Chapter 6

Inequalities involving generalized Konhauser polynomial

6.1 Introduction

In this chapter, a generalized structure of the well known Konhauser polynomial

$$Z_m^{\sigma}(x;k) = \frac{\Gamma(km+\sigma+1)}{\Gamma(m+1)} \sum_{n=0}^{m} (-1)^n \begin{pmatrix} m \\ n \end{pmatrix} \frac{x^{kn}}{\Gamma(kn+\sigma+1)}, \ \Re(\sigma) > -1, \tag{6.1.1}$$

suggested by the function ((2.1.5) of Chapter-2):

$$E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(z;s,r) = \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s \ z^n}{\Gamma(\alpha n+\beta) \ [(\lambda)_{\mu n}]^r \ n!} \tag{6.1.2}$$

is studied. In particular, the differential equation and inverse inequality relation are obtained. Moreover, several other properties involving inequalities are also derived which yield as the particular cases, the inverse series relation, the generating function relations and finite summation formulas. An integral representation is also derived for this polynomial. The fractional integral operator, fractional differential operator, Laplace transform and Euler(Beta) transform are also applied on this polynomial.
In fact, by taking $\gamma = -m$, a negative integer, replacing β by $\sigma + 1$ and z by real variable x^k, $k \in \mathbb{N}$ in (6.1.2), the generalized structure of Konhauser polynomial

is obtained which is denoted here by $B_{m^*}^{(\alpha,\sigma,\lambda,\mu)}(x^k;s,r)$. The explicit representation is as follows.mm

$$B_{m^*}^{(\alpha,\sigma,\lambda,\mu)}(x^k;s,r) = \frac{\Gamma(\alpha m+\sigma+1)}{(m!)^s}\sum_{n=0}^{m^*}\frac{[(-m)_{\delta n}]^s\ x^{kn}}{\Gamma(\alpha n+\sigma+1)\ [(\lambda)_{\mu n}]^r\ n!}, \tag{6.1.3}$$

where $\alpha,\sigma,\lambda>0,\ m,\delta,\mu,k,s\in\mathbb{N},\ r\in\mathbb{N}\cup\{0\}$, and $m^*=[\frac{m}{\delta}]$ denotes the integral part of $\frac{m}{\delta}$.

This generalized polynomial will be henceforth referred to as the Generalized Konhauser polynomial, briefly by GKP.

6.2 Generalized Konhauser polynomial

For $\alpha,\beta,\lambda,\sigma>0,\ m,\delta,\mu,k,s\in\mathbb{N},\ r\in\mathbb{N}\cup\{0\}$ and putting $\beta=\sigma+1,\gamma=-m$ and replacing z by x^k in (6.1.2), one gets

$$\begin{aligned}E_{\alpha,\,\sigma+1,\,\lambda,\,\mu}^{-m,\delta}(x^k;s,r) &= \sum_{n=0}^{m^*}\frac{[(-m)_{\delta n}]^s\ x^{kn}}{\Gamma(\alpha n+\sigma+1)\ [(\lambda)_{\mu n}]^r\ n!}\\ &= \frac{(m!)^s}{\Gamma(\alpha m+\sigma+1)}\frac{\Gamma(\alpha m+\sigma+1)}{(m!)^s}\\ &\quad\times\sum_{n=0}^{m^*}\frac{[(-m)_{\delta n}]^s\ x^{kn}}{\Gamma(\alpha n+\sigma+1)\ [(\lambda)_{\mu n}]^r\ n!}\\ &= \frac{(m!)^s}{\Gamma(\alpha m+\sigma+1)}B_{m^*}^{(\alpha,\sigma,\lambda,\mu)}(x^k;s,r).\end{aligned}$$

If $\alpha=k\in\mathbb{N},s=1,r=0$ then this further reduces to another generalization of Konhauser polynomial due to Ajudia, Prajapati, Agarwal [58]:

$$E_{k,\,\sigma+1,\,\lambda,\,\mu}^{-m,\delta}(x^k;1,0) = \frac{\Gamma(m+1)}{\Gamma(km+\sigma+1)}L_{m^*}^{(k,\sigma)}(x^k), \tag{6.2.1}$$

where

$$L_{m^*}^{(k,\sigma)}(x^k)=Z_{m^*}^{\sigma}(x;k)=\frac{\Gamma(km+\sigma+1)}{\Gamma(m+1)}\sum_{n=0}^{m^*}\frac{(-m)_{\delta n}}{\Gamma(kn+\sigma+1)}\frac{x^{kn}}{n!}. \tag{6.2.2}$$

The reducibility of (6.1.3) to (6.2.2) is given by

$$B_{m^*}^{(k,\sigma,\lambda,\mu)}(x^k;1,0)=Z_{m^*}^{\sigma}(x;k). \tag{6.2.3}$$

The classical Konhauser polynomial given in (6.1.1) is

$$Z_m^{\sigma}(x;k) = B_m^{(k,\sigma,\lambda,\mu)}(x^k;1,0) = \frac{\Gamma(km+\sigma+1)}{\Gamma(m+1)}\; E_{k,\,\sigma+1,\,\lambda,\,\mu}^{-m,1}(x^k;1,0). \quad (6.2.4)$$

Evidently,

$$\begin{aligned} B_m^{(1,\sigma,\lambda,\mu)}(x;1,0) &= \frac{\Gamma(m+\sigma+1)}{\Gamma(m+1)}\; E_{1,\,\sigma+1,\,\lambda,\,\mu}^{-m,1}(x;1,0) \\ &= L_m^{(\sigma)}(x) \qquad (6.2.5) \end{aligned}$$

is (generalized) Laguerre polynomial (**??**).

6.3 Differential equation

For the polynomial (6.1.3), the differential equation is obtained as follows.

Theorem 6.3.1. *If* $\alpha,\beta,\lambda,m,\delta,\mu,k,s \in \mathbb{N}$, $r \in \mathbb{N}\cup\{0\}$ *and the operator* Θ *is defined by* $\Theta f(x) = x\dfrac{d}{dx}f(x)$ *then* $U = B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r)$ *satisfies the equation*

$$\begin{aligned} &\left[\left\{\prod_{j=0}^{\alpha-1}\left(\frac{1}{k}\Theta + \frac{\beta+j}{\alpha} - 1\right)\right\}\left\{\prod_{i=0}^{\mu-1}\left(\frac{1}{k}\Theta + \frac{\lambda+i}{\mu} - 1\right)^r\right\}\Theta \right. \\ &\left. -\frac{\delta^{s\delta}\; k}{\alpha^{\alpha}\;\mu^{r\mu}}\; x^k \left\{\prod_{l=0}^{\delta-1}\left(\frac{1}{k}\Theta + \frac{-m+l}{\delta}\right)^s\right\}\right] U = 0. \qquad (6.3.1) \end{aligned}$$

Proof. The first task is to convert the coefficient of x^{kn} in (6.1.3) in factorial function notations with the aid of the formula [63, Eq.2, p.22]:

$$(\alpha)_{kn} = k^{nk}\left(\frac{\alpha}{k}\right)_n \left(\frac{\alpha+1}{k}\right)_n \cdots \left(\frac{\alpha+k-1}{k}\right)_n = k^{nk}\prod_{s=0}^{k-1}\left(\frac{\alpha+s}{k}\right)_n, \; k\in\mathbb{N}.$$

Then

$$\begin{aligned} U &= \frac{\Gamma(\alpha m+\beta+1)}{(m!)^s}\sum_{n=0}^{m^*}\frac{[(-m)_{\delta n}]^s\; x^{kn}}{\Gamma(\alpha n+\beta+1)\;[(\lambda)_{\mu n}]^r\; n!} \\ &= \frac{\Gamma(\alpha m+\beta+1)}{(m!)^s\;\Gamma(\beta+1)}\sum_{n=0}^{m^*}\frac{[(-m)_{\delta n}]^s\; x^{kn}}{(\beta+1)_{\alpha n}\;[(\lambda)_{\mu n}]^r\; n!} \\ &= \frac{\Gamma(\alpha m+\beta+1)}{(m!)^s\;\Gamma(\beta+1)}\sum_{n=0}^{m^*}\frac{\delta^{s\delta n}\left[\left(\frac{-m}{\delta}\right)_n\right]^s\;\left[\left(\frac{-m+1}{\delta}\right)_n\right]^s\cdots\left[\left(\frac{-m+\delta-1}{\delta}\right)_n\right]^s\; x^{kn}}{\alpha^{\alpha n}\left(\frac{\beta+1}{\alpha}\right)_n\;\left(\frac{\beta+2}{\alpha}\right)_n\cdots\left(\frac{\beta+\alpha}{\alpha}\right)_n} \end{aligned}$$

$$\times \frac{1}{\mu^{r\mu n}\left[\left(\frac{\lambda}{\mu}\right)_n\right]^r \left[\left(\frac{\lambda+1}{\mu}\right)_n\right]^r \cdots \left[\left(\frac{\lambda+\mu-1}{\mu}\right)_n\right]^r n!}$$

$$= \frac{\Gamma(\alpha m+\beta+1)}{(m!)^s\,\Gamma(\beta+1)} \sum_{n=0}^{m^*} \frac{\delta^{s\delta n}}{\alpha^{\alpha n}\,\mu^{r\mu n}} \frac{\left\{\prod_{l=0}^{\delta-1}\left[\left(\frac{-m+l}{\delta}\right)_n\right]^s\right\}}{\left\{\prod_{j=0}^{\alpha-1}\left(\frac{\beta+j+1}{\alpha}\right)_n\right\}\left\{\prod_{i=0}^{\mu-1}\left[\left(\frac{\lambda+i}{\mu}\right)_n\right]^r\right\} n!}\, x^{kn}.$$

Using the symbols:

$$\prod_{j=0}^{\alpha-1}\left(\frac{\beta+j+1}{\alpha}\right)_n = A_n,\ \prod_{i=0}^{\mu-1}\left[\left(\frac{\lambda+i}{\mu}\right)_n\right]^r = B_n, \prod_{l=0}^{\delta-1}\left[\left(\frac{-m+l}{\delta}\right)_n\right]^s = C_n,$$

$$\prod_{j=0}^{\alpha-1}\left(\frac{1}{k}\Theta+\frac{\beta+j+1}{\alpha}-1\right) = \Phi_1, \prod_{i=0}^{\mu-1}\left(\frac{1}{k}\Theta+\frac{\lambda+i}{\mu}-1\right)^r = \Phi_2,$$

$$\prod_{l=0}^{\delta-1}\left(\frac{1}{k}\Theta+\frac{-m+l}{\delta}\right)^s = \Psi,\ \frac{\delta^{s\delta}}{\alpha^{\alpha}\,\mu^{r\mu}} = p,$$

it takes the form

$$\begin{aligned}\Theta U &= \frac{\Gamma(\alpha m+\beta+1)}{(m!)^s\,\Gamma(\beta+1)} \sum_{n=0}^{m^*} p^n \frac{C_n}{A_n\,B_n\,n!}\,\Theta x^{kn}\\ &= \frac{\Gamma(\alpha m+\beta+1)}{(m!)^s\,\Gamma(\beta+1)} \sum_{n=0}^{m^*} p^n \frac{C_n}{A_n\,B_n\,n!}\,kn\,x^{kn}\\ &= \frac{\Gamma(\alpha m+\beta+1)\,k}{(m!)^s\,\Gamma(\beta+1)} \sum_{n=1}^{m^*} p^n \frac{C_n\,x^{kn}}{A_n\,B_n\,(n-1)!}.\end{aligned}$$

