infty}\frac{(-1)^i q^{i(i-1)/2}}{(q;q)_i(q;q)_j}\ {}^*E^{\gamma,\delta,\rho+i+j}_{\alpha,\beta,\lambda,\mu}(z;s,r|q). \tag{3.2.33}$$

Proof. As in Theorem 3.2.1, take

$$V_n=\frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_\infty]^r\ (q^{\alpha n+\beta};q)_\infty}{[(q^{\gamma+\delta n};q)_\infty]^s\ (q;q)_n}$$

and put

$$^*E^{\gamma,\delta,\rho}_{\alpha,\beta,\lambda,\mu}(z;s,r|q)=\sum_{n=0}^{\infty}\frac{(-1)^n\ q^{-n(n-1)/2}\ V_n}{(q^{\rho+n};q)_\infty}z^n$$

which is valid under the condition $\Re(\alpha^2+r\mu^2-s\delta^2)>0$.
Now, introducing the function $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r|q)$ in the integrand of the integral (1.2.44) and applying this integral to the identity [18, p. 9]: $e_q(y)E_q(-y)=1$, one finds the integral

$$\int_0^1 t^{\rho-1}\ E_q(tq)\ e_q(xt)\ E_q(-xt)\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(zt;s,r|q)\ \mathrm{d}_qt$$
$$=\int_0^1 t^{\rho-1}\ E_q(tq)\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(zt;s,r|q)\ \mathrm{d}_qt. \tag{3.2.34}$$

Here substituting the corresponding series representations for $e_q(xt)$ and $E_q(-xt)$ on the left hand side, one gets

$$\begin{aligned}
l.h.s.\ &=\int_0^1 t^{\rho-1}E_q(tq)\ e_q(xt)\ E_q(-xt)\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(zt;s,r|q)\ \mathrm{d}_qt\\
&=\int_0^1 t^{\rho-1}\left\{\sum_{i=0}^{\infty}(-1)^i q^{i(i-1)/2}\frac{x^it^i}{(q;q)_i}\right\}\left\{\sum_{j=0}^{\infty}\frac{x^jt^j}{(q;q)_j}\right\}\\
&\quad\times E_q(tq)\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(zt;s,r|q)\ \mathrm{d}_qt\\
&=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^iq^{i(i-1)/2}}{(q;q)_i(q;q)_j}x^{i+j}\int_0^1 t^{\rho+i+j-1}\ E_q(tq)\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(zt;s,r|q)\ \mathrm{d}_qt\\
&=(1-q)(q;q)_\infty\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^iq^{i(i-1)/2}}{(q;q)_i(q;q)_j}\sum_{n=0}^{\infty}\frac{(-1)^n\ q^{-n(n-1)/2}\ V_n}{(q^{\rho+i+j+n};q)_\infty}z^n
\end{aligned}$$

$$= (1-q)(q;q)_\infty \sum_{i,j=0}^{\infty} \frac{(-1)^i q^{i(i-1)/2}}{(q;q)_i (q;q)_j} \, {}^*E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta,\rho+i+j}(z;s,r|q). \tag{3.2.35}$$

Likewise,

$$\begin{aligned} r.h.s. &= \int_0^1 t^{\rho-1} E_q(tq) E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(zt;s,r|q)\ d_q t \\ &= \sum_{n=0}^{\infty} u_n z^n \int_0^1 t^{\rho+n-1}\ E_q(tq)\ d_q t \\ &= (1-q)(q;q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n\ q^{-n(n-1)/2}\ V_n}{(q^{\rho+n};q)_\infty} z^n \\ &= (1-q)\ (q;q)_\infty\ {}^*E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta,\rho}(z;s,r|q). \end{aligned} \tag{3.2.36}$$

Consequently from (3.2.35) and (3.2.36), (3.2.34) yields (3.2.33). □

Next, the double series representation of the function $e_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(z;s,r|q)$ can be obtained similarly as above. This is given in

Theorem 3.2.13. *With the restrictions to the parameters as in earlier properties,*

$${}^*e_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta,\rho}(z;s,r|q) = \sum_{i,j=0}^{\infty} \frac{(-1)^i q^{i(i-1)/2}}{(q;q)_i (q;q)_j} \, {}^*e_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta,\rho+i+j}(z;s,r|q), \tag{3.2.37}$$

where

$$*e_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta,\rho}(z;s,r|q) = \sum_{n=0}^{\infty} \frac{U_n}{(q^{\rho+n};q)_\infty} z^n.$$

Note 3.2.1. *This property does not hold if $r \in \mathbb{Z}_{<0}$. Hence, q- Dotsenko function and q- Elliptic function fail to satisfy (3.2.37).*

3.3 Other results

3.3.1 Differentiation

Theorem 3.3.1. *If $m \in \mathbb{N}, \alpha,\ \beta,\ \gamma,\ \lambda \in \mathbb{C}$ with $\Re(\alpha,\beta,\gamma,\lambda) > 0$ and $\delta,\ \mu > 0$ then*

$$\left(\frac{d_q}{d_q z}\right)^m E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(z;s,r|q) = \frac{z^{-m}}{(1-q)^m} E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(z;s,r|q), \tag{3.3.1}$$

and

$$\left(\frac{d_q}{d_q z}\right)^m z^{\beta-1}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega z^{\alpha}; s, r|q) \;=\; z^{\beta-m-1}\, E^{\gamma,\delta}_{\alpha,\beta-m,\lambda,\mu}(\omega z^{\alpha}; s, r|q), \quad (3.3.2)$$

wherein $Re(\beta - m) > 0$.

Proof. In order to prove (3.3.1), consider the

$$\begin{aligned}
l.h.s. &= \left(\frac{d_q}{d_q z}\right)^m \sum_{n=0}^{\infty} \frac{(-1)^{pn}\, q^{pn(n-1)/2}\, (q^{\alpha n+\beta}; q)_\infty\, [(q^{\lambda+\mu n}; q)_\infty]^r\, (q^{n+1}; q)_\infty\, z^n}{[(q^{\gamma+\delta n}; q)_\infty]^s} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{pn}\, q^{pn(n-1)/2}\, (q^{\alpha n+\beta}; q)_\infty\, [(q^{\lambda+\mu n}; q)_\infty]^r\, (q^{n+1}; q)_\infty}{[(q^{\gamma+\delta n}; q)_\infty]^s} \\
&\quad \times \sum_{j=0}^{m} (-1)^j\, q^{j(j-1)/2} \begin{bmatrix} m \\ j \end{bmatrix}_q \frac{(q^{m-j} z)^n}{z^m (q-1)^m q^{m(m-1)/2}} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{pn}\, q^{pn(n-1)/2}\, (q^{\alpha n+\beta}; q)_\infty\, [(q^{\lambda+\mu n}; q)_\infty]^r\, (q^{n+1}; q)_\infty}{[(q^{\gamma+\delta n}; q)_\infty]^s\, (1-q)^m} \\
&\quad \times (-1)^m\, z^{n-m} q^{mn-m(m-1)/2} \prod_{j=1}^{m} (1 - q^{-n+j-1}) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{pn}\, q^{pn(n-1)/2}\, (-1)^m (q^{\alpha n+\beta}; q)_\infty\, [(q^{\lambda+\mu n}; q)_\infty]^r\, (q^{n+1}; q)_\infty}{(1-q)^m [(q^{\gamma+\delta n}; q)_\infty]^s} z^{n+1-m-1} \\
&\quad \times q^{mn-m(m-1)/2}\, (-1)^m\, q^{-nm-m+\frac{m^2}{2}+\frac{m}{2}-1} \frac{(q^{n+1-m}; q)_\infty}{(q^{n+1}; q)_\infty} \\
&= \frac{z^{-m}}{(1-q)^m}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z; s, r|q) \\
&= r.h.s.
\end{aligned}$$

