Chapter 7

Inequalities involving generalized q-Konhauser polynomial

7.1 Introduction

In this chapter, two q-extensions of the GKP (6.1.3) are considered and their properties corresponding to those obtained in Chapter-6, are derived. Having suggested by the q-analogues (3.1.1) and (3.1.2) of the function (2.1.5), the two q-GKP are defined as follows.

Definition 7.1.1. *For* $\alpha, \beta, \lambda > 0,\ m, \delta, \mu, k, s \in \mathbb{N},\ r \in \mathbb{N} \cup \{0\},\ m^* = [\frac{m}{\delta}]$, *the integral part of* $\frac{m}{\delta}$, *define*

$$B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q) = \frac{(q^{\beta+1}; q)_{\alpha m}}{[(q^k; q^k)_m]^s} \sum_{n=0}^{m^*} \frac{q^{sk\delta n(m+(\delta nk-1)/2)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1}; q)_{\alpha n}\ [(q^\lambda; q)_{\mu n}]^r} \times \frac{[(q^{-mk}; q^k)_{\delta n}]^s\ x^{kn}}{(q^k; q^k)_n}. \tag{7.1.1}$$

Definition 7.1.2. *For* $\alpha, \beta, \lambda > 0,\ m, \delta, \mu, k, s \in \mathbb{N},\ r \in \mathbb{N} \cup \{0\},\ m^* = [\frac{m}{\delta}]$, *the integral part of* $\frac{m}{\delta}$, *define*

$$b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q) = \frac{(q^{\beta+1}; q)_{\alpha m}}{[(q^k; q^k)_m]^s} \sum_{n=0}^{m^*} \frac{[(q^{-mk}; q^k)_{\delta n}]^s\ x^{kn}}{(q^{\beta+1}; q)_{\alpha n}\ [(q^\lambda; q)_{\mu n}]^r\ (q^k; q^k)_n}. \tag{7.1.2}$$

The polynomials in (7.1.2) and (7.1.1) will be referred to as q-GKP.

7.2 Generalized q-Konhauser polynomial

If $s = 1, r = 0$ then (7.1.1) reduces to q-analogue of another generalization of the Konhauser polynomial (6.2.2) in the form considered by [58]:

$$\begin{aligned} B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k;1,0|q) &= \frac{(q^{\beta+1};q)_{\alpha m}}{(q^k;q^k)_m}\sum_{n=0}^{m^*}\frac{q^{k\delta n(k\delta n-1)/2+\delta nm}\ q^{\delta n\alpha(\beta+1)}}{(q^{\beta+1};q)_{\alpha n}} \\ &\quad\times\frac{(q^{-mk};q^k)_{\delta n}\ x^{kn}}{(q^k;q^k)_n} \\ &= Z_{m^*}^{\beta}(x;k|q). \qquad (7.2.1)\end{aligned}$$

A q-analogue of the classical Konhauser polynomial (6.1.1) is obtained from (7.2.1) by taking $\delta = 1$ and $\alpha = k$, that is

$$\begin{aligned} B_{m}^{(k,\beta,\lambda,\mu)}(x^k;1,0|q) &= \frac{(q^{\beta+1};q)_{km}}{(q^k;q^k)_m}\sum_{n=0}^{m}\frac{q^{kn(kn-1)/2+kn(m+\beta+1)}(q^{-mk};q^k)_n\ x^{kn}}{(q^{\beta+1};q)_{kn}\ (q^k;q^k)_n} \\ &= Z_{m}^{\beta}(x;k|q). \qquad (7.2.2)\end{aligned}$$

Further, with $k = 1$,

$$\begin{aligned} B_{m}^{(1,\beta,\lambda,\mu)}(x;1,0|q) &= \frac{(q^{\beta+1};q)_{m}}{(q;q)_m}\sum_{n=0}^{m}\frac{q^{n(n+1)/2+n(m+\beta)}(q^{-m};q)_n\ x^{n}}{(q^{\beta+1};q)_{n}\ (q;q)_n} \\ &= L_{m}^{(\beta)}(x|q) \qquad (7.2.3)\end{aligned}$$

is a q-analogue of the generalized Laguerre polynomial. If $k = 1$ then (7.1.2) reduces to the special case of the function (3.1.1) with z replaced by x,

$$\begin{aligned} b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x;s,r|q) &= \frac{(q^{\beta+1};q)_{\alpha m}}{[(q;q)_m]^s}\sum_{n=0}^{m^*}\frac{[(q^{-m};q)_{\delta n}]^s\ x^n}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r\ (q;q)_n} \\ &= \frac{(q^{\beta+1};q)_{\alpha m}}{[(q;q)_m]^s}\ e_{\alpha,\beta+1,\lambda,\mu}^{-m,\delta}(x;s,r|q). \qquad (7.2.4)\end{aligned}$$

Theorem 7.2.1. *Let*

$$\begin{aligned} B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r|q) &= \frac{(q^{\beta+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s}\sum_{n=0}^{m^*}\frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r} \\ &\quad\times\frac{[(q^{-mk};q^k)_{\delta n}]^s\ x^{kn}}{(q^k;q^k)_n}. \qquad (7.2.5)\end{aligned}$$

Then as limit $m \to \infty$, $B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q)$ approaches to the entire function

$$B_{\infty}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q) = \frac{(q^{\beta+1}; q)_\infty}{[(q^k; q^k)_\infty]^s} \sum_{n=0}^{\infty} \frac{(-1)^{s\delta n}\ q^{s(k\delta n(k\delta n-1)/2+k\delta n(\delta n-1)/2)}}{(q^{\beta+1}; q)_{\alpha n}\ [(q^\lambda; q)_{\mu n}]^r} \times \frac{q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\ x^{kn}}{(q^k; q^k)_n} \tag{7.2.6}$$

in any bounded domain.

Proof. It will be shown first that the series in (7.2.6) has an infinite radius of convergence.
Taking

$$\begin{aligned} \upsilon_n &= \frac{(-1)^{s\delta n}\ q^{s(k\delta n(k\delta n-1)/2+k\delta n(\delta n-1)/2)}}{(q^{\beta+1}; q)_{\alpha n}\ [(q^\lambda; q)_{\mu n}]^r} \frac{q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^k; q^k)_n} \\ &= \frac{(-1)^{s\delta n}\ q^{s(k\delta n(k\delta n-1)/2+k\delta n(\delta n-1)/2)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1}; q)_\infty\ [(q^\lambda; q)_\infty]^r} \\ &\quad \times \frac{(q^{\alpha n+\beta+1}; q)_\infty\ [(q^{\mu n+\lambda}; q)_\infty]^r}{(q^k; q^k)_n}. \end{aligned}$$

Then using D'Albert's Ratio test, the radius of convergence R is given by

$$\begin{aligned} R &= \lim_{n\to\infty} \left| \frac{\upsilon_n}{\upsilon_{n+1}} \right| \\ &= \lim_{n\to\infty} \left| \frac{(-1)^{s\delta n}\ q^{s(k\delta n(k\delta n-1)/2+k\delta n(\delta n-1)/2)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1}; q)_\infty\ [(q^\lambda; q)_\infty]^r\ (q^k; q^k)_n} \right. \\ &\quad \times \frac{(q^{\beta+1}; q)_\infty\ [(q^\lambda; q)_\infty]^r\ (q^{\alpha n+\beta+1}; q)_\infty\ [(q^{\mu n+\lambda}; q)_\infty]^r}{(-1)^{s\delta(n+1)}\ q^{sk\delta(n+1)(\delta(n+1)-1)/2+(k\delta(n+1)-1)/2}} \\ &\quad \left. \times \frac{(q^k; q^k)_{n+1}}{q^{\delta(n+1)(\alpha(\beta+1)+r\mu\lambda)}\ (q^{\alpha(n+1)+\beta+1}; q)_\infty\ [(q^{\mu(n+1)+\lambda}; q)_\infty]^r} \right| \\ &= \lim_{n\to\infty} \left| \frac{q^{sk\delta-sk^2\delta^2}\ (1-q^{(n+1)k})}{q^{ns\delta^2(k(k+1))}\ q^{\delta(\alpha(\beta+1)+r\mu\lambda)}} \right. \\ &\quad \left. \times \frac{(1-q^{\alpha n+\beta+1})\ (1-q^{\alpha n+\beta+2}) \dots (1-q^{\alpha n+\beta+\alpha})}{[(1-q^{\mu n+\lambda+1})\ (1-q^{\mu n+\lambda+2}) \dots (1-q^{\mu n+\lambda+\mu})]^{-r}} \right| \\ &= \infty. \end{aligned}$$

Here it suffices to show that for m sufficiently large,

$$\sum_{n=0}^{m^*} \frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1}; q)_{\alpha n}\ [(q^\lambda; q)_{\mu n}]^r} \frac{[(q^{-mk}; q^k)_{\delta n}]^s\ x^{kn}}{(q^k; q^k)_n} \tag{7.2.7}$$

tends to

$$\sum_{n=0}^{\infty} \frac{(-1)^{s\delta n}\ q^{s(k\delta n(k\delta n-1)/2+k\delta n(\delta n-1)/2)}}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r} \frac{q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\ x^{kn}}{(q^k;q^k)_n}. \tag{7.2.8}$$

In fact,

$$\begin{aligned}
&\Bigg| \sum_{n=0}^{\infty} \frac{(-1)^{s\delta n}\ q^{s(k\delta n(k\delta n-1)/2+k\delta n(\delta n-1)/2)}}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r} \frac{q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\ x^{kn}}{(q^k;q^k)_n} \\
&- \sum_{n=0}^{m^*} \frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r} \frac{[(q^{-mk};q^k)_{\delta n}]^s\ x^{kn}}{(q^k;q^k)_n} \Bigg| \\
= &\Bigg| \sum_{n=0}^{m^*} \left\{ q^{sk\delta n(\delta n-1)/2} - [(q^{-mk};q^k)_{\delta n}]^s\ q^{sk\delta nm}\ (-1)^{s\delta n} \right\} \\
&\times \frac{q^{sk\delta n(k\delta n-1)/2}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\ (-1)^{s\delta n}\ x^{kn}}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r\ (q^k;q^k)_n} \Bigg| \\
= &\Bigg| \sum_{n=0}^{m^*} \Bigg\{ q^{sk\delta n(\delta n-1)/2} - \Big[(1-q^{-mk})\ (1-q^{-mk+k})\ (1-q^{-mk+2k}) \dots \\
&\times (1-q^{-mk+(\delta n-1)k}) \Big]^s\ q^{sk\delta nm}\ (-1)^{s\delta n} \Bigg\} \\
&\times \frac{q^{sk\delta n(k\delta n-1)/2}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\ (-1)^{s\delta n}\ x^{kn}}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r\ (q^k;q^k)_n} \Bigg| \\
= &\Bigg| \sum_{n=0}^{m^*} \Bigg\{ q^{sk\delta n(\delta n-1)/2} - \Big[(q^{-mk}-1)\ (q^{-mk+k}-1)\ (q^{-mk+2k}-1) \dots \\
&\times (q^{-mk+(\delta n-1)k}-1) \Big]^s\ q^{sk\delta nm} \Bigg\} \\
&\times \frac{q^{sk\delta n(k\delta n-1)/2}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\ (-1)^{s\delta n}\ x^{kn}}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r\ (q^k;q^k)_n} \Bigg| \\
= &\Bigg| \sum_{n=0}^{m^*} \Bigg\{ q^{sk\delta n(\delta n-1)/2} - \Big[(1-q^{mk})\ (1-q^{mk-k})\ (1-q^{mk-2k}) \dots \\
&\times (1-q^{mk-(\delta n-1)k}) \Big]^s\ q^{sk\delta nm}\ q^{sk\delta n(\delta n-1)/2-sk\delta nm} \Bigg\} \\
&\times \frac{q^{sk\delta n(k\delta n-1)/2}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\ (-1)^{s\delta n}\ x^{kn}}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r\ (q^k;q^k)_n} \Bigg| \\
\leq &\sum_{n=0}^{m^*} \Bigg| q^{sk\delta n(\delta n-1)/2} - \Big[(1-q^{mk})\ (1-q^{mk-k})\ (1-q^{mk-2k}) \dots \\
&\times (1-q^{mk-(\delta n-1)k}) \Big]^s\ q^{sk\delta n(\delta n-1)/2} \Bigg|
\end{aligned}$$

$$\times\frac{q^{sk\delta n(k\delta n-1)/2}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\ |x|^{kn}}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r\ (q^k;q^k)_n}. \tag{7.2.9}$$