Further,

$$\begin{aligned}\Phi_2\Theta U &= \frac{\Gamma(\alpha m+\beta+1)\,k}{(m!)^s\,\Gamma(\beta+1)} \sum_{n=1}^{m^*} p^n \frac{C_n}{A_n\,B_n\,(n-1)!}\,\Phi_2 x^{kn}\\ &= \frac{\Gamma(\alpha m+\beta+1)\,k}{(m!)^s\,\Gamma(\beta+1)} \sum_{n=1}^{m^*} p^n \frac{C_n}{A_n\,B_n\,(n-1)!}\\ &\quad \times\left\{\prod_{j=0}^{\alpha-1}\left(n+\frac{\beta+j+1}{\alpha}-1\right)\right\} x^{kn}\\ &= \frac{\Gamma(\alpha m+\beta+1)\,k}{(m!)^s\,\Gamma(\beta+1)} \sum_{n=1}^{m^*} p^n \frac{C_n\,x^{kn}}{A_{n-1}\,B_n\,(n-1)!}.\end{aligned}$$

Finally,

$$\begin{aligned}
\Phi_1\Phi_2\Theta U &= \frac{\Gamma(\alpha m+\beta+1)\ k}{(m!)^s\ \Gamma(\beta+1)}\sum_{n=1}^{m^*} p^n \frac{C_n}{A_{n-1}\ B_n\ (n-1)!}\ \Phi_1 x^{kn} \\
&= \frac{\Gamma(\alpha m+\beta+1)\ k}{(m!)^s\ \Gamma(\beta+1)}\sum_{n=1}^{m^*} p^n \frac{C_n}{A_{n-1}\ B_n\ (n-1)!} \\
&\quad \times \left\{\prod_{i=0}^{\mu-1}\left(n+\frac{\lambda+i}{\mu}-1\right)^r\right\} x^{kn} \\
&= \frac{\Gamma(\alpha m+\beta+1)\ k}{(m!)^s\ \Gamma(\beta+1)}\sum_{n=1}^{m^*} p^n \frac{C_n\ x^{kn}}{A_{n-1}\ B_{n-1}\ (n-1)!}.
\end{aligned}$$

Thus,

$$\Phi_1\Phi_2\Theta U = \frac{\Gamma(\alpha m+\beta+1)\ k}{(m!)^s\ \Gamma(\beta+1)}\sum_{n=0}^{m^*} p^{n+1} \frac{C_{n+1}\ x^{kn+k}}{A_n\ B_n\ n!}. \tag{6.3.2}$$

On the other hand,

$$\begin{aligned}
\Psi U &= \frac{\Gamma(\alpha m+\beta+1)}{(m!)^s\ \Gamma(\beta+1)}\sum_{n=0}^{m^*} p^n \frac{C_n}{A_n\ B_n\ n!}\Psi\ x^{kn} \\
&= \frac{\Gamma(\alpha m+\beta+1)}{(m!)^s\ \Gamma(\beta+1)}\sum_{n=0}^{m^*} p^n \frac{C_n}{A_n\ B_n\ n!}\left\{\prod_{l=0}^{\delta-1}\left(n+\frac{-m+l}{\delta}\right)^s\right\} x^{kn} \\
&= \frac{\Gamma(\alpha m+\beta+1)}{(m!)^s\ \Gamma(\beta+1)}\sum_{n=0}^{m^*} p^n \frac{C_{n+1}\ x^{kn}}{A_n\ B_n\ n!},
\end{aligned}$$

hence

$$kpx^k\Psi U = \frac{\Gamma(\alpha m+\beta+1)\ k}{(m!)^s\ \Gamma(\beta+1)}\sum_{n=0}^{m^*} p^{n+1} \frac{C_{n+1}\ x^{kn+k}}{A_n\ B_n\ n!}. \tag{6.3.3}$$

The differential equation (6.3.1) now follows from (6.3.2) and (6.3.3). □

6.4 Inverse series and inequality relations

If the real valued functions $f(x,n;s)$ and $g(x,n;s)$, $s \in \mathbb{N}\setminus\{1\}$ are such that $f(x,n;s) < B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r)$ and $g(x,n;s) > B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r)$, then one finds the following inequality relations.

The presence of parameter "s" yields an *unusual* inverse series relations involving the polynomial (6.1.3). In fact for $s = 1$, the usual inverse series relation occurs whereas for other values of s, the series relations involve the inequality. This is proved in the following theorems.

Theorem 6.4.1. *Let $f(x,n;s)$ and $g(x,n;s)$ be real valued functions, $\alpha,\beta,\lambda > 0$, and $\mu, k \in \mathbb{N}$, $r \in \mathbb{N}\cup\{0\}$. If s is odd positive integer and $m, (n-$a non negative integer) are even positive integers, then*

$$f(x,n;s) < B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r) \tag{6.4.1}$$

implies

$$x^{kn} > \frac{\Gamma(\alpha n+\beta+1)\,[(\lambda)_{\mu n}]^r\, n!}{(mn!)^s}\sum_{j=0}^{mn}\frac{[(-mn)_j]^s}{\Gamma(\alpha j+\beta+1)\ j!}\ f(x,j;s); \tag{6.4.2}$$

and

$$x^{kn} < \frac{\Gamma(\alpha n+\beta+1)\,[(\lambda)_{\mu n}]^r\, n!}{(mn!)^s}\sum_{j=0}^{mn}\frac{[(-mn)_j]^s}{\Gamma(\alpha j+\beta+1)\ j!}\ g(x,j;s), \tag{6.4.3}$$

implies

$$g(x,n;s) > B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r). \tag{6.4.4}$$

Proof. Since the inequality (6.4.1) holds, putting

$$\frac{\Gamma(\alpha n+\beta+1)\ [(\lambda)_{\mu n}]^r\ n!}{[(mn)!]^s}\sum_{j=0}^{mn}\frac{[(-mn)_j]^s}{\Gamma(\alpha j+\beta+1)}\ f(x,j;s) \quad = \quad \xi_n$$

and substituting the series inequality (6.4.1) for $f(x,j;s)$, one gets

$$\begin{aligned}
\xi_n \quad &< \quad \frac{\Gamma(\alpha n+\beta+1)\ [(\lambda)_{\mu n}]^r\ n!}{[(mn)!]^s}\sum_{j=0}^{mn}\frac{[(-mn)_j]^s}{\Gamma(\alpha j+\beta+1)}\frac{\Gamma(\alpha j+\beta+1)}{(j!)^s}\\
&\quad\times\sum_{i=0}^{[\frac{j}{m}]}\frac{[(-j)_{mi}]^s\ x^{ki}}{\Gamma(\alpha i+\beta+1)\ [(\lambda)_{\mu i}]^r\ i!}\\
&= \quad \frac{\Gamma(\alpha n+\beta+1)\ [(\lambda)_{\mu n}]^r\ n!}{[(mn)!]^s}\sum_{j=0}^{mn}\frac{(-1)^{sj}\,[(mn)!]^s}{[(mn-j)!]^s}\\
&\quad\times\sum_{i=0}^{[\frac{j}{m}]}\frac{(-1)^{smi}\ x^{ki}}{[(j-mi)!]^s\,\Gamma(\alpha i+\beta+1)\ [(\lambda)_{\mu i}]^r\ i!}
\end{aligned}$$

$$= \sum_{j=0}^{mn} \sum_{i=0}^{\left[\frac{j}{m}\right]} \frac{(-1)^{sj+smi}\ \Gamma(\alpha n+\beta+1)\ [(\lambda)_{\mu n}]^r\ n!\ x^{ki}}{[(j-mi)!]^s\ \ [(mn-j)!]^s\ \ \Gamma(\alpha i+\beta+1)\ [(\lambda)_{\mu i}]^r\ i!}.$$

Here the double series relation (1.2.21),

$$\sum_{i=0}^{mn} \sum_{j=0}^{\left[\frac{i}{m}\right]} f(i,j) = \sum_{j=0}^{n} \sum_{i=0}^{mn-mj} f(i+mj,j)$$

further simplifies this to

$$\begin{aligned}
\xi_n &< \sum_{i=0}^{n} \sum_{j=0}^{mn-mi} \frac{(-1)^{sj}\ \Gamma(\alpha n+\beta+1)\ [(\lambda)_{\mu n}]^r\ n!\ x^{ki}}{(j!)^s\,[(mn-mi-j)!]^s\ \Gamma(\alpha i+\beta+1)\ [(\lambda)_{\mu i}]^r\ i!} \\
&= x^{kn} + \sum_{i=0}^{n-1} \frac{\Gamma(\alpha n+\beta+1)\ [(\lambda)_{\mu n}]^r n!\ x^{ki}}{[(mn-mi)!]^s\ \Gamma(\alpha i+\beta+1)\ [(\lambda)_{\mu i}]^r\ i!} \\
&\quad \times \sum_{j=0}^{mn-mi} (-1)^{sj} \binom{mn-mi}{j}^s \\
&\le x^{kn} + \sum_{i=0}^{n-1} \frac{\Gamma(\alpha n+\beta+1)\ [(\lambda)_{\mu n}]^r\ n!\ x^{ki}}{[(mn-mi)!]^s\ \Gamma(\alpha i+\beta+1)\ [(\lambda)_{\mu i}]^r\ i!} \\
&\quad \times \left(\sum_{j=0}^{mn-mi} (-1)^{j} \binom{mn-mi}{j}\right)^s .
\end{aligned}$$

Since the inner most series on the right hand side vanishes, the inequality (6.4.2) follows.

The other series inequality relation occurs as follows.

