Chapter 4

Fractional integration and differentiation

4.1 Introduction

In this chapter, an operator involving $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r)$ will be introduced in the space $L(a,b)$, a space of Lebesgue measurable real or complex functions. Certain properties of Riemann - Liouville fractional integral and differential operators associated with the function $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r)$ are studied and the integral representations are derived. Some properties of a special case of this function are also studied by means of fractional calculus.

Definition 4.1.1. *For* $\alpha,\beta,\gamma,\lambda,\omega \in \mathbb{C}$; $\Re(\alpha,\beta,\gamma,\lambda) > 0$; $\delta,\mu > 0$, $r \in \mathbb{N} \cup \{-1,0\}$, $s \in \mathbb{N} \cup \{0\}$ *and* $x > a$

$$\left(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0_+} f\right)(x) = \int_a^x (x-t)^{\beta-1}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\left(\omega(x-t)^{\alpha}; s,r\right) f(t)\ \mathrm{d}t. \tag{4.1.1}$$

4.2 Main results

4.2.1 Fractional operators associated with $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r)$

In this section the following results are proved.

Theorem 4.2.1. *Let* $0 \leq a < \infty$, $\alpha,\ \beta,\ \gamma,\ \lambda,\ \eta \in \mathbb{C}$, $\Re(\alpha, \beta, \gamma, \lambda, \eta) > 0$; $\delta,\ \mu > 0$ *for* $x > a$, *then*

$$\begin{aligned} &\left(I^{\eta}_{a+}\ (t-a)^{\beta-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega(t-a)^{\alpha}; s, r\right)\ (x) \\ =\ & (x-a)^{(\eta+\beta-1)}\ E^{\gamma,\delta}_{\alpha,\beta+\eta,\lambda,\mu}(\omega(x-a)^{\alpha}; s, r), \end{aligned} \tag{4.2.1}$$

and

$$\begin{aligned} &\left(D^{\eta}_{a+}\ (t-a)^{\beta-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega(c(t-a))^{\alpha}; s, r)\right)\ (x) \\ =\ & (x-a)^{(\beta-\eta-1)}\ E^{\gamma,\delta}_{\alpha,\beta-\eta,\lambda,\mu}(\omega(x-a)^{\alpha}; s, r). \end{aligned} \tag{4.2.2}$$

Proof. Applying Riemann-Liouville fractional integral operator (1.6.3) given by

$$I^{\mu}_{a+}[(t-a)^{\beta-1}](x) = \frac{\Gamma(\beta)}{\Gamma(\mu+\beta)}(x-a)^{\mu+\beta-1}$$

on the function (2.1.5), one gets

$$\begin{aligned} &\left(I^{\eta}_{a+}\ (t-a)^{\beta-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega(t-a)^{\alpha}; s, r\right)\ (x) \\ =\ & \left(I^{\eta}_{a+}\ (t-a)^{\beta-1}\ \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^{s}\ (\omega\ (t-a)^{\alpha})^{n}}{\Gamma(\alpha n+\beta)\ [(\lambda)_{\mu n}]^{r}\ n!}\right)(x) \\ =\ & \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^{s}\ \omega^{n}}{\Gamma(\alpha n+\beta)\ [(\lambda)_{\mu n}]^{r}\ n!} \left(I^{\eta}_{a+}\ (t-a)^{\alpha n+\beta-1}\right)(x), \\ =\ & \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^{s}\ \omega^{n}}{\Gamma(\alpha n+\beta)\ [(\lambda)_{\mu n}]^{r}\ n!} \frac{\Gamma(\alpha n+\beta)}{\Gamma(\alpha n+\beta+\eta)}(x-a)^{\alpha n+\beta+\eta-1} \\ =\ & (x-a)^{\beta+\eta-1} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^{s}\ (\omega\ (x-a)^{\alpha})^{n}}{\Gamma(\alpha n+\beta+\eta)\ [(\lambda)_{\mu n}]^{r}\ n!} \end{aligned}$$

which is (4.2.1).
Similarly using Riemann-Liouville fractional derivative operator (1.6.4) given by

$$(D^{\alpha}_{a+}\ f)(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n}\ (I^{n-\alpha}_{a+}\ f)(x).$$

on the function (2.1.5), one obtains

$$\begin{aligned} &\left(D^{\eta}_{a+}\ (t-a)^{\beta-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega(t-a)^{\alpha}; s, r)\right)\ (x) \\ =\ & \left(\frac{d}{dx}\right)^{n} \left(I^{n-\eta}_{a+}(t-a)^{\beta-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega(t-a)^{\alpha}; s, r)\right)(x) \end{aligned}$$

$$= \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \Big[(x-a)^{n+\beta-\eta-1}\, E^{\gamma,\delta}_{\alpha,\,\beta-\eta+n,\,\lambda,\,\mu}(\omega(x-a)^\alpha; s, r)\Big]$$
$$= (x-a)^{\beta-\eta-1}\, E^{\gamma,\delta}_{\alpha,\beta-\eta,\lambda,\mu}(\omega(x-a)^\alpha; s, r).$$

□

Theorem 4.2.2. *Let* $\alpha,\gamma,\lambda,\eta \in \mathbb{C},\ \Re(\alpha,\gamma,\lambda,\eta) > 0,\ \delta,\mu > 0$ *then*

$$I^\eta_{0+}\left[E^{1,\delta}_{\alpha,1,\lambda,\mu}(x^\alpha; s, r)\right] = x^\eta\, E^{1,\delta}_{\alpha,\eta+1,\lambda,\mu}(x^\alpha; s, r). \tag{4.2.3}$$

Proof. Applying the integral operator (1.6.1) given by

$$_xI^\mu_a f(x) = I^\mu_{a+} f(x) = \frac{1}{\Gamma(\mu)} \int_a^x \frac{f(t)}{(x-t)^{1-\mu}} dt, \qquad x > a,$$

to $E^{1,\delta}_{\alpha,1,\lambda,\mu}(x^\alpha; s, r)$, one gets

$$I^\eta_{0+}\left[E^{1,\delta}_{\alpha,1,\lambda,\mu}(x^\alpha; s, r)\right]$$
$$= I^\eta_{0+}\left[\sum_{n=0}^\infty \frac{[(1)_{\delta n}]^s\, x^{\alpha n}}{\Gamma(\alpha n+1)\,[(\lambda)_{\mu n}]^r\, n!}\right]$$
$$= \sum_{n=0}^\infty \frac{[(1)_{\delta n}]^s}{\Gamma(\alpha n+1)\,[(\lambda)_{\mu n}]^r\, n!} I^\eta_{0+}(x^{\alpha n})$$
$$= \frac{1}{\Gamma(\eta)} \sum_{n=0}^\infty \frac{[(1)_{\delta n}]^s}{\Gamma(\alpha n+1)\,[(\lambda)_{\mu n}]^r\, n!} \int_0^x t^{\alpha n}(x-t)^{\eta-1}\, \mathrm{d}t.$$

The substitution $t = xu$ together with (1.2.10) further simplifies this as follows.

$$\frac{1}{\Gamma(\eta)} \sum_{n=0}^\infty \frac{[(1)_{\delta n}]^s}{\Gamma(\alpha n+1)\,[(\lambda)_{\mu n}]^r\, n!}\, x^{\alpha n+\eta} \int_0^1 u^{\alpha n+1-1}(1-u)^{\eta-1}\, \mathrm{d}u$$
$$= \frac{1}{\Gamma(\eta)} \sum_{n=0}^\infty \frac{[(1)_{\delta n}]^s}{\Gamma(\alpha n+1)\,[(\lambda)_{\mu n}]^r\, n!}\, x^{\alpha n+\eta}\, \mathfrak{B}(\alpha n+1,\eta)$$
$$= \frac{1}{\Gamma(\eta)} \sum_{n=0}^\infty \frac{[(1)_{\delta n}]^s}{\Gamma(\alpha n+1)\,[(\lambda)_{\mu n}]^r\, n!}\, x^{\alpha n+\eta}\, \frac{\Gamma(\alpha n+1)\Gamma(\eta)}{\Gamma(\alpha n+\eta+1)}$$
$$= \sum_{n=0}^\infty \frac{[(1)_{\delta n}]^s}{\Gamma(\alpha n+\eta+1)\,[(\lambda)_{\mu n}]^r\, n!}\, x^{\alpha n+\eta}$$
$$= x^\eta\, E^{1,\delta}_{\alpha,\eta+1,\lambda,\mu}(x^\alpha; s, r).$$

□

Theorem 4.2.3. *Let* $0 \leq a < \infty, \alpha,\ \beta,\ \gamma,\ \lambda,\ \eta \in \mathbb{C},\ \Re(\alpha, \beta, \gamma, \lambda, \eta) > 0;\ \delta,\ \mu > 0$ *for* $x > a$, *then*

$$\begin{aligned} &\Big(D^{\eta,\nu}_{a+}\ (t-a)^{\beta-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega(t-a)^{\alpha}; s, r)\Big)(x) \\ = \ & (x-a)^{\beta-\eta-1}\ E^{\gamma,\delta}_{\alpha,\beta-\eta,\lambda,\mu}(\omega(x-a)^{\alpha}; s, r). \end{aligned} \tag{4.2.4}$$