And (3.3.2) may be proved as follows.

$$\begin{aligned}
&\left(\frac{d_q}{d_q z}\right)^m z^{\beta-1}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega z^{\alpha}; s, r|q) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{pn}\, q^{pn(n-1)/2}\, (q^{\alpha n+\beta}; q)_\infty\, [(q^{\lambda+\mu n}; q)_\infty]^r\, (q^{n+1}; q)_\infty\, \omega^n}{[(q^{\gamma+\delta n}; q)_\infty]^s} \\
&\quad \times \left(\frac{d_q}{d_q z}\right)^m z^{\alpha n+\beta-1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{pn}\, q^{pn(n-1)/2}\, (q^{\alpha n+\beta}; q)_\infty\, [(q^{\lambda+\mu n}; q)_\infty]^r\, (q^{n+1}; q)_\infty\, \omega^n}{[(q^{\gamma+\delta n}; q)_\infty]^s} \\
&\quad \times \sum_{j=0}^{m} q^{j(j-1)/2} \begin{bmatrix} m \\ j \end{bmatrix}_q \frac{(q^{m-j} z)^{\alpha n+\beta-1}}{z^m\, (q-1)^m\, q^{m(m-1)/2}}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ (q^{\alpha n+\beta};q)_\infty\ [(q^{\lambda+\mu n};q)_\infty]^r\ (q^{n+1};q)_\infty\ \omega^n}{[(q^{\gamma+\delta n};q)_\infty]^s}$$

$$\times(-1)^m\ (1-q)^{-m}\ z^{\alpha n+\beta-m-1}\ q^{m(\alpha n+\beta)-m(m-1)/2}\ \prod_{j=1}^{m}(1-q^{-\alpha n-\beta+j})$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ (q^{\alpha n+\beta};q)_\infty\ [(q^{\lambda+\mu n};q)_\infty]^r\ (q^{n+1};q)_\infty\ \omega^n}{[(q^{\gamma+\delta n};q)_\infty]^s} z^{\alpha n+\beta-m-1}$$

$$\times(-1)^m(1-q)^{-m}q^{m(\alpha n+\beta-m/2-1/2)}\ (q^{-(\alpha n+\beta-1)};q)_m$$

$$= \frac{z^{\beta-m-1}}{(1-q)^m}\ E^{\gamma,\delta}_{\alpha,\beta-m,\lambda,\mu}(\omega z^\alpha;s,r|q).$$

□

3.3.2 Integral representations

Taking $f(z) = E^{\gamma;\delta}_{\alpha,\beta,\lambda,\mu}(zu^\alpha;s,r|q)$ in the q-Euler (Beta) transform (1.5.7), one gets

Theorem 3.3.2. *If* $\alpha,\ \beta,\ \gamma,\ \lambda,\ \sigma,\ \eta,\ \nu\ \in\mathbb{C},\ \Re(\alpha,\beta,\gamma,\lambda,\sigma,\eta,\nu)>0,\ \delta,\ \mu>0$ *then*

$$\frac{(1-q)^{-\eta}}{\Gamma_q(\eta)}\int_0^1 u^{\beta-1}\frac{(uq;q)_\infty}{(uq^\eta;q)_\infty}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(zu^\alpha;s,r|q)\ \mathrm{d}_q u = E^{\gamma,\delta}_{\alpha,\beta+\eta,\lambda,\mu}(z;s,r|q), \quad (3.3.3)$$

$$\int_0^z t^{\beta-1}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\ (\omega t^\alpha;s,r|q)\ \mathrm{d}_q t = \frac{z^\beta}{(1-q)^{-1}}E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu}\ (\omega z^\alpha;s,r|q), \qquad (3.3.4)$$

and

$$\frac{(1-q)^{-a}}{\Gamma_q(a)}\int_0^1 z^{a-1}(z;q)_{\beta-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(x(zq^{\beta-1};q)_\alpha;s,r|q)\ \mathrm{d}_q z = E^{\gamma,\,\delta}_{\alpha,\beta+a,\lambda,\mu}(x;s,r|q). \quad (3.3.5)$$

Proof. In order to prove (3.3.3), take

$$l.h.s. \ = \sum_{n=0}^{\infty}\frac{(q^{\alpha n+\beta};q)_\infty\ [(q^{\lambda+\mu n};q)_\infty]^r\ (q^{n+1};q)_\infty\ z^n}{[(q^{\gamma+\delta n};q)_\infty]^s}$$

$$\times\int_0^1 u^{\alpha n+\beta-1}\frac{(uq;q)_\infty}{(uq^\eta;q)_\infty}\ \mathrm{d}_q u$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ (q^{\alpha n+\beta};q)_\infty\ [(q^{\lambda+\mu n};q)_\infty]^r\ (q^{n+1};q)_\infty\ z^n}{[(q^{\gamma+\delta n};q)_\infty]^s} \\
&\quad \times (1-q)\ (q;q)_\infty \frac{(q^{\alpha n+\beta+\eta};q)_\infty}{(q^{\alpha n+\beta};q)_\infty (q^\eta;q)_\infty} \\
&= \frac{\Gamma_q(\eta)}{(1-q)^{-\eta}}\ E^{\gamma,\delta}_{\alpha,\beta+\eta,\lambda,\mu}(z;s,r|q) \\
&= r.h.s.
\end{aligned}$$

Next, it is easy to prove (3.3.4). In fact,

$$\begin{aligned}
l.h.s. &= \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ (q^{\alpha n+\beta};q)_\infty\ [(q^{\lambda+\mu n};q)_\infty]^r\ (q^{n+1};q)_\infty\ \omega^n}{[(q^{\gamma+\delta n};q)_\infty]^s} \\
&\quad \times \int_0^z t^{\alpha n+\beta-1}\ \mathrm{d}_q t \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ (q^{\alpha n+\beta};q)_\infty (q^{n+1};q)_\infty\ \omega^n}{[(q^{\gamma+\delta n};q)_\infty]^s [(q^{\lambda+\mu n};q)_\infty]^{-r}} z \\
&\quad \times (1-q)\ \sum_{k=0}^{\infty} (zq^k)^{\alpha n+\beta-1}\ q^k \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ (q^{\alpha n+\beta};q)_\infty (q^{n+1};q)_\infty\ \omega^n}{[(q^{\gamma+\delta n};q)_\infty]^s [(q^{\lambda+\mu n};q)_\infty]^{-r}} z^{\alpha n+\beta} \\
&\quad \times (1-q)\ \sum_{k=0}^{\infty} q^{k(\alpha n+\beta)} \\
&= \frac{z^\beta}{(1-q)^{-1}}\ E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu}(\omega z^\alpha;s,r|q) \\
&= r.h.s.
\end{aligned}$$