The absolute difference may be simplified with the aid of the inequality

$$\prod_{j=1}^{k}(1-x_j)\geq 1-\sum_{j=1}^{k}x_j,\quad 0\leq x_j\leq 1,\ j=1,2,\ldots,k,$$

to get

$$\begin{aligned}
&\Bigg|q^{sk\delta n(\delta n-1)/2}-\Big[(1-q^{mk})\ (1-q^{mk-k})\ (1-q^{mk-2k})\ldots \\
&\quad\times(1-q^{mk-(\delta n-1)k})\Big]^s\ q^{sk\delta n(\delta n-1)/2}\Bigg| \\
= \ & q^{sk\delta n(\delta n-1)/2} \\
&\times\left|1-\Big[(1-q^{mk})\ (1-q^{mk-k})\ (1-q^{mk-2k})\ldots(1-q^{mk-(\delta n-1)k})\Big]^s\right| \\
= \ & q^{sk\delta n(\delta n-1)/2}\left|1-\left[\prod_{j=1}^{\delta n}(1-q^{mk-jk+k})\right]^s\right| \\
\leq \ & q^{sk\delta n(\delta n-1)/2}\left|1-\left(1-\sum_{j=1}^{\delta n}q^{mk-jk+k}\right)^s\right| \\
\leq \ & q^{sk\delta n(\delta n-1)/2}\left|\sum_{j=1}^{\delta n}q^{mk-jk+k}\right|^s \\
= \ & q^{sk\delta n(\delta n-1)/2}\left(\sum_{j=1}^{\delta n}q^{mk-jk+k}\right)^s \\
= \ & q^{sk\delta n(\delta n-1)/2+skm}\left(\sum_{j=0}^{\delta n-1}q^{-jk}\right)^s \\
= \ & q^{sk\delta n(\delta n-1)/2+skm}\frac{(1-q^{-\delta nk})^s}{(1-q^{-k})^s} \\
= \ & q^{sk\delta n(\delta n-1)/2-sk\delta n+smk+sk}\frac{(1-q^{\delta nk})^s}{(1-q^{k})^s} \\
\leq \ & \frac{q^{sk\delta n(\delta n-1)/2-sk\delta n+smk+sk}}{(1-q^k)^s},
\end{aligned}$$

because $\delta n\leq m$. Therefore,

$$\Bigg|q^{sk\delta n(\delta n-1)/2}-\Big[(1-q^{mk})\ (1-q^{mk-k})\ (1-q^{mk-2k})\ldots$$

$$\times(1-q^{mk-(\delta n-1)k})\Big]^s \, q^{sk\delta n(\delta n-1)/2}\Bigg|$$
$$\leq \frac{q^{sk\delta n(\delta n-1)/2-sn\delta k+smk+sk}}{(1-q^k)^s}. \tag{7.2.10}$$

This last inequality is valid for all non negative values of δn. Substituting this into (7.2.9), one gets

$$\Bigg|\sum_{n=0}^{\infty}\frac{(-1)^{s\delta n}\, q^{s(k\delta n(k\delta n-1)/2+k\delta n(\delta n-1)/2)}}{(q^{\beta+1};q)_{\alpha n}\,[(q^{\lambda};q)_{\mu n}]^r}\frac{q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\, x^{kn}}{(q^k;q^k)_n}$$
$$-\sum_{n=0}^{m^*}\frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\, q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{\alpha n}\,[(q^{\lambda};q)_{\mu n}]^r}\frac{[(q^{-mk};q^k)_{\delta n}]^s\, x^{kn}}{(q^k;q^k)_n}\Bigg|$$
$$\leq \frac{q^{smk+sk}}{(1-q^k)^s}\sum_{n=0}^{\infty}\frac{q^{sk\delta n(\delta n-1)/2-sn\delta k}q^{sk\delta n(k\delta n-1)/2}\, q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\, |x|^{kn}}{(q^{\beta+1};q)_{\alpha n}\,[(q^{\lambda};q)_{\mu n}]^r\,(q^k;q^k)_n}. \tag{7.2.11}$$

Thus the last series (7.2.11) has an infinite radius of convergence and is therefore bounded in every bounded domain. It follows that the left hand side in (7.2.9) $\to 0$ as $n \to \infty$ uniformly in any bounded domain. Hence the series (7.2.7) converges to (7.2.8) uniformly on any bounded domain. □

7.3 Difference equations

The operators considered in Chapter 3, subsection 3.2.3 are used once again in this chapter. They are relisted below.

$$\Lambda_q f(x) = f(x) - f(xq^{-1}), \quad \Theta f(x) = f(x) - f(xq),$$

$$\mathcal{D}_q\, f(x) = (1-q)\, D_q f(x) := (1-q)\,\frac{f(x)-f(xq)}{x-xq} = \frac{f(x)-f(xq)}{x},$$

$$\frac{\left\{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[\Theta + c^{-u}q^{1-(b+v)/a} - 1]^m\right\}}{\left\{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[c^{-u}q^{1-(b+v)/a}]^m\right\}} = \Phi_{u,v}^{(a,b,c;m)}$$

and

$$\frac{\left\{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[\Theta + c^{-u}q^{(b+v)/a} - 1]^m\right\}}{\left\{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[c^{-u}q^{-(b+v)/a}]^m\right\}} = \Psi_{u,v}^{(a,b,c;m)}.$$

In the notations of these operators, the difference equation satisfied by the polynomial $B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q)$ is derived in the following theorem.

Theorem 7.3.1. *Let* $\alpha, \beta, \lambda, m, \delta, \mu, k, s \in \mathbb{N},\ r \in \mathbb{N} \cup \{0\},\ m^* = [\frac{m}{\delta}]$ *then* $B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q)$ *satisfies the equation*

$$\begin{aligned}&\left[\Phi_{\ell,\kappa}^{(\mu,\lambda,\eta;r)}\ \Phi_{h,g}^{(\alpha,\beta+1,\sigma;1)}\ \Theta\right] B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q)\\ &-x^k\ q^{s(k\delta(k\delta-1)/2)+sk\delta m}\ \Psi_{j,i}^{(\delta k,-mk,\chi;s)}\ B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k q^{s(k\delta)^2}; s, r|q) = 0, \qquad (7.3.1)\end{aligned}$$

where χ *is* $(\delta k)^{th}$ *root of unity,* η *is* μ^{th} *root of unity,* σ *is* α^{th} *root of unity.*

Proof. The coefficient of x^{nk} in (7.1.1) will be first expressed in q-factorial notation with the aid of the formulas [18, Appendix I]:

$$(a; q)_{kn} = (a, aq, \ldots, aq^{k-1}; q^k)_n,$$

$$(a^k; q^k)_n = (a, a\omega_k, \ldots, a\omega_k^{k-1}; q^k)_n \ \ ; \omega_k = e^{(2\pi i)/k},$$

$$(A; q^n)_{\nu k} = (A^{1/n}; q)_{\nu k}(A^{1/n}\omega; q)_{\nu k} \ldots (A^{1/n}\omega^{n-1}; q)_{\nu k},\ \omega^n = 1,$$

and

$$(q^\gamma; q^\delta)_n = (q^{\gamma/\delta}; q)_n\ (\varpi q^{\gamma/\delta}; q)_n \ldots (\varpi^{\delta-1} q^{\gamma/\delta}; q)_n = \prod_{i=0}^{\delta-1}(\varpi^i q^{\gamma/\delta}; q)_n,\ \varpi^\delta = 1.$$

Now if

$$\begin{aligned}&\sum_{n=0}^{[n/\delta]}\left\{\prod_{j=0}^{\delta k-1}\ \prod_{i=0}^{\delta k-1}[(\chi^j\ q^{(-mk+i)/(\delta k)}; q)_n]^s\right\}\left\{\prod_{\ell=0}^{\mu-1}\prod_{\kappa=0}^{\mu-1}[(\eta^\ell\ q^{(\lambda+\kappa)/\mu}; q)_n]^r\right\}^{-1}\\ &\times\frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{\left\{\prod_{h=0}^{\alpha-1}\prod_{g=0}^{\alpha-1}(\sigma^h\ q^{(\beta+g)/\alpha}; q)_n\right\}}\frac{x^{nk}}{(q^k; q^k)_n} = \mathcal{Y}, \qquad (7.3.2)\end{aligned}$$

$$\prod_{j=0}^{\delta k-1}\ \prod_{i=0}^{\delta k-1}[(\chi^j\ q^{(-mk+i)/(\delta k)}; q)_n]^s = \mathcal{H}_n,\ \prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1}[(\eta^\ell\ q^{(\lambda+k)/\mu}; q)_n]^r = \mathcal{B}_n,$$

$$\prod_{h=0}^{\alpha-1}\prod_{g=0}^{\alpha-1}(\sigma^h\ q^{(\beta+g)/\alpha};q)_n=\mathcal{C}_n,\ \ q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}=\mathcal{G}_n,$$

then (7.3.2) will assume the elegant form:

$$\mathcal{Y}=\sum_{n=0}^{[n/\delta]}\frac{\mathcal{H}_n\ \mathcal{G}_n}{\mathcal{B}_n\ \mathcal{C}_n\ (q^k;q^k)_n}x^{nk}.$$

Now

$$\Theta\mathcal{Y}\ =\ \sum_{n=0}^{[n/\delta]}\frac{\mathcal{H}_n\ \mathcal{G}_n}{\mathcal{B}_n\ \mathcal{C}_n\ (q^k;q^k)_n}\Theta x^{nk}=\sum_{n=1}^{[n/\delta]}\frac{\mathcal{H}_n\ \mathcal{G}_n}{\mathcal{B}_n\ \mathcal{C}_n}\frac{x^{nk}}{(q^k;q^k)_{n-1}}.$$

Next operating by $\Phi_{h,g}^{(\alpha,\beta,\sigma;1)}$, one gets

$$\begin{aligned}\Phi_{h,g}^{(\alpha,\beta,\sigma;1)}\Theta\mathcal{Y}\ &=\ \sum_{n=1}^{[n/\delta]}\frac{\mathcal{H}_n\ \mathcal{G}_n}{\mathcal{B}_n\ (q^k;q^k)_{n-1}}\frac{\left\{\prod_{h=0}^{\alpha-1}\prod_{g=0}^{\alpha-1}(\Theta+\sigma^{-h}q^{1-(\beta+g)/\alpha}-1)\right\}}{\left\{\prod_{h=0}^{\alpha-1}\prod_{g=0}^{\alpha-1}(\sigma^{-h}q^{1-(\beta+g)/\alpha})\right\}}\\&\quad\times\left\{\prod_{h=0}^{\alpha-1}\prod_{g=0}^{\alpha-1}(\sigma^h q^{(\beta+g)/\alpha};q)_n\right\}^{-1}x^{nk}\\&=\ \sum_{n=1}^{[n/\delta]}\frac{\mathcal{H}_n\ \mathcal{G}_n}{\mathcal{B}_n\ (q^k;q^k)_{n-1}}\frac{\left\{\prod_{h=0}^{\alpha-1}\prod_{g=0}^{\alpha-1}(1-\sigma^h q^{n-1+(\beta+g)/\alpha})\right\}}{\left\{\prod_{h=0}^{\alpha-1}\prod_{g=0}^{\alpha-1}(\sigma^h q^{(\beta+g)/\alpha};q)_n\right\}}x^{nk}\\&=\ \sum_{n=1}^{[n/\delta]}\frac{\mathcal{H}_n\ \mathcal{G}_n}{\mathcal{B}_n\ \mathcal{C}_{n-1}\ (q^k;q^k)_{n-1}}x^{nk}.\end{aligned}$$

Finally,

$$\begin{aligned}\Phi_{\ell,\kappa}^{(\mu,\lambda,\eta;r)}\Phi_{h,g}^{(\alpha,\beta,\sigma;1)}\Theta\mathcal{Y}\ &=\ \sum_{n=1}^{[n/\delta]}\frac{\mathcal{H}_n\ \mathcal{G}_n}{\mathcal{C}_{n-1}\ (q^k;q^k)_{n-1}}\frac{\left\{\prod_{\ell=0}^{\mu-1}\prod_{\kappa=0}^{\mu-1}[(\Theta+\eta^{-\ell}q^{1-(\lambda+\kappa)/\mu}-1)]^r\right\}}{\left\{\prod_{\ell=0}^{\mu-1}\prod_{\kappa=0}^{\mu-1}[(\eta^{-\ell}q^{1-(\lambda+\kappa)/\mu})]^r\right\}}\\&\quad\times\left\{\prod_{\ell=0}^{\mu-1}\prod_{\kappa=0}^{\mu-1}[(\eta^\ell\ q^{(\lambda+\kappa)/\mu};q)_n]^r\right\}^{-1}x^{nk}\end{aligned}$$

$$= \sum_{n=1}^{[n/\delta]} \frac{\mathcal{H}_n \ \mathcal{G}_n}{\mathcal{C}_{n-1} \ (q^k;q^k)_{n-1}} \frac{\left\{\prod_{\ell=0}^{\mu-1}\prod_{\kappa=0}^{\mu-1}[(-q^n+\eta^{-\ell}q^{1-(\lambda+\kappa)/\mu})]^r\right\}}{\left\{\prod_{\ell=0}^{\mu-1}\prod_{\kappa=0}^{\mu-1}[(\eta^{-\ell}q^{1-(\lambda+\kappa)/\mu})]^r\right\}}$$

$$\times\left\{\prod_{\ell=0}^{\mu-1}\prod_{\kappa=0}^{\mu-1}[(\eta^{\ell}\ q^{(\lambda+\kappa)/\mu};q)_n]^r\right\}^{-1} x^{nk}$$

$$= \sum_{n=1}^{[n/\delta]} \frac{\mathcal{H}_n \ \mathcal{G}_n}{\mathcal{B}_{n-1} \ \mathcal{C}_{n-1} \ (q^k;q^k)_{n-1}} x^{nk}.$$