Proof. With the use of Hilfer's generalized Riemann-Liouville fractional derivative operator(1.6.6) given by

$$(D^{\mu,\,\nu}_{a+} f)(x) = \big(I^{\nu(1-\mu)}_{a+}\ \frac{d}{dx}\ (I^{(1-\nu)(1-\mu)}_{a+} f)\big)(x),$$

one gets

$$\begin{aligned} &\Big(D^{\eta,\nu}_{a+}\ (t-a)^{\beta-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega(t-a)^{\alpha}; s, r)\Big)\ (x) \\ = \ & \left(D^{\eta,\nu}_{a+}\ \left[\sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^{s}\ \omega^{n}\ (t-a)^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)\ [(\lambda)_{\mu n}]^{r}\ n!}\right]\right)(x) \\ = \ & \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^{s}\ \omega^{n}}{\Gamma(\alpha n+\beta)\ [(\lambda)_{\mu n}]^{r}\ n!}\ \big(D^{\eta,\nu}_{a+}\ \ [(t-a)^{\alpha n+\beta-1}]\big)\ (x) \\ = \ & \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^{s}\ \omega^{n}}{\Gamma(\alpha n+\beta)\ [(\lambda)_{\mu n}]^{r}\ n!} \frac{\Gamma(\alpha n+\beta)}{\Gamma(\alpha n+\beta-\eta)} (x-a)^{\alpha n+\beta-\eta-1} \\ = \ & (x-a)^{\beta-\eta-1}\ E^{\gamma,\delta}_{\alpha,\beta-\eta,\lambda,\mu}\left(\omega(x-a)^{\alpha}; s, r\right). \end{aligned}$$

□

Theorem 4.2.4. *Let* $\alpha, \beta, \gamma, \lambda, \nu, \omega \in \mathbb{C};\ \Re(\alpha, \beta, \gamma, \lambda, \nu) > 0;\ \delta, \mu > 0$ *then*

$$\big(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0_{+}}(t-a)^{\nu-1}\big)(x) = (x-a)^{\beta+\nu-1}\ \Gamma(\nu)\ E^{\gamma,\delta}_{\alpha,\beta+\nu,\lambda,\mu}\big(\omega(x-a)^{\alpha}\big). \tag{4.2.5}$$

Proof. If $f(t) = (t-a)^{\nu-1}$, then (4.1.1) gives

$$\begin{aligned} &\left(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0_{+}}\ (t-a)^{\nu-1}\right)(x) \\ = \ & \int_{a}^{x} (x-t)^{\beta-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\big(\omega(x-t)^{\alpha}\big)(t-a)^{\nu-1}\ \mathrm{d}t \\ = \ & \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\, n}]^{s}\ \omega^{n}}{\Gamma(\alpha n+\beta)\ [(\lambda)_{\mu\, n}]^{r}\ n!} \int_{a}^{x}\ (x-t)^{\alpha n+\beta-1}\ (t-a)^{\nu-1}\ \mathrm{d}t \\ = \ & \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\, n}]^{s}\ \omega^{n}}{\Gamma(\alpha n+\beta)[(\lambda)_{\mu\, n}]^{r} n!}\ \mathfrak{B}(\alpha n+\beta-1, \nu). \end{aligned}$$

Now using (1.2.10), the simplification of the above equation gives (4.2.5). It is now shown that the operator defined by (4.1.1) is bounded. □

Theorem 4.2.5. *Let the function* ϕ *be in the space* $L(a,b)$ *of Lebesgue measurable functions on a finite interval* $[a,b]$ *of* $\mathbb{R}$ *given by*

$$L(a,b) \;=\; \Big\{f:\ \|f\|_1 = \int_a^b |f(t)|\ \mathrm{d}t < \infty\Big\}. \tag{4.2.6}$$

Then the integral operator $\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0_+}$ *is bounded on* $L(a,b)$ *and*

$$\|\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0_+}\phi\|_1 \;\le\; \mathfrak{M}\ \|\phi\|_1, \tag{4.2.7}$$

where the constant $\mathfrak{M}$ $(0<\mathfrak{M}<\infty)$ *given by*

$$\begin{aligned}\mathfrak{M} \;&=\; (b-a)^{\Re(\beta)} \sum_{k=0}^{\infty} \frac{|(\gamma)_{\delta k}|^s}{|\Gamma(\alpha k+\beta)|\ (\Re(\alpha k+\beta))} \\ &\quad\times \frac{|\omega\ (b-a)^{\alpha}|^k}{|(\lambda)_{\mu k}|^r\ k!}\end{aligned} \tag{4.2.8}$$

in which the series is converges for $\Re(\alpha)+r\mu-s\delta+1>0$.

Proof. Using (4.1.1) and (4.2.6) and interchanging the order of integration by applying the Dirichlet formula [67], one gets

$$\begin{aligned}&\|\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0_+}\phi\|_1 \\ &= \int_a^b \left| \int_a^x (x-t)^{\beta-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega(x-t)^{\alpha};s,r))\ \phi(t)\ \mathrm{d}t\right|\ \mathrm{d}x \\ &\le \int_a^b \left[\int_t^b (x-t)^{\Re(\beta)-1} \left|E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega(x-t)^{\alpha};s,r)\right|\ \mathrm{d}x\right]\ |\phi(t)|\ \mathrm{d}t.\end{aligned}$$

On substituting $x-t=u$, this gives

$$\begin{aligned}&\|\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0_+}\phi\|_1 \\ &\le \int_a^b \left[\int_0^{b-t} u^{\Re(\beta)-1} \left|E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega u^{\alpha};s,r)\right|\ \mathrm{d}u\right]\ |\phi(t)|\ \mathrm{d}t, \\ &\le \int_a^b \sum_{k=0}^{\infty} \frac{|(\gamma)_{\delta k}|^s\ \omega^k}{|\Gamma(\alpha k+\beta)|\ |(\lambda)_{\mu k}|^r\ k!} \left[\int_0^{b-a} u^{\Re(\alpha k+\beta-1)}\ \mathrm{d}u\right] |\phi(t)|\ \mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&= \int_a^b \sum_{k=0}^{\infty} \frac{|(\gamma)_{\delta k}|^s\ (\omega)^k\ |b-a|^{\Re(\alpha k+\beta)}\ |\phi(t)|}{|\Gamma(\alpha k+\beta)|\ |(\lambda)_{\mu k}|^r\ k!\ \Re(\alpha k+\beta)}\ \mathrm{d}t \\
&= (b-a)^{\Re(\beta)} \sum_{k=0}^{\infty} \frac{|(\gamma)_{\delta k}|^s}{|\Gamma(\alpha k+\beta)|\ |(\lambda)_{\mu k}|^r\ k!} \frac{|\omega\ (b-a)^{\Re(\alpha)}|^k}{\Re(\alpha k+\beta)} \int_a^b |\phi(t)|\ \mathrm{d}t \\
&= (b-a)^{\Re(\beta)} \sum_{k=0}^{\infty} \frac{|(\gamma)_{\delta\, k}|^s\ |\omega\ (b-a)^{\alpha}|^k}{|\Gamma(\alpha k+\beta)|\ \Re(\alpha k+\beta)\ |(\lambda)_{\mu k}|^r\ k!} \|\phi\|_1 \\
&= \mathfrak{M}\ \|\phi\|_1, \quad say
\end{aligned}
$$

where, $\mathfrak{M}$ is finite. □

Theorem 4.2.6. *Let* $\alpha, \beta, \gamma, \lambda, \omega \in \mathbb{C}$; $\Re(\alpha, \beta, \gamma, \lambda) > 0$; $\delta, \mu > 0$ *then the relations*

$$
I^{\eta}_{a+}\ \mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\, a+} f = \mathcal{E}^{\gamma,\delta}_{\alpha,\beta+\eta,\lambda,\mu,\omega;\, a+} f = \mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\, a+} I^{\eta}_{a+} f \tag{4.2.9}
$$

hold for any summable function $f \in L(a,b)$.