Finally,

$$\begin{aligned}
&\int_0^1 z^{a-1}\ (z;q)_{\beta-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(x(zq^{\beta-1};q)_\alpha;s,r|q)\ \mathrm{d}_q z \\
&= \int_0^1 z^{a-1}\ (z;q)_{\beta-1}\ \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ (q^{\alpha n+\beta};q)_\infty\ [(q^{\lambda+\mu n};q)_\infty]^r}{[(q^{\gamma+\delta n};q)_\infty]^s [(q^{\lambda+\mu n};q)_\infty]^{-r}} \\
&\quad \times (q^{n+1};q)_\infty\ (zq^{\beta-1};q)_{\alpha n}\ x^n\ \mathrm{d}_q z \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ (q^{\alpha n+\beta};q)_\infty (q^{n+1};q)_\infty\ x^n\ (zq^{\beta-1};q)_{\alpha n}}{[(q^{\gamma+\delta n};q)_\infty]^s [(q^{\lambda+\mu n};q)_\infty]^{-r}}
\end{aligned}$$

$$
\begin{aligned}
&\times \int_0^1 z^{a-1}\,(z;q)_{\alpha n+\beta-1}\,\mathrm{d}_q z\\
= &\sum_{n=0}^{\infty} \frac{(-1)^{pn}\; q^{pn(n-1)/2}\;(q^{\alpha n+\beta};q)_\infty (q^{n+1};q)_\infty\, x^n\,(zq^{\beta-1};q)_{\alpha n}}{[(q^{\gamma+\delta n};q)_\infty]^s[(q^{\lambda+\mu n};q)_\infty]^{-r}}\\
&\times \int_0^1 z^{a-1}\,\frac{(zq;q)_\infty}{(zq^{\alpha n+\beta};q)_\infty}\,\mathrm{d}_q z\\
= &\sum_{n=0}^{\infty} \frac{(-1)^{pn}\; q^{pn(n-1)/2}\;(q^{\alpha n+\beta};q)_\infty\,(q^{n+1};q)_\infty\,(zq^{\beta-1};q)_{\alpha n}}{x^{-n}\,[(q^{\lambda+\mu n};q)_\infty]^{-r}[(q^{\gamma+\delta n};q)_\infty]^s}\,B_q(a,\alpha n+\beta)\\
= &\sum_{n=0}^{\infty} \frac{(-1)^{pn}\; q^{pn(n-1)/2}\;(q^{\alpha n+\beta};q)_\infty\,[(q^{\lambda+\mu n};q)_\infty]^r\,(q^{n+1};q)_\infty\, x^n\,(zq^{\beta-1};q)_{\alpha n}}{[(q^{\gamma+\delta n};q)_\infty]^s}\\
&\times (1-q)\,(q;q)_\infty \frac{(q^{\alpha n+\beta+a};q)_\infty}{(q^{\alpha n+\beta};q)_\infty (q^a;q)_\infty}\\
= &\frac{(1-q)(q;q)_\infty}{(q^a;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^{pn}q^{pn(n-1)/2}\;(q^{\alpha n+\beta+a};q)_\infty[(q^{\lambda+\mu n};q)_\infty]^r(q^{n+1};q)_\infty x^n}{[(q^{\gamma+\delta n};q)_\infty]^s}.
\end{aligned}
$$

□

3.4 Special cases

The special cases of above obtained properties in section 3.2 are now illustrated below by taking one special case from Table-3, section 3.1 for each property.

- **Contour integral**-q-Elliptic function:

$$K(\sqrt{z}|q) = \frac{1}{4i}\int_L \frac{\Gamma_q(S)[\Gamma_q(\frac{1}{2}-S)]^2\,(-z)^{-S}}{\Gamma_q(1-S)}\mathrm{d}_q S.$$

- q**- Difference equation**-q-analogue of Shukla and Prajapati's function: For $\alpha,\delta\in\mathbb{N},\ y = E^{\gamma,\delta}_{\alpha,\beta}(z|q)$ satisfies the equation

$$\left[\Phi^{(\alpha,\beta,\sigma;1)}_{h,m}\;\Theta\right] E^{\gamma,\delta}_{\alpha,\beta}(z|q) - \left[(-1)^{\alpha^2-\delta^2+1}\; z\Psi^{(\delta,\gamma,\zeta;1)}_{j,i}\right] E^{\gamma,\delta}_{\alpha,\beta}(zq^{\alpha^2-\delta^2+1}|q) = 0,$$

in which ζ is δ^{th} root of unity, σ is α^{th} root of unity.

- **Eigen function property**-q-Dotsenko function:

$$\left\{\mathcal{D}_q\;\frac{(\Lambda_q+q^{1-b}-1)^{-1}}{q^{b-1}}\;\Omega^{(\frac{\omega}{\nu},b,\eta;-1)}_{\ell,k}\;\Phi^{(c,\frac{\omega}{\nu},\sigma;1)}_{h,m}\right\}\;{}_2R_1(a,b;c,\omega;\nu;\lambda z;q)$$

$$= \lambda\ {}_2R_1(a,b;c,\omega;\nu;\lambda z;q)$$

- **Mixed relations**-q-SNF:

  1. $D_q\left(z^{\beta_1}E_{\gamma,K}\left[(\alpha_1,\beta_1+k,\alpha_2,\beta_2);z^{\alpha_1}|q\right]\right)$
  $=(1-q^{k-1})\ D_q\left(z^{\beta_1}E_{\gamma,K}\left[(\alpha_1,\beta_1+k+1,\alpha_2,\beta_2);z^{\alpha_1}|q\right]\right)$
  $+q^{k-1}\ (1-q)\ D_q^2\left(z^{\beta_1+1}E_{\gamma,K}\left[(\alpha_1,\beta_1+k+1,\alpha_2,\beta_2);z^{\alpha_1}|q\right]\right).$

  2. $(1-q^{\beta_1})\ E_{\gamma,K}[(\alpha_1,\beta_1+1,\alpha_2,\beta_2);z^{\alpha_1}|q]$
  $+(1-q)\ q^{\beta_1}\ z\ D_qE_{\gamma,K}[(\alpha_1,\beta_1+1,\alpha_2,\beta_2);z^{\alpha_1}|q]$
  $=E_{\gamma,K}[(\alpha_j,\beta_j)_{1,2};z^{\alpha_1}|q].$

- **Inverse series relation**-q-BMF:

$$z^mJ^{\mu}_{\nu+m}(zq^{m(\mu^2+1)};q) = \frac{(-1)^{m(\mu^2+1)}q^{-m(m-1)(\mu^2+1)}}{(q-1)^m} \times\sum_{k=0}^{m}(-1)^kq^{(\mu^2+1)k(k-2m+1)/2}\begin{bmatrix}m\\k\end{bmatrix}_{\mu^2+1}J^{\mu}_{\nu}(zq^{k(\mu^2+1)};q).$$

if and only if

$$J^{\mu}_{\nu}(zq^{m(\mu^2+1)};q) = \sum_{k=0}^{m}\frac{(-1)^kq^{3k(k-1)(\mu^2+1)/2}}{(1-q)^k}\begin{bmatrix}m\\k\end{bmatrix}_{\mu^2+1}z^kJ^{\mu}_{\nu+k}(zq^{k(\mu^2+1)};q).$$

- **Double series relation**-q-SNF:

$${}^*E_{\gamma,K,\rho,1}[(\alpha_m,\beta_m)_{1,2};z|q]=\sum_{i,j=0}^{\infty}\frac{(-1)^iq^{i(i-1)/2}}{(q;q)_i(q;q)_j}\ {}^*E_{\gamma,K,\rho+i+j,1}[(\alpha_m,\beta_m)_{1,2};z|q].$$

3.5 q-Bessel function family

Definition 3.5.1. *If* $\alpha,\beta,\gamma,\lambda,z\in\mathbb{C}$ *with* $\Re(\alpha,\beta,\gamma,\lambda)>0,\ \delta,\mu>0,\ r,s\in\mathbb{N}\cup\{0\}$ *then*

$$\left(\frac{z}{2}\right)^{\xi}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\left(-\frac{z^2}{4};s,r|q\right) = \sum_{n=0}^{\infty}\frac{(-1)^{(p+1)n}\ q^{pn(n-1)/2}\ [\Gamma_q(\gamma+\delta n)]^s}{2^{2n+\xi}\ \Gamma_q(\beta+\alpha n)\ [\Gamma_q(\lambda+\mu n)]^r\ (q;q)_n}z^{2n+\xi}, \tag{3.5.1}$$

where $p=\alpha^2+r\mu^2-s\delta^2+1$ *with* $\Re(p)>0$.