Thus,

$$\Phi_{\ell,\kappa}^{(\mu,\lambda,\eta;r)}\Phi_{h,g}^{(\alpha,\beta,\sigma;1)}\Theta\mathcal{Y} \ = \ \sum_{n=0}^{[n/\delta]} \frac{\mathcal{H}_{n+1} \ \mathcal{G}_{n+1}}{\mathcal{B}_n \ \mathcal{C}_n \ (q^k;q^k)_n} x^{nk+k}. \tag{7.3.3}$$

Further,

$$\Psi_{j,i}^{(\delta k,-mk,\chi;s)} \ B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k q^{s(k\delta)^2};s,r|q)$$

$$= \sum_{n=0}^{[n/\delta]} \frac{\mathcal{H}_n \ \mathcal{G}_n \ q^{s(k\delta)^2 n}}{\mathcal{B}_n \ \mathcal{C}_n \ (q^k;q^k)_n} \frac{\left\{\prod_{j=0}^{\delta k-1}\ \prod_{i=0}^{\delta k-1}[(\Theta+\chi^{-j}q^{-(-mk+i)/(\delta k)}-1)]^s\right\}}{\left\{\prod_{j=0}^{\delta k-1}\ \prod_{i=0}^{\delta k-1}[(\chi^{-j}q^{-(-mk+i)/(\delta k)})]^s\right\}} x^{nk}$$

$$= \sum_{n=0}^{[n/\delta]} \frac{\mathcal{G}_n \ q^{s(k\delta)^2}}{\mathcal{B}_n \ \mathcal{C}_n \ (q^k;q^k)_n} \left\{\prod_{j=0}^{\delta k-1}\ \prod_{i=0}^{\delta k-1}[(\chi^{j}\ q^{(-mk+i)/(\delta k)};q)_n]^s\right\}$$

$$\times\left\{\prod_{j=0}^{\delta k-1}\ \prod_{i=0}^{\delta k-1}[(1-\chi^{j}q^{n+(-mk+i)/(\delta k)})]^s\right\} x^{nk},$$

and hence

$$x^k \ q^{s(k\delta(k\delta-1)/2)+sk\delta m} \ \Psi_{j,i}^{(\delta k,-mk,\chi;s)} \ B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k q^{s(k\delta)^2};s,r|q)$$

$$= \sum_{n=0}^{[n/\delta]} \frac{\mathcal{H}_{n+1} \ \mathcal{G}_{n+1}}{\mathcal{B}_n \ \mathcal{C}_n \ (q^k;q^k)_n} x^{nk+k}. \tag{7.3.4}$$

The equation (7.3.1) now follows by comparing (7.3.3) and (7.3.4). □

In this proof, if the factor $\mathcal{G}_n = q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}$ is dropped then the resultant equation is satisfied by the function $\mathcal{W} = b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r|q)$. This is stated as

Theorem 7.3.2. *Let $\alpha, \beta, \lambda, m, \delta, \mu, k, s \in \mathbb{N}$, $r \in \mathbb{N} \cup \{0\}$, $m^* = [\frac{m}{\delta}]$ then $\mathcal{W} = b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q)$ satisfies the equation*

$$\left[\Phi_{\ell,\kappa}^{(\mu,\lambda,\eta;r)}\ \Phi_{h,g}^{(\alpha,\beta+1,\sigma;1)}\ \Theta - x^k\ \Psi_{j,i}^{(\delta k,-mk,\chi;s)}\right]\mathcal{W} = 0, \tag{7.3.5}$$

where χ is $(\delta k)^{th}$ root of unity, η is μ^{th} root of unity, σ is α^{th} root of unity.

7.4 Inverse series and inequality relations

In parallel to the inverse series inequality relations of section 6.4 and the results obtained in subsequent sections, the parameter $'s'$ is again exploited here to obtain analogues results. As mentioned in section 6.4, here also for $s = 1$ the usual inverse series relations occurs and for other values of s the inverse series inequality relations occur.

If the real valued functions $F(x, n; s|q)$, $G(x, n; s|q)$, $f(x, n; s|q)$, $g(x, n; s|q)$, where $s \in \mathbb{N} \setminus \{1\}$ are such that $F(x, n; s|q) < B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q)$, $G(x, n; s|q) > B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q)$, $f(x, n; s|q) < b_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q)$ and $g(x, n; s|q) > b_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q)$ then there hold the following inequality relations.

Theorem 7.4.1. *Let $F(x, n; s|q)$ and $G(x, n; s|q)$ be real valued functions, $\alpha, \beta, \lambda > 0$, and $\mu, k \in \mathbb{N}$, $r \in \mathbb{N}\cup\{0\}$. If s is odd positive integer and $m, (n-$a non negative integer) are even positive integers, then*

$$F(x, n; s|q) < B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q) \tag{7.4.1}$$

implies

$$\begin{aligned} x^{kn} \quad > \quad & \frac{q^{-mn(\alpha(\beta+1)+r\mu\lambda)}\ q^{-skmn(kmn-1)/2}\ (q^{\beta+1}; q)_{\alpha n}\ [(q^{\lambda}; q)_{\mu n}]^r\ (q^k; q^k)_n}{[(q^k; q^k)_{mn}]^s} \\ & \times \sum_{j=0}^{mn} \frac{q^{skj}\ [(q^{-kmn}; q^k)_j]^s}{(q^{\beta+1}; q)_{\alpha j}} F(x, j; s|q); \end{aligned} \tag{7.4.2}$$

and

$$\begin{aligned} x^{kn} \quad < \quad & \frac{q^{-mn(\alpha(\beta+1)+r\mu\lambda)}\ q^{-skmn(kmn-1)/2}\ (q^{\beta+1}; q)_{\alpha n}\ [(q^{\lambda}; q)_{\mu n}]^r\ (q^k; q^k)_n}{[(q^k; q^k)_{mn}]^s} \\ & \times \sum_{j=0}^{mn} \frac{q^{skj}\ [(q^{-kmn}; q^k)_j]^s}{(q^{\beta+1}; q)_{\alpha j}} G(x, j; s|q) \end{aligned} \tag{7.4.3}$$

implies

$$G(x,n;s|q) \quad > \quad B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r|q). \tag{7.4.4}$$

Proof. Here the inequality (7.4.1) holds. Putting

$$\begin{aligned}\omega_n \quad = \quad & \frac{q^{-mn(\alpha(\beta+1)+r\mu\lambda)}\ q^{-skmn(kmn-1)/2}\ (q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r\ (q^k;q^k)_n}{[(q^k;q^k)_{mn}]^s} \\ & \times \sum_{j=0}^{mn} \frac{q^{skj}\ [(q^{-kmn};q^k)_j]^s}{(q^{\beta+1};q)_{\alpha j}} F(x,j;s|q)\end{aligned}$$

and substituting the series inequality (7.4.1) for $F(x,j;s|q)$, one gets

$$\begin{aligned}\omega_n \quad < \quad & \frac{q^{-mn(\alpha(\beta+1)+r\mu\lambda)}\ q^{-skmn(kmn-1)/2}\ (q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r\ (q^k;q^k)_n}{[(q^k;q^k)_{mn}]^s} \\ & \times \sum_{j=0}^{mn} \frac{q^{skj}\ [(q^{-kmn};q^k)_j]^s}{(q^{\beta+1};q)_{\alpha j}} \frac{(q^{\beta+1};q)_{\alpha j}}{[(q^k;q^k)_j]^s} \\ & \times \sum_{i=0}^{[\frac{j}{m}]} \frac{q^{s(kmi(kmi-1)/2+kmij)}\ q^{mi(\alpha(\beta+1)+r\mu\lambda)}[(q^{-kj};q^k)_{mi}]^s\ x^{ki}}{(q^{\beta+1};q)_{\alpha i}\ [(q^\lambda;q)_{\mu i}]^r\ (q^k;q^k)_i} \\ = \quad & \frac{q^{-mn(\alpha(\beta+1)+r\mu\lambda)}\ q^{-skmn(mn-1)/2}\ (q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r\ (q^k;q^k)_n}{[(q^k;q^k)_{mn}]^s} \\ & \times \sum_{j=0}^{mn} \frac{q^{skj}\ (-1)^{sj} q^{skj(j-1)/2-skmnj}\ \left[(q^k;q^k)_{mn}\right]^s}{[(q^k;q^k)_{mn-j}]^s\ (q^{\beta+1};q)_{\alpha j}} \\ & \times \frac{(q^{\beta+1};q)_{\alpha j}}{((q^k;q^k)_j)^s} \sum_{i=0}^{[\frac{j}{m}]} \frac{(-1)^{smi}\ q^{s(kmi(kmi-1)/2+kmij)}\ q^{mi(\alpha(\beta+1)+r\mu\lambda)}}{[(q^k;q^k)_{j-mi}]^s\ (q^{\beta+1};q)_{\alpha i}\ [(q^\lambda;q)_{\mu i}]^r} \\ & \times \frac{q^{skmi(kmi-1)/2-skjmi}\ \left((q^k;q^k)_j\right)^s\ x^{ki}}{(q^k;q^k)_i} \\ = \quad & \sum_{j=0}^{mn} \sum_{i=0}^{[\frac{j}{m}]} \frac{(-1)^{sj+smi}\ q^{s(kmi(kmi-1)/2+kmij)}\ q^{mi(\alpha(\beta+1)+r\mu\lambda)}\ q^{skmi(mi-1)/2-skjmi}}{[(q^k;q^k)_{j-mi}]^s\ [(q^k;q^k)_{mn-j}]^s} \\ & \times \frac{q^{-mn(\alpha(\beta+1)+r\mu\lambda)}\ q^{-skmn(kmn-1)/2}\ q^{skj}\ q^{skj(j-1)/2-skmnj}\ (q^{\beta+1};q)_{\alpha n}}{(q^{\beta+1};q)_{\alpha i}\ [(q^\lambda;q)_{\mu i}]^r\ (q^k;q^k)_i} \\ & \times [(q^\lambda;q)_{\mu n}]^r\ (q^k;q^k)_n\ x^{ki}.\end{aligned}$$

Now in view of the double series relation (1.2.21)

$$\sum_{i=0}^{mn} \sum_{j=0}^{[\frac{i}{m}]} f(i,j) = \sum_{j=0}^{n} \sum_{i=0}^{mn-mj} f(i+mj,j),$$

one gets

$$
\begin{aligned}
\omega_n &< \sum_{i=0}^{n}\sum_{j=0}^{mn-mi} \frac{(-1)^{sj}\ q^{skmi(kmi-1)/2}\ q^{mi(\alpha(\beta+1)+r\mu\lambda)}\ q^{skj(mi-mn+1)+skmi(mi-mn)}}{((q^k;q^k)_j)^s\,[(q^k;q^k)_{mn-mi-j}]^s} \\
&\quad\times\frac{q^{skj(j-1)/2}\ q^{-mn(\alpha(\beta+1)+r\mu\lambda)}\ q^{-skmn(kmn-1)/2}\ (q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r}{(q^{\beta+1};q)_{\alpha i}\ [(q^\lambda;q)_{\mu i}]^r} \\
&\quad\times\frac{(q^k;q^k)_n\ x^{ki}}{(q^k;q^k)_i} \\
&= x^{kn}+\sum_{i=0}^{n-1}\frac{q^{s(kmi(kmi-1)/2+skmi(mi-mn)}\ q^{(mi-mn)(\alpha(\beta+1)+r\mu\lambda)}}{[(q^k;q^k)_{mn-mi}]^s\ (q^{\beta+1};q)_{\alpha i}\ [(q^\lambda;q)_{\mu i}]^r} \\
&\quad\times\frac{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r(q^k;q^k)_n\ x^{ki}}{(q^k;q^k)_i} \\
&\quad\times\sum_{j=0}^{mn-mi}(-1)^{sj}\ q^{skj(j-1)/2}\ q^{skj(mi-mn+1)}\begin{bmatrix} mn-mi \\ j\end{bmatrix}_{q^k}^{s} \\
&\le x^{kn}+\sum_{i=0}^{n-1}\frac{q^{skmi(kmi-1)/2+skmi(mi-mn)}\ q^{(mi-mn)(\alpha(\beta+1)+r\mu\lambda)}}{[(q^k;q^k)_{mn-mi}]^s\ (q^{\beta+1};q)_{\alpha i}\ [(q^\lambda;q)_{\mu i}]^r} \\
&\quad\times\frac{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r(q^k;q^k)_n\ x^{ki}}{(q^k;q^k)_i} \\
&\quad\times\left(\sum_{j=0}^{mn-mi}(-1)^{j}\ q^{kj(j-1)/2}\ q^{skj(mi-mn+1)}\begin{bmatrix} mn-mi \\ j\end{bmatrix}_{q^k}\right)^{s} \\
&= x^{kn}+\sum_{i=0}^{n-1}\frac{q^{s(kmi(kmi-1)/2+skmi(mi-mn)}\ (q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r}{[(q^k;q^k)_{mn-mi}]^s\ (q^{\beta+1};q)_{\alpha i}\ [(q^\lambda;q)_{\mu i}]^r} \\
&\quad\times\frac{(q^k;q^k)_n\ x^{ki}}{(q^k;q^k)_i}\left\{\prod_{j=1}^{mn-mi}(1-q^{k(mi-mn+j)})\right\}^{s}.
\end{aligned}
$$

Here the product on the right hand side vanishes, hence $\omega_n < x^{kn}$.