Proof. From (4.1.1),

$$
I^{\eta}_{a+}\ \mathcal{E}^{\gamma,\,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\, a+} f(x) \ = \ I^{\eta}_{a+} \left(\int_a^u (u-t)^{\beta-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\big(\omega(u-t)^{\alpha}; s, r\big) f(t)\ \mathrm{d}t \right)(x).
$$

In the view of (1.6.1), this further gives

$$
\begin{aligned}
I^{\eta}_{a+}\ \mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\, a+} f(x) \ &= \ \frac{1}{\Gamma(\eta)} \int_a^x \int_a^u (u-t)^{\beta-1}\ (x-u)^{\eta-1} \\
&\quad \times E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\big(\omega(u-t)^{\alpha}; s, r\big) f(u)\ \mathrm{d}t\ \mathrm{d}u.
\end{aligned}
$$

Now applying Dirichlet formula [67] to interchange the order of integrations, one obtains

$$
\begin{aligned}
I^{\eta}_{a+}\ \mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\, a+} f(x) \ &= \ \int_a^x \left[\frac{1}{\Gamma(\eta)} \int_t^x (u-t)^{\beta-1}\ (x-u)^{\eta-1} \right. \\
&\quad \left. \times E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\big(\omega(u-t)^{\alpha}; s, r\big)\ \mathrm{d}u \right] f(t)\ \mathrm{d}t.
\end{aligned}
$$

Here the substitution $u - t = \tau$ gives

$$I^{\eta}_{a+}\ \mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,a+} f(x) = \int_a^x \left[\frac{1}{\Gamma(\eta)} \int_0^{x-t} \tau^{\beta-1}\ (x-t-\tau)^{\eta-1} \times E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\big(\omega\tau^{\alpha}; s, r\big)\ \mathrm{d}\tau \right] f(t)\ \mathrm{d}t.$$

Again use of (1.6.1) gives

$$I^{\eta}_{a+}\ \mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,a+} f(x) = \int_a^x \left(I^{\eta}_{0+} \left[\tau^{\beta-1} E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\big(\omega\tau^{\alpha}; s, r\big) \right] \right) (x-t)\ f(t)\ \mathrm{d}t.$$

On making use of (4.2.1), this yields

$$\begin{aligned} I^{\eta}_{a+}\ \mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,a+} f(x) &= \int_a^x \left[(u-t)^{\beta+\eta-1} E^{\gamma,\delta}_{\alpha,\beta+\eta,\lambda,\mu} \left(\omega(u-t)^{\alpha}; s, r \right) \right] f(t)\ \mathrm{d}t \\ &= \mathcal{E}^{\gamma,\delta}_{\alpha,\beta+\eta,\lambda,\mu,\omega;\,a+} f(x) \end{aligned}$$

from (4.1.1). To prove another relation, consider

$$\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,a+} I^{\eta}_{a+}\ f(x) = \int_a^x (x-t)^{\beta-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\big(\omega(x-t)^{\alpha}; s, r\big)\ I^{\eta}_{a+} f(t)\ \mathrm{d}t.$$

With the help of (1.6.1), this is written as

$$\begin{aligned} &\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,a+} I^{\eta}_{a+}\ f(x) \\ &= \int_a^x (x-t)^{\beta-1}\ (x-u)^{\eta-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\big(\omega(x-t)^{\alpha}; s, r\big) I^{\eta}_{a+}\ f(t)\ \mathrm{d}t \\ &= \int_a^x (x-t)^{\beta-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\big(\omega(x-t)^{\alpha}; s, r\big)\ \frac{1}{\Gamma(\eta)} \int_a^t\ (t-u)^{\eta-1}\ f(u)\ \mathrm{d}u\ \mathrm{d}t \\ &= \frac{1}{\Gamma(\eta)} \int_a^x \int_a^t (x-t)^{\beta-1}\ (t-u)^{\eta-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\big(\omega(x-t)^{\alpha}; s, r\big)\ f(u)\ \mathrm{d}u\ \mathrm{d}t \\ &= \frac{1}{\Gamma(\eta)} \int_a^x \int_u^x (x-t)^{\beta-1}\ (t-u)^{\eta-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\big(\omega(x-t)^{\alpha}; s, r\big)\ f(u)\ \mathrm{d}t\ \mathrm{d}u. \end{aligned}$$

Here substituting $x-t=\tau$, one gets

$$\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,a_+} I^{\eta}_{a+}\ f(x) \ = \ \int\limits_a^x \frac{1}{\Gamma(\eta)} \int\limits_0^{x-u} \Big[\tau^{\beta-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\big(\omega\tau^{\alpha};s,r\big)\ (x-u-\tau)^{\eta-1}\ \mathrm{d}\tau\Big]\ f(u)\ \mathrm{d}u.$$

Again use of (1.6.1) and then (4.2.1) and (4.1.1) in turn, one obtains

$$\begin{aligned}\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,a_+} I^{\eta}_{a+}\ f(x) \ &= \ \int\limits_a^x I^{\eta}_{0+}\Big[\tau^{\beta-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\big(\omega\tau^{\alpha};s,r\big)\ (x-u)\Big] f(u)\ \mathrm{d}u\\ &= \int\limits_a^x \tau^{\beta+\eta-1}\ E^{\gamma,\delta}_{\alpha,\beta+\eta,\lambda,\mu}\big(\omega\tau^{\alpha};s,r\big)\ f(u)\ \mathrm{d}u\\ &= \mathcal{E}^{\gamma,\delta}_{\alpha,\beta+\eta,\lambda,\mu,\omega;\,a+} f(x).\end{aligned}$$

□

Theorem 4.2.7. *Let* $\alpha,\beta,\gamma,\lambda,\omega\in\mathbb{C};\ \Re(\alpha,\beta,\gamma,\lambda)>0;\ \delta,\mu>0$ *then the relation*

$$D^{\eta}_{a+}\ \mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,a_+} f = \mathcal{E}^{\gamma,\delta}_{\alpha,\beta-\eta,\lambda,\mu,\omega;\,a+} f \tag{4.2.10}$$

holds for any summable function $f\in L(a,b)$.

Proof. From (4.1.1),

$$D^{\eta}_{a+}\ \mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,a_+} f(x) = D^{\eta}_{a+}\left(\int\limits_a^u (u-t)^{\beta-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\big(\omega(u-t)^{\alpha};s,r\big) f(t)\ \mathrm{d}t\right)(x).$$

Now using (1.6.4), it gives

$$\begin{aligned}&D^{\eta}_{a+}\ \mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,a_+} f(x)\\ &= \left(\frac{d}{dx}\right)^n \left(I^{n-\eta}_{a+} \int\limits_a^x (x-t)^{\beta-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\big(\omega(x-t)^{\alpha};s,r\big) f(t)\ \mathrm{d}t\right).\end{aligned}$$

Further using Theorem 4.2.6 and then making an appeal to (2.3.2), one finds

$$\begin{aligned}&D^{\eta}_{a+}\ \mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,a_+} f(x)\\ &= \left(\frac{d}{dx}\right)^n \int\limits_a^x (x-t)^{\beta-n-1}\ E^{\gamma,\delta}_{\alpha,\beta+n-\eta,\lambda,\mu}\big(\omega(x-t)^{\alpha};s,r\big) f(t)\ \mathrm{d}t\end{aligned}$$

$$= \int_a^x (x-t)^{\beta-1}\, E^{\gamma,\delta}_{\alpha,\beta-\eta,\lambda,\mu}\big(\omega(x-t)^{\alpha}; s, r\big) f(t)\, \mathrm{d}t$$
$$= \mathcal{E}^{\gamma,\delta}_{\alpha,\beta-\eta,\lambda,\mu,\omega;\, a+} f(x).$$

□

Theorem 4.2.8. *Let* $\alpha, \beta, \gamma, \lambda, \omega \in \mathbb{C}$; $\Re(\alpha, \beta, \gamma, \lambda) > 0$; $\delta, \mu > 0$ *then the relation*

$$D^{\eta,\nu}_{a+}\, \mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\, a_+} f = \mathcal{E}^{\gamma,\delta}_{\alpha,\beta-\eta,\lambda,\mu,\omega;\, a_+} f, \quad 0 < \eta < 1,\ 0 \le \nu \le 1 \tag{4.2.11}$$

holds for any summable function $f \in L(a, b)$.

Proof. From (1.6.6), that is,

$$(D^{\mu,\,\nu}_{a+} f)(x) = \big(I^{\nu(1-\mu)}_{a+} \frac{\mathrm{d}}{\mathrm{d}x}\, (I^{(1-\nu)(1-\mu)}_{a+} f)\big)(x),$$

one gets

$$\begin{aligned} D^{\eta,\nu}_{a+} \mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\, a_+} f(x) &= \left(I^{\nu(1-\eta)}_{a+} \frac{\mathrm{d}}{\mathrm{d}x} \left(I^{(1-\eta)(1-\nu)}_{a+} \left(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\, a+} f \right)\right)\right)(x) \\ &= \left(I^{\eta(1-\nu)}_{a+} \frac{\mathrm{d}}{\mathrm{d}x} \left(I^{(1-\eta)(1-\nu)}_{a+} \left(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\, a_+} f \right)\right)\right)(x) \\ &= \left(I^{\nu(1-\eta)}_{a+} \left(D^{\eta+\nu-\eta\nu}_{a+} \left(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\, a_+} f \right)\right)\right)(x). \end{aligned}$$

This in view of Theorem 4.2.7, gives

$$D^{\eta,\nu}_{a+} \mathcal{E}^{\gamma,\,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu,\,\omega;\, a_+} f(x) = \left(I^{\nu(1-\eta)}_{a+} \left(\mathcal{E}^{\gamma,\,\delta}_{\alpha,\,\beta-\eta-\nu+\eta\nu,\,\lambda,\,\mu,\,\omega;\, a+} f \right)\right)(x).$$

Finally from Theorem 4.2.6, it follows that

$$D^{\eta,\nu}_{a+} \mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\, a+} f(x) = \left(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta-\eta,\lambda,\mu,\omega;\, a+} f \right)(x).$$

□

4.2.2 Gauss multiplication type formula of $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z)$

If m is a positive integer and $z \in \mathbb{C}$ then the Gauss multiplication theorem [63] states that

$$\prod_{k=1}^{m} \Gamma\left(z + \frac{k-1}{m}\right) = (2\pi)^{\frac{m-1}{2}} \; m^{\frac{1}{2}-mz} \; \Gamma(mz). \tag{4.2.12}$$