Alternatively in the view of the definition of q-Gamma function (1.2.31) the q-form (3.5.1) can also be put in the form:

$$\left(\frac{z}{2}\right)^{\xi} E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}\left(-\frac{z^2}{4}; s, r|q\right) = \sum_{n=0}^{\infty} \frac{(-1)^{(p+1)n}\ q^{pn(n-1)/2}\ (q^{\alpha n+\beta};q)_\infty}{2^{2n+\xi}\ [(q^{\gamma+\delta n};q)_\infty]^s} \times \frac{[(q^{\lambda+\mu n};q)_\infty]^r\ z^{2n+\xi}}{(q;q)_n}, \tag{3.5.2}$$

The q-analogues of the those functions listed above in Chapter 2 in Table-2 are all yielded by the function (3.5.1).
They are tabulated below.

Table-4

q-Function of	r	s	α	β	γ	δ	λ	μ	ξ
Bessel	0	0	1	$\nu+1$	-	-	-	-	ν
Generalized Bessel-Maitland	1	0	σ	$\nu+\eta+1$	-	-	$\eta+1$	1	$\nu+2\eta$
Lommel	1	1	1	$\frac{\eta-\nu+3}{2}$	1	1	$\frac{\eta+\nu+3}{2}$	1	$\eta+1$
Struve	1	1	1	$3/2$	1	1	$3/2+\nu$	1	$\nu+1$

The explicit forms of the functions mentioned in this table are as stated below.

- q-Bessel function [18, Ex.1.24, p.25]:

$$J_\nu^{(1)}(z;q) = \sum_{n=0}^{\infty}(-1)^n\ q^{n(n-1)}\ (q^{n+\nu+1};q)_\infty\ (q^{n+1};q)_\infty\ \left(\frac{z}{2}\right)^{\nu+2n}.$$

- q-Analogue of generalized Bessel-Maitland function:

$$J_{\nu,\eta}^{\sigma}(z;q) = \sum_{n=0}^{\infty}(-1)^{(\sigma^2+1)n}\ q^{n(n-1)(\sigma^2+2)/2}\ (q^{\eta+n+1};q)_\infty \times (q^{\sigma n+\nu+\eta+1};q)_\infty\ (z/2)^{\nu+2\eta+2n}.$$

- q-Lommel function:

$$2^{-(\eta+1)}\ S_{\eta,\nu}(z;q) = \sum_{n=0}^{\infty}(-1)^n\ q^{n(n-1)}\ (q^{(\eta+\nu+3)/2+n};q)_\infty \times (q^{(\eta-\nu+3)/2+n};q)_\infty\ (z/2)^{2n+\eta+1}.$$

- q-Struve function:

$$H_\nu(z;q) = \sum_{n=0}^{\infty}(-1)^n \, q^{n(n-1)} \, (q^{(3/2+\nu)+n};q)_\infty \, (q^{(3/2+n)};q)_\infty \, (z/2)^{2n+\nu+1}.$$

3.5.1 Convergence

Theorem 3.5.1. *Let* $\Re(\alpha,\beta,\gamma,\lambda)>0,\ \Re(\alpha^2+r\mu^2-s\delta^2+1)>0,\ \delta,\mu>0,\ r,s\in\mathbb{N}\cup\{0\}$ *and* $0<q<1$. *Then* $\left(\frac{z}{2}\right)^{\xi}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\left(-\frac{z^2}{4};s,r|q\right)$ *is an entire function of order zero.*

The proof is simillar to the proof of the Theorem 3.2.1 in which

$$V_{n,\xi} = \frac{(-1)^{(p+1)n} \, q^{pn(n-1)/2} \, [\Gamma_q(\gamma+\delta n)]^s}{2^{2n+\xi} \, \Gamma_q(\beta+\alpha n) \, [\Gamma_q(\lambda+\mu n)]^r \, \Gamma_q(n+1)} \tag{3.5.3}$$

instead of V_n given in (3.2.1).

3.5.2 Contour integral

Theorem 3.5.2. *Let* $\alpha>0;\beta,\gamma,\lambda\in\mathbb{C}$ *with* $\Re(\beta,\gamma,\lambda)>0$ *and* $\delta,\mu>0$. *Then the function* $\left(\frac{z}{2}\right)^{\xi}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\left(-\frac{z^2}{4};s,r|q\right)$ *is expressible as the Mellin - Barnes q-integral:*

$$\left(\frac{z}{2}\right)^{\xi}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\left(-\frac{z^2}{4};s,r|q\right) = \frac{\left(\frac{z}{2}\right)^{\xi}}{2\pi i}\int_L \frac{(-1)^{(p+1)S}\, q^{-pS(-S-1)/2}\, \Gamma_q(S)\, [\Gamma_q(\gamma-\delta S)]^s \, z^{-2S}}{\Gamma_q(\beta-\alpha S)\, [\Gamma_q(\lambda-\mu S)]^r}\, \mathrm{d}_qS, \tag{3.5.4}$$

where $|argz|<\pi$. *The contour* L *of integration begins from* $-i\infty$ *and proceeds towards* $+i\infty$, *and is indented to keep the poles of integrand at* $S=-n$ *to the left; and the poles at* $S=(\gamma+n)/\delta$ *to the right of the path for all* $n\in\mathbb{N}\cup\{0\}$.

The proof is same as the proof of the Theorem 3.2.3.

3.5.3 Difference equation

Here with the help of the following operators, the q-difference equation of (3.5.1) will be derived. Define

$$\delta_q f(x) = f(xq^{1/2}), \quad \Theta f(x) = f(x) - f(xq), \quad \big(\Theta + q^{\xi/2}\delta_q - 1\big) = \Phi_\xi,$$

$$\frac{\left\{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[\Theta + c^{-u}q^{1-(b+v)/a}q^{\xi/2}\delta_q - 1]^m\right\}}{\left\{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[c^{-u}q^{1-(b+v)/a}]^m\right\}} = \Phi^{(a,b,c;m)}_{\xi,u,v},$$

and

$$\frac{\left\{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[\Theta + c^{-u}q^{(b+v)/a}q^{\xi/2}\delta_q - 1]^m\right\}}{\left\{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[c^{-u}q^{-(b+v)/a}]^m\right\}} = \Psi^{(a,b,c;m)}_{\xi,u,v}.$$

In these notations, the q-difference equation satisfied by (3.5.1) is obtain in

Theorem 3.5.3. *Let $\alpha, \mu, \delta \in \mathbb{N}$ then $\left(\frac{z}{2}\right)^\xi E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\left(-\frac{z^2}{4}; s, r|q\right)$ satisfies the equation*

$$\begin{aligned} &q^{s\xi}\,\Phi^{(\mu,\lambda,\eta;r)}_{\xi,\ell,k}\,\Phi^{(\alpha,\beta,\sigma;1)}_{\xi,h,m}\,\Phi_\xi\left(\frac{z}{2}\right)^\xi E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\left(-\frac{z^2}{4}q^s; s, r|q\right) \\ &+\frac{(-1)^p\,q^{(r+2)(\xi+1)+s}\,z^2}{4}\,\Psi^{(\delta,\gamma,\zeta;s)}_{\xi,j,i}\left(\frac{z}{2}\right)^\xi E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\left(-\frac{z^2}{4}q^{p+r+2}; s, r|q\right) = 0, \end{aligned} \tag{3.5.5}$$

in which ζ is δ^{th} root of unity, η is μ^{th} root of unity, σ is α^{th} root of unity.