Next, the proof of another inequality relations stated above runs as follows. Here (7.4.3) holds true. Now if

$$
\begin{aligned}
\nu_n &= \frac{(q^{\beta+1};q)_{\alpha n}}{[(q^k;q^k)_n]^s}\sum_{j=0}^{\left[\frac{n}{m}\right]}\frac{q^{s(kmj(kmj-1)/2+skmjn)}\ q^{mj(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{\alpha j}\ [(q^\lambda;q)_{\mu j}]^r} \\
&\quad\times\frac{[(q^{-nk};q^k)_{mj}]^s\ x^{kj}}{(q^k;q^k)_j}
\end{aligned}
$$

then substituting the series inequality (7.4.3) for x^{kn} , one gets

$$\begin{aligned}
\nu_n &< \frac{(q^{\beta+1};q)_{\alpha n}}{[(q^k;q^k)_n]^s}\sum_{j=0}^{\left[\frac{n}{m}\right]}\frac{q^{s(kmj(kmj-1)/2+kmjn)}\ q^{mj(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{\alpha j}\ [(q^\lambda;q)_{\mu j}]^r}\frac{[(q^{-nk};q^k)_{mj}]^s}{(q^k;q^k)_j}\\
&\quad\times\frac{q^{-mj(\alpha(\beta+1)+r\mu\lambda)}\ q^{-skmj(kmj-1)/2}\ (q^{\beta+1};q)_{\alpha j}\ [[q^\lambda]_{\mu j}]^r\ (q^k;q^k)_j}{[(q^k;q^k)_{mj}]^s}\\
&\quad\times\sum_{i=0}^{mj}\frac{q^{ski}\ [(q^{-kmj};q^k)_i]^s}{(q^{\beta+1};q)_{\alpha i}}G(x,i;s|q)\\
&= \frac{(q^{\beta+1};q)_{\alpha n}}{((q^k;q^k)_n)^s}\sum_{j=0}^{\left[\frac{n}{m}\right]}\frac{(-1)^{smj}\ q^{skmjn}\ q^{skmj(mj-1)/2-snmj}\ ((q^k;q^k)_n)^s}{\left[(q^k;q^k)_{(n-mj)}\right]^s\ [(q^k;q^k)_{mj}]^s}\\
&\quad\times\sum_{i=0}^{mj}\frac{(-1)^{is}\ q^{ski}\ q^{ski(i-1)/2-skimj}\ \left[(q^k;q^k)_{mj}\right]^s}{\left[(q^k;q^k)_{(n-mj)}\right]^s\ (q^{\beta+1};q)_{\alpha i}}G(x,i;s|q)\\
&= \sum_{mj=0}^{n}\sum_{i=0}^{mj}\frac{(-1)^{smj+is}\ q^{ski(i+1)/2}\ q^{skmj(mj-1)/2-skimj}\ (q^{\beta+1};q)_{\alpha n}}{\left[(q^k;q^k)_{(n-mj)}\right]^s\ \left[(q^k;q^k)_{(mj-i)}\right]^s\ (q^{\beta+1};q)_{\alpha i}}\\
&\quad\times G(x,i;s|q).
\end{aligned}$$

In view of double series relation (1.2.22), this takes the form:

$$\begin{aligned}
\nu_n &< \sum_{i=0}^{n}\sum_{mj=i}^{n}\frac{(-1)^{smj+is}\ q^{ski(i+1)/2}\ q^{skmj(mj-1)/2-skimj}\ (q^{\beta+1};q)_{\alpha n}}{\left[(q^k;q^k)_{(n-mj)}\right]^s\ \left[(q^k;q^k)_{(mj-i)}\right]^s\ (q^{\beta+1};q)_{\alpha i}}\\
&\quad\times G(x,i;s|q)\\
&= G(x,n;s|q)+\sum_{i=0}^{n-1}\frac{(-1)^{is}\ q^{ski(i+1)/2}\ (q^{\beta+1};q)_{\alpha n}}{(q^{\beta+1};q)_{\alpha i}}G(x,i;s|q)\\
&\quad\times\sum_{mj=i}^{n}\frac{(-1)^{smj}\ q^{skmj(mj-1)/2-skimj}}{\left[(q^k;q^k)_{(n-mj)}\right]^s\ \left[(q^k;q^k)_{(mj-i)}\right]^s}\\
&= G(x,n;s|q)+\sum_{i=0}^{n-1}\frac{(q^{\beta+1};q)_{\alpha n}}{(q^{\beta+1};q)_{\alpha i}}G(x,i;s|q)\\
&\quad\times\sum_{mj=0}^{n-i}\frac{(-1)^{smj}\ q^{skmj(mj-1)/2}}{\left[(q^k;q^k)_{(n-i-mj)}\right]^s\ [(q^k;q^k)_{mj}]^s}\\
&= G(x,n;s|q)+\sum_{i=0}^{n-1}\frac{q^{ski(i+1)/2}\ (q^{\beta+1};q)_{\alpha n}}{(q^{\beta+1};q)_{\alpha i}\left[(q^k;q^k)_{(n-i)}\right]^s}G(x,i;s|q)\\
&\quad\times\sum_{mj=0}^{n-i}(-1)^{smj}\ q^{skmj(mj-1)/2}\begin{bmatrix}n-i\\ mj\end{bmatrix}_k^s\\
&\le G(x,n;s|q)+\sum_{i=0}^{n-1}\frac{q^{ski(i+1)/2}\ (q^{\beta+1};q)_{\alpha n}}{(q^{\beta+1};q)_{\alpha i}\left[(q^k;q^k)_{(n-i)}\right]^s}B_{i^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r|q)
\end{aligned}$$

$$\times\left(\sum_{mj=0}^{n-i}(-1)^{mj}\quad q^{kmj(mj-1)/2}\begin{bmatrix}n-i\\ mj\end{bmatrix}_k\right)^s \tag{7.4.5}$$

$$= \quad G(x,n;s|q)+\sum_{i=0}^{n-1}\frac{q^{ski(i+1)/2}\ (q^{\beta+1};q)_{\alpha n}}{(q^{\beta+1};q)_{\alpha i}\left[(q^k;q^k)_{(n-i)}\right]^s}G(x,i;s|q)$$

$$\times\left\{\prod_{mj=1}^{n-i}\left(1-q^{kmj-k}\right)\right\}^s.$$

This gives $\nu_n < G(x,n;s|q)$. □

Towards the converse of these inequality relations, one can obtain the following theorem.

Theorem 7.4.2. *Let $F(x,n;s|q)$ and $G(x,n;s|q)$ be real valued functions, $\alpha,\beta,\lambda>0$, and $\mu,k\in\mathbb{N}$, $\;r\in\mathbb{N}\cup\{0\}$. If either s is an even positive integer or $s,m,(n-a$ non negative integer) are all odd positive integers, then*

$$x^{kn} \quad > \quad \frac{q^{-mn(\alpha(\beta+1)+r\mu\lambda)}\ q^{-skmn(kmn-1)/2}\ (q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r\ (q^k;q^k)_n}{[(q^k;q^k)_{mn}]^s}$$

$$\times\sum_{j=0}^{mn}\frac{q^{skj}\ [(q^{-kmn};q^k)_j]^s}{(q^{\beta+1};q)_{\alpha j}}F(x,j;s|q) \tag{7.4.6}$$

implies

$$F(x,n;s|q) < B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r|q); \tag{7.4.7}$$

and

$$G(x,n;s|q) \quad > \quad B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r|q)m \tag{7.4.8}$$

implies

$$x^{kn} \quad < \quad \frac{q^{-mn(\alpha(\beta+1)+r\mu\lambda)}\ q^{-skmn(kmn-1)/2}\ (q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r\ (q^k;q^k)_n}{[(q^k;q^k)_{mn}]^s}$$

$$\times\sum_{j=0}^{mn}\frac{q^{skj}\ [(q^{-kmn};q^k)_j]^s}{(q^{\beta+1};q)_{\alpha j}}G(x,j;s|q)m. \tag{7.4.9}$$

The proof runs parallel to that of Theorem 7.4.1, hence is omitted.
For $s=1$, the polynomial (7.1.1) yields the following inverse series relation.

Theorem 7.4.3. *For $\alpha, \beta, \lambda > 0,\ m, \mu, k \in \mathbb{N},\ r \in \mathbb{N} \cup \{0\}$,*

$$B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q) = \frac{(q^{\beta+1};q)_{\alpha n}}{(q^k;q^k)_n} \sum_{j=0}^{\left[\frac{n}{m}\right]} \frac{q^{k(mj(mj-1)/2+mjn)}\ q^{mj(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{\alpha j}\ [(q^\lambda;q)_{\mu j}]^r} \times \frac{(q^{-nk};q^k)_{mj}\ x^{kj}}{(q^k;q^k)_j} \tag{7.4.10}$$

if and only if

$$\frac{x^{kn}}{(q^k;q^k)_n} = \frac{q^{-mn(\alpha(\beta+1)+r\mu\lambda)}\ q^{-kmn(kmn-1)/2}\ (q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r}{(q^k;q^k)_{mn}} \times \sum_{j=0}^{mn} \frac{q^{kj}\ (q^{-kmn};q^k)_j}{(q^{\beta+1};q)_{\alpha j}} B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q), \tag{7.4.11}$$

and for $n \neq ml,\ l \in \mathbb{N}$,

$$\sum_{j=0}^{n} \frac{q^{kj}\ (q^{-kn};q^k)_j}{(q^{\beta+1};q)_{\alpha j}} B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q), = 0. \tag{7.4.12}$$