By taking $z = n + \frac{\beta}{m}$, this takes the form

$$\prod_{k=0}^{m-1} \Gamma\left(n + \frac{\beta}{m} + \frac{k}{m}\right) = (2\pi)^{\frac{m-1}{2}} \; m^{\frac{1}{2}-mn-\beta} \; \Gamma(mn+\beta),$$

that is,

$$\frac{1}{\Gamma(mn+\beta)} = \frac{(2\pi)^{\frac{m-1}{2}}}{m^{mn+\beta-\frac{1}{2}}} \left\{\prod_{k=0}^{m-1} \Gamma\left(n + \frac{\beta}{m} + \frac{k}{m}\right)\right\}^{-1}.$$

Using this, the function (2.1.5) for $\alpha = m \in \mathbb{N}$, takes the form

$$E^{\gamma,\delta}_{m,\beta,\lambda,\mu}(z;s,r) = \frac{(2\pi)^{\frac{m-1}{2}}}{m^{\beta-\frac{1}{2}}} \left\{\prod_{k=0}^{m-1} \Gamma\left(\frac{\beta+k}{m}\right)\right\}^{-1} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^{s} \; z^{n}}{\left(\frac{\beta+k}{m}\right)_{n} \; [(\lambda)_{\mu n}]^{r} \; m^{mn} \; n!}. \tag{4.2.13}$$

Here substituting $\delta = q, \; s = 1, \; r = 0$ in (4.2.13), it reduces to

$$E^{\gamma,q}_{m,\beta}(z) = \frac{(2\pi)^{\frac{m-1}{2}}}{m^{\beta-\frac{1}{2}}} \left\{\prod_{k=0}^{m-1} \Gamma\left(\frac{\beta+k}{m}\right)\right\}^{-1} \sum_{n=0}^{\infty} \frac{[(\gamma)_{qn}]}{\left(\frac{\beta+k}{m}\right)_{n}} \frac{\left(\frac{z}{m^{m}}\right)^{n}}{n!}. \tag{4.2.14}$$

This generalizes the result due to Kilbas [31] when $q = 1$.

4.2.3 Fractional differential equations based upon the Hilfer derivative operator

Consider the following fractional differential equation due to Kilbas [32, p. 144, Eq.(1.2)].

$$\left(D^{\alpha}_{a+} \, y\right)(x) = \lambda \; \left(\mathcal{E}^{\gamma,1}_{\alpha,\beta,\omega;\, a+}\right)(x) \; + \; f(x), \tag{4.2.15}$$

with the initial condition

$$\left(D_{a+}^{\alpha-k}\, y\right)(0+) \;=\; b_k,$$

where $a < x \leq b;\ \Re(\alpha) > 0; \lambda, \beta, \gamma \in \mathbb{C},\ b_k \in \mathbb{C};\ k = 1, 2, \ldots n.$
Here the general integral operator $\left(\mathcal{E}_{\alpha,\beta,\omega;\, a+}^{\gamma,1}\right)(x)$ is a special case $\delta = 1, s = 1, r = 0$ of (4.1.1). Indeed, as pointed out by Kilbas et al. [32, p. 144], the homogeneous differentail equation corresponding to (4.2.15) when $f(x) = 0$ is a generalization of a certain first order Volterra-type integro-differential equation governing the unsaturated behaviour of the free electron laser (see also [89]). Later, by using the Laplace transform method, H. M. Srivastava and Z. Tomovski [90, Theorem-8, p.10 and Theorem-8, p.11] obtained an explicit solution in the $L(0,\infty)$, of a more general fractional diffrential equation than (4.2.15) which contains the generalized Riemann-Liouville fractional derivative operator (1.6.6). In the light of these theorems, the following theorems are proved which involve the operator defined in (4.1.1).

Theorem 4.2.9. *If* $0 < \eta < 1,\ 0 \leq \nu \leq 1,\ \omega, \xi \in \mathbb{C},\ \Re(\alpha) > max\{0, \Re(\delta) - 1\}$ *and* $min\{\Re(\beta, \gamma, \lambda, \mu)\} > 0$ *then*

$$\left(D_{0+}^{\eta,\,\nu}\, y\right)(x) \;=\; \xi\, \left(\mathcal{E}_{\alpha,\beta,\lambda,\mu,\omega;\,0+}^{\gamma,\delta}\right)(x) \;+\; f(x) \tag{4.2.16}$$

with the initial condition

$$\left(I_{0+}^{(1-\nu)(1-\eta)}\, y\right)(0+) \;=\; C,$$

has solution in the space $L(0,\infty)$ *given by*

$$\begin{aligned} y(x) \;&=\; C\, \frac{x^{\eta-\nu(1-\eta)-1}}{\Gamma(\eta-\nu+\eta\nu)} + \xi\, x^{\eta+\beta}\, E_{\alpha,\beta+\eta+1,\lambda,\mu}^{\gamma,\delta}\left(\omega x^{\alpha}\right) \\ &\quad + \frac{1}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1}\, f(t)\, \mathrm{d}t, \end{aligned} \tag{4.2.17}$$

where C *is arbitrary constant.*

Proof. Applying the Laplace transform on both sides of (4.2.16), it gives

$$\mathcal{L}\left(D_{0+}^{\eta;\nu}\, y(x)\right)(S) \;=\; \mathcal{L}\left(\xi\, \left(\mathcal{E}_{\alpha,\beta,\lambda,\mu,\omega;\,0+}^{\gamma,\delta}\right)(x) \;+\; f(x)\right)(S),$$

and further with the help of (4.1.1),

$$\mathcal{L}\left(D_{0+}^{\eta,\nu}\, y(x)\right)(S) = \xi\,\mathcal{L}\left(\int_0^x x^{\beta-1}\, E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}\big(\omega x^{\alpha};s,r\big)(1)(t)\,\mathrm{d}t\right)(S) + F(S). \tag{4.2.18}$$

Now

$$\begin{aligned}
&\mathcal{L}\left(x^{\beta-1}\, E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}\big(\omega x^{\alpha};s,r\big)\right)(S)\\
&= \mathcal{L}\left(x^{\beta-1}\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^{s}\,\omega^{n}\,x^{\alpha n}}{\Gamma(\alpha n+\beta)\,[(\lambda)_{\mu n}]^{r}\,n!}\right)(S)\\
&= \sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^{s}\,\omega^{n}}{\Gamma(\alpha n+\beta)\,[(\lambda)_{\mu n}]^{r}\,n!}\mathcal{L}(x^{\alpha n+\beta-1})\\
&= \sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^{s}\,\omega^{n}}{\Gamma(\alpha n+\beta)\,[(\lambda)_{\mu n}]^{r}\,n!}\frac{\Gamma(\alpha n+\beta)}{S^{\alpha n+\beta}}.
\end{aligned} \tag{4.2.19}$$

Using the Laplace convolution Theorem 1.4.4 in (4.2.18), one gets

$$\mathcal{L}\left(D_{0+}^{\eta,\nu}\, y(x)\right)(S) = \xi\,\mathcal{L}\left(x^{\beta-1}\, E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}\big(\omega x^{\alpha};s,r\big)\right)(S)\mathcal{L}(1)(S) + F(S).$$

Using (1.6.7), which is

$$\mathcal{L}[D_{0+}^{\mu,\nu} f(x)](S) = S^{\mu}\,\mathcal{L}[f(x)](S) - S^{\nu(1-\mu)}(I_{0+}^{(1-\nu)(1-\mu)} f)(0+),$$

and then using (4.2.19), this becomes

$$S^{\eta}\,\mathcal{L}(y(x))(S) - S^{\nu(1-\eta)}\left(I_{0+}^{(1-\nu)(1-\eta)}\, y\right)(0+) = \xi\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^{s}\,\omega^{n}}{[(\lambda)_{\mu n}]^{r}\,n!}\frac{\Gamma(\alpha n+\beta)}{S^{\alpha n+\beta+1}}.$$

Hence,

$$\begin{aligned}
S^{\eta}\,Y(S) - C\,S^{\nu(1-\eta)} &= \xi\,\mathcal{L}[x^{\beta-1}\,E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(\omega\,x^{\alpha})](S)\,\mathcal{L}(1)(S) + F(S)\\
&= \xi\,S^{-\beta-1}\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^{s}\,(\omega\,S^{-\alpha})^{n}}{[(\lambda)_{\mu n}]^{r}\,n!} + F(S).
\end{aligned}$$

From this, it follows that

$$Y(S) = C\,S^{\nu(1-\eta)-\eta} + \xi S^{-\beta-\eta-1}\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^{s}(\omega\,S^{\alpha})^{n}}{[(\lambda)_{\mu n}]^{r} n!} + F(S)S^{-\eta}. \tag{4.2.20}$$

Now, taking the inverse Laplace transform on both sides of equation (4.2.20), one obtains

$$\begin{aligned}
y(x) &= C\,\mathcal{L}^{-1}\big(S^{\nu(1-\eta)-\eta}\big)(x) + \xi \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^S\ \omega^n}{[(\lambda)_{\mu n}]^r\ n!}\ \mathcal{L}^{-1}(S^{-\alpha n-\beta-\eta-1})(x) \\
&\quad + \mathcal{L}^{-1}(S^{-\eta}F(S)) \\
&= C\frac{x^{\eta-\nu(1-\eta)-1}}{\Gamma(\eta-\nu+\eta\nu)} + \xi\ x^{\eta+\beta}\sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\omega^n x^{\alpha n}}{\Gamma(\alpha n+\beta+\eta+1)[(\lambda)_{\mu n}]^r n!} \\
&\quad + \frac{1}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1} f(t)\ \mathrm{d}t \\
&= C\frac{x^{\eta-\nu(1-\eta)-1}}{\Gamma(\eta-\nu+\eta\,\nu)}\ +\ \xi\ x^{\eta+\beta}\ E^{\gamma,\delta}_{\alpha,\beta+\eta+1,\lambda,\mu}\big(\omega x^{\alpha}\big) \\
&\quad + \frac{1}{\Gamma(\eta)}\int_0^x\ (x-t)^{\eta-1}\ f(t)\ \mathrm{d}t.
\end{aligned}$$

□

The instance $f(y) = E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu}(y;s,r)$ is the following theorem.