Proof. With the aid of the formulas [18, Appendix I], the coefficient $V_{n,\xi}$ of z^n in (3.5.3) is first expressed in q-factorial form. Then for $\alpha, \mu, \delta \in \mathbb{N}$,

$$\begin{aligned} V_{n,\xi} &= \frac{(-1)^{(p+1)n}\,q^{pn(n-1)/2}\,[(q^\gamma;q)_{\delta n}]^s}{2^{2n+\xi}\,[(q^\lambda;q)_{\mu n}]^r\,(q^\beta;q)_{\alpha n}\,(q;q)_n} \\ &= \frac{(-1)^{(p+1)n}\,q^{pn(n-1)/2}\,[(q^\gamma;q)_{\delta n}]^s}{2^{2n+\xi}\,[(q^\lambda;q)_{\mu n}]^r\,(q^\beta;q)_{\alpha n}\,(q;q)_n} \\ &= \frac{(-1)^{(p+1)n}\,q^{pn(n-1)/2}\,[(q^\gamma;q^\delta)_n]^s\,[(q^{\gamma+1};q^\delta)_n]^s\cdots[(q^{\gamma+\delta-1};q^\delta)_n]^s}{2^{2n+\xi}\,[(q^\lambda;q^\mu)_n]^r\,[(q^{\lambda+1};q^\mu)_n]^r\cdots[(q^{\lambda+\mu-1};q^\mu)_n]^r} \end{aligned}$$

$$\times\frac{1}{(q^{\beta};q^{\alpha})_n\ (q^{\beta+1};q^{\alpha})_n\cdots(q^{\beta+\alpha-1};q^{\alpha})_n(q;q)_n}$$

$$=\frac{(-1)^{(p+1)n}\ q^{pn(n-1)/2}\ \left\{\prod_{j=0}^{\delta-1}\prod_{i=0}^{\delta-1}[(\zeta^j\ q^{(\gamma+i)/\delta};q)_n]^s\right\}}{2^{2n+\xi}\ \left\{\prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1}[(\eta^{\ell}\ q^{(\lambda+k)/\mu};q)_n]^r\right\}}$$

$$\times\frac{1}{\left\{\prod_{h=0}^{\alpha-1}\prod_{m=0}^{\alpha-1}(\sigma^h\ q^{(\beta+m)/\alpha};q)_n\right\}(q;q)_n},\tag{3.5.6}$$

where ζ is δ^{th} root of unity, η is μ^{th} root of unity, σ is α^{th} root of unity.
Now take

$$\prod_{j=0}^{\delta-1}\prod_{i=0}^{\delta-1}[(\zeta^j\ q^{(\gamma+i)/\delta};q)_n]^s=\mathcal{A}_n,\ \prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1}[(\eta^{\ell}\ q^{(\lambda+k)/\mu};q)_n]^r=\mathcal{B}_n,\tag{3.5.7}$$

$$\prod_{h=0}^{\alpha-1}\prod_{m=0}^{\alpha-1}(\sigma^h\ q^{(\beta+m)/\alpha};q)_n=\mathcal{C}_n,\ \frac{(-1)^{(p+1)n}\ q^{pn(n-1)/2}}{2^{2n+\xi}}=\mathcal{D}_n.\tag{3.5.8}$$

Since the series in (3.5.1) has infinite radius of convergent, term-by-term operation is permitted, hence

$$\begin{aligned}
&\Phi_{\xi}\ \left(\frac{z}{2}\right)^{\xi}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\left(-\frac{z^2}{4}q^s;s,r|q\right)\\
=&\sum_{n=0}^{\infty}\frac{\mathcal{A}_n\ \mathcal{D}_n}{\mathcal{B}_n\ \mathcal{C}_n\ (q;q)_n}\ \Phi_{\xi}\ z^{2n+\xi}\\
=&\sum_{n=0}^{\infty}\frac{q^{sn}\ \mathcal{A}_n\ \mathcal{D}_n}{\mathcal{B}_n\ \mathcal{C}_n\ (q;q)_n}\left(\Theta+q^{\xi/2}\delta_q-1\right)z^{2n+\xi}\\
=&\sum_{n=0}^{\infty}\frac{q^{sn}\ \mathcal{A}_n\ \mathcal{D}_n}{\mathcal{B}_n\ \mathcal{C}_n\ (q;q)_n}\left(z^{2n+\xi}-(zq)^{2n+\xi}+q^{\xi/2}z^{2n+\xi}q^{n+\xi/2}-z^{2n+\xi}\right)\\
=&\sum_{n=0}^{\infty}\frac{q^{sn}\ \mathcal{A}_n\ \mathcal{D}_n}{\mathcal{B}_n\ \mathcal{C}_n\ (q;q)_n}\left((zq)^{2n+\xi}+q^{\xi/2}z^{2n+\xi}q^{n+\xi/2}\right)\\
=&\sum_{n=0}^{\infty}\frac{q^{sn}\ \mathcal{A}_n\ \mathcal{D}_n}{\mathcal{B}_n\ \mathcal{C}_n\ (q;q)_n}z^{2n+\xi}(q^{n+\xi}-q^{2n+\xi})\\
=&\sum_{n=0}^{\infty}\frac{q^{sn}\ \mathcal{A}_n\ \mathcal{D}_n}{\mathcal{B}_n\ \mathcal{C}_n\ (q;q)_n}z^{2n+\xi}q^{n+\xi}(1-q^n)\\
=&\sum_{n=1}^{\infty}\frac{q^{n+\xi}\ q^{sn}\ \mathcal{A}_n\ \mathcal{D}_n}{\mathcal{B}_n\ \mathcal{C}_n}\frac{z^{2n+\xi}}{(q;q)_{n-1}}.
\end{aligned}$$