Proof. The proof of (7.4.10) implies (7.4.11) runs as follows.
Here the equality (7.4.10) holds. Putting

$$J_n = \frac{q^{-mn(\alpha(\beta+1)+r\mu\lambda)}\ q^{-kmn(kmn-1)/2}\ (q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r\ (q^k;q^k)_n}{(q^k;q^k)_{mn}} \times \sum_{j=0}^{mn} \frac{q^{kj}\ (q^{-kmn};q^k)_j}{(q^{\beta+1};q)_{\alpha j}} B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q)$$

and substituting the series equality (7.4.10) for $B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q)$, one gets

$$\begin{aligned} J_n &= \frac{q^{-mn(\alpha(\beta+1)+r\mu\lambda)}\ q^{-kmn(kmn-1)/2}\ (q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r\ (q^k;q^k)_n}{(q^k;q^k)_{mn}} \\ &\quad \times \sum_{j=0}^{mn} \frac{q^{kj}\ (q^{-kmn};q^k)_j}{(q^{\beta+1};q)_{\alpha j}} \frac{(q^{\beta+1};q)_{\alpha j}}{(q^k;q^k)_j} \\ &\quad \times \sum_{i=0}^{\left[\frac{j}{m}\right]} \frac{q^{kmi(kmi-1)/2+kmij}\ q^{mi(\alpha(\beta+1)+r\mu\lambda)}\ (q^{-kj};q^k)_{mi}\ x^{ki}}{(q^{\beta+1};q)_{\alpha i}\ [(q^\lambda;q)_{\mu i}]^r\ (q^k;q^k)_i} \\ &= \frac{q^{-mn(\alpha(\beta+1)+r\mu\lambda)}\ q^{-kmn(mn-1)/2}\ (q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r\ (q^k;q^k)_n}{(q^k;q^k)_{mn}} \\ &\quad \times \sum_{j=0}^{mn} \frac{q^{kj}\ (-1)^j q^{kj(j-1)/2-kmnj}(q^k;q^k)_{mn}}{(q^k;q^k)_{mn-j}(q^{\beta+1};q)_{\alpha j}} \end{aligned}$$

$$\times \frac{(q^{\beta+1};q)_{\alpha j}}{(q^k;q^k)_j} \sum_{i=0}^{\left[\frac{j}{m}\right]} \frac{(-1)^{mi}\ q^{kmi(kmi-1)/2+kmij}\ q^{mi(\alpha(\beta+1)+r\mu\lambda)}}{(q^k;q^k)_{j-mi}(q^{\beta+1};q)_{\alpha i}\ [(q^\lambda;q)_{\mu i}]^r}$$
$$\times \frac{q^{kmi(kmi-1)/2-kjmi}(q^k;q^k)_j\ x^{ki}}{(q^k;q^k)_i}$$
$$= \sum_{j=0}^{mn} \sum_{i=0}^{\left[\frac{j}{m}\right]} \frac{(-1)^{j+mi}\ q^{kmi(kmi-1)/2+kmij} q^{mi(\alpha(\beta+1)+r\mu\lambda)} q^{kmi(mi-1)/2-kjmi}}{(q^k;q^k)_{j-mi}(q^k;q^k)_{mn-j}}$$
$$\times \frac{q^{-mn(\alpha(\beta+1)+r\mu\lambda)}\ q^{-kmn(kmn-1)/2}\ q^{kj}\ q^{kj(j-1)/2-kmnj}\ (q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r}{(q^{\beta+1};q)_{\alpha i}\ [(q^\lambda;q)_{\mu i}]^r\ (q^k;q^k)_i}$$
$$\times (q^k;q^k)_n\ x^{ki}.$$

Now in view of the double series relation (1.2.21)

$$\sum_{i=0}^{mn} \sum_{j=0}^{\left[\frac{i}{m}\right]} f(i,j) = \sum_{j=0}^{n} \sum_{i=0}^{mn-mj} f(i+mj,j),$$

one gets

$$J_n = \sum_{i=0}^{n} \sum_{j=0}^{mn-mi} \frac{(-1)^j\ q^{kmi(kmi-1)/2}\ q^{mi(\alpha(\beta+1)+r\mu\lambda)}\ q^{kj(mi-mn+1)+kmi(mi-mn)}}{(q^k;q^k)_j\ (q^k;q^k)_{mn-mi-j}}$$
$$\times \frac{q^{kj(j-1)/2} q^{-mn(\alpha(\beta+1)+r\mu\lambda)} q^{-kmn(kmn-1)/2} (q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r\ (q^k;q^k)_n}{(q^{\beta+1};q)_{\alpha i}\ [(q^\lambda;q)_{\mu i}]^r}$$
$$\times \frac{x^{ki}}{(q^k;q^k)_i}$$
$$= \frac{x^{kn}}{(q^k;q^k)_n} + \sum_{i=0}^{n-1} \frac{q^{kmi(kmi-1)/2+kmi(mi-mn)}\ q^{(mi-mn)(\alpha(\beta+1)+r\mu\lambda)}}{(q^k;q^k)_{mn-mi}\ (q^{\beta+1};q)_{\alpha i}\ [(q^\lambda;q)_{\mu i}]^r}$$
$$\times \frac{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r (q^k;q^k)_n\ x^{ki}}{(q^k;q^k)_i}$$
$$\times \sum_{j=0}^{mn-mi} (-1)^j\ q^{kj(j-1)/2}\ q^{kj(mi-mn+1)} \begin{bmatrix} mn-mi \\ j \end{bmatrix}_{q^k}$$
$$= \frac{x^{kn}}{(q^k;q^k)_n} + \sum_{i=0}^{n-1} \frac{q^{kmi(kmi-1)/2+kmi(mi-mn)}\ q^{(mi-mn)(\alpha(\beta+1)+r\mu\lambda)}}{(q^k;q^k)_{mn-mi}\ (q^{\beta+1};q)_{\alpha i}\ [(q^\lambda;q)_{\mu i}]^r}$$
$$\times \frac{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r (q^k;q^k)_n\ x^{ki}}{(q^k;q^k)_i}$$
$$\times \sum_{j=0}^{mn-mi} (-1)^j\ q^{kj(j-1)/2}\ q^{kj(mi-mn+1)} \begin{bmatrix} mn-mi \\ j \end{bmatrix}_{q^k}$$
$$= \frac{x^{kn}}{(q^k;q^k)_n} + \sum_{i=0}^{n-1} \frac{q^{kmi(kmi-1)/2+kmi(mi-mn)}\ (q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r}{(q^k;q^k)_{mn-mi}\ (q^{\beta+1};q)_{\alpha i}\ [(q^\lambda;q)_{\mu i}]^r\ (q^k;q^k)_i}$$

$$\begin{aligned}
&\times (q^k;q^k)_n \; x^{ki} \prod_{j=1}^{mn-mi} (1-q^{k(mi-mn+j)}) \\
= &\frac{x^{kn}}{(q^k;q^k)_n}
\end{aligned}$$

as the product on the right hand side vanishes. To show further that (7.4.10) also implies (7.4.12), one may substitute for $B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q)$ from (7.4.10) to the left hand side of (7.4.12), to get

$$\begin{aligned}
&\sum_{j=0}^{n} \frac{q^{kj}\;(q^{-kn};q^k)_j}{(q^{\beta+1};q)_{\alpha j}} B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q) \\
= &\sum_{j=0}^{n} \frac{q^{kj}\;(-1)^j\;q^{kj(j-1)/2-knj}(q^k;q^k)_n}{(q^k;q^k)_{n-j}(q^{\beta+1};q)_{\alpha j}} \\
&\times \sum_{i=0}^{[\frac{j}{m}]} \frac{(-1)^{mi}\;q^{kmi(kmi-1)/2+kmij}q^{mi(\alpha(\beta+1)+r\mu\lambda)}q^{kmi(kmi-1)/2-kjmi}(q^k;q^k)_j\;x^{ki}}{(q^{\beta+1};q)_{\alpha i}\;[(q^\lambda;q)_{\mu i}]^r\;(q^k;q^k)_{j-mi}\;(q^k;q^k)_i} \\
= &\sum_{i=0}^{[\frac{n}{m}]} \frac{q^{3kmi(kmi-1)/2+kmi-knmi}\;q^{mi(\alpha(\beta+1)+r\mu\lambda)}\;(q^k;q^k)_n\;x^{ki}}{(q^{\beta+1};q)_{\alpha i}\;[(q^\lambda;q)_{\mu i}]^r\;(q^k;q^k)_{n-mi}\;(q^k;q^k)_i} \\
&\times \sum_{j=0}^{n-mi} (-1)^j\;q^{kj(j-1)/2}\;q^{kj(mi-n+1)} \begin{bmatrix} n-mi \\ j \end{bmatrix}_{q^k} \\
= &\sum_{i=0}^{[\frac{n}{m}]} \frac{q^{3kmi(kmi-1)/2+kmi-knmi}\;q^{mi(\alpha(\beta+1)+r\mu\lambda)}\;(q^k;q^k)_n\;x^{ki}}{(q^{\beta+1};q)_{\alpha i}\;[(q^\lambda;q)_{\mu i}]^r\;(q^k;q^k)_{n-mi}\;(q^k;q^k)_i} \prod_{j=1}^{n-mi} (1-q^{k(mi-n+j)}) \\
= &\;0
\end{aligned}$$

if $n \neq ml,\; l \in \mathbb{N}$. Thus completing the first part. The proof of converse part runs as follows [13]. In order to show that both the series (7.4.11) and the condition (7.4.12) together imply the series (7.4.10), the following simplest inverse series relations [64, Eq.(1), p.43] will be used.

$$\Delta_n = \sum_{j=0}^{n} \frac{q^{knj}\;(q^{-kn};q^k)_j}{(q^k;q^k)_j}\;\Psi_j \;\Leftrightarrow\; \Psi_n = \sum_{j=0}^{n} \frac{q^{kj}\;(q^{-kn};q^k)_j}{(q^k;q^k)_j}\;\Delta_j.$$

Here putting

$$\Psi_j = \frac{q^{kj}\;(q^k;q^k)_j}{(q^{\beta+1};q)_{\alpha j}}\;B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q),$$

and considering one sided relation that is, the series on the left hand side implies the series on the right side, one gets

$$\Delta_n = \sum_{j=0}^{n} \frac{q^{kj}\ (q^{-kn};q^k)_j}{(q^{\beta+1};q)_{\alpha j}}\ B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q) \tag{7.4.13}$$

$\Rightarrow$

$$B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q) = \frac{(q^{\beta+1};q)_{\alpha j}}{(q^k;q^k)_n} \sum_{j=0}^{n} \frac{(q^{-kn};q^k)_j}{(q^k;q^k)_j}\ \Delta_j. \tag{7.4.14}$$

Since the condition (7.4.12) holds, $\omega_n = 0$ for $n \neq ml,\ l \in \mathbb{N}$, whereas

$$\Delta_{mn} = \sum_{j=0}^{mn} \frac{q^{kj}\ (q^{-kmn};q^k)_j}{(q^{\beta+1};q)_{\alpha j}}\ B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q).$$

But since the series (7.4.11) holds true.

$$\Delta_{mn} = \frac{q^{mn(\alpha(\beta+1)+r\mu\lambda)}\ q^{kmn(kmn-1)/2}\ (q^k;q^k)_{mn}\ x^{kn}}{(q^k;q^k)_n\ (q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r}.$$

Consequently, the inverse pair (7.4.13) and (7.4.14) assume the form:

$$\begin{aligned} B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q) \quad &= \quad \frac{(q^{\beta+1};q)_{\alpha n}}{(q^k;q^k)_n} \sum_{j=0}^{\left[\frac{n}{m}\right]} \frac{q^{k(mj(mj-1)/2+mjn)}\ q^{mj(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{\alpha j}\ [(q^\lambda;q)_{\mu j}]^r} \\ &\quad \times \frac{(q^{-nk};q^k)_{mj}\ x^{kj}}{(q^k;q^k)_j} \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} \frac{x^{kn}}{(q^k;q^k)_n} \quad &= \quad \frac{q^{-mn(\alpha(\beta+1)+r\mu\lambda)}\ q^{-kmn(kmn-1)/2}\ (q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r}{(q^k;q^k)_{mn}} \\ &\quad \times \sum_{j=0}^{mn} \frac{q^{kj}\ (q^{-kmn};q^k)_j}{(q^{\beta+1};q)_{\alpha j}} B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q), \end{aligned}$$

subject to the condition (7.4.12). □

The following theorems are in parallel to the Theorem 7.4.1, Theorem 7.4.2 and Theorem 7.4.3.

Theorem 7.4.4. *Let* $f(x,n;s|q)$ *and* $g(x,n;s|q)$ *be real valued functions,* $\alpha,\beta,\lambda >$ 0*, and* $\mu,k \in \mathbb{N},\ r \in \mathbb{N}\cup\{0\}$*. If* s *is odd positive integer and* $m, (n-$*a non negative* *integer) are even positive integers, then*

$$f(x,n;s|q) < b_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r|q) \tag{7.4.15}$$

implies

$$x^{kn} > \frac{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r\ (q^k;q^k)_n}{[(q^k;q^k)_{mn}]^s} \sum_{j=0}^{mn} \frac{q^{skj}\ [(q^{-kmn};q^k)_j]^s}{(q^{\beta+1};q)_{\alpha j}} f(x,j;s|q); \tag{7.4.16}$$

and

$$x^{kn} < \frac{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r\ (q^k;q^k)_n}{[(q^k;q^k)_{mn}]^s} \sum_{j=0}^{mn} \frac{q^{skj}\ [(q^{-kmn};q^k)_j]^s}{(q^{\beta+1};q)_{\alpha j}} g(x,j;s|q) \tag{7.4.17}$$

implies

$$g(x,n;s|q) > b_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r|q). \tag{7.4.18}$$

Theorem 7.4.5. *Let $f(x,n;s|q)$ and $g(x,n;s|q)$ be real valued functions, $\alpha,\beta,\lambda >$ 0, and $\mu,k \in \mathbb{N}$, $r \in \mathbb{N}\cup\{0\}$. If either s is an even positive integer or $s,m,(n-a$ non negative integer) are all odd positive integers, then*

$$x^{kn} > \frac{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r\ (q^k;q^k)_n}{[(q^k;q^k)_{mn}]^s} \sum_{j=0}^{mn} \frac{q^{skj}\ [(q^{-kmn};q^k)_j]^s}{(q^{\beta+1};q)_{\alpha j}} f(x,j;s|q) \tag{7.4.19}$$

implies

$$f(x,n;s|q) < b_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r|q); \tag{7.4.20}$$

and

$$g(x,n;s|q) > b_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r|q)m \tag{7.4.21}$$

implies

$$x^{kn} < \frac{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r\ (q^k;q^k)_n}{[(q^k;q^k)_{mn}]^s} \sum_{j=0}^{mn} \frac{q^{skj}\ [(q^{-kmn};q^k)_j]^s}{(q^{\beta+1};q)_{\alpha j}} g(x,j;s|q). \tag{7.4.22}$$

For $s=1$, the polynomial (7.1.2) yields the following inverse series relation.