Theorem 4.2.10. *If* $0<\eta<\ 1,\ 0\le\nu\le 1,\ \omega,\xi\in\mathbb{C},\ R(\alpha)>max\{0,R(\delta)-1\}$

and $min\{R(\beta,\gamma,\lambda,\mu)\}\ >\ 0$ *then*

$$\Big(D^{\eta,\nu}_{0+}y\Big)(x) = \xi\Big(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0+}\Big)(x) + x^{\beta}E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu}\big(\omega(ax)^{\alpha};s,r\big) \qquad (4.2.21)$$

with the initial condition

$$\Big(I^{(1-\nu)(1-\eta)}_{0+}\ y\Big)(0+)\ =\ C,$$

has solution in the space $L(0,\infty)$ *given by*

$$y(x) = C\ \frac{x^{\eta-\nu(1-\eta)-1}}{\Gamma(\eta-\nu+\eta\nu)}\ +\ (\xi+1)\ x^{\eta+\beta}\ E^{\gamma,\delta}_{\alpha,\beta\eta+1,\lambda,\mu}\big(\omega(ax)^{\alpha};s,r\big), \qquad (4.2.22)$$

where C *is arbitrary constant.*

Proof. Put

$$f(t) = t^{\beta}\ E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu}\big(\omega(at)^{\alpha};s,r\big)$$

in Theorem 4.2.9, to get

$$y(x) = C \frac{x^{\eta-\nu(1-\eta)-1}}{\Gamma(\eta-\nu+\eta\,\nu)} + \xi\, x^{\eta+\beta}\, E^{\gamma,\delta}_{\alpha,\beta+\eta+1,\lambda,\mu}\big(\omega(ax)^{\alpha}; s, r\big)$$
$$+ \frac{1}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1}\, t^{\beta}\, E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu}\big(\omega(ax)^{\alpha}; s, r\big)\, \mathrm{d}t. \tag{4.2.23}$$

Here,

$$\int_0^x (x-t)^{\eta-1}\, t^{\beta}\, E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu}\big(\omega(at)^{\alpha}; s, r\big)\, \mathrm{d}t$$
$$= \int_0^x (x-t)^{\eta-1}\, t^{\beta} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\,n}]^s\, (\omega\,(at)^{\alpha})^n}{\Gamma(\alpha n+\beta+1)\, [(\lambda)_{\mu n}]^r\, n!}\, \mathrm{d}t$$
$$= \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\, \omega^n}{\Gamma(\alpha n+\beta+1)\, [(\lambda)_{\mu n}]^r\, n!} \int_0^x (x-t)^{\eta-1}\, t^{\alpha n+\beta}\, \mathrm{d}t.$$

Taking $t = xu$, $dt = xdu$, $u \to 0$ as $t \to 0$ and $u \to 1$ as $t \to x$, hence

$$\int_0^x (x-t)^{\eta-1}\, t^{\beta}\, E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu}\big(\omega(at)^{\alpha}; s, r\big)\, \mathrm{d}t$$
$$= \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\, \omega^n\, x^{\alpha n+\eta+\beta}}{\Gamma(\alpha n+\beta)\, [(\lambda)_{\mu n}]^r\, n!} \int_0^1 (1-u)^{\eta-1}\, u^{\alpha n+\beta}\, \mathrm{d}t$$
$$= \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\, \omega^n\, x^{\alpha n+\eta+\beta}\, \Gamma(\eta)\, \Gamma(\alpha n+\beta)}{\Gamma(\alpha n+\beta)\, [(\lambda)_{\mu n}]^r\, n!\, \Gamma(\alpha n+\beta+\eta+1)}$$
$$= \Gamma(\eta) \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\,n}]^s\, (\omega\,(ax)^{\alpha})^n\, x^{(\eta+\beta)}}{\Gamma(\alpha n+\beta+\eta+1)\, [(\lambda)_{\mu n}]^r\, n!}$$
$$= x^{(\eta+\beta)}\, \Gamma(\eta)\, E^{\gamma,\delta}_{\alpha,\beta+\eta+1,\lambda,\mu}\, \big(\omega x^{\alpha}; s, r\big).$$

Using this in (4.2.23), (4.2.22) is obtained. □

Theorem 4.2.11. *If* $0 < \eta < 1$, $0 \le \nu \le 1$, $\omega, \xi \in \mathbb{C}$, $\Re(\alpha) > max\{0, \Re(\delta) - 1\}$ *and* $min\{\Re(\beta, \gamma, \lambda, \mu)\} > 0$ *then*

$$x\left(D^{\eta,\,\nu}_{0+}\, y\right)(x) = \xi\left(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0+}\right)(x) \tag{4.2.24}$$

with the initial condition

$$\left(I_{0+}^{(1-\nu)(1-\eta)}\, y\right)(0+) \;=\; C_1,$$

has solution in the space $L(0,\infty)$ given by

$$y(x) \;=\; C_2\frac{x^{\eta-1}}{\Gamma(\eta)}+C_1\frac{x^{\eta-\nu(1-\eta)-1}}{\Gamma(\eta-\nu(1-\eta))}+\xi\; E_{\alpha,\beta+\eta,\lambda,\mu}^{\gamma,\delta}\big(\omega x^{\alpha};s,r\big), \quad (4.2.25)$$

where C_1 and C_2 are arbitrary constants.

Proof. Applying Laplace transform on both the sides of (4.2.24), gives

$$\mathcal{L}\left(x\left(D_{0+}^{\eta,\,\nu}\, y\right)(x)\right)(S) \;=\; \mathcal{L}\left(\xi\left(\mathcal{E}_{\alpha,\beta,\lambda,\mu,\omega;\,0+}^{\gamma,\delta}\right)(x)\right)(S)$$

and using (4.1.1), this gives

$$\mathcal{L}\left(x\; D_{0+}^{\eta,\nu}\; y(x)\right)(S) \;=\; \xi\;\mathcal{L}\left(\int_0^x x^{\beta-1}\; E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}\big(\omega x^{\alpha};s,r\big)(1)(t)\;\mathrm{d}t\right)(S). \quad (4.2.26)$$

Now using the Laplace convolution Theorem 1.4.4 in (4.2.18), one gets

$$\mathcal{L}\left(x\; D_{0+}^{\eta,\nu}\; y(x)\right)(S) \;=\; \xi\;\mathcal{L}\left(x^{\beta-1}\; E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}\big(\omega x^{\alpha};s,r\big)\right)(S)\;\mathcal{L}(1)(S).$$

Using the derivative formula (1.4.5) with $n=1$, and using (1.6.7) and (4.2.19), this yields

$$\frac{\partial}{\partial S}\left(S^{\eta}\;\mathcal{L}(y(x))(S)-S^{\nu(1-\eta)}\left(I_{0+}^{(1-\nu)(1-\eta)}\; y\right)(0+)\right)$$
$$=\; -\xi\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^{s}\;\omega^{n}}{\Gamma(\alpha n+\beta)\;[(\lambda)_{\mu n}]^{r}\;n!}\frac{\Gamma(\alpha n+\beta+1)}{S^{\alpha n+\beta+1}}.$$