Next operating by $\Phi^{(\alpha,\beta,\sigma;1)}_{\xi,h,m}$,

$$\Phi^{(\alpha,\beta,\sigma;1)}_{\xi,h,m}\ \Phi_{\xi}\ \left(\frac{z}{2}\right)^{\xi}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\left(-\frac{z^2}{4}q^s;s,r|q\right)$$

$$= \sum_{n=1}^{\infty}\frac{q^{n+\xi}\ q^{sn}\ \mathcal{A}_n\ \mathcal{D}_n}{\mathcal{B}_n\ (q;q)_{n-1}}\frac{\left\{\prod_{h=0}^{\alpha-1}\prod_{m=0}^{\alpha-1}(\Theta+\sigma^{-h}q^{1-(\beta+m)/\alpha}q^{\xi/2}\delta_q-1)\right\}}{\left\{\prod_{h=0}^{\alpha-1}\prod_{m=0}^{\alpha-1}(\sigma^{-h}q^{1-(\beta+m)/\alpha})\right\}}$$

$$\times\frac{z^{2n+\xi}}{\left\{\prod_{h=0}^{\alpha-1}\prod_{m=0}^{\alpha-1}(\sigma^{h}q^{(\beta+m)/\alpha};q)_n\right\}}$$

$$= \sum_{n=1}^{\infty}\frac{q^{2(n+\xi)}\ q^{sn}\ \mathcal{A}_n\ \mathcal{D}_n}{\mathcal{B}_n\ (q;q)_{n-1}}\frac{\left\{\prod_{h=0}^{\alpha-1}\prod_{m=0}^{\alpha-1}(1-\sigma^{h}q^{n-1+(\beta+m)/\alpha})\right\}}{\left\{\prod_{h=0}^{\alpha-1}\prod_{m=0}^{\alpha-1}(\sigma^{h}q^{(\beta+m)/\alpha};q)_n\right\}}z^{2n+\xi}$$

$$= \sum_{n=1}^{\infty}\frac{q^{2(n+\xi)}\ q^{sn}\ \mathcal{A}_n\ \mathcal{D}_n}{\mathcal{B}_n\ \mathcal{C}_{n-1}\ (q;q)_{n-1}}z^{2n+\xi}.$$

Finally,

$$\Phi^{(\mu,\lambda,\eta;r)}_{\xi,\ell,k}\ \Phi^{(\alpha,\beta,\sigma;1)}_{\xi,h,m}\ \Phi_{\xi}\ \left(\frac{z}{2}\right)^{\xi}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\left(-\frac{z^2}{4}q^s;s,r|q\right)$$

$$= \sum_{n=1}^{\infty}\frac{q^{2(n+\xi)}\ q^{sn}\ \mathcal{A}_n\ \mathcal{D}_n}{\mathcal{C}_{n-1}\ (q;q)_{n-1}}\frac{\left\{\prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1}[(\Theta+\eta^{-\ell}q^{1-(\lambda+k)/\mu}q^{\xi/2}\delta_q-1)]^r\right\}}{\left\{\prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1}[(\eta^{-\ell}q^{1-(\lambda+k)/\mu})]^r\right\}}$$

$$\times\frac{z^{2n+\xi}}{\left\{\prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1}[(\eta^{\ell}\ q^{(\lambda+k)/\mu};q)_n]^r\right\}}$$

$$= \sum_{n=1}^{\infty}\frac{q^{2(n+\xi)}\ q^{sn}\ \mathcal{A}_n\ \mathcal{D}_n}{\mathcal{C}_{n-1}\ (q;q)_{n-1}}\frac{\left\{\prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1}[(-q^{n}+\eta^{-\ell}q^{1-(\lambda+k)/\mu})]^r\right\}}{\left\{\prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1}[(\eta^{-\ell}q^{1-(\lambda+k)/\mu})]^r\right\}}$$

$$\times\frac{z^{(2+r)n+\xi}}{\left\{\prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1}[(\eta^{\ell}\ q^{(\lambda+k)/\mu};q)_n]^r\right\}}$$

$$= \sum_{n=1}^{\infty} \frac{q^{(2+r)(n+\xi)}\; q^{sn}\; \mathcal{A}_n\; \mathcal{D}_n}{\mathcal{B}_{n-1}\; \mathcal{C}_{n-1}\; (q;q)_{n-1}} z^{2n+\xi}.$$

Thus,

$$q^{s\xi}\; \Phi^{(\mu,\lambda,\eta;r)}_{\xi,\ell,k}\; \Phi^{(\alpha,\beta,\sigma;1)}_{\xi,h,m}\; \Phi_{\xi}\; \left(\frac{z}{2}\right)^{\xi}\; E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\left(-\frac{z^2}{4}q^s; s, r|q\right)$$
$$= \sum_{n=0}^{\infty} q^{(r+s+2)n}\; q^{(r+2)(\xi+1)+s(\xi+1)} \frac{A_{n+1}\; D_{n+1}}{\mathcal{B}_n\; \mathcal{C}_n\; (q;q)_n} z^{2(n+1)+\xi}. \tag{3.5.9}$$

On the other hand,

$$\Psi^{(\delta,\gamma,\zeta;s)}_{\xi,j,i}\; \left(\frac{z}{2}\right)^{\xi}\; E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\left(-\frac{z^2}{4}q^{p+r+2}; s, r|q\right)$$
$$= \sum_{n=0}^{\infty} \frac{\mathcal{A}_n\; \mathcal{D}_n\; q^{(p+r+2)n}}{\mathcal{B}_n\; \mathcal{C}_n\; (q;q)_n} \frac{\left\{\prod_{j=0}^{\delta-1}\prod_{i=0}^{\delta-1}[(\Theta+\zeta^{-j}q^{-(\gamma+i)/\delta}q^{\xi/2}\delta_q-1)]^s\right\}}{\left\{\prod_{j=0}^{\delta-1}\prod_{i=0}^{\delta-1}[(\zeta^{-j}q^{-(\gamma+i)/\delta})]^s\right\}} z^{2n+\xi}$$
$$= \sum_{n=0}^{\infty} \frac{q^{s(n+\xi)}\; \mathcal{D}_n\; q^{(p+r+2)n}}{\mathcal{B}_n\; \mathcal{C}_n\; (q;q)_n} \left\{\prod_{j=0}^{\delta-1}\prod_{i=0}^{\delta-1}[(\zeta^j\; q^{(\gamma+i)/\delta}; q)_n]^s\right\}$$
$$\times \left\{\prod_{j=0}^{\delta-1}\prod_{i=0}^{\delta-1}[(1-\zeta^j q^{n+(\gamma+i)/\delta})]^s\right\} z^{2n+\xi},$$

whence one finds

$$\frac{(-1)^{p+1}\; q^{(r+2)(\xi+1)+s}\; z^2}{4}\; \Psi^{(\delta,\gamma,\zeta;s)}_{\xi,j,i}\; \left(\frac{z}{2}\right)^{\xi}\; E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\left(-\frac{z^2}{4}q^{p+r+2}; s, r|q\right)$$
$$= \sum_{n=0}^{\infty} q^{(r+s+2)n}\; q^{(r+2)(\xi+1)+s(\xi+1)} \frac{\mathcal{A}_{n+1}\; \mathcal{D}_{n+1}}{\mathcal{B}_n\; \mathcal{C}_n\; (q;q)_n} z^{2(n+1)+\xi}. \tag{3.5.10}$$

Thus, (3.5.5) follows by comparing (3.5.9) and (3.5.10). □

3.5.4 Mixed relations

Theorem 3.5.4. *The mixed relation for* $k = 2, 3, 4 \ldots,$ *is given by*

$$D_q\left(z^{\beta}\left(\frac{z}{2}\right)^{\alpha\xi} E^{\gamma,\delta}_{2\alpha,\,\beta+\alpha\xi+k,\,\lambda,\,\mu}\left(-\left(\frac{z^2}{4}\right)^{\alpha}; s, r|q\right)\right)$$
$$= (1-q^{k-1})\; D_q\left(z^{\beta}\left(\frac{z}{2}\right)^{\alpha\xi} E^{\gamma,\delta}_{2\alpha,\,\beta+\alpha\xi+k+1,\,\lambda,\,\mu}\left(-\left(\frac{z^2}{4}\right)^{\alpha}; s, r|q\right)\right)$$

$$+ \, q^{k-1} \, (1-q) \; D_q \left(z^{\beta+1} \left(\frac{z}{2}\right)^{\alpha\xi} E^{\gamma,\delta}_{2\alpha,\, \beta+\alpha\xi+k+1,\, \lambda,\, \mu} \left(-\left(\frac{z^2}{4}\right)^{\alpha} ; s, r|q \right) \right). \quad (3.5.11)$$

The proof differs from that of Theorem 3.2.8 only in the operator considered. As suggested by Theorem 3.2.9, the following relation may be proved in a straight forward manner.