Here (6.4.3) holds true. Taking

$$\frac{\Gamma(\alpha n+\beta+1)}{(n!)^s} \sum_{j=0}^{\left[\frac{n}{m}\right]} \frac{[(-n)_{mj}]^s\ x^{kj}}{\Gamma(\alpha j+\beta+1)\ [(\lambda)_{\mu j}]^r\ j!} \quad = \quad \psi_n$$

and then substituting the series inequality (6.4.3) for x^{kj} , one gets

$$\begin{aligned}
\psi_n &< \frac{\Gamma(\alpha n+\beta+1)}{(n!)^s} \sum_{j=0}^{\left[\frac{n}{m}\right]} \frac{[(-n)_{mj}]^s}{\Gamma(\alpha j+\beta+1)\ [(\lambda)_{\mu j}]^r\ j!} \\
&\quad \times \frac{\Gamma(\alpha j+\beta+1)\ [(\lambda)_{\mu j}]^r\ j!}{[(mj)!]^s} \sum_{i=0}^{mj} \frac{[(-mj)_i]^s}{\Gamma(\alpha i+\beta+1)}\ g(x,i;s)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\Gamma(\alpha n+\beta+1)}{(n!)^s} \sum_{j=0}^{\left[\frac{n}{m}\right]} \frac{(-1)^{smj}\ (n!)^s}{[(n-mj)!]^s\,[(mj)!]^s} \\
&\quad \times \sum_{i=0}^{mj} \frac{(-1)^{is}\ [(mj)!]^s}{[(mj-i)!]^s\ \Gamma(\alpha i+\beta+1)} g(x,i;s) \\
&= \sum_{mj=0}^{n} \sum_{i=0}^{mj} \frac{(-1)^{smj+is}\ \Gamma(\alpha n+\beta+1)}{[(n-mj)!]^s\ \ [(mj-i)!]^s\ \ \Gamma(\alpha i+\beta+1)}\ g(x,i;s).
\end{aligned}
$$

In view of double series relation (1.2.22):

$$
\sum_{k=0}^{n}\sum_{j=0}^{k} f(k,j) = \sum_{j=0}^{n}\sum_{k=j}^{n} f(k,j),
$$

this takes the form:

$$
\begin{aligned}
\psi_n &< \sum_{i=0}^{n} \sum_{mj=i}^{n} \frac{(-1)^{smj+is}\ \Gamma(\alpha n+\beta+1)}{[(n-mj)!]^s\ \ [(mj-i)!]^s\ \ \Gamma(\alpha i+\beta+1)}\ g(x,i;s) \\
&= g(x,n;s) + \sum_{i=0}^{n-1} \frac{(-1)^{is}\ \Gamma(\alpha n+\beta+1)}{\Gamma(\alpha i+\beta+1)}\ g(x,i;s) \\
&\quad \times \sum_{mj=i}^{n} \frac{(-1)^{smj}}{[(n-mj)!]^s\ \ [(mj-i)!]^s} \\
&= g(x,n;s) + \sum_{i=0}^{n-1} \frac{\Gamma(\alpha n+\beta+1)}{\Gamma(\alpha i+\beta+1)}\ g(x,i;s) \\
&\quad \times \sum_{mj=0}^{n-i} \frac{(-1)^{smj}}{[(n-i-mj)!]^s\ \ [(mj)!]^s} \\
&= g(x,n;s) + \sum_{i=0}^{n-1} \frac{\Gamma(\alpha n+\beta+1)}{\Gamma(\alpha i+\beta+1)\,[(n-i)!]^s}\ g(x,i;s) \\
&\quad \times \sum_{mj=0}^{n-i} (-1)^{smj} \binom{n-i}{mj}^s \\
&\leq g(x,n;s) + \sum_{i=0}^{n-1} \frac{\Gamma(\alpha n+\beta+1)}{\Gamma(\alpha i+\beta+1)\,[(n-i)!]^s}\ g(x,i;s) \\
&\quad \times \left(\sum_{mj=0}^{n-i} (-1)^{mj} \binom{n-i}{mj}\right)^s.
\end{aligned}
$$

Once again the inner most series on the right hand side vanishes, hence

$$
\psi_n < g(x,n;s).
$$

□

Towards the converse of these inequality relations, one can obtain the following theorem.

Theorem 6.4.2. *Let $f(x,n;s)$ and $g(x,n;s)$ be real valued functions, $\alpha,\beta,\lambda>0$, and $\mu,k\in\mathbb{N}$, $\;r\in\mathbb{N}\cup\{0\}$. If either s is an even positive integer or $s,m,(n-$a non negative integer) are all odd positive integers, then*

$$x^{kn} > \frac{\Gamma(\alpha n+\beta+1)\,[(\lambda)_{\mu n}]^r\, n!}{(mn!)^s}\sum_{j=0}^{mn}\frac{[(-mn)_j]^s}{\Gamma(\alpha j+\beta+1)\ j!}\ f(x,j;s) \tag{6.4.5}$$

implies

$$f(x,n;s) < B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r); \tag{6.4.6}$$

and

$$g(x,n;s) > B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r) \tag{6.4.7}$$

implies

$$x^{kn} < \frac{\Gamma(\alpha n+\beta+1)\,[(\lambda)_{\mu n}]^r\, n!}{(mn!)^s}\sum_{j=0}^{mn}\frac{[(-mn)_j]^s}{\Gamma(\alpha j+\beta+1)\ j!}\ g(x,j;s). \tag{6.4.8}$$

The proof runs parallel to that of Theorem 6.4.1, hence is omitted.
Now, for $s=1$, one obtains the inverse series relations for the polynomial (6.1.3) which is stated as

Theorem 6.4.3. *For $\alpha,\beta,\lambda>0, m,\mu,k\in\mathbb{N},\ r\in\mathbb{N}\cup\{0\}$,*

$$B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r) \;=\; \frac{\Gamma(\alpha n+\beta+1)}{n!}\sum_{j=0}^{[n/m]}\frac{(-n)_{mj}\ x^{kj}}{\Gamma(\alpha j+\beta+1)\,[(\lambda)_{\mu j}]^r\, j!} \tag{6.4.9}$$

if and only if

$$\frac{x^{kn}}{n!} \;=\; \frac{\Gamma(\alpha n+\beta+1)\,[(\lambda)_{\mu n}]^r}{(mn)!}\sum_{j=0}^{mn}\frac{(-mn)_j}{\Gamma(\alpha j+\beta+1)}B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r), \tag{6.4.10}$$

and for $n \neq ml,\ l \in \mathbb{N}$,

$$\sum_{j=0}^{n} \frac{(-n)_j}{\Gamma(\alpha j+\beta+1)} B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r) = 0. \tag{6.4.11}$$

Proof. The proof of (6.4.9) implies (6.4.10) runs as follows.
Denoting the right hand side of (6.4.10) by Ω_n, and then substituting for $B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r)$ from (6.4.9), one gets

$$\begin{aligned}
\Omega_n &= \frac{\Gamma(\alpha n+\beta+1)\,[(\lambda)_{\mu n}]^r}{(mn)!} \sum_{j=0}^{mn} \frac{(-mn)_j}{\Gamma(\alpha j+\beta+1)}\, B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r) \\
&= \frac{\Gamma(\alpha n+\beta+1)\,[(\lambda)_{\mu n}]^r}{(mn)!} \sum_{j=0}^{mn} (-mn)_j \sum_{i=0}^{[j/m]} \frac{(-j)_{mi}\, x^{ki}}{\Gamma(\alpha i+\beta+1)\,[(\lambda)_{\mu i}]^r\, i!}.
\end{aligned}$$

This in view of the double series relation (1.2.21), further takes the form

$$\begin{aligned}
\Omega_n &= \sum_{j=0}^{mn} \sum_{i=0}^{[j/m]} \frac{(-1)^{j+mi}\,\Gamma(\alpha n+\beta+1)\ [(\lambda)_{\mu n}]^r\ x^{ki}}{(mn-j)!\,(j-mi)!\,\Gamma(\alpha i+\beta+1)\ [(\lambda)_{\mu i}]^r\ i!} \\
&= \sum_{i=0}^{n} \sum_{j=0}^{mn-mi} \frac{(-1)^j\ \Gamma(\alpha n+\beta+1)\ [(\lambda)_{\mu n}]^r}{(mn-mi-j)!\ j!\ \Gamma(\alpha i+\beta+1)\ [(\lambda)_{\mu i}]^r\ i!}\, x^{ki} \\
&= \frac{x^{kn}}{n!} + \sum_{i=0}^{n-1} \frac{\Gamma(\alpha n+\beta+1)\ [(\lambda)_{\mu n}]^r\ x^{ki}}{\Gamma(\alpha i+\beta+1)\ [(\lambda)_{\mu i}]^r\ (mn-mi)!\ i!} \sum_{j=0}^{mn-mi} (-1)^j \binom{mn-mi}{j}.
\end{aligned}$$

Here the inner sum in the second term on the right hand side vanishes, consequently, one may arrive at $\Omega_n = \frac{x^{kn}}{n!}$.
To show further that (6.4.9) also implies (6.4.11), one may substitute for $B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r)$ from (6.4.9) to the left hand of (6.4.11), to get

$$\begin{aligned}
&\sum_{j=0}^{n} \frac{(-n)_j}{\Gamma(\alpha j+\beta+1)}\ B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r) \\
&= \sum_{j=0}^{n} \frac{(-1)^j\ n!}{(n-j)!} \sum_{i=0}^{[j/mi]} \frac{(-1)^{mi}\ x^{ki}}{\Gamma(\alpha i+\beta+1)\ ((\lambda)_{\mu i})^r\ (j-mi)!\ i!} \\
&= \sum_{i=0}^{[n/m]} \frac{n!\ x^{ki}}{\Gamma(\alpha i+\beta+1)\ ((\lambda)_{\mu i})^r\ (n-mi)!\ i!} \sum_{j=0}^{n-mi} (-1)^j \binom{n-mi}{j} \\
&= 0
\end{aligned}$$

if $n \neq ml,\ l \in \mathbb{N}$. Thus completing the first part. The proof of converse part runs as follows [13]. In order to show that both the series (6.4.10) and the condition (6.4.11) together imply the series (6.4.9), the following simplest inverse series relations [64, Eq.(1), p.43] will be used.

$$\omega_n = \sum_{j=0}^{n} \frac{(-n)_j}{j!}\ \rho_j \ \Leftrightarrow\ \rho_n = \sum_{j=0}^{n} \frac{(-n)_j}{j!}\ \omega_j.$$

Here putting

$$\rho_j = \frac{j!}{\Gamma(\alpha j+\beta+1)}\ B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r),$$

and considering one sided relation that is, the series on the left hand side implies the series on the right side, one gets

$$\omega_n = \sum_{j=0}^{n} \frac{(-n)_j}{\Gamma(\alpha j+\beta+1)}\ B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r) \tag{6.4.12}$$

$$\Rightarrow$$

$$B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r) = \frac{\Gamma(\alpha n+\beta+1)}{n!}\ \sum_{j=0}^{n} \frac{(-n)_j}{j!}\ \omega_j. \tag{6.4.13}$$

Since the condition (6.4.11) holds, $\omega_n = 0$ for $n \neq ml,\ l \in \mathbb{N}$, whereas

$$\omega_{mn} = \sum_{j=0}^{mn} \frac{(-mn)_j}{\Gamma(\alpha j+\beta+1)}\ B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r).$$

But since the series (6.4.10) holds true,

$$\omega_{mn} = \frac{(mn)!\ x^{kn}}{n!\ \Gamma(\alpha n+\beta+1)\ ((\lambda)_{\mu n})^r}.$$

Consequently, the inverse pair (6.4.12) and (6.4.13) assume the form:

$$\begin{aligned} \frac{x^{kn}}{n!} &= \frac{\Gamma(\alpha n+\beta+1)((\lambda)_{\mu n})^r}{(mn)!}\ \sum_{j=0}^{mn} \frac{(-mn)_j}{\Gamma(\alpha j+\beta+1)} \\ &\quad \times B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r) \\ \Rightarrow & \\ B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r) &= \frac{\Gamma(\alpha n+\beta+1)}{n!}\ \sum_{j=0}^{[n/m]} \frac{(-n)_{mj}}{(mj)!}\ \omega_{mj} \end{aligned}$$

$$= \frac{\Gamma(\alpha n+\beta+1)}{n!}\sum_{j=0}^{[n/m]}\frac{(-n)_{mj}\ x^{kj}}{\Gamma(\alpha j+\beta+1)\ ((\lambda)_{\mu j})^r\ j!},$$

subject to the condition (6.4.11). □

6.5 Some inequalities

In this section certain inequalities connecting GKP and GHF are deduced.