Theorem 7.4.6. *For $\alpha,\beta,\lambda>0,\ m,\mu,k\in\mathbb{N},\ r\in\mathbb{N}\cup\{0\}$, the polynomial*

$$b_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q) \;=\; \frac{(q^{\beta+1};q)_{\alpha n}}{(q^k;q^k)_n}\sum_{j=0}^{\left[\frac{n}{m}\right]}\frac{(q^{-nk};q^k)_{mj}\;x^{kj}}{(q^{\beta+1};q)_{\alpha j}\;[(q^\lambda;q)_{\mu j}]^r\;(q^k;q^k)_j} \tag{7.4.23}$$

if and only if

$$x^{kn}=\frac{(q^{\beta+1};q)_{\alpha n}\;[(q^\lambda;q)_{\mu n}]^r\;(q^k;q^k)_n}{(q^k;q^k)_{mn}}\sum_{j=0}^{mn}\frac{q^{kmn(j+1)}\;(q^{-mnk};q^k)_j}{(q^{\beta+1};q)_{\alpha j}}b_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q), \tag{7.4.24}$$

and for $n\neq ml,\ l\in\mathbb{N}$,

$$\sum_{j=0}^{n}\frac{q^{kj}\;(q^{-kn};q^k)_j}{(q^{\beta+1};q)_{\alpha j}}B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q),=0. \tag{7.4.25}$$

7.5 Some inequalities

In this section certain inequalities containing q-GKP are obtained.

Theorem 7.5.1. *If $\alpha,\beta,\lambda>0,\ m,\delta,\mu,k,s\in\mathbb{N},\ r\in\mathbb{N}\cup\{0\},\ 0<st<1$ then the following series inequality holds.*

$$\sum_{m=0}^{\infty}\frac{B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r|q)}{(q^{\beta+1};q)_{\alpha m}}\;t^{ms}\;\leq\;\left(e_{q^k}(t)\right)^s\;\frac{[(q^k;q^k)_\infty]^s}{(q^{\beta+1};q)_\infty}\times B_{\infty}^{(\alpha,\beta,\lambda,\mu)}(x^k\,t^{s\delta};s,r|q). \tag{7.5.1}$$

Proof. From left hand side of (7.5.1),

$$\begin{aligned}
&\sum_{m=0}^{\infty}\frac{B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r|q)}{(q^{\beta+1};q)_{\alpha m}}\;t^{ms}\\
=&\sum_{m=0}^{\infty}\frac{1}{(q^{\beta+1};q)_{\alpha m}}\frac{(q^{\beta+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s}\\
&\times\sum_{n=0}^{m^*}\frac{q^{sk\delta n(m+(\delta nk-1)/2)}\;q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\;[(q^{-mk};q^k)_{\delta n}]^s\;x^{kn}}{(q^{\beta+1};q)_{\alpha n}\;[(q^\lambda;q)_{\mu n}]^r\;(q^k;q^k)_n}\;t^{ms}\\
=&\sum_{m=0}^{\infty}\sum_{n=0}^{m^*}\frac{q^{sk\delta n(m+(\delta nk-1)/2)}\;q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\;x^{kn}}{[(q^k;q^k)_m]^s\;(q^{\beta+1};q)_{\alpha n}\;[(q^\lambda;q)_{\mu n}]^r\;(q^k;q^k)_n}\\
&\times\frac{(-1)^{s\delta n}\;q^{sk\delta n(\delta n-1)/2-skm\delta n}\;[(q^k;q^k)_m]^s}{[(q^k;q^k)_{(m-\delta n)}]^s}\;t^{ms}
\end{aligned}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{m^*} \frac{(-1)^{s\delta n}\ q^{sk\delta n(k\delta n-1)/2+sk\delta n(\delta n-1)/2}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\ x^{kn}}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r\ (q^k;q^k)_n} \frac{t^{ms}}{[(q^k;q^k)_{(m-\delta n)}]^s}$$

$$= \sum_{m=0}^{\infty} \frac{t^{ms}}{[(q^k;q^k)_m]^s} \sum_{n=0}^{\infty} \frac{(-1)^{s\delta n}\ q^{sk\delta n(k\delta n-1)/2+sk\delta n(\delta n-1)/2}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\ t^{s\delta n}\ x^{kn}}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r\ (q^k;q^k)_n}.$$

Here the inner sum is obtained by making limit $m \to \infty$ in

$$B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k\, t^{s\delta}; s, r|q)$$
$$= \frac{(q^{\beta+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s} \sum_{n=0}^{m^*} \frac{t^{s\delta n}\ q^{sk\delta n(m+(\delta nk-1)/2)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\ [(q^{-mk};q^k)_{\delta n}]^s\ x^{kn}}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r\ (q^k;q^k)_n},$$

and since $0 < t < 1$,

$$\sum_{m=0}^{\infty} \frac{B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q)}{(q^{\beta+1};q)_{\alpha m}}\ t^{ms}$$
$$= \sum_{m=0}^{\infty} \frac{t^{ms}}{[(q^k;q^k)_m]^s} \frac{[(q^k;q^k)_\infty]^s}{(q^{\beta+1};q)_\infty}\ B_{\infty}^{(\alpha,\beta,\lambda,\mu)}(x^k\, t^{s\delta}; s, r|q)$$
$$\leq \left(\sum_{m=0}^{\infty} \frac{t^m}{(q^k;q^k)_m}\right)^s \frac{[(q^k;q^k)_\infty]^s}{(q^{\beta+1};q)_\infty}\ B_{\infty}^{(\alpha,\beta,\lambda,\mu)}(x^k\, t^{s\delta}; s, r|q)$$
$$= \left(e_{q^k}(t)\right)^s \frac{[(q^k;q^k)_\infty]^s}{(q^{\beta+1};q)_\infty}\ B_{\infty}^{(\alpha,\beta,\lambda,\mu)}(x^k\, t^{s\delta}; s, r|q).$$

□

Theorem 7.5.2. *If* $\alpha, \beta, \lambda > 0, m, \delta, \mu, k, s \in \mathbb{N},\ r \in \mathbb{N}\cup\{0\}, 0 < t < 1, 0 < st < 1$ *then*

$$\sum_{m=0}^{\infty} \frac{q^{skm(m-1)/2}\ b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q)}{(q^{\beta+1};q)_{\alpha m}}\ t^{ms}$$
$$\leq \left(E_{q^k}(t)\right)^s \sum_{n=0}^{\infty} \frac{(-tq^{-k})^{s\delta n}\ x^{kn}}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r\ (q^k;q^k)_n}. \qquad (7.5.2)$$

Proof. From left hand side of (7.5.2),

$$\sum_{m=0}^{\infty} \frac{q^{skm(m-1)/2}\ b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q)}{(q^{\beta+1};q)_{\alpha m}}\ t^{ms}$$
$$= \sum_{m=0}^{\infty} \frac{q^{skm(m-1)/2}}{(q^{\beta+1};q)_{\alpha m}} \frac{(q^{\beta+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s} \sum_{n=0}^{m^*} \frac{[(q^{-mk};q^k)_{\delta n}]^s\ x^{kn}}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r\ (q^k;q^k)_n}\ t^{ms}$$
$$= \sum_{m=0}^{\infty} q^{skm(m-1)/2} \sum_{n=0}^{m^*} \frac{(-1)^{s\delta n}\ q^{sk\delta n(\delta n-1)/2-skm\delta n}}{[(q^k;q^k)_{(m-\delta n)}]^s\ (q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r} \frac{x^{kn}\ t^{ms}}{(q^k;q^k)_n}$$

$$
\begin{aligned}
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{s\delta n}\, q^{skm(m-1)/2-sk\delta n}\, x^{kn}\, t^{s(m+\delta n)}}{[(q^k;q^k)_m]^s\, (q^{\beta+1};q)_{\alpha n}\, [(q^\lambda;q)_{\mu n}]^r\, (q^k;q^k)_n} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\left(\frac{q^{km(m-1)/2}\, t^m}{(q^k;q^k)_m}\right)^s\right) \frac{(-1)^{s\delta n}\, (tq^{-k})^{s\delta n}\, x^{kn}}{(q^{\beta+1};q)_{\alpha n}\, [(q^\lambda;q)_{\mu n}]^r\, (q^k;q^k)_n} \\
&\le \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\frac{q^{km(m-1)/2}\, t^m}{(q^k;q^k)_m}\right)^s \frac{(-tq^{-k})^{s\delta n}\, x^{kn}}{(q^{\beta+1};q)_{\alpha n}\, [(q^\lambda;q)_{\mu n}]^r\, (q^k;q^k)_n} \\
&= \left(E_{q^k}(t)\right)^s \sum_{n=0}^{\infty} \frac{(-tq^{-k})^{s\delta n}\, x^{kn}}{(q^{\beta+1};q)_{\alpha n}\, [(q^\lambda;q)_{\mu n}]^r\, (q^k;q^k)_n} \\
&= r.h.s,
\end{aligned}
$$

when $0 < t < 1$. □

7.5.1 Special cases - Generating function relations

For $s = 1$, the series inequality relations in Theorem 7.5.1 will yield the generating function relation. Their various specializations are deduced here.

(i) Taking $\alpha = k \in \mathbb{N},\ r = 0$ in (7.5.1) leads to

$$
\sum_{m=0}^{\infty} \frac{L_{m^*}^{(k,\beta)}(x^k|q)}{(q^{\beta+1};q)_{km}}\, t^m = e_{q^k}(t)\, \frac{(q^k;q^k)_\infty}{(q^{\beta+1};q)_\infty}\, L_\infty^{(k,\beta)}(x^k\, t^\delta|q).
$$

Further the case $\delta = 1$, gives

$$
\sum_{m=0}^{\infty} \frac{Z_m^{\beta}(x;k|q)}{(q^{\beta+1};q)_{km}}\, t^m = e_{q^k}(t)\, \frac{(q^k;q^k)_\infty}{(q^{\beta+1};q)_\infty}\, Z_\infty^{\beta}(x\, t^{\frac{1}{k}};k|q).
$$

Finally, for $k = 1$ this reduces to (cf. [63, Eq. (1), p. 201])

$$
\sum_{m=0}^{\infty} \frac{L_m^{(\beta)}(x|q)}{(q^{\beta+1};q)_m}\, t^m = e_q(t)\, \frac{(q;q)_\infty}{(q^{\beta+1};q)_\infty}\, L_\infty^{(\beta)}(xt|q).
$$

7.5.2 Special cases-Inequalities

If $\alpha = k \in \mathbb{N}$ in (7.5.1) then

$$
\sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k;s,r|q)}{(q^{\beta+1};q)_{km}}\, t^{ms} \le \left(e_{q^k}(t)\right)^s \frac{[(q^k;q^k)_\infty]^s}{(q^{\beta+1};q)_\infty}\, B_\infty^{(k,\beta,\lambda,\mu)}(x^k\, t^{s\delta};s,r|q). \quad (7.5.3)
$$

This will be used in the next section.
Further for $\delta = 1, r = 0$, this reduces to

$$\sum_{m=0}^{\infty} \frac{Z^{\beta}_{m,s}(x^k|q)}{(q^{\beta+1};q)_{km}}\, t^{ms} \;\; \leq \;\; \big(e_{q^k}(t)\big)^s \, \frac{[(q^k;q^k)_\infty]^s}{(q^{\beta+1};q)_\infty}\, Z^{\beta}_{\infty}(x^k\, t^s;|q).$$

Consequently, the generalized Laguerre polynomial case $k = 1$, is

$$\sum_{m=0}^{\infty} \frac{L^{(\beta)}_{m,s}(x|q)}{(q^{\beta+1};q)_{m}}\, t^{ms} \;\; \leq \;\; (e_q(t))^s \, \frac{[(q;q)_\infty]^s}{(q^{\beta+1};q)_\infty}\, L^{(\beta)}_{\infty,s}(x\, t^s|q).$$

Here

$$Z^{\beta}_{m,s}(x^k|q) \;\; = \;\; \frac{(q^{\beta+1};q)_{km}}{[(q^k;q^k)_m]^s} \sum_{n=0}^{m} \frac{q^{s(kn(kn-1)/2+knm)}\;\, q^{n(k(\beta+1))}}{(q^{\beta+1};q)_{kn}} \times \frac{[(q^{-mk};q^k)_n]^s\;\, x^{kn}}{(q^k;q^k)_n}, \quad \Re(\beta) > -1, \tag{7.5.4}$$

is q-analogue of Konhauser polynomial (6.5.5). And

$$L^{(\beta)}_{m,s}(x|q) \;\; = \;\; Z^{\beta}_{m,s}(x|q).$$

is q-extended Laguerre polynomial.
If we take $\alpha = k,\;\; k \in \mathbb{N}$ then (7.5.2) gives

$$\sum_{m=0}^{\infty} \frac{q^{skm(m-1)/2}\;\, b^{(k,\beta,\lambda,\mu)}_{m^*}(x^k;s,r|q)}{(q^{\beta+1};q)_{km}}\, t^{ms} \leq \;\; \big(E_{q^k}(t)\big)^s \sum_{n=0}^{\infty} \frac{(-tq^{-k})^{s\delta n}\;\, x^{kn}}{(q^{\beta+1};q)_{kn}\;\, [(q^{\lambda};q)_{\mu n}]^r\;\, (q^k;q^k)_n}. \tag{7.5.5}$$

This will be used in the next section.