Theorem 3.5.5. *For* $2\alpha, \beta+\alpha\xi, \gamma, \lambda \in \mathbb{C}; \Re(\alpha, \beta+\alpha\xi, \gamma, \lambda) > 0,\; \delta, \mu > 0$ *there holds the mixed recurrence relation:*

$$\begin{aligned} & (1-q^{\beta}) \; \left(\frac{z}{2}\right)^{\alpha\xi} E^{\gamma,\delta}_{2\alpha,\beta+\alpha\xi+1,\lambda,\mu} \left(-\left(\frac{z^2}{4}\right)^{\alpha} ; s, r|q \right) \\ & + (1-q) \; q^{\beta} \; z \, D_q \left(\frac{z}{2}\right)^{\alpha\xi} E^{\gamma,\delta}_{2\alpha,\beta+\alpha\xi+1,\lambda,\mu} \left(-\left(\frac{z^2}{4}\right)^{\alpha} ; s, r|q \right) \\ = \; & \left(\frac{z}{2}\right)^{\alpha\xi} E^{\gamma,\delta}_{2\alpha,\beta+\alpha\xi,\lambda,\mu} \left(-\left(\frac{z^2}{4}\right)^{\alpha} ; s, r|q \right). \end{aligned}$$

3.5.5 Finite summation formulas

Here the function (3.5.2) is expressed as a finite series of the same function. For the sake of simplicity, this function will be symbolized as ${}_{\xi}E(z)$. Now define the difference operator by

$${}_{\xi}\Lambda_q f(x) \;=\; \frac{f(x) - q^{\frac{-\xi}{2}} f(xq^{1/2})}{x^2 - x^2 q}, \quad (3.5.12)$$

then one gets

$$\begin{aligned} {}_{\xi}\Lambda_q \, {}_{\xi}E(z) \;&=\; \sum_{n=0}^{\infty} \frac{(-1)^{(p+1)n} \; q^{pn(n-1)/2} [(q^{\lambda+\mu n}; q)_\infty]^r (q^{\beta+\alpha n}; q)_\infty}{[(q^{\gamma+\delta n}; q)_\infty]^s \; 2^{2n+\xi} \; (q;q)_n} \, {}_{\xi}\Lambda_q \, z^{2n+\xi} \\ &=\; \sum_{n=0}^{\infty} \frac{(-1)^{(p+1)(n+1)} \; q^{pn(n+1)/2} \; [(q^{\lambda+\mu+\mu n}; q)_\infty]^r \; (q^{\beta+\alpha+\alpha n}; q)_\infty}{(1-q) \; [(q^{\gamma+\delta+\delta n}; q)_\infty]^s \; 2^{2(n+1)+\xi} \; (q;q)_n} \; z^{2n+\xi}. \end{aligned}$$

Likewise,

$$\begin{aligned} {}_{\xi}\Lambda_q^2 \, {}_{\xi}E(z) \;&=\; \frac{1}{(1-q)^2} \sum_{n=0}^{\infty} \frac{(-1)^{(p+1)(n+2)} \; q^{p(n+1)(n+2)/2} [(q^{\lambda+2\mu+n\mu}; q)_\infty]^r}{[(q^{\gamma+2\delta+\delta n}; q)_\infty]^s \; 2^{2(n+2)+\xi} \; (q;q)_n} \\ &\quad \times (q^{\beta+2\alpha+\alpha n}; q)_\infty \; z^{2n+\xi}. \end{aligned}$$

In general,

$$\begin{aligned} {}_{\xi}\Lambda_q^m \, {}_{\xi}E(z) &= \frac{1}{(1-q)^m} \sum_{n=0}^{\infty} \frac{(-1)^{(p+1)(n+m)} \; q^{p(n+m)(n+m-1)/2} [(q^{\lambda+m\mu+n\mu};q)_\infty]^r}{[(q^{\gamma+m\delta+n\delta};q)_\infty]^s \; 2^{2(n+m)+\xi} \; (q;q)_n} \\ &\quad \times (q^{m\alpha+\sigma(n)};q)_\infty \; z^n \\ &= \frac{(-1)^{pm} \; q^{pm(m-1)/2}}{2^{2m} \; (1-q)^m} E_{\alpha,\beta+m\alpha,\lambda+m\mu,\mu}^{\gamma+m\delta,\delta}(zq^{pm};s,r|q). \end{aligned}$$

Here in view of the formula (3.2.24), one finds the formula for the operator (3.5.12) in the form:

$$x^{2m}(1-q)^m \, {}_{\xi}\Lambda_q^m f(x) = (-1)^m q^{-m(m-1)/2} \sum_{k=0}^{m} (-1)^k q^{k(k-1)/2} \begin{bmatrix} m \\ k \end{bmatrix} f(xq^{m-k}).$$

For the choice $f(x) = {}_{\xi}E(z) = \left(\frac{z}{2}\right)^{\xi} E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}\left(-\frac{z^2}{4};s,r|q\right)$, this gives

$$\begin{aligned} &\left(\frac{z}{2}\right)^{2m+\xi} E_{\alpha,\beta+m\alpha,\lambda+m\mu,\mu}^{\gamma+m\delta,\delta}\left(-\frac{z^2}{4}q^{pm};s,r|q\right) \\ &= (-1)^{(p+1)m} \; q^{-pm(m-1)/2} \sum_{k=0}^{m} (-1)^k q^{pk(k-2m+1)/2} \begin{bmatrix} m \\ k \end{bmatrix}_{q^p} E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}\left(-\frac{z^2}{4}q^{pk};s,r|q\right). \end{aligned} \tag{3.5.13}$$

This last identity enables one to derive one more such form; the method of obtaining this uses inverse series relations [10] which is already proved as Lemma 3.2.1. The comparison of (6.4.4) with (3.5.13) suggests that

$$g_k = \left(\frac{z}{2}\right)^{\xi} E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}\left(-\frac{z^2}{4}q^{pk};s,r|q\right)$$

and

$$G_m = \frac{(-1)^{(p+1)m} \; (q-1)^m}{q^{-pm(m-1)}} \; z^{2m} \left(\frac{z}{2}\right)^{\xi} E_{\alpha,\beta+m\alpha,\lambda+m\mu}^{\gamma+m\delta,\delta}\left(-\frac{z^2}{4}q^{pm};s,r|q\right).$$

With this choice, the inverse series (6.4.3) provides the finite series formula:

$$\begin{aligned} E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}\left(-\frac{z^2}{4}q^{pm};s,r|q\right) &= \sum_{k=0}^{m} (-1)^k q^{pk(k-1)/2} \begin{bmatrix} m \\ k \end{bmatrix}_{q^p} (-1)^{pk} \; (q-1)^k \\ &\quad \times \left(\frac{z}{2}\right)^{2k+\xi} E_{\alpha,\beta+k\alpha,\lambda+k\mu,\mu}^{\gamma+k\delta,\delta}\left(-\frac{z^2}{4}q^{pk};s,r|q\right). \end{aligned} \tag{3.5.14}$$

3.5.6 Double series representation

Next, the double series representation of the function $\left(\frac{z}{2}\right)^{\xi} E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\left(-\frac{z^2}{4}; s, r|q\right)$ is derived as follows.
As in Theorem 3.5.1, take

$$V_{n,\xi} = \frac{(-1)^{(p+1)n}\ q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_\infty]^r\ (q^{\alpha n+\beta};q)_\infty}{[(q^{\gamma+\delta n};q)_\infty]^s\ 2^{2n+\xi}\ (q;q)_n}$$

and put

$$\left(\frac{z}{2}\right)^{\xi}\ {}^*E^{\gamma,\delta,\rho}_{\alpha,\beta,\lambda,\mu}\left(-\frac{z^2}{4}; s, r|q\right) = \sum_{n=0}^{\infty} \frac{(-1)^n\ q^{-n(n-1)/2}\ V_{n,\xi}}{(q^{\rho+n};q)_\infty} z^{2n+\xi}$$

which is valid under the condition $\Re(\alpha^2 + r\mu^2 - s\delta^2) > 0$.