Theorem 6.5.1. *If* $\alpha,\beta,\lambda,\sigma,x>0,\ \delta,\mu,m,k,s\in\mathbb{N},\ r\in\mathbb{N}\cup\{0\},\ 0<t<1,$ *and* $p=\frac{\delta^{s\delta}}{\alpha^{\alpha}\ \mu^{r\mu}}$ *as before, then the following series inequalities hold.*

$$\sum_{m=0}^{\infty}\frac{B_{m^*}^{(\alpha,\beta,\lambda,m)}(x^k;s,r)}{(\beta+1)_{\alpha m}}\ t^{ms} \le e^{ts}\ {}_0F_{\alpha+r\mu}\left[\begin{matrix} -; & \\ \Delta(\alpha;\beta+1),\quad \Delta(\mu;\lambda)^r, & \end{matrix}\ \frac{x^k}{\alpha^{\alpha}\ \mu^{\mu}}\ (-t)^{s\delta}\right], \tag{6.5.1}$$

$$\sum_{m=0}^{\infty}\frac{((\sigma)_m)^s}{(\beta+1)_{\alpha m}}B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r)\ t^{ms} \le (1-t)^{-s\sigma}\ {}_{s\delta}F_{\alpha+r\mu}\left[\begin{matrix} \Delta(\delta;\sigma)^s; & \\ \Delta(\alpha;\beta+1),\quad \Delta(\mu;\lambda)^r; & \end{matrix}\ px^k\left(\tfrac{-t}{1-t}\right)^{s\delta}\right], \tag{6.5.2}$$

and

$$\sum_{m=0}^{\infty}\frac{((\sigma)_m)^s}{\Gamma(\alpha m+\beta+1)}B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r)\ t^{ms} \le (1-t)^{-s\sigma}E_{\alpha,\,\beta+1,\,\lambda,\,\mu}^{\sigma,\delta}\left(x^k\left(\frac{-t}{1-t}\right)^{s\delta};s,r\right). \tag{6.5.3}$$

Proof. (of (6.5.1))
Here,

$$\begin{aligned} l.h.s. &= \sum_{m=0}^{\infty}\frac{B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r)}{(\beta+1)_{\alpha m}}\ t^{ms} \\ &= \sum_{m=0}^{\infty}\frac{1}{\Gamma(\alpha m+\beta+1)}\frac{(\beta+1)_{\alpha m}}{(m!)^s}\sum_{n=0}^{m^*}\frac{[(-m)_{\delta n}]^s\ x^{kn}}{\Gamma(\alpha n+\beta+1)\ [(\lambda)_{\mu n}]^r\ n!}\ t^{ms} \end{aligned}$$

$$\begin{aligned}
&= \sum_{m=0}^{\infty} \frac{(\beta+1)_{\alpha m}}{(\beta+1)_{\alpha m}\ (m!)^s} \sum_{n=0}^{m^*} \frac{(-1)^{s\delta n}\ (m!)^s\ x^{kn}\ t^{ms}}{((m-\delta n)!)^s\ (\beta+1)_{\alpha n}\ [(\lambda)_{\mu n}]^r\ n!} \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{s\delta n}\ x^{kn}\ t^{(m+\delta n)s}}{(m!)^s\ (\beta+1)_{\alpha n}\ [(\lambda)_{\mu n}]^r\ n!} \\
&= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{\infty} \left(\frac{t^m}{m!}\right)^s \right\} \frac{(-1)^{s\delta n}\ t^{s\delta n}\ x^{kn}}{(\beta+1)_{\alpha n}\ [(\lambda)_{\mu n}]^r\ n!} \\
&\le \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} \frac{t^m}{m!} \right)^s \frac{(-1)^{s\delta n}\ t^{s\delta n}\ x^{kn}}{(\beta+1)_{\alpha n}\ [(\lambda)_{\mu n}]^r\ n!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{s\delta n}\ t^{s\delta n}\ x^{kn}}{(\beta+1)_{\alpha n}\ [(\lambda)_{\mu n}]^r\ n!}\ e^{ts} \\
&= e^{ts} \sum_{n=0}^{\infty} \left[\prod_{u=1}^{\mu} \left(\frac{\lambda+u-1}{\mu}\right)_n\right]^{-r} \left[\prod_{j=1}^{\alpha} \left(\frac{\beta+j-1}{\alpha}\right)_n\right]^{-1} \frac{(-t)^{\delta s n}\ x^{kn}}{\mu^{\mu r n}\ \alpha^{\alpha n}\ n!} \\
&= e^{ts}\ {}_0F_{\alpha+r\mu}\left[\begin{matrix} -; \\ \Delta(\alpha;\beta+1), \quad \Delta(\mu;\lambda)^r, \end{matrix} \quad \frac{x^k}{\alpha^\alpha\ \mu^{r\mu}}\ (-t)^{s\delta}\right] \\
&= r.h.s.
\end{aligned}$$

□

Proof. (of (6.5.2))
The left hand side

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{((\sigma)_m)^s}{(\beta+1)_{\alpha m}} B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r)\ t^{ms} \\
&= \sum_{m=0}^{\infty} \frac{((\sigma)_m)^s}{(\beta+1)_{\alpha m}} \frac{\Gamma(\alpha m+\beta+1)}{(m!)^s} \sum_{n=0}^{m^*} \frac{[(-m)_{\delta n}]^s\ x^{kn}\ t^{ms}}{\Gamma(\alpha n+\beta+1)[(\lambda)_{\mu n}]^r\ n!} \\
&= \sum_{m=0}^{\infty} \frac{((\sigma)_m)^s}{(\beta+1)_{\alpha m}} \frac{(\beta+1)_{\alpha m}}{(m!)^s} \sum_{n=0}^{m^*} \frac{(-1)^{s\delta n}\ (m!)^s\ x^{kn}\ t^{ms}}{((m-\delta n)!)^s\ (\beta+1)_{\alpha n}\ [(\lambda)_{\mu n}]^r\ n!} \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{s\delta n}\ ((\sigma)_{m+\delta n})^s\ x^{kn}\ t^{(m+\delta n)s}}{(m!)^s\ (\beta+1)_{\alpha n}\ [(\lambda)_{\mu n}]^r\ n!} \\
&= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{\infty} \left(\frac{(\sigma+\delta n)_m\ t^m}{m!} \right)^s \right\} \frac{(-1)^{s\delta n}\ ((\sigma)_{\delta n})^s\ t^{s\delta n}\ x^{kn}}{(\beta+1)_{\alpha n}\ [(\lambda)_{\mu n}]^r\ n!} \\
&\le \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} \frac{(\sigma+\delta n)_m\ t^m}{m!} \right)^s \frac{(-1)^{s\delta n}\ ((\sigma)_{\delta n})^s\ t^{s\delta n}\ x^{kn}}{(\beta+1)_{\alpha n}\ [(\lambda)_{\mu n}]^r\ n!} \\
&= \sum_{n=0}^{\infty} (1-t)^{-s\sigma-s\delta n} \frac{(-1)^{s\delta n}\ ((\sigma)_{\delta n})^s\ t^{s\delta n}\ x^{kn}}{(\beta+1)_{\alpha n}\ [(\lambda)_{\mu n}]^r\ n!}
\end{aligned}$$

$$= \sum_{n=0}^{\infty}(1-t)^{-s\sigma-\delta sn}\sum_{n=0}^{\infty}\frac{\left\{\prod_{i=1}^{\delta}\left(\frac{\sigma+i-1}{\delta}\right)_n\right\}^s}{\left\{\prod_{u=1}^{\mu}\left(\frac{\lambda+u-1}{\mu}\right)_n\right\}^r\left\{\prod_{j=1}^{\alpha}\left(\frac{\beta+j-1}{\alpha}\right)_n\right\}\ n!}\frac{(-\delta\ t)^{\delta sn}\ x^{kn}}{\mu^{\mu rn}\ \alpha^{\alpha n}}$$

$$= (1-t)^{-s\sigma}\ {}_{s\delta}F_{\alpha+r\mu}\left[\begin{matrix}\Delta(\delta;\sigma)^s; & & px^k\left(\frac{-t}{1-t}\right)^{s\delta}\\ \Delta(\alpha;\beta+1), & \Delta(\mu;\lambda)^r; & \end{matrix}\right].$$

□

Proof. (of (6.5.3)) Here

$$\sum_{m=0}^{\infty}\frac{((\sigma)_m)^s}{(\beta+1)_{\alpha m}}B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r)\ t^{ms}$$

$$= \sum_{m=0}^{\infty}\frac{((\sigma)_m)^s}{\Gamma(\alpha m+\beta+1)}\frac{\Gamma(\alpha m+\beta+1)}{(m!)^s}\sum_{n=0}^{m^*}\frac{[(-m)_{\delta n}]^s\ x^{kn}\ t^{ms}}{\Gamma(\alpha n+\beta+1)\ [(\lambda)_{mn}]^r\ n!}$$

$$= \sum_{m=0}^{\infty}\frac{((\sigma)_m)^s}{\Gamma(\alpha m+\beta+1)}\frac{\Gamma(\alpha m+\beta+1)}{(m!)^s}$$

$$\times\sum_{n=0}^{m^*}\frac{(-1)^{s\delta n}(m!)^s\ ((m-\delta n)!)^{-s}\ x^{kn}\ t^{ms}}{\Gamma(\alpha n+\beta+1)\ [(\lambda)_{mn}]^r\ n!}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{s\delta n}\ ((\sigma)_{m+\delta n})^s\ x^{kn}\ t^{ms+s\delta n}}{(m!)^s\ \Gamma(\alpha n+\beta+1)\ [(\lambda)_{mn}]^r\ n!}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{(\sigma+\delta n)_m\ t^{m+\delta n}}{m!}\right)^s\frac{(-1)^{s\delta n}\ ((\sigma)_{\delta n})^s\ x^{kn}}{\Gamma(\alpha n+\beta+1)\ [(\lambda)_{mn}]^r\ n!}$$

$$\leq \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\frac{(\sigma+\delta n)_m\ t^m}{m!}\right)^s\frac{(-1)^{s\delta n}\ ((\sigma)_{\delta n})^s\ x^{kn}}{\Gamma(\alpha n+\beta+1)\ [(\lambda)_{mn}]^r\ n!}$$

$$= \sum_{n=0}^{\infty}(1-t)^{-s\sigma-s\delta n}\frac{(-1)^{s\delta n}((\sigma)_{\delta n})^s\ x^{kn}\ t^{s\delta n}}{\Gamma(\alpha n+\beta+1)\ [(\lambda)_{mn}]^r\ n!}$$

$$= (1-t)^{-s\sigma}E_{\alpha,\,\beta+1,\,\lambda,\,\mu}^{\sigma,\delta}\left(x^k\left(\frac{-t}{1-t}\right)^{s\delta};s,r\right).$$

Hence (6.5.3) holds. □

Note 6.5.1. *Equality occurs in the theorem when* $s=1$.