Hence,

$$\begin{aligned}\frac{\partial}{\partial S}\left(S^{\eta}\;Y(S)-C_1\;S^{\nu(1-\eta)}\right) \;&=\; -\xi\;\mathcal{L}\left[x^{\beta-1}\;E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(\omega\; x^{\alpha})\right](S)\;\mathcal{L}(1)(S)+F(S)\\ &=\; -\xi\;S^{-\beta-1}\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^{s}\;(\alpha n+\beta)\;(\omega\;S^{-\alpha})^{n}}{[(\lambda)_{\mu n}]^{r}\;n!}\end{aligned}$$

which leads to the first order ordinary linear differential equation:

$$Y'(S)+\frac{\eta}{S}\;Y(S)-C_1\;\nu(1-\eta)S^{\nu(1-\eta)-1-\eta}$$

$$+\xi\ S^{-\beta-\eta-1}\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^{s}\ (\alpha n+\beta)\ (\omega\ S^{-\alpha})^{n}}{[(\lambda)_{\mu n}]^{r}\ n!}=0$$

for $Y(S)$. Now, by solving this ordinary differential equation, one finds

$$\begin{aligned} Y(S) &= \exp\left(-\int^{S}\frac{\eta}{S}dS\right)\Bigg[C_2+\int^{S}\Big(C_1\ \nu(1-\eta)\ S^{\nu(1-\eta)-1-\eta} \\ &\quad -\xi\ S^{-\beta-\eta-1}\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^{s}\ (\alpha n+\beta)\ (\omega\ S^{-\alpha})^{n}}{[(\lambda)_{\mu n}]^{r}\ n!}\Big)\exp\left(\int^{S}\frac{\eta}{S}\ \mathrm{d}S\right)\ \mathrm{d}S\Bigg] \\ &= C_2\ S^{-\eta}+C_1\ S^{\nu(1-\eta)-\eta}-\xi\ S^{-\eta}\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^{s}\ (\alpha n+\beta)\ \omega^{n}}{[(\lambda)_{\mu n}]^{r}\ n!}\int^{S}\frac{1}{S^{\alpha n+\beta+1}}\ \mathrm{d}S \\ &= C_2\ S^{-\eta}+C_1\ S^{\nu(1-\eta)-\eta}+\xi\ S^{-\eta}\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^{s}\ (\alpha n+\beta)\ \omega^{n}}{[(\lambda)_{\mu n}]^{r}\ n!\ (\alpha n+\beta)\ S^{\alpha n+\beta}}. \end{aligned}$$

Finally applying inverse Laplace transform, it gives

$$\begin{aligned} y(x) &= C_2\frac{x^{\eta-1}}{\Gamma(\eta)}+C_1\frac{x^{\eta-\nu(1-\eta)-1}}{\Gamma(\eta-\nu(1-\eta))}+\xi\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^{s}\ \omega^{n}}{[(\lambda)_{\mu n}]^{r}\ n!}\frac{x^{\alpha n+\beta+\eta-1}}{\Gamma(\alpha n+\beta+\eta)} \\ &= C_2\frac{x^{\eta-1}}{\Gamma(\eta)}+C_1\frac{x^{\eta-\nu(1-\eta)-1}}{\Gamma(\eta-\nu(1-\eta))}+\xi\ E^{\gamma,\delta}_{\alpha,\beta+\eta,\lambda,\mu}\big(\omega x^{\alpha};s,r\big). \end{aligned}$$

□

4.2.4 Integral transforms of $\left(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0_+}f\right)(x)$

An application of Mellin transform is considered on $\left(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0_+}f\right)(x)$ in the following theorem.

Theorem 4.2.12. *Let* $\alpha,\beta,\gamma,\lambda,\omega\in\mathbb{C},\Re(\alpha,\beta,\gamma,\lambda)>0;\delta,\mu>0,\Re(1-S-\beta)>0$ *then*

$$\begin{aligned} &\mathcal{M}\Big\{(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0_+}f)(x);S\Big\}=\frac{[\Gamma(\lambda)]^{r}}{2\pi i\ [\Gamma(\gamma)]^{s}\ \Gamma(1-S)} \\ &\times H^{r+3,s+1}_{s+1,r+3}\left[-wt^{\alpha}\ \middle|\ \begin{matrix} [(1-\gamma,\ \delta)]^{s}, & (0,1) & & \\ (0,1), & (1-S-\beta,\alpha), & [(1-\lambda,\mu)]^{r}, & (0,1)\end{matrix}\right] \\ &\times\mathcal{M}\{t^{\beta}f(t);S\}. \end{aligned}$$

Proof. By the definition of the Mellin transform,

$$\mathcal{M}\Big\{(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0_+}f)(x);S\Big\}$$
$$= \int_0^\infty x^{S-1}\int_0^x (x-t)^{\beta-1}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\big(\omega(x-t)^{\alpha};s,r\big)f(t)\ \mathrm{d}t\ \mathrm{d}x.$$

Interchanging the order of integration, which is permissible under the given conditions, one finds that

$$\mathcal{M}\Big\{(E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0_+}f)(x);S\Big\}$$
$$= \int_0^\infty f(t)\int_t^\infty x^{S-1}(x-t)^{\beta-1}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\big(\omega(x-t)^{\alpha};s,r\big)\ \mathrm{d}x\ \mathrm{d}t.$$

For $x=t+u$, then above integral takes the form

$$\mathcal{M}\Big\{(\mathcal{E}^{\gamma,\,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0_+}f)(x);S\Big\}$$
$$= \int_0^\infty f(t)\int_0^\infty (t+u)^{S-1}\ u^{\beta-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega u^{\alpha};s,r)\ \mathrm{d}u\ \mathrm{d}t.$$

To evaluate the u-integral, the Mittag-Leffler function is expressed in terms of its Mellin-Barnes contour integral by means of the formula (2.2.16), then the above expression gets transformed into the form

$$\mathcal{M}\Big\{(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0_+}f)(x);S\Big\}$$
$$= \int_0^\infty f(t)\ \frac{[\Gamma(\lambda)]^r}{2\pi i\,[\Gamma(\gamma)]^s}\int_{-i\infty}^{i\infty}\frac{\Gamma(-\xi)\,\Gamma(1+\xi)\,[\Gamma(\gamma+\delta\xi)]^s\,(-\omega)^{\xi}}{\Gamma(\beta+\alpha\xi)\,[\Gamma(\lambda+\mu\xi)]^r\,\Gamma(1+\xi)}$$
$$\times\int_0^\infty (t+u)^{S-1}\ u^{\alpha\xi+\beta-1}\ \mathrm{d}u\ \mathrm{d}\xi\ \mathrm{d}t.$$

Now put $u=tv$, to get

$$\mathcal{M}\Big\{(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0_+}f)(x);S\Big\}$$
$$= \int_0^\infty f(t)\frac{[\Gamma(\lambda)]^r}{2\pi i\,[\Gamma(\gamma)]^s}\int_{-i\infty}^{i\infty}\frac{\Gamma(-\xi)\,\Gamma(1+\xi)\,[\Gamma(\gamma+\delta\xi)]^s\,(-\omega)^{\xi}}{\Gamma(\beta+\alpha\xi)\,[\Gamma(\lambda+\mu\xi)]^r\,\Gamma(1+\xi)}$$

$$\times \int_0^\infty (1+v)^{S-1}\ v^{\alpha\xi+\beta-1}\ \mathrm{d}v\ t^{\alpha\xi+S+\beta-1}\ \mathrm{d}\xi\ \mathrm{d}t.$$

On evaluating the u- integral with the help of the formula

$$\int_0^\infty x^{\nu-1}\ (x+a)^{-\rho}\ \mathrm{d}x\ =\ \frac{\Gamma(\nu)\ \Gamma(\rho-\nu)}{\Gamma(\rho)};\ \ \Re(\rho) > \Re(\nu) > 0,$$

then the right hand side of above equation simplifies to

$$\mathcal{M}\left\{(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0_+} f)(x); S\right\}$$

$$= \frac{[\Gamma(\lambda)]^r}{2\pi i\ [\Gamma(\gamma)]^s\ \Gamma(1-S)} \int_0^\infty t^{\beta+S-1}\ f(t)$$

$$\times \int_{-i\infty}^{i\infty} \frac{\Gamma(-\xi)\ \Gamma(1+\xi)\ \Gamma(1-S-\alpha\xi-\beta)}{[\Gamma(\gamma+\delta\xi)]^{-s}\ [\Gamma(\lambda+\mu\xi)]^r\ \Gamma(1+\xi)} (-\omega t^\alpha)^\xi\ \mathrm{d}\xi\ \mathrm{d}t$$

$$= \frac{[\Gamma(\lambda)]^r}{2\pi i\ [\Gamma(\gamma)]^s\ \Gamma(1-S)} \int_0^\infty t^{\beta+S-1}\ f(t)$$

$$\times H^{1,s+3}_{s+3,r+1}\left[-wt^\alpha \left| \begin{matrix} (1,-1), & (0,1), & [(1-\gamma,\ \delta)]^s, & (S+\beta,-\alpha) \\ (0,1), & [(1-\lambda,\mu)]^r & & \end{matrix} \right.\right] \mathrm{d}t$$

$$= \frac{[\Gamma(\lambda)]^r}{2\pi i\ [\Gamma(\gamma)]^s\ \Gamma(1-S)}$$

$$\times \mathcal{M}\left\{ f(t)\ H^{1,s+3}_{s+3,r+1}\left[-wt^\alpha \left| \begin{matrix} (1,-1),\ (0,1),\ [(1-\gamma,\ \delta)]^s, (S+\beta,-\alpha) \\ (0,1),\ [(1-\lambda,\mu)]^r \end{matrix} \right.\right]; S+\beta \right\}$$

In view of the definition (1.2.12) of H-function, this yields the desired result. □

For $s=1,\ r=0,\ \delta=q$ the Theorem 4.2.12 reduces to the following form.