Theorem 3.5.6. *If* $\Re(\alpha^2 + r\mu^2 - s\delta^2) > 0$, *then*

$$\left(\frac{z}{2}\right)^{\xi}\ {}^*E^{\gamma,\delta,\rho}_{\alpha,\beta,\lambda,\mu}\left(-\frac{z^2}{4}; s, r|q\right) = \sum_{i,j=0}^{\infty} \frac{(-1)^i q^{i(i-1)/2}}{(q;q)_i (q;q)_j} \left(\frac{z}{2}\right)^{\xi}\ {}^*E^{\gamma,\delta,\rho+i+j}_{\alpha,\beta,\lambda,\mu}\left(-\frac{z^2}{4}; s, r|q\right). \tag{3.5.15}$$

The proof of this theorem is same as the proof of Theorem 3.2.12. Hence it is omitted.

3.6 Special cases

The properties obtained above will now be illustrated by taking one special case from Table-4, section 3.5, for each property.

- **Contour integral**-q-Generalized Bessel-Maitland function:

$$J^{\sigma}_{\nu,\eta}(z;q) \quad = \quad \frac{\left(\frac{z}{2}\right)^{\nu+2\eta}}{2\pi i} \int_L \frac{(-1)^{(\sigma^2+3)S}\ q^{-(\sigma^2+2)S(-S-1)/2}\ \Gamma_q(S)\ z^{-2S}}{\Gamma_q(\nu+\eta+1-\sigma S)\ [\Gamma_q(\eta+1-S)]^r} \mathrm{d}_q S.$$

- q-**Difference equation**-Bessel function:

$$q^{s\nu}\ \Phi^{(1,\nu+1,1;1)}_{\nu,h,m}\ \Phi_\nu\ J^{(1)}_\nu(z;q) + \frac{(-1)^2\ q^{2(\xi+1)}\ z^2}{4} J^{(1)}_\nu(zq^4;q) = 0.$$

- **Mixed relations**-q-Lommel function:

1. $D_q\left(z^{\frac{\eta-\nu+3}{2}}\left(\frac{z}{2}\right)^{\eta+1}E^{1,1}_{2,\,\frac{\eta-\nu+3}{2}+\eta+1+k,\,\frac{\eta+\nu+3}{2},\,1}\left(-\frac{z^2}{4};1,1|q\right)\right)$
$= (1-q^{k-1})\,D_q\left(z^{\frac{\eta-\nu+3}{2}}\left(\frac{z}{2}\right)^{\eta+1}E^{1,1}_{2,\,\frac{\eta-\nu+3}{2}+\eta+k+2,\,\frac{\eta+\nu+3}{2},\,1}\left(-\frac{z^2}{4};1,1|q\right)\right)$
$+q^{k-1}(1-q)\,D_q\left(z^{\frac{\eta-\nu+3}{2}+1}\left(\frac{z}{2}\right)^{\eta+1}E^{1,1}_{2,\,\frac{\eta-\nu+3}{2}+\eta+k+2,\,\frac{\eta+\nu+3}{2},1}\left(-\frac{z^2}{4};1,1|q\right)\right).$

2. $(1-q^{\frac{\eta-\nu+3}{2}})\,\left(\frac{z}{2}\right)^{\eta+1}E^{1,1}_{2,\frac{\eta-\nu+3}{2}+\eta+2,\frac{\eta+\nu+3}{2},1}\left(-\frac{z^2}{4};1,1|q\right)$
$+(1-q)\,q^{\frac{\eta-\nu+3}{2}}\,z\,D_q\left(\frac{z}{2}\right)^{\eta+1}E^{1,1}_{2,\frac{\eta-\nu+3}{2}+\eta+2,\frac{\eta+\nu+3}{2},1}\left(-\frac{z^2}{4};1,1|q\right)$
$=\left(\frac{z}{2}\right)^{\eta+1}E^{1,1}_{2,\frac{\eta-\nu+3}{2}+\eta+1,\frac{\eta+\nu+3}{2},1}\left(-\frac{z^2}{4};1,1|q\right).$

- **Inverse series relation**-q-Generalized Bessel-Maitland funtion:

$$\begin{aligned}
&z^{2m}\left(\frac{z}{2}\right)^{\nu+2\eta}E^{\gamma+m\delta,\delta}_{\sigma,\nu+\eta+1+m\sigma,\eta+1+m,\mu}\left(-\frac{z^2}{4}q^{(\sigma^2+2)m};0,1|q\right)\\
=\;&(-1)^{(\sigma^2+3)m}\;q^{-(\sigma^2+2)m(m-1)/2}\sum_{k=0}^{m}(-1)^k q^{(\sigma^2+2)k(k-2m+1)/2}\\
&\times\begin{bmatrix}m\\k\end{bmatrix}_{q^{(\sigma^2+2)}}E^{\gamma,\delta}_{\sigma,\nu+\eta+1,\eta+1,1}\left(-\frac{z^2}{4}q^{(\sigma^2+2)k};0,1|q\right).
\end{aligned}$$

if and only if

$$\begin{aligned}
&E^{\gamma,\delta}_{\sigma,\nu+\eta+1,\eta+1,1}\left(-\frac{z^2}{4}q^{(\sigma^2+2)m};0,1|q\right)\\
=\;&\sum_{k=0}^{m}(-1)^k q^{(\sigma^2+2)k(k-1)/2}\begin{bmatrix}m\\k\end{bmatrix}_{q^{(\sigma^2+2)}}\;(-1)^{(\sigma^2+2)k}\\
&\times z^{2k}\left(\frac{z}{2}\right)^{\nu+2\eta}\;E^{\gamma+k\delta,\delta}_{\sigma,\nu+\eta+1+k\alpha,\eta+1+k,1}\left(-\frac{z^2}{4}q^{(\sigma^2+2)k};0,1|q\right).
\end{aligned}$$

- **Double series relation**-q-Sturve function:

$$\begin{aligned}
\left(\frac{z}{2}\right)^{\nu+1}\;{}^{*}E^{1,1,\rho}_{1,3/2,3/2+\nu,1}\left(-\frac{z^2}{4};1,1|q\right)\;=\;&\sum_{i,j=0}^{\infty}\frac{(-1)^i q^{i(i-1)/2}}{(q;q)_i(q;q)_j}\left(\frac{z}{2}\right)^{\nu+1}\\
&{}^{*}E^{1,1,\rho+i+j}_{1,3/2,3/2+\nu,1}\left(-\frac{z^2}{4};1,1|q\right).
\end{aligned}$$