6.5.1 Special cases-inequalities

(i) If $\alpha = k \in \mathbb{N}$ then (6.5.1) becomes

$$\sum_{m=0}^{\infty} \frac{1}{(\beta+1)_{km}} B_{m^*}^{(k,\beta,\lambda,m)}(x^k; s, r)\ t^{ms}$$
$$\leq e^{ts}\ {}_0F_{k+r\mu}\left[\begin{array}{ll} -; & \frac{x^k}{k^k\ \mu^{r\mu}}\,(-t)^{s\delta} \\ \Delta(k;\beta+1),\quad \Delta(\mu;\lambda)^r, & \end{array}\right]. \tag{6.5.4}$$

The special case $\delta = 1, r = 0$ of GKP $B_{n^*}^{(\alpha,\sigma,\lambda,\mu)}(x; s, r)$ will be denoted by $Z_{m,s}^{\beta}(x; k)$. Thus

$$Z_{m,s}^{\beta}(x;k) = \frac{\Gamma(km+\beta+1)}{\Gamma(m+1)} \sum_{n=0}^{m} \frac{[(-m)_n]^s\ x^{kn}}{\Gamma(kn+\beta+1)\ n!},\ \Re(\beta) > -1 \tag{6.5.5}$$

is another extended form of the Konhauser polynomial (6.1.1).
With these substitutions and notation (6.5.4) gets reduces to

$$\sum_{m=0}^{\infty} \frac{Z_{m,s}^{\beta}(x;k)}{(\beta+1)_{km}}\ t^m\ \leq e^{st}\ {}_0F_k\left[\begin{array}{ll} -; & \frac{x^k}{k^k}(-t)^s \\ \Delta(k;\beta+1); & \end{array}\right].$$

Finally, $k = 1$ yields

$$\sum_{m=0}^{\infty} \frac{L_{m,s}^{(\beta)}(x)}{(\beta+1)_m}\ t^m\ \leq e^{st}\ {}_0F_1\left[\begin{array}{ll} -; & x(-t)^s \\ \beta+1; & \end{array}\right]$$

involving "extended" generalized Laguerre polynomial $L_{m,s}^{(\beta)}(x)$.
(ii) If $\delta = 1, r = 0, \alpha = k \in \mathbb{N}$ then (6.5.2) reduces to

$$\sum_{m=0}^{\infty} \frac{((\sigma)_m)^s}{(\beta+1)_{km}} Z_{m,s}^{\beta}(x;k)\ t^{ms}$$
$$\leq (1-t)^{-s\sigma}\ {}_sF_k\left[\begin{array}{ll} \sigma^s; & \frac{x^k}{k^k}\left(\frac{-t}{1-t}\right)^s \\ \Delta(k;\beta+1); & \end{array}\right],\ (k \geq s). \tag{6.5.6}$$

The associated series inequality occurs for $k = 1$, in the form:

$$\sum_{m=0}^{\infty} \frac{((\sigma)_m)^s}{(\beta+1)_m} L_{m,s}^{(\beta)}(x) t^{ms} \leq (1-t)^{-s\sigma}\ {}_sF_1\left[\begin{array}{ll} \sigma^s; & x\left(\frac{-t}{1-t}\right)^s \\ \beta+1; & \end{array}\right].$$

The function ${}_sF_1[*]$ exists only for $s = 1, 2$.

6.5.2 Special cases - Generating function relations

For $s = 1$, the series inequality relations in Theorem 6.5.1 provide the generating function relations.

(i) Taking as before $\alpha = k \in \mathbb{N},\ r = 0$ in (6.5.1), one gets

$$\sum_{m=0}^{\infty} \frac{L_{m^*}^{(k,\beta)}(x^k)}{(\beta+1)_{km}}\, t^m = e^t\, {}_0F_k\left[\begin{array}{cc} -; & \frac{x^k}{k^k}(-t)^{\delta} \\ \Delta(k;\beta+1); & \end{array}\right],$$

involving the polynomial (6.2.2).

Further for $\delta = 1$, it yields the generating function relation:

$$\sum_{m=0}^{\infty} \frac{Z_m^{\beta}(x;k)}{(\beta+1)_{km}}\, t^m \ = e^t\, {}_0F_k\left[\begin{array}{cc} -; & -\frac{x^k\ t}{k^k} \\ \Delta(k;\beta+1); & \end{array}\right].$$

Finally, if $k = 1$ then it gives the known generating function relation [63, Eq. (1), p. 201] for Laguerre polynomial:

$$\sum_{m=0}^{\infty} \frac{L_m^{(\beta)}(x)}{(\beta+1)_m}\, t^m \ = e^t\, {}_0F_1\left[\begin{array}{cc} -; & -xt \\ \beta+1; & \end{array}\right].$$

(ii) If $\delta = 1, r = 0, \alpha = k \in \mathbb{N}$ then (6.5.2) provides the generating function relation:

$$\sum_{m=0}^{\infty} \frac{(\sigma)_m}{(\beta+1)_{km}} Z_m^{\beta}(x;k)\ t^m = (1-t)^{-\sigma}\ {}_1F_k\left[\begin{array}{cc} \sigma; & \frac{x^k}{k^k}\left(\frac{-t}{1-t}\right) \\ \Delta(k;\beta+1); & \end{array}\right]. \quad (6.5.7)$$

And with $k = 1$, it yields the known relation [63, Eq.(3), p.202]

$$\sum_{m=0}^{\infty} \frac{(\sigma)_m}{(\beta+1)_m} L_m^{(\beta)}(x) t^m = (1-t)^{-\sigma}\ {}_1F_1\left[\begin{array}{cc} \sigma; & \frac{-xt}{1-t} \\ \beta+1; & \end{array}\right].$$

(iii) From (6.5.3), one immediately gets

$$\sum_{m=0}^{\infty} \frac{(\sigma)_m\ t^m}{(\beta+1)_{\alpha m}} B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x;1,r) = (1-t)^{-\sigma} E_{\alpha,\,\beta+1,\,\lambda,\,\mu}^{\sigma,\delta}\left(x\left(\frac{-t}{1-t}\right)^{\delta};1,r\right). \quad (6.5.8)$$

With $r = 0, \alpha = k \in \mathbb{N}$, this would reduce to

$$\sum_{m=0}^{\infty} \frac{(\sigma)_m}{(\beta+1)_{km}} L_{m^*}^{(k,\sigma)}(x^k)\ t^m = (1-t)^{-\sigma}\ E_{k,\,\beta+1}^{\sigma,\delta}\left(x^k\left(\frac{-t}{1-t}\right)^{\delta}\right).$$

Further putting $\delta = 1$, it gives

$$\sum_{m=0}^{\infty} \frac{(\sigma)_m}{(\beta+1)_{km}} Z_m^{\sigma}(x;k)\ t^m = (1-t)^{-\sigma} E_{k,\,\beta+1}^{\sigma}\left(\frac{-x^k\ t}{1-t}\right).$$

The case $k = 1$ is the relation:

$$\sum_{m=0}^{\infty} \frac{(\sigma)_m}{(\beta+1)_m} L_m^{(\sigma)}(x)\ t^m = (1-t)^{-\sigma} E_{1,\,\beta+1}^{\sigma}\left(\frac{-x\ t}{1-t}\right).$$

6.6 Finite series inequalities

In this section, the following inequalities involving finite series and GKP are established.

Theorem 6.6.1. *If* $\beta, \lambda > 0,\ \delta, \mu, k, m, s, w \in \mathbb{N},\ r \in \mathbb{N} \cup \{0\}$, $-1 < w\left(\left(\frac{y}{k}\right)^{\frac{k}{s\delta}} - \left(\frac{x}{k}\right)^{\frac{k}{s\delta}}\right) < 0$ *then*

$$\begin{aligned} B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k;s,r) \ \leq\ & \left(\frac{x}{y}\right)^{\frac{km}{\delta}} \sum_{j=0}^{m} \begin{pmatrix} \beta+km \\ kj \end{pmatrix} \frac{(kj)!}{j!} \left(\left(\frac{y}{x}\right)^{\frac{k}{\delta}} - 1\right)^{j} \\ & \times B_{(m-j)^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r). \end{aligned} \tag{6.6.1}$$

Proof. The inequality (6.5.1):

$$\begin{aligned} &\sum_{m=0}^{\infty} \frac{1}{(\beta+1)_{km}} B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k;s,r)\ t^{ms} \\ &\leq e^{ts}\ {}_0F_{k+r\mu}\left[\begin{matrix} -; & \frac{x^k}{k^k \mu^{r\mu}}(-t\delta)^{s\delta} \\ \Delta(k;\beta+1),\quad \Delta(\mu;\lambda)^r; & \end{matrix}\right] \end{aligned}$$

with $t = \left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w$, takes the form

$$\begin{aligned} &\sum_{m=0}^{\infty} \frac{1}{(\beta+1)_{km}} B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k;s,r) \left(\left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)^{ms} \leq \exp\left(sw\left(\frac{y}{k}\right)^{\frac{k}{s\delta}}\right) \\ &\times {}_0F_{k+r\mu}\left[\begin{matrix} -; & \frac{x^k}{k^k \mu^{r\mu}}\left(\left(\frac{y}{k}\right)^{\frac{k}{s\delta}}(-w)\right)^{s\delta} \\ \Delta(k;\beta+1),\quad \Delta(\mu;\lambda)^r, & \end{matrix}\right]. \end{aligned}$$

That is,

$$\exp\left(-sw\left(\frac{y}{k}\right)^{\frac{k}{s\delta}}\right)\sum_{m=0}^{\infty}\frac{1}{(\beta+1)_{km}}B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k;s,r)\ \left(\left(\frac{y}{k}\right)^{\frac{k}{s\delta}}w\right)^{ms}$$
$$\leq\ {}_0F_{k+r\mu}\left[\begin{matrix}-; & \frac{x^k\ y^k}{k^{2k}\mu^{r\mu}}(-w)^{s\delta}\\ \Delta(k;\beta+1),\quad \Delta(\mu;\lambda)^r; & \end{matrix}\right]. \tag{6.6.2}$$