Corollary 4.2.1. *The Mellin transform*

$$\mathcal{M}\left\{(\mathcal{E}^{\gamma,q}_{\alpha,\beta,\omega;\,0_+} f)(x); S\right\}$$

$$= \frac{1}{2\pi i\ \Gamma(\gamma)\ \Gamma(1-S)}$$

$$\times \mathcal{M}\left\{ f(t)\ H^{1,4}_{4,1}\left[-wt^\alpha \left| \begin{matrix} (1,-1), (0,1), (1-\gamma,q), (S+\beta,-\alpha) \\ (0,1) \end{matrix} \right.\right]; S+\beta \right\},$$

where $\Re(\alpha,\beta,\gamma)>0;\ q\in(0,1)\cup\mathbb{N}, \Re(1-S-\beta)>0$ *and* $H^{2,1}_{1,2}(*)$ *is the H-function defined by (1.2.12).*

Theorem 4.2.13. *The Laplace transform of the operator* $\left(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0_+}f\right)(x)$ *is given by*

$$\mathcal{L}\left\{(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0_+}f)(x);P\right\}$$
$$=\frac{[\Gamma(\lambda)]^r}{[\Gamma(\gamma)]^s\ P^{\beta}}\ {}_{s+1}\psi_{r+1}\left[\begin{array}{lll}[(\gamma,q)]^s, & (1,1); & \omega/P^{\alpha}\\ [(\lambda,\mu)]^r, & (1,1); & \end{array}\right]F(P),$$

where $\Re(\alpha,\beta,\gamma)>0;\Re(p)>|\omega|^{1/\Re(\alpha)}$ *and F(P) is the Laplace transform of* $f(t)$*, defined by*

$$L\{f(t);P\}=F(P)=\int_0^{\infty}e^{-Pt}f(t)\,\mathrm{d}t,\quad \Re(P)>0.$$

Proof. By virtue of the definition of Laplace transform, and interchanging the order of integration which is permissible under the conditions given in the theorem, one finds that

$$\mathcal{L}\left\{\left(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0_+}f\right)(x);P\right\}$$
$$=\int_0^{\infty}e^{-Px}\int_0^{x}(x-t)^{\beta-1}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}[\omega(x-t)^{\alpha}]\ f(t)\ \mathrm{d}t\ \mathrm{d}x$$
$$=\int_0^{\infty}e^{-Px}\ f(t)\int_t^{\infty}(x-t)^{\beta-1}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}[\omega(x-t)^{\alpha}]\ \mathrm{d}x\ \mathrm{d}t.$$

With $x=t+u$, this gives

$$\mathcal{L}\left\{\left(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0_+}f\right)(x);P\right\}$$
$$=\int_0^{\infty}e^{-Pt}f(t)\ \mathrm{d}t\int_0^{\infty}e^{-Pu}\ u^{\beta-1}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}[\omega u^{\alpha}]\ \mathrm{d}u$$
$$=\sum_{k=0}^{\infty}\frac{[(\gamma)_{\delta k}]^s\ \omega^k}{\Gamma(\alpha k+\beta)\ [(\lambda)_{\mu k}]^r\ k!}\int_0^{\infty}e^{-Pu}\ u^{\beta+\alpha k-1}\ \mathrm{d}u\int_0^{\infty}e^{-Pt}\ f(t)\mathrm{d}t$$
$$=\sum_{k=0}^{\infty}\frac{[(\gamma)_{\delta k}]^s\ \omega^k}{\Gamma(\alpha k+\beta)\ [(\lambda)_{\mu k}]^r\ k!}\ \frac{\Gamma(\alpha k+\beta)}{P^{\beta+\alpha k}}\int_0^{\infty}e^{-Pt}f(t)\mathrm{d}t$$

$$= \sum_{k=0}^{\infty} \frac{[(\gamma)_{\delta k}]^s \, \omega^k}{P^{\beta+\alpha k} \, [(\lambda)_{\mu k}]^r \, k!} \int_0^{\infty} e^{-Pt} f(t) \mathrm{d}t$$

$$= \frac{[\Gamma(\lambda)]^r}{[\Gamma(\gamma)]^s \, P^{\beta}} \, {}_s\psi_r \left[\begin{array}{ll} [(\gamma,\delta)]^s, & \omega/P^{\alpha} \\ [(\lambda,\mu)]^r, & \end{array} \right] F(P).$$

□

The special case $s = 1, \; r = 0, \; \delta = q$ of Theorem 4.2.13 is

Corollary 4.2.2. *In the notations of the above theorem,*

$$\mathcal{L}\left\{ (\mathcal{E}^{\gamma,\,q}_{\alpha,\beta,\omega;\,0_+} f)(x); P \right\} = \frac{1}{\Gamma(\gamma)} P^{-\beta} \, {}_1\psi_0 \left[\begin{array}{ll} (\gamma,q); & \omega/P^{\alpha} \\ -; & \end{array} \right] F(P).$$

4.2.5 Properties of $E_t(c,\nu,\gamma,\delta,\lambda,\mu)$ and $E_t(c,-\eta,\gamma,\delta,\lambda,\mu)$

In this section, certain properties of the functions $E_t(c,\nu,\gamma,\delta,\lambda,\mu)$ and $E_t(c,-\eta,\gamma,\delta,\lambda,\mu)$ will be obtained.
Consider the function

$$f(t) \;=\; \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s \, (ct)^n}{(n!)^2 \, [(\lambda)_{\mu n}]^r},$$

where $\gamma, \lambda \in \mathbb{C}, \Re(\gamma,\lambda) > 0, \delta,\mu > 0, \; r \in \mathbb{N} \cup \{-1, 0\}, \; s \in \mathbb{N} \cup \{0\}$, and c is arbitrary constant.
Now, using (1.6.1),

$${}_xI^{\nu}_a f(x) = I^{\nu}_{a+} f(x) = \frac{1}{\Gamma(\nu)} \int_a^x \frac{f(t)}{(x-t)^{1-\nu}} \mathrm{d}t, \qquad x > a.$$

The fractional integral operator of order ν, gives

$$I^{\nu}_{0+} f(t) \;=\; \frac{1}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu-1} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s \, (c\xi)^n}{(n!)^2 \, [(\lambda)_{\mu n}]^r} \, \mathrm{d}\xi$$

$$= \frac{1}{\Gamma(\nu)} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s \, c^n}{(n!)^2 \, [(\lambda)_{\mu n}]^r} \int_0^t \xi^n (t-\xi)^{\nu-1} \, \mathrm{d}\xi.$$

Taking $\xi = tu$, this gives

$$
\begin{aligned}
I^{\nu}_{0+}f(t) &= \frac{1}{\Gamma(\nu)}\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s\ t^{\nu+n}\ c^n}{(n!)^2\ [(\lambda)_{\mu n}]^r}\int_0^1 u^n\ (1-u)^{\nu-1}\mathrm{d}u \\
&= \frac{1}{\Gamma(\nu)}\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s\ t^{\nu+n}\ c^n}{(n!)^2\ [(\lambda)_{\mu n}]^r}\ \mathfrak{B}(n+1,\nu) \\
&= \frac{1}{\Gamma(\nu)}\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s\ t^{\nu+n}\ c^n}{(n!)^2\ [(\lambda)_{\mu n}]^r}\ \frac{\Gamma(n+1)\ \Gamma(\nu)}{\Gamma(n+\nu+1)} \\
&= t^{\nu}\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s\ (ct)^n}{\Gamma(n+\nu+1)\ n!\ [(\lambda)_{\mu n}]^r} \\
&= t^{\nu}\ E^{\gamma,\delta}_{1,\nu+1,\lambda,\mu}(ct;s,r).
\end{aligned}
$$

Put

$$
t^{\nu}\ E^{\gamma,\delta}_{1,\nu+1,\lambda,\mu}(ct;s,r) = E_t(c,\nu,\gamma,\delta,\lambda,\mu). \tag{4.2.27}
$$

Now using (1.6.4), the fractional differential operator of order η is given as

$$
\begin{aligned}
D^{\eta}_{0+}f(t) &= D^k\left[I^{k-\eta}\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s\ (ct)^n}{(n!)^2\ [(\lambda)_{\mu n}]^r}\right] \\
&= D^k\left[t^{k-\eta}\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s\ (ct)^n}{\Gamma(n+k-\eta+1)\ n!\ [(\lambda)_{\mu n}]^r}\right] \\
&= \sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s\ c^n}{\Gamma(n+k-\eta+1)\ n!\ [(\lambda)_{\mu n}]^r}D^k(t^{k-\eta+n}).
\end{aligned}
$$

After some simplification and using (2.1.5), it yields

$$
\begin{aligned}
D^{\eta}_{0+}f(t) &= t^{-\eta}\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s\ (ct)^n}{\Gamma(n+(1-\eta))\ [(\lambda)_{\mu n}]^r\ n!}. \\
&= t^{-\eta}\ E^{\gamma,\delta}_{1,1-\eta,\lambda,\mu}(ct;s,r).
\end{aligned} \tag{4.2.28}
$$

Put

$$
E_t(c,-\eta,\gamma,\delta,\lambda,\mu) = t^{-\eta}\ E^{\gamma,\delta}_{1,1-\eta,\lambda,\mu}(ct;s,r). \tag{4.2.29}
$$

In the following theorems, the act of fractional integral and differential operators are taken in to account.