7.6 Finite q-series inequalities

In this section, certain ieqalities involving finite q-series are derived. The first inequality below involves finite q-series and q-GKP.

Theorem 7.6.1. *If* $\beta, \lambda \in \mathbb{R}_{>0},\;\; m, \delta, \mu, k, s \in \mathbb{N},\;\; r \in \mathbb{N} \cup \{0\}$, *then*

$$B^{(k,\beta,\lambda,\mu)}_{m^*}(x^k;s,r|q) \;\; \leq \;\; (q^{\beta+1};q)_{km} \left(\frac{x}{y}\right)^{\frac{km}{\delta}} \sum_{j=0}^{m} \frac{(-1)^j\;\, q^{kj(j-1)/2}}{(q^k;q^k)_j}$$

$$\times \left(\left(\frac{y}{x}\right)^{\frac{k}{\delta}} q^{k(-j-s+1)}; q^k\right)_j \frac{B_{(m-j)^*}^{(k,\beta,\lambda,\mu)}(y^k; s, r|q)}{(q^{\beta+1}; q)_{k(m-j)}}. \quad (7.6.1)$$

$$b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q) \leq \left(\frac{x}{y}\right)^{\frac{km}{\delta}} \sum_{j=0}^{m} \frac{(-1)^j q^{skj(j+1)/2-skmj} \ (q^{\beta+1}; q)_{km}}{(q^{\beta+1}; q)_{k(m-j)} \ (q^k; q^k)_j} \left(\frac{x}{k}\right)^{\frac{kj}{\delta}}$$
$$\times \left(\left(\frac{y}{x}\right)^{\frac{k}{\delta}} q^{k(1-s)}; q^k\right)_j \ b_{(m-j)^*}^{(\alpha,\beta,\lambda,\mu)}(y^k; s, r|q). \quad (7.6.2)$$

Proof. From the inequality (7.5.3), one gets

$$\sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k; s, r|q)}{(q^{\beta+1}; q)_{km}} t^{ms} \leq \left(e_{q^k}(t)\right)^s \frac{[(q^k; q^k)_\infty]^s}{(q^{\beta+1}; q)_\infty} B_\infty^{(k,\beta,\lambda,\mu)}(x^k t^{s\delta}; s, r|q).$$

With $t = \left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w$, it gives

$$\sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k; s, r|q)}{(q^{\beta+1}; q)_{km}} \left(\frac{y}{k}\right)^{\frac{km}{\delta}} w^{ms}$$
$$\leq \left(e_{q^k}\left(\left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^s \frac{[(q^k; q^k)_\infty]^s}{(q^{\beta+1}; q)_\infty} B_\infty^{(k,\beta,\lambda,\mu)}\left(\left(\frac{xy}{k}\right)^k w^{s\delta}; s, r|q\right).$$

Hence,

$$\sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k; s, r|q)}{(q^{\beta+1}; q)_{km}} \left(\frac{y}{k}\right)^{\frac{km}{\delta}} w^{ms} \left(e_{q^k}\left(\left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{-s}$$
$$\leq \frac{[(q^k; q^k)_\infty]^s}{(q^{\beta+1}; q)_\infty} B_\infty^{(k,\beta,\lambda,\mu)}\left(\left(\frac{xy}{k}\right)^k w^{s\delta}; s, r|q\right). \quad (7.6.3)$$

Now interchanging the role of x and y in (7.6.3), it yields

$$\sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k; s, r|q)}{(q^{\beta+1}; q)_{km}} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} w^{ms} \left(e_{q^k}\left(\left(\frac{x}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{-s}$$
$$\leq \frac{[(q^k; q^k)_\infty]^s}{(q^{\beta+1}; q)_\infty} B_\infty^{(k,\beta,\lambda,\mu)}\left(\left(\frac{xy}{k}\right)^k w^{s\delta}; s, r|q\right). \quad (7.6.4)$$

Here from (7.6.3) and (7.6.4), either

$$\sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k; s, r|q)}{(q^{\beta+1}; q)_{km}} \left(\frac{y}{k}\right)^{\frac{km}{\delta}} w^{ms} \left(e_{q^k}\left(\left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{-s}$$

$$\leq \sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} w^{ms} \left(e_{q^k}\left(\left(\frac{x}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{-s} \tag{7.6.5}$$

or

$$\sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} w^{ms} \left(e_{q^k}\left(\left(\frac{x}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{-s}$$
$$\leq \sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k;s,r|q)}{(q^{\beta+1};q)_{km}} \left(\frac{y}{k}\right)^{\frac{km}{\delta}} w^{ms} \left(e_{q^k}\left(\left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{-s} \tag{7.6.6}$$

Now rewriting the inequality (7.6.5) in the form

$$\sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k;s,r|q)}{(q^{\beta+1};q)_{km}} \left(\frac{y}{k}\right)^{\frac{km}{\delta}} w^{ms}$$
$$\leq \sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} w^{ms} \left(e_{q^k}\left(\left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{s} \left(e_{q^k}\left(\left(\frac{x}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{-s}$$

and using the easily verifiable identities and inequalities ($sx, sy \in (0,1),\ s \in \mathbb{N}$), ([11], [18]):

$$\begin{aligned}
e_q(x)E_q(-x) &= 1,\\
(E_q(-x))^s &\leq E_q(-x^s),\\
(1+x)\,E_q(qx) &= E_q(x),\\
e_{q^{-1}}(x) &= E_q(-xq),\\
(1-x)\,e_q(x) &= e_q(qx),\\
\left(e_{q^{-1}}(-xq^{-1})\right)^{-s} &\leq e_q(x^s q^{-s}),\\
(e_q(-x))^s &\leq e_q(-x^s),\\
\left(e_{q^{-1}}(-xq^{-1})\right)^{s} &\leq e_q(-x^s q^{-s}),
\end{aligned}$$

the above inequality can easily be written as

$$\sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k;s,r|q)}{(q^{\beta+1};q)_{km}} \left(\frac{y}{k}\right)^{\frac{km}{\delta}} w^{ms}$$
$$\leq \sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} w^{ms} \left(E_{q^k}\left(-\left(\frac{x}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{s}$$

$$\times \left(E_{q^k}\left(-\left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{-s}$$

$$\leq \sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} w^{ms}\ E_{q^k}\left(-\left(\frac{x}{k}\right)^{\frac{k}{\delta}} w^s\right)$$

$$\times \left(E_{q^k}\left(-\left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{-s}$$

$$= \sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} w^{ms}\ E_{q^k}\left(-\left(\frac{x}{k}\right)^{\frac{k}{\delta}} w^s\right)$$

$$\times \left(1-\left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)^{-s} \left(E_{q^k}\left(-q^k\ \left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{-s}$$

$$= \sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} w^{ms}\ E_{q^k}\left(-\left(\frac{x}{k}\right)^{\frac{k}{\delta}} w^s\right)$$

$$\times \left(1-\left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)^{-s} \left(e_{q^{-k}}\left(\left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{-s}$$

$$= \sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} w^{ms}\ E_{q^k}\left(-\left(\frac{x}{k}\right)^{\frac{k}{\delta}} w^s\right)$$

$$\times \left(e_{q^{-k}}\left(q^{-k}\ \left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{-s}$$

$$\leq \sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} w^{ms}\ E_{q^k}\left(-\left(\frac{x}{k}\right)^{\frac{k}{\delta}} w^s\right)$$

$$\times e_{q^k}\left(q^{-sk}\ \left(\frac{y}{k}\right)^{\frac{k}{\delta}} w^s\right)$$

$$= \sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} w^{ms} \sum_{j=0}^{\infty} \frac{(-1)^j\ q^{kj(j-1)/2}}{(q^k;q^k)_j} \left(\frac{x}{k}\right)^{\frac{kj}{\delta}} w^{sj}$$

$$\times \sum_{i=0}^{\infty} \frac{\left(\frac{y}{k}\right)^{\frac{ki}{\delta}}\ q^{-ski}\ w^{si}}{(q^k;q^k)_i}$$

$$= \sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} w^{ms} \sum_{j=0}^{\infty}\sum_{i=0}^{j} \frac{(-1)^{j-i}\ q^{k(j-i)(j-i-1)/2}}{(q^k;q^k)_{j-i}}$$

$$\times \left(\frac{x}{k}\right)^{\frac{k(j-i)}{\delta}} \frac{w^{s(j-i)}\ \left(\frac{y}{k}\right)^{\frac{ki}{\delta}}\ q^{-ski}\ w^{si}}{(q^k;q^k)_i}$$

$$= \sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} w^{ms} \sum_{j=0}^{\infty} \frac{(-1)^j\ q^{kj(j-1)/2}\ w^{sj}}{(q^k;q^k)_j} \left(\frac{x}{k}\right)^{\frac{kj}{\delta}}$$

$$\times \sum_{i=0}^{j} (-1)^i\ q^{ki(i-1)/2} \begin{bmatrix} j \\ i \end{bmatrix}_k \left(\frac{y}{x}\right)^{\frac{ki}{\delta}} q^{(1-j-s)ki}$$

$$= \sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k; s, r|q)}{(q^{\beta+1}; q)_{km}} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} w^{ms} \sum_{j=0}^{\infty} \frac{(-1)^j \; q^{kj(j-1)/2} \; w^{sj}}{(q^k; q^k)_j} \left(\frac{x}{k}\right)^{\frac{kj}{\delta}}$$
$$\times \prod_{i=1}^{j} \left(1 - \left(\frac{y}{x}\right)^{\frac{k}{\delta}} q^{k(i-j-s)}\right)$$
$$= \sum_{m=0}^{\infty} \sum_{j=0}^{m} \frac{B_{(m-j)^*}^{(k,\beta,\lambda,\mu)}(y^k; s, r|q)}{(q^{\beta+1}; q)_{k(m-j)}} \left(\frac{x}{k}\right)^{\frac{k(m-j)}{\delta}} w^{(m-j)s} \frac{q^{kj(j-1)/2} \; w^{sj}}{(q^k; q^k)_j} \left(\frac{x}{k}\right)^{\frac{kj}{\delta}}$$
$$\times \prod_{i=1}^{j} \left(1 - \left(\frac{y}{x}\right)^{\frac{k}{\delta}} q^{k(i-j)}\right)$$
$$= \sum_{m=0}^{\infty} \sum_{j=0}^{m} \frac{B_{(m-j)^*}^{(k,\beta,\lambda,\mu)}(y^k; s, r|q)}{(q^{\beta+1}; q)_{k(m-j)}} \frac{q^{kj(j-1)/2}}{(q^k; q^k)_j} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} \left(\left(\frac{y}{x}\right)^{\frac{k}{\delta}} q^{k(-j-s+1)}; q^k\right)_j w^{ms}.$$

Now comparing the coefficients of w^{ms} both the sides, one arrives at (7.6.1).
Next, the inequality (7.6.2) may be proved by using the particular case (7.5.5) of Theorem 7.5.2, that is

$$\sum_{m=0}^{\infty} \frac{q^{skm(m-1)/2}}{(q^{\beta+1}; q)_{km}} b_{m^*}^{(k,\beta,\lambda,\mu)}(x^k; s, r|q) \; t^{ms}$$
$$\leq \left(E_{q^k}(t)\right)^s \sum_{n=0}^{\infty} \frac{(-tq^{-k})^{s\delta n} \; x^{kn}}{(q^{\beta+1}; q)_{kn} \; [(q^{\lambda}; q)_{\mu n}]^r \; (q^k; q^k)_n}.$$