In this, interchanging x and y, gives

$$\exp\left(-sw\left(\frac{x}{k}\right)^{\frac{k}{s\delta}}\right)\sum_{m=0}^{\infty}\frac{1}{(\beta+1)_{km}}B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r)\ \left(\left(\frac{x}{k}\right)^{\frac{k}{s\delta}}w\right)^{ms}$$
$$\leq\ {}_0F_{k+r\mu}\left[\begin{matrix}-; & \frac{x^k\ y^k}{k^{2k}\mu^{r\mu}}(-w)^{s\delta}\\ \Delta(k;\beta+1),\quad \Delta(\mu;\lambda)^r; & \end{matrix}\right]. \tag{6.6.3}$$

From (6.6.2) and (6.6.3), it follows that either

$$\sum_{m=0}^{\infty}\frac{1}{(\beta+1)_{km}}B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k;s,r)\ \left(\left(\frac{y}{k}\right)^{\frac{k}{s\delta}}w\right)^{ms}\ \exp\left(-sw\left(\frac{y}{k}\right)^{\frac{k}{s\delta}}\right)$$
$$\leq\sum_{m=0}^{\infty}\frac{1}{(\beta+1)_{km}}B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r)\ \left(\left(\frac{x}{k}\right)^{\frac{k}{s\delta}}w\right)^{ms}\ \exp\left(-sw\left(\frac{x}{k}\right)^{\frac{k}{s\delta}}\right) \tag{6.6.4}$$

or

$$\sum_{m=0}^{\infty}\frac{1}{(\beta+1)_{km}}B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r)\ \left(\left(\frac{x}{k}\right)^{\frac{k}{s\delta}}w\right)^{ms}\ \exp\left(-sw\left(\frac{x}{k}\right)^{\frac{k}{s\delta}}\right)$$
$$\leq\sum_{m=0}^{\infty}\frac{1}{(\beta+1)_{km}}B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k;s,r)\ \left(\left(\frac{y}{k}\right)^{\frac{k}{s\delta}}w\right)^{ms}\ \exp\left(-sw\left(\frac{y}{k}\right)^{\frac{k}{s\delta}}\right). \tag{6.6.5}$$

Rewritting the inequality (6.6.4) by transfering the exponential function from left side and proceeding further, one obtains

$$\sum_{m=0}^{\infty}\frac{1}{(\beta+1)_{km}}B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k;s,r)\left(\left(\frac{y}{k}\right)^{\frac{k}{s\delta}}\ w\right)^{ms}$$
$$\leq\ \sum_{m=0}^{\infty}\frac{1}{(\beta+1)_{km}}B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r)\ \left(\frac{x}{k}\right)^{\frac{mk}{\delta}}\ w^{ms}$$
$$\times\exp\left(sw\left(\left(\frac{y}{k}\right)^{\frac{k}{s\delta}}-\left(\frac{x}{k}\right)^{\frac{k}{s\delta}}\right)\right)$$
$$\leq\ \sum_{m=0}^{\infty}\frac{1}{(\beta+1)_{km}}B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r)\ \left(\frac{x}{k}\right)^{\frac{mk}{\delta}}\ w^{ms}$$

$$
\begin{aligned}
&\times \exp\left(w^s\left(\left(\frac{y}{k}\right)^{\frac{k}{\delta}} - \left(\frac{x}{k}\right)^{\frac{k}{\delta}}\right)\right)\\
= &\sum_{m=0}^{\infty} \frac{1}{(\beta+1)_{km}}\, B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k; s, r)\, \left(\frac{x}{k}\right)^{\frac{mk}{\delta}}\, w^{ms}\\
&\times \sum_{j=0}^{\infty} \frac{1}{j!}\left(w^s\left(\left(\frac{y}{k}\right)^{\frac{k}{\delta}} - \left(\frac{x}{k}\right)^{\frac{k}{\delta}}\right)\right)^j\\
= &\sum_{m=0}^{\infty}\sum_{j=0}^{m} \frac{1}{(\beta+1)_{k(m-j)}\, j!}\, B_{(m-j)^*}^{(k,\beta,\lambda,\mu)}(y^k; s, r)\, \left(\frac{x}{k}\right)^{\frac{(m-j)k}{\delta}}\, w^{(m-j)s}\\
&\times \left(w^s\left(\left(\frac{y}{k}\right)^{\frac{k}{\delta}} - \left(\frac{x}{k}\right)^{\frac{k}{\delta}}\right)\right)^j\\
= &\sum_{m=0}^{\infty}\sum_{j=0}^{m} \frac{1}{(\beta+1)_{k(m-j)}\, j!}\, B_{(m-j)^*}^{(k,\beta,\lambda,\mu)}(y^k; s, r)\\
&\times \left(\frac{x}{k}\right)^{\frac{mk}{\delta}}\, \left(\left(\frac{y}{x}\right)^{\frac{k}{\delta}} - 1\right)^j\, w^{ms}.
\end{aligned}
$$

Now comparing the coefficients of w^{ms}, one arrives at (6.6.1). The inequality in (6.6.1) with y and x interchanged, may be proved by using (6.6.5). □

The equality occurs in the theorem when $s = 1$. This is recorded in the following.

6.6.1 Special cases

(i) From (6.6.1) , the finite summation formula occurs for $s = 1$:

$$
\begin{aligned}
B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k; 1, r) \; = \; &\left(\frac{x}{y}\right)^{\frac{km}{\delta}} \sum_{j=0}^{m} \begin{pmatrix} \beta + km \\ kj \end{pmatrix} \frac{(kj)!}{j!}\\
&\times \left(\left(\frac{y}{x}\right)^{\frac{k}{\delta}} - 1\right)^j B_{(m-j)^*}^{(k,\beta,\lambda,\mu)}(y^k; 1, r). \qquad (6.6.6)
\end{aligned}
$$

If $r = 0$ then this provides the following summation formula involving the generalized Laguerre polynomial (6.2.2).

$$
L_{m^*}^{(k,\beta)}(x^k) = \left(\frac{x}{y}\right)^{\frac{km}{\delta}} \sum_{j=0}^{m} \begin{pmatrix} \beta + km \\ kj \end{pmatrix} \frac{(kj)!}{j!} \left(\left(\frac{y}{x}\right)^{\frac{k}{\delta}} - 1\right)^j L_{(m-j)^*}^{(k,\beta)}(y^k). \qquad (6.6.7)
$$

Further, $\delta = 1$ yields the summation formula:

$$Z_m^{\beta}(x;k) = \left(\frac{x}{y}\right)^{km} \sum_{j=0}^{m} \begin{pmatrix} \beta + km \\ kj \end{pmatrix} \frac{(kj)!}{j!} \left(\left(\frac{y}{x}\right)^k - 1\right)^j Z_{m-j}^{\beta}(y;k). \quad (6.6.8)$$

The Laguerre polynomial case follows immediately with $k = 1$ as given below.

$$L_m^{(\beta)}(x) = \left(\frac{x}{y}\right)^m \sum_{j=0}^{m} \begin{pmatrix} \beta + m \\ j \end{pmatrix} \left(\frac{y}{x} - 1\right)^j L_{m-j}^{(\beta)}(y). \quad (6.6.9)$$

6.7 Mixed relation

In the following, the relation involving the derivative of GKP is obtained.

Theorem 6.7.1. *If* $\alpha,\ \beta,\ \lambda > 0,\ \delta, \mu, k, m, s \in \mathbb{N},\ r \in \mathbb{N} \cup \{0\}$ *then*

$$\frac{\alpha}{k}\ x\ DB_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r) = (\alpha m + \beta)B_{m^*}^{(\alpha,\beta-1,\lambda,\mu)}(x^k;s,r) - \beta B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r), \quad (6.7.1)$$

where $D = \frac{d}{dx}$.

Proof. Consider

$$\begin{aligned} \left(\frac{\alpha}{k}\ x\ D + \beta\right) B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r) &= \frac{\Gamma(\alpha m + \beta + 1)}{(m!)^s} \\ &\quad \times \sum_{n=0}^{m^*} \frac{(\alpha n + \beta)\ [(-m)_{\delta n}]^s\ x^{kn}}{\Gamma(\alpha n + \beta + 1)\ [(\lambda)_{\mu n}]^r\ n!} \\ &= \frac{\Gamma(\alpha m + \beta + 1)}{(m!)^s} \sum_{n=0}^{m^*} \frac{[(-m)_{\delta n}]^s\ x^{kn}}{\Gamma(\alpha n + \beta)\ [(\lambda)_{\mu n}]^r\ n!} \\ &= (\alpha m + \beta)B_{m^*}^{(\alpha,\beta-1,\lambda,\mu)}(x^k;s,r). \end{aligned}$$

The relation (7.9.5) now follows. □

6.8 Integral representations

Theorem 6.8.1. *If* $\alpha, \beta, \sigma, \lambda > 0,\ \delta, \mu, k, m, s \in \mathbb{N},\ r \in \mathbb{N} \cup \{0\}$ *then*

$$B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^{\alpha};s,r) = \frac{\Gamma(\alpha m + \beta + 1)}{\Gamma(\alpha m + \sigma + 1)\ \Gamma(\beta - \sigma)x^{\beta}}$$

$$\times \int_0^x (x-u)^{\beta-\sigma-1} u^{\sigma} B_{m^*}^{(\alpha,\sigma,\lambda,\mu)}(u^{\alpha}; s, r) \mathrm{d}u. \quad (6.8.1)$$

Proof. It is straight forward to see that

$$\begin{aligned}
&\int_0^x (x-u)^{\beta-\sigma-1} u^{\sigma} B_{m^*}^{(\alpha,\sigma,\lambda,\mu)}(u^{\alpha}; s, r)\ \mathrm{d}u \\
&= \int_0^x (x-u)^{\beta-\sigma-1} u^{\sigma} \frac{\Gamma(\alpha m+\sigma+1)}{(m!)^s} \sum_{n=0}^{m^*} \frac{[(-m)_{\delta n}]^s u^{\alpha n}}{\Gamma(\alpha n+\sigma+1)[(\lambda)_{\mu n}]^r\ n!} \mathrm{d}u \\
&= \frac{\Gamma(\alpha m+\sigma+1)}{(m!)^s} \sum_{n=0}^{m^*} \frac{[(-m)_{\delta n}]^s}{\Gamma(\alpha n+\sigma+1)[(\lambda)_{\mu n}]^r\ n!} \int_0^x (x-u)^{\beta-\sigma-1} u^{\alpha n+\sigma} \mathrm{d}u \\
&= \frac{\Gamma(\alpha m+\sigma+1)}{(m!)^s} \sum_{n=0}^{m^*} \frac{[(-m)_{\delta n}]^s x^{\alpha n+\sigma+\beta-\sigma-1+1}}{\Gamma(\alpha n+\sigma+1)[(\lambda)_{\mu n}]^r\ n!} \mathfrak{B}(\alpha n+\sigma+1, \beta-\sigma) \\
&= \frac{\Gamma(\alpha m+\sigma+1)\Gamma(\beta-\sigma) x^{\beta}}{(m!)^s} \sum_{n=0}^{m^*} \frac{[(-m)_{\delta n}]^s}{\Gamma(\alpha n+\beta+1)[(\lambda)_{\mu n}]^r n!} x^{\alpha n} \\
&= \frac{\Gamma(\alpha m+\sigma+1)\Gamma(\beta-\sigma) x^{\beta}}{\Gamma(\alpha m+\beta+1)} B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x; s, r).
\end{aligned}$$

□

Another integral representation is derived in the following theorem.