Theorem 4.2.14. *If* $\gamma, \lambda, \nu \in \mathbb{C}, \Re(\gamma, \lambda, \nu) > 0, \delta, \mu > 0,\ r \in \mathbb{N} \cup \{-1, 0\},\ s \in \mathbb{N} \cup \{0\},$ *c is arbitrary constant and fractional integral and differential operators are of order* σ *then*

$$I_{0+}^{\sigma}\ E_t(c, \nu, \gamma, \delta, \lambda, \mu) = E_t(c, \sigma + \nu, \gamma, \delta, \lambda, \mu). \tag{4.2.30}$$

$$D_{0+}^{\sigma} E_t(c, \nu, \gamma, \delta, \lambda, \mu) = E_t(c, \nu - \sigma, \gamma, \delta, \lambda, \mu). \tag{4.2.31}$$

Proof. From (1.6.1),

$$I_{0+}^{\sigma}\ E_t(c, \nu, \gamma, \delta, \lambda, \mu) = \frac{1}{\Gamma(\sigma)} \int_0^t (t - \xi)^{\sigma - 1}\ E_\xi(c, \nu, \gamma, \delta, \lambda, \mu)\ d\xi.$$

Using (4.2.27) this gives

$$\begin{aligned} I_{0+}^{\sigma}\ E_\xi(c, \nu, \gamma, \delta, \lambda, \mu) &= \frac{1}{\Gamma(\sigma)} \int_0^t (t - \xi)^{\sigma - 1}\ \xi^{\nu} \\ &\quad \times \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\ (c\xi)^n}{\Gamma(n + \nu + 1)\ n!\ [(\lambda)_{\mu n}]^r}\ d\xi. \end{aligned}$$

Here substituting $\xi = xt$, then after some simplification, one gets

$$\begin{aligned} I_{0+}^{\sigma}\ E_t(c, \nu, \gamma, \delta, \lambda, \mu) &= \frac{1}{\Gamma(\sigma)} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\ c^n\ t^{\sigma+\nu+n}}{\Gamma(n + \nu + 1)\ n!\ [(\lambda)_{\mu n}]^r} \int_0^1 x^{\sigma - 1}\ x^{\nu + n}\ dx \\ &= \frac{1}{\Gamma(\sigma)} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\ c^n\ t^{\sigma+\nu+n}}{\Gamma(n + \nu + 1)\ n!\ [(\lambda)_{\mu n}]^r}\ \mathfrak{B}(\sigma, \nu + n + 1) \\ &= \frac{1}{\Gamma(\sigma)} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\ c^n\ t^{\sigma+\nu+n}}{\Gamma(n + \nu + 1)\ n!\ [(\lambda)_{\mu n}]^r}\ \frac{\Gamma(\sigma)\ \Gamma(\nu + n + 1)}{\Gamma(\sigma + \nu + n + 1)}. \end{aligned}$$

Once again use of (4.2.27) gives (4.2.30).
Now, in view of (1.6.4) and using (4.2.30),

$$\begin{aligned} D_{0+}^{\sigma}\ \ E_t(c, \nu, \gamma, \delta, \lambda, \mu) &= D^k \Big[I^{k-\sigma}\ E_t(c, \nu, \gamma, \delta, \lambda, \mu) \Big] \\ &= D^k \Big[E_t(c, k - \sigma + \nu, \gamma, \delta, \lambda, \mu) \Big] \\ &= D^k\ \{ t^{k-\sigma+\nu}\ E_{1, k-\sigma+\nu+1, \lambda, \mu}^{\gamma, \delta}(ct; s, r) \} \\ &= D^k \left[t^{k-\sigma+\nu} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\ (ct)^n}{\Gamma(n + k - \sigma + \nu + 1)\ n!\ [(\lambda)_{\mu n}]^r} \right] \end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s \; c^n}{\Gamma(n+k-\sigma+\nu+1)\; n! \; [(\lambda)_{\mu n}]^r} D^k(t^{k-\sigma+\nu+n}).$$

On simplifying this and using (2.1.5), one arrives at

$$\begin{aligned} D^{\sigma}_{0+}\, E_t(c,\nu,\gamma,\delta,\lambda,\mu) &= t^{\nu-\sigma} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s \; (ct)^n}{\Gamma(n+(1+\nu-\sigma))\; [(\lambda)_{\mu n}]^r \; n!}. \\ &= t^{\nu-\sigma}\; E^{\gamma,\delta}_{1,1+\nu-\sigma,\lambda,\mu}(ct;s,r). \end{aligned}$$

Finally using (4.2.27), (4.2.31) occurs. □

Using this Theorem 4.2.14, one may prove

Theorem 4.2.15. *If $\gamma,\lambda,\eta \in \mathbb{C}, \Re(\gamma,\lambda,\eta) > 0, \delta,\mu > 0,\;\; s,r \in \mathbb{Z},\;\; c$ is arbitrary constant and fractional integral as well as differential operators are of order σ then*

$$I^{\sigma}_{0+}\, E_t(c,-\eta,\gamma,\delta,\lambda,\mu) \;=\; E_t(c,\sigma-\eta,\gamma,\delta,\lambda,\mu), \tag{4.2.32}$$

$$D^{\sigma}_{0+}\, E_t(c,-\eta,\gamma,\delta,\lambda,\mu) \;=\; E_t(c,-\sigma-\eta,\gamma,\delta,\lambda,\mu). \tag{4.2.33}$$

Theorem 4.2.16. *If $\gamma,\lambda,\nu,\eta \in \mathbb{C}, \Re(\gamma,\lambda,\nu,\eta) > 0, \delta,\mu > 0,\;\; r \in \mathbb{N}\cup\{-1,0\},\;\; s \in \mathbb{N}\cup\{0\}$, and c is arbitrary constant then*

$$\frac{1}{\Gamma(\lambda)}\; \mathcal{L}\left(E_t(c,\nu,\gamma,\delta,\lambda,\mu)\right) \;=\; \frac{1}{S^{\nu+1}} E^{\gamma,\delta}_{\mu,\lambda,\lambda,\mu}(ct/S;s,r-1) \tag{4.2.34}$$

$$\frac{1}{\Gamma(\lambda)}\; \mathcal{L}\left(E_t(c,-\eta,\gamma,\delta,\lambda,\mu)\right) \;=\; \frac{1}{S^{-\eta+1}} E^{\gamma,\delta}_{\mu,\lambda,\lambda,\mu}(ct/S;s,r-1) \tag{4.2.35}$$

Proof. Here

$$\begin{aligned} l.h.s. &= \frac{1}{\Gamma(\lambda)}\; \mathcal{L}\left(E_t(c,\nu,\gamma,\delta,\lambda,\mu)\right) \\ &= \frac{1}{\Gamma(\lambda)}\; \mathcal{L}\left(t^{\nu}\; E^{\gamma,\delta}_{1,\nu+1,\lambda,\mu}(ct;s,r)\right) \\ &= \frac{1}{\Gamma(\lambda)}\; \mathcal{L}\left(t^{\nu} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s \; (ct)^n}{\Gamma(n+\nu+1)\; n! \; [(\lambda)_{\mu n}]^r}\right) \\ &= \frac{1}{\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s \; c^n}{\Gamma(n+\nu+1)\; n! \; [(\lambda)_{\mu n}]^r} \mathcal{L}\left(t^{n+\nu}\right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\ c^n}{\Gamma(n+\nu+1)\ n!\ [(\lambda)_{\mu n}]^r} \frac{\Gamma(\nu+n+1)}{S^{n+\nu+1}} \\
&= \frac{1}{S^{\nu+1}} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\ c^n}{\Gamma(\mu n+\lambda)\ [(\lambda)_{\mu n}]^{r-1}\ S^n\ n!} \\
&= \frac{1}{S^{\nu+1}} E^{\gamma,\delta}_{\mu,\lambda,\lambda,\mu}(ct/S; s, r-1) \\
&= r.h.s.\ of\ (4.2.34).
\end{aligned}$$

Similarly,

$$\begin{aligned}
l.h.s. &= \frac{1}{\Gamma(\lambda)}\, \mathcal{L}\left(E_t(c,-\eta,\gamma,\delta,\lambda,\mu)\right) \\
&= \frac{1}{\Gamma(\lambda)}\, \mathcal{L}\left(t^{-\eta}\ E^{\gamma,\delta}_{1,-\eta+1,\lambda,\mu}(ct; s, r)\right) \\
&= \frac{1}{\Gamma(\lambda)}\, \mathcal{L}\left(t^{-\eta} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\ (ct)^n}{\Gamma(n-\eta+1)\ n!\ [(\lambda)_{\mu n}]^r}\right) \\
&= \frac{1}{\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\ c^n}{\Gamma(n-\eta+1)\ n!\ [(\lambda)_{\mu n}]^r} \mathcal{L}\left(t^{n-\eta}\right) \\
&= \frac{1}{\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\ c^n}{\Gamma(n-\eta+1)\ n!\ [(\lambda)_{\mu n}]^r} \frac{\Gamma(-\eta+n+1)}{S^{n-\eta+1}} \\
&= \frac{1}{\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\ c^n}{n!\ [(\lambda)_{\mu n}]^r\ S^{n+\nu+1}} \\
&= \frac{1}{S^{-\eta+1}} E^{\gamma,\delta}_{\mu,\lambda,\lambda,\mu}(ct/S; s, r-1).
\end{aligned}$$

Hence (4.2.35) holds. □

(A piece of the content of this work has been published in "Ukrainian Mathematical Journal, Vol. 66, No. 8, January, 2015, 1267-1280")
MR ♯ 3334434