Theorem 6.8.2. *If* $\alpha, \beta, \sigma, \lambda > 0,\ \delta, \mu, k, m, s \in \mathbb{N},\ r \in \mathbb{N} \cup \{0\}$ *then*

$$(x-t)^{\beta} B_{m^*}^{(\alpha,\beta,\lambda,\mu)}((x-t)^{\alpha}; s, r) = \frac{\Gamma(\alpha m+\beta+1)}{\Gamma(\alpha m+\beta-\sigma+1)\Gamma(\sigma)} \times \int_t^x (x-u)^{\sigma-1} (u-t)^{\beta-\sigma} B_{m^*}^{(\alpha,\beta-\sigma,\lambda,\mu)}((u-t)^{\alpha}; s, r)\ \mathrm{d}u. \quad (6.8.2)$$

Proof. Consider

$$\mathfrak{I} = \int_t^x (x-u)^{\sigma-1} (u-t)^{\beta-\sigma} B_{m^*}^{(\alpha,\beta-\sigma,\lambda,\mu)}((u-t)^{\alpha}; s, r)\ \mathrm{d}u.$$

By changing the variable u to $s = \frac{u-t}{x-t}$, then

$$\mathfrak{I} = \frac{\Gamma(\alpha m+\beta-\sigma+1)}{(m!)^s} \sum_{n=0}^{m^*} \frac{[(-m)_{\delta n}]^s (x-t)^{\alpha n+\beta}}{\Gamma(\alpha n+\beta-\sigma+1)[(\lambda)_{\mu n}]^r n!} \int_0^1 s^{\alpha n+\beta-\sigma} (1-s)^{\sigma-1} \mathrm{d}s$$

$$= \frac{\Gamma(\alpha m+\beta-\sigma+1)}{(m!)^s}\sum_{n=0}^{m^*}\frac{[(-m)_{\delta n}]^s(x-t)^{\alpha n+\beta}}{\Gamma(\alpha n+\beta-\sigma+1)[(\lambda)_{\mu n}]^r n!}\mathfrak{B}(\alpha n+\beta-\sigma+1,\sigma).$$

This after some simplification, leads to

$$\mathfrak{I} = \frac{\Gamma(\alpha m+\beta-\sigma+1)\Gamma(\sigma)(x-t)^{\beta}}{\Gamma(\alpha m+\beta+1)} B_{m^*}^{(\alpha,\beta,\lambda,\mu)}((x-t)^{\alpha};s,r).$$

□

6.8.1 Special cases

Here, putting $\delta = s = 1, r = 0, \alpha = k \in \mathbb{N}$ in (6.8.1), gives (Ajudia, Prajapati, Agarwal [58])

$$Z_m^{\beta}(x;k) = \frac{\Gamma(km+\beta+1)}{\Gamma(km+\sigma+1)\,\Gamma(\beta-\sigma)x^{\beta}}\int_0^x (x-u)^{\beta-\sigma-1}u^{\sigma}Z_m^{\sigma}(u;k)\,\mathrm{d}u.$$

The special case $k = 1$, further gives

$$L_m^{(\beta)}(x) = \frac{\Gamma(m+\beta+1)}{\Gamma(m+\sigma+1)\,\Gamma(\beta-\sigma)x^{\beta}}\int_0^x (x-u)^{\beta-\sigma-1}u^{\sigma}L_m^{\sigma}(u)\,\mathrm{d}u.$$

By substituting $t = 0$ in (6.8.2), one obtains

$$x^{\beta}B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^{\alpha};s,r) = \frac{\Gamma(\alpha m+\beta+1)}{\Gamma(\alpha m+\beta-\sigma+1)\Gamma(\sigma)} \times\int_0^x (x-u)^{\sigma-1}u^{\beta-\sigma}B_{m^*}^{(\alpha,\beta-\sigma,\lambda,\mu)}(u^{\alpha};s,r)\,\mathrm{d}u.$$

Taking $\sigma = 1$, this reduces to

$$x^{\beta}B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^{\alpha};s,r) = (\alpha m+\beta)\int_0^x u^{\beta-1}B_{m^*}^{(\alpha,\beta-1,\lambda,\mu)}(u^{\alpha};s,r)\,\mathrm{d}u.$$

Further, for $x = 1$, it gives

$$\frac{1}{(\alpha m+\beta)}B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(1;s,r) = \int_0^1 u^{\beta-1}B_{m^*}^{(\alpha,\beta-1,\lambda,\mu)}(u^{\alpha};s,r)\,\mathrm{d}u.$$

6.9 Fractional operators

In this section, the fractional operators are applied on GKP and the following results are obtained.

6.9.1 Fractional integral operator

In the first place consider the fractional integral operator (1.6.1)

$$I_{0+}^{\nu} f(x) \;=\; \frac{1}{\Gamma(\nu)} \int_0^x \frac{f(t)}{(x-t)^{1-\nu}}\, \mathrm{d}u, \quad x > 0,$$

Theorem 6.9.1. *If* $\alpha, \beta, \lambda, \nu > 0,\ \delta, \mu, k, m, s \in \mathbb{N},\ r \in \mathbb{N} \cup \{0\}$, *then*

$$I_{0+}^{\nu}\left[x^{\beta} B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^{\alpha}; s, r)\right] = \frac{\Gamma(\alpha m+\beta+1)}{\Gamma(\alpha m+\beta+\nu+1)} x^{\beta+\nu} B_{m^*}^{(\alpha,\beta+\nu,\lambda,\mu)}(x^{\alpha}; s, r). \quad (6.9.1)$$

Proof. Here beginning with

$$\begin{aligned}
& I_{0+}^{\nu}\left[x^{\beta} B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^{\alpha}; s, r)\right] \\
&= \int_0^x \frac{(x-t)^{\nu-1}}{\Gamma(\nu)} t^{\beta} B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(t^{\alpha}; s, r) \\
&= \int_0^x \frac{(x-t)^{\nu-1}\ t^{\beta}}{\Gamma(\nu)} \frac{\Gamma(\alpha m+\beta+1)}{(m!)^s} \sum_{n=0}^{m^*} \frac{[(-m)_{\delta n}]^s\ t^{\alpha n}}{\Gamma(\alpha n+\beta+1)[(\lambda)_{\mu n}]^r n!} \\
&= \frac{\Gamma(\alpha m+\beta+1)}{(m!)^s\ \Gamma(\nu)} \sum_{n=0}^{m^*} \frac{[(-m)_{\delta n}]^s}{\Gamma(\alpha n+\beta+1)[(\lambda)_{\mu n}]^r n!} \int_0^x (x-t)^{\nu-1}\ t^{\alpha n+\beta}.
\end{aligned}$$

And replacing t by xu, one gets

$$\begin{aligned}
& I_{0+}^{\nu}\left[x^{\beta} B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^{\alpha}; s, r)\right] \\
&= \frac{\Gamma(\alpha m+\beta+1)}{(m!)^s\ \Gamma(\nu)} \sum_{n=0}^{m^*} \frac{[(-m)_{\delta n}]^s x^{\alpha n+\beta+\nu}}{\Gamma(\alpha n+\beta+1)[(\lambda)_{\mu n}]^r n!} \int_0^1 (1-u)^{\nu-1}\ u^{\alpha n+\beta} \mathrm{d}u \\
&= \frac{\Gamma(\alpha m+\beta+1)}{(m!)^s\ \Gamma(\nu)} \sum_{n=0}^{m^*} \frac{[(-m)_{\delta n}]^s x^{\alpha n+\beta+\nu}}{\Gamma(\alpha n+\beta+1)[(\lambda)_{\mu n}]^r n!} \frac{\Gamma(\alpha n+\beta+1)\Gamma(\nu)}{\Gamma(\alpha n+\beta+\nu+1)} \\
&= \frac{\Gamma(\alpha m+\beta+1)}{(m!)^s} \sum_{n=0}^{m^*} \frac{[(-m)_{\delta n}]^s x^{\alpha n+\beta+\nu}}{\Gamma(\alpha n+\beta+\nu+1)[(\lambda)_{\mu n}]^r\ n!}
\end{aligned}$$

$$= \frac{\Gamma(\alpha m+\beta+1)\ x^{\beta+\nu}}{\Gamma(\alpha m+\beta+\nu+1)} B_{m^*}^{(\alpha,\beta+\nu,\lambda,\mu)}(x^\alpha; s, r).$$

□

6.9.1.1 Special cases

Taking $r = 0, s = 1$ in (6.9.1), gives

$$I_{0+}^{\nu}\left[x^\beta L_{m^*}^{(\alpha,\beta)}(x^\alpha)\right] = \frac{\Gamma(\alpha m+\beta+1)\ x^{\beta+\nu}}{\Gamma(\alpha m+\beta+\nu+1)} L_{m^*}^{(\alpha,\beta+\nu)}(x^\alpha).$$

For $\alpha = k \in \mathbb{N}, \delta = 1$, this reduces to

$$I_{0+}^{\nu}\left[x^\beta Z_m^{\beta}(x;k)\right] = \frac{\Gamma(km+\beta+1)\ x^{\beta+\nu}}{\Gamma(km+\beta+\nu+1)} Z_m^{\beta+\nu}(x;k).$$

Putting $k = 1$, one obtains

$$I_{0+}^{\nu}\left[x^\beta L_m^{\beta}(x)\right] = \frac{\Gamma(m+\beta+1)\ x^{\beta+\nu}}{\Gamma(m+\beta+\nu+1)} L_m^{\beta+\nu}(x).$$

6.9.2 Fractional differential operator

Now the fractional differential operator D_{0+}^{ν} (1.6.4) will be considered. This is given by

$$(D_{0+}^{\nu} f)(x) = \left(\frac{d}{dx}\right)^n (I_{0+}^{n-\nu} f)(x).$$

Theorem 6.9.2. *If* $\alpha,\ \beta,\ \lambda, \nu > 0,\ \delta, \mu, k, m, s \in \mathbb{N},\ r \in \mathbb{N} \cup \{0\}$, *then*

$$D_{0+}^{\nu}\left[x^\beta B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^\alpha; s, r)\right] = \frac{\Gamma(\alpha m+\beta+1)\quad x^{\beta-\nu}}{\Gamma(\alpha m+\beta+m-\nu+1)\ \Gamma(\alpha m+\beta-\nu)} \times B_{m^*}^{(\alpha,\beta-\nu-1,\lambda,\mu)}(x^\alpha; s, r). \tag{6.9.2}$$

Proof. With the use of (1.6.4) and Theorem 6.9.1,

$$\begin{aligned} l.h.s &= D_{0+}^{\nu}\left[x^\beta B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^\alpha; s, r)\right] \\ &= \left(\frac{d}{dx}\right)^n \left(I^{n-\nu}\left[x^\beta B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^\alpha; s, r)\right]\right) \end{aligned}$$

$$= \left(\frac{d}{dx}\right)^n \left(\frac{\Gamma(\alpha m+\beta+1)}{\Gamma(\alpha m+\beta+m-\nu+1)} x^{\beta+n-\nu} B_{m^*}^{(\alpha,\beta+n-\nu,\lambda,\mu)}(x^\alpha;s,r)\right)$$

$$= \left(\frac{d}{dx}\right)^n \left(\frac{\Gamma(\alpha m+\beta+1)}{\Gamma(\alpha m+\beta+m-\nu+1)} \sum_{n=0}^{m^*} \frac{[(-m)_{\delta n}]^s \; x^{\alpha n+\beta+n-\nu}}{\Gamma(\alpha n+\beta+n-\nu+1)\;[(\lambda)_{\mu n}]^r \; n!}\right)$$

$$= \frac{\Gamma(\alpha m+\beta+1)\;\; x^{\beta-\nu}}{\Gamma(\alpha m+\beta+n-\nu+1)} \sum_{n=0}^{m^*} \frac{[(-m)_{\delta n}]^s \; x^{\alpha n}}{\Gamma(\alpha n+\beta-\nu)\;[(\lambda)_{\mu n}]^r \; n!}$$

$$= \frac{\Gamma(\alpha m+\beta+1)\;\; x^{\beta-\nu}}{\Gamma(\alpha m+\beta+m-\nu+1)\;\Gamma(\alpha m+\beta-\nu)} B_{m^*}^{(\alpha,\beta-\nu-1,\lambda,\mu)}(x^\alpha;s,r).$$

□

6.9.2.1 Special cases

Taking $\beta,\nu \in \mathbb{C}$ with $\Re(\beta) > -1, r = 0, s = 1, \alpha = k \in \mathbb{N}$ and $\delta = 1$ in (6.9.2), gives

$$D_{0+}^{\nu}\left[x^{\beta}\; Z_m^{\beta}(x;k)\right] \;=\; \frac{\Gamma(\alpha m+\beta+1)\; x^{\beta-\nu-1}}{\Gamma(\alpha m+\beta+m-\nu+1)\;\Gamma(\alpha m+\beta-\nu)} Z_m^{\beta+\nu}(x;k).$$

And for the generalized Laguerre polynomial, it gives

$$D_{0+}^{\nu}\left[x^{\beta} L_m^{(\beta)}(x)\right] \;=\; \frac{\Gamma(\alpha m+\beta+1)\; x^{\beta-\nu-1}}{\Gamma(\alpha m+\beta+m-\nu+1)\;\Gamma(\alpha m+\beta-\nu)} L_m^{(\beta-\nu-1)}(x),$$

when $k = 1$.

6.10 Integral transforms

6.10.1 Laplace transform:

Theorem 6.10.1. *If* $\alpha,\beta,\lambda > 0,\; \delta,\mu,k,m,s \in \mathbb{N},\; r,\; \sigma \in \mathbb{N}\cup\{0\},\; \nu \in \mathbb{C}$ *then*

$$\mathfrak{L}\left\{t^{\nu} B_{m^*}^{(\alpha,\beta,\lambda,\mu)}((xt^{\sigma});s,r)\right\} = \frac{(\beta+1)_{\alpha m}\Gamma(\nu+1)}{s^{\nu+1}(m!)^s}$$

$$\times {}_{s\delta}F_{k+r\mu}\left[\begin{array}{lll} \Delta(\delta;-m)^s, & \Delta(\sigma;\nu+1); & \dfrac{\delta^{s\delta}\sigma^{\sigma}x}{\alpha^{\alpha}\mu^{r\mu}s^{\sigma}} \\ \Delta(\alpha;\beta+1), & \Delta(\mu;\lambda)^r; & \end{array}\right]. \qquad (6.10.1)$$

Proof. The left hand member

$$
\begin{aligned}
& \mathfrak{L}\left\{t^{\nu} B_{m^*}^{(\alpha,\beta,\lambda,\mu)}((xt^{\sigma}); s, r)\right\} \\
= & \int_0^{\infty} e^{-st} t^{\nu} B_{m^*}^{(\alpha,\beta,\lambda,\mu)}((xt^{\sigma}); s, r)\, \mathrm{dt} \\
= & \int_0^{\infty} e^{-st} t^{\nu} \frac{\Gamma(\alpha m+\beta+1)}{(m!)^s} \sum_{n=0}^{m^*} \frac{[(-m)_{\delta n}]^s\, (xt^{\sigma})^n}{\Gamma(\alpha n+\beta+1)[(\lambda)_{\mu n}]^r n!}\, \mathrm{dt} \\
= & \frac{\Gamma(\alpha m+\beta+1)}{(m!)^s} \sum_{n=0}^{m^*} \frac{[(-m)_{\delta n}]^s\, x^n}{\Gamma(\alpha n+\beta+1)[(\lambda)_{\mu n}]^r n!} \int_0^{\infty} e^{-st} t^{\sigma n+\nu}\, \mathrm{dt} \\
= & \frac{\Gamma(\alpha m+\beta+1)}{(m!)^s} \sum_{n=0}^{m^*} \frac{[(-m)_{\delta n}]^s\, x^n}{\Gamma(\alpha n+\beta+1)[(\lambda)_{\mu n}]^r n!} \frac{\Gamma(\sigma n+\nu+1)}{s^{\sigma n+\nu+1}} \\
= & \frac{\Gamma(\alpha m+\beta+1)}{(m!)^s} \sum_{n=0}^{m^*} \frac{[(-m)_{\delta n}]^s\, x^n}{\Gamma(\alpha n+\beta+1)[(\lambda)_{\mu n}]^r n!} \\
& \times \frac{\Gamma(\sigma n+\nu+1)}{s^{\sigma n+\nu+1}} \frac{\Gamma(\nu+1)}{\Gamma(\nu+1)} \frac{\Gamma(\beta+1)}{\Gamma(\sigma-\alpha+\beta+1)} \\
= & \frac{(\beta+1)_{\alpha m}\, \Gamma(\nu+1)}{s^{\nu+1}\, (m!)^s} \sum_{n=0}^{m^*} \frac{[(-m)_{\delta n}]^s\, x^n}{(\beta+1)_{\alpha n}\, [(\lambda)_{\mu n}]^r\, n!} \frac{(\nu+1)_{\sigma n}\, (1)_n}{s^{\sigma n}\, n!} \\
= & \frac{(\beta+1)_{\alpha m}\, \Gamma(\nu+1)}{s^{\nu+1}\, (m!)^s} \\
& \times {}_{s\delta}F_{k+r\mu}\left[\begin{array}{lll} \Delta(\delta;-m)^s, & \Delta(\sigma;\nu+1); & \dfrac{\delta^{s\delta}\sigma^{\sigma} x}{\alpha^{\alpha}\mu^{r\mu} s^{\sigma}} \\ \Delta(\alpha;\beta+1), & \Delta(\mu;\lambda)^r; & \end{array}\right].
\end{aligned}
\tag{6.10.2}
$$

□

6.10.2 Euler(Beta) transform:

Theorem 6.10.2. *If* $\alpha, \beta, \lambda > 0,\ \delta, \mu, k, m, s, a, b \in \mathbb{N},\ r \in \mathbb{N} \cup \{0\}$, *then*

$$
\begin{aligned}
\mathfrak{B}\left\{B_{m^*}^{(\alpha,\beta,\lambda,m)}((tx^{\alpha}); s, r) : a, b\right\} = & \frac{(\beta+1)_{\alpha m}\Gamma(b)}{(m!)^s\, \Gamma(a)} \\
\times\, {}_{s\delta}F_{\alpha+r\mu} & \left[\begin{array}{lll} \Delta(\delta;-m)^s, & \Delta(\alpha;a); & \dfrac{\delta^{s\delta}\, t}{\alpha^{\alpha}\mu^{r\mu}} \\ \Delta(\alpha;\beta+1), & \Delta(\mu;\lambda)^r, & \Delta(\alpha;a+b); \end{array}\right].
\end{aligned}
\tag{6.10.3}
$$

Proof. Here

$$\mathfrak{B}\left\{B_{m^*}^{(\alpha,\beta,\lambda,\mu)}((tx^\alpha);s,r):a,b\right\} \tag{6.10.4}$$

$$= \int_0^1 x^{a-1}(1-x)^{b-1} B_{m^*}^{(\alpha,\beta,\lambda,\mu)}((tx^\alpha);s,r)\,\mathrm{dx}$$

$$= \int_0^1 x^{a-1}(1-x)^{b-1} \frac{\Gamma(\alpha m+\beta+1)}{(m!)^s} \sum_{n=0}^{m^*} \frac{[(-m)_{\delta n}]^s\ (tx^\alpha)^n}{\Gamma(\alpha n+\beta+1)[(\lambda)_{\mu n}]^r n!}\,\mathrm{dx}$$

$$= \frac{\Gamma(\alpha m+\beta+1)}{(m!)^s} \sum_{n=0}^{m^*} \frac{[(-m)_{\delta n}]^s\ t^n}{\Gamma(\alpha n+\beta+1)[(\lambda)_{\mu n}]^r n!} \int_0^1 x^{\alpha n+a-1}(1-x)^{b-1}\,\mathrm{dx}$$

$$= \frac{\Gamma(\alpha m+\beta+1)}{(m!)^s} \sum_{n=0}^{m^*} \frac{[(-m)_{\delta n}]^s\ t^n}{\Gamma(\alpha n+\beta+1)[(\lambda)_{\mu n}]^r n!} \mathfrak{B}(\alpha n+a,b)$$

$$= \frac{\Gamma(\alpha m+\beta+1)}{(m!)^s} \sum_{n=0}^{m^*} \frac{[(-m)_{\delta n}]^s\ t^n}{\Gamma(\alpha n+\beta+1)[(\lambda)_{\mu n}]^r n!} \frac{\Gamma(\alpha n+a)\ \Gamma(b)}{\Gamma(\alpha n+a+b)}$$

$$= \frac{(\beta+1)_{\alpha m}\ \Gamma(b)}{(m!)^s\ \Gamma(a)} \sum_{n=0}^{m^*} \frac{[(-m)_{\delta n}]^s\ (a)_{\alpha n} t^n}{(\beta+1)_{\alpha n}[(\lambda)_{\mu n}]^r\ (a+b)_{\alpha n}\ n!}$$

$$= \frac{(\beta+1)_{\alpha m}\ \Gamma(b)}{(m!)^s\ \Gamma(a)}$$

$$\times\ {}_{s\delta}F_{\alpha+r\mu}\left[\begin{array}{lll} \Delta(\delta;-m)^s, & \Delta(\alpha;a); & \dfrac{\delta^{s\delta}\ t}{\alpha^\alpha \mu^{r\mu}} \\ \Delta(\alpha;\beta+1), & \Delta(\mu;\lambda)^r, \quad \Delta(\alpha;a+b); & \end{array}\right]. \tag{6.10.5}$$

□

(A piece of the content of this work is accepted in “The Mathematics Student” for publication)