Taking $t = \left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w$, this gives

$$\sum_{m=0}^{\infty} \frac{q^{skm(m-1)/2} \; b_{m^*}^{(k,\beta,\lambda,\mu)}(x^k; s, r|q)}{(q^{\beta+1}; q)_{km}} \left(\frac{y}{k}\right)^{\frac{km}{\delta}} w^{ms}$$
$$\leq \left(E_{q^k}\left(\left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^s \sum_{n=0}^{\infty} \left(\frac{xy}{kq^{s\delta}}\right)^{kn} \frac{(-w)^{s\delta n}}{(q^{\beta+1}; q)_{kn} \; [(q^{\lambda}; q)_{\mu n}]^r \; (q^k; q^k)_n}. \quad (7.6.7)$$

and interchanging the role of x and y, it becomes

$$\sum_{m=0}^{\infty} \frac{q^{skm(m-1)/2} \; b_{m^*}^{(k,\beta,\lambda,\mu)}(y^k; s, r|q)}{(q^{\beta+1}; q)_{km}} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} w^{ms}$$
$$\leq \left(E_{q^k}\left(\left(\frac{x}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^s \sum_{n=0}^{\infty} \left(\frac{xy}{kq^{s\delta}}\right)^{kn} \frac{(-w)^{s\delta n}}{(q^{\beta+1}; q)_{kn} \; [(q^{\lambda}; q)_{\mu n}]^r \; (q^k; q^k)_n}. \quad (7.6.8)$$

From (7.6.7) and (7.6.8), it follows that either

$$\sum_{m=0}^{\infty} \frac{q^{skm(m-1)/2} \; b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q)}{(q^{\beta+1}; q)_{km}} \left(\frac{y}{k}\right)^{\frac{km}{\delta}} w^{ms} \left(E_{q^k}\left(\left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{-s}$$

$$\leq \sum_{m=0}^{\infty} \frac{q^{skm(m-1)/2}\ b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}} \ \left(\frac{x}{k}\right)^{\frac{km}{\delta}} w^{ms} \left(E_{q^k}\left(\left(\frac{x}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{-s} \tag{7.6.9}$$

or

$$\sum_{m=0}^{\infty} \frac{q^{skm(m-1)/2}\ b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}} \ \left(\frac{x}{k}\right)^{\frac{km}{\delta}} w^{ms} \left(E_{q^k}\left(\left(\frac{x}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{-s}$$
$$\leq \sum_{m=0}^{\infty} \frac{q^{skm(m-1)/2}\ b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r|q)}{(q^{\beta+1};q)_{km}} \ \left(\frac{y}{k}\right)^{\frac{km}{\delta}} w^{ms} \left(E_{q^k}\left(\left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{-s}. \tag{7.6.10}$$

Here considering (7.6.9), and rewriting it as

$$\sum_{m=0}^{\infty} \frac{q^{skm(m-1)/2}\ b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r|q)}{(q^{\beta+1};q)_{km}} \ \left(\frac{y}{k}\right)^{\frac{km}{\delta}} w^{ms}$$
$$\leq \sum_{m=0}^{\infty} \frac{q^{skm(m-1)/2}\ b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}} \ \left(\frac{x}{k}\right)^{\frac{km}{\delta}} w^{ms}$$
$$\times \left(E_{q^k}\left(\left(\frac{x}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{-s} \left(E_{q^k}\left(\left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{s}$$

and using the above listed q-exponential functions' identities, it take the form

$$\sum_{m=0}^{\infty} \frac{q^{skm(m-1)/2}\ b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r|q)}{(q^{\beta+1};q)_{km}} \ \left(\frac{y}{k}\right)^{\frac{km}{\delta}} w^{ms}$$
$$\leq \sum_{m=0}^{\infty} \frac{q^{skm(m-1)/2}\ b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}} \ \left(\frac{x}{k}\right)^{\frac{km}{\delta}} w^{ms}$$
$$\times \left(e_{q^k}\left(-\left(\frac{x}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{s} \left(E_{q^k}\left(\left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{s}$$
$$\leq \sum_{m=0}^{\infty} \frac{q^{skm(m-1)/2}\ b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}} \ \left(\frac{x}{k}\right)^{\frac{km}{\delta}} w^{ms}$$
$$\times \left(e_{q^k}\left(-\left(\frac{x}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{s} \left(1+\left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)^{s} \left(E_{q^k}\left(\left(\frac{y}{k}\right)^{\frac{k}{s\delta}} q^k\ w\right)\right)^{s}$$
$$\leq \sum_{m=0}^{\infty} \frac{q^{skm(m-1)/2}\ b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}} \ \left(\frac{x}{k}\right)^{\frac{km}{\delta}} w^{ms}$$
$$\times \left(e_{q^k}\left(-\left(\frac{x}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{s} \left(1+\left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)^{s} \left(e_{q^{-k}}\left(-\left(\frac{y}{k}\right)^{\frac{k}{s\delta}}\ w\right)\right)^{s}$$
$$\leq \sum_{m=0}^{\infty} \frac{q^{skm(m-1)/2}\ b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}} \ \left(\frac{x}{k}\right)^{\frac{km}{\delta}} w^{ms}$$

$$\times\left(e_{q^k}\left(-\left(\frac{x}{k}\right)^{\frac{k}{s\delta}}w\right)\right)^s\left(e_{q^{-k}}\left(-\left(\frac{y}{k}\right)^{\frac{k}{s\delta}}q^{-k}\,w\right)\right)^s$$

$$\leq\sum_{m=0}^{\infty}\frac{q^{skm(m-1)/2}\;b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}}\left(\frac{x}{k}\right)^{\frac{km}{\delta}}w^{ms}$$

$$\times\left(e_{q^k}\left(-\left(\frac{x}{k}\right)^{\frac{k}{s\delta}}w\right)\right)^s\left(e_{q^{-k}}\left(-\left(\frac{y}{k}\right)^{\frac{k}{s\delta}}q^{-k}\,w\right)\right)^s$$

$$\leq\sum_{m=0}^{\infty}\frac{q^{skm(m-1)/2}\;b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}}\left(\frac{x}{k}\right)^{\frac{km}{\delta}}w^{ms}$$

$$\times e_{q^k}\left(-\left(\frac{x}{k}\right)^{\frac{k}{\delta}}w^s\right)\left(e_{q^{-k}}\left(-\left(\frac{y}{k}\right)^{\frac{k}{s\delta}}q^{-k}\,w\right)\right)^s$$

$$\leq\sum_{m=0}^{\infty}\frac{q^{skm(m-1)/2}\;b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}}\left(\frac{x}{k}\right)^{\frac{km}{\delta}}w^{ms}$$

$$\times e_{q^k}\left(-\left(\frac{x}{k}\right)^{\frac{k}{\delta}}w^s\right)e_{q^{-k}}\left(-\left(\frac{y}{k}\right)^{\frac{k}{\delta}}q^{-sk}\,w^s\right)$$

$$\leq\sum_{m=0}^{\infty}\frac{q^{skm(m-1)/2}\;b_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}}\left(\frac{x}{k}\right)^{\frac{km}{\delta}}w^{ms}$$

$$\times\sum_{j=0}^{\infty}\frac{(-1)^j\left(\frac{x}{k}\right)^{\frac{kj}{\delta}}w^{sj}}{(q^k;q^k)_j}\sum_{i=0}^{\infty}\frac{(-1)^i\left(\frac{y}{k}\right)^{\frac{ki}{\delta}}q^{-ski}\,w^{si}}{(q^{-k};q^{-k})_i}.$$

From the definition of q-exponential function and with the help of the formula:

$$(q^{-k};q^{-k})_n\;=\;(q^k;q^k)_n\,(-q^{-k})^n\,q^{-kn(n-1)/2},$$

one finds

$$\sum_{m=0}^{\infty}\frac{q^{skm(m-1)/2}\;b_{m^*}^{(k,\beta,\lambda,\mu)}(x^k;s,r|q)}{(q^{\beta+1};q)_{km}}\left(\frac{y}{k}\right)^{\frac{km}{\delta}}w^{ms}$$

$$\leq\sum_{m=0}^{\infty}\frac{q^{skm(m-1)/2}\;b_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}}\left(\frac{x}{k}\right)^{\frac{km}{\delta}}w^{ms}$$

$$\times\sum_{j=0}^{\infty}\frac{(-1)^j}{(q^k;q^k)_j}\left(\frac{x}{k}\right)^{\frac{kj}{\delta}}w^{sj}\sum_{i=0}^{\infty}\frac{q^{ki-ski+ki(i-1)/2}}{(q^k;q^k)_i}\left(\frac{y}{k}\right)^{\frac{ki}{\delta}}w^{si}$$

$$=\sum_{m=0}^{\infty}\frac{q^{skm(m-1)/2}\;b_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}}\left(\frac{x}{k}\right)^{\frac{km}{\delta}}w^{ms}$$

$$\times\sum_{j=0}^{\infty}\sum_{i=0}^{j}\frac{(-1)^j}{(q^k;q^k)_{j-i}}\frac{(-1)^i\,q^{ki-ski+ki(i-1)/2}\,w^{si}}{(q^k;q^k)_i}\left(\frac{x}{k}\right)^{\frac{kj}{\delta}}\left(\frac{y}{x}\right)^{\frac{ki}{\delta}}w^{sj}$$

$$=\sum_{m=0}^{\infty}\frac{q^{skm(m-1)/2}\;b_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}}\left(\frac{x}{k}\right)^{\frac{km}{\delta}}w^{ms}$$

$$\times \sum_{j=0}^{\infty} \frac{(-1)^j}{(q^k;q^k)_j} \left(\frac{x}{k}\right)^{\frac{kj}{\delta}} w^{sj} \sum_{i=0}^{j} (-1)^i\, q^{ki(i-1)/2} \begin{bmatrix} j \\ i \end{bmatrix}_k q^{-ski+ki} \left(\frac{y}{x}\right)^{\frac{ki}{\delta}}$$

$$= \sum_{m=0}^{\infty} \frac{q^{skm(m-1)/2}\, b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{\alpha m}} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} w^{ms}$$

$$\times \sum_{j=0}^{\infty} \frac{(-1)^j}{(q^k;q^k)_j} \left(\frac{x}{k}\right)^{\frac{kj}{\delta}} w^{sj} \prod_{i=1}^{j} \left(1 - \left(\frac{y}{x}\right)^{\frac{k}{\delta}} q^{k(i-s)}\right)$$

$$= \sum_{m=0}^{\infty} \sum_{j=0}^{m} \frac{(-1)^j\, q^{sk(m-j)(m-j-1)/2}}{(q^{\beta+1};q)_{\alpha(m-j)}\, (q^k;q^k)_j}\, b_{(m-j)^*}^{(\alpha,\beta,\lambda,\mu)}(y^k;s,r|q)$$

$$\times \left(\frac{x}{k}\right)^{\frac{km}{\delta}} w^{ms} \left(\left(\frac{y}{x}\right)^{\frac{k}{\delta}} q^{-sk+k};q^k\right)_j$$

$$= \sum_{m=0}^{\infty} q^{skm(m-1)/2} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} \sum_{j=0}^{m} \frac{(-1)^j q^{skj(j+1)/2-skmj}}{(q^{\beta+1};q)_{\alpha(m-j)}\, (q^k;q^k)_j} \left(\frac{x}{k}\right)^{\frac{kj}{\delta}}$$

$$\times \left(\left(\frac{y}{x}\right)^{\frac{k}{\delta}} q^{-sk+k};q^k\right)_j w^{ms}.$$

Thus,

$$\sum_{m=0}^{\infty} \frac{q^{skm(m-1)/2}\, b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r|q)}{(q^{\beta+1};q)_{\alpha m}} \left(\frac{y}{k}\right)^{\frac{km}{\delta}} w^{ms}$$

$$\leq \sum_{m=0}^{\infty} (-1)^{sm}\, q^{skm(m-1)/2} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} \sum_{j=0}^{m} \frac{(-1)^j q^{skj(j+1)/2-skmj} \left(\frac{x}{k}\right)^{\frac{kj}{\delta}}}{(q^{\beta+1};q)_{\alpha(m-j)}\, (q^k;q^k)_j}$$

$$\times \left(\left(\frac{y}{x}\right)^{\frac{k}{\delta}} q^{-sk+k};q^k\right)_j b_{(m-j)^*}^{(\alpha,\beta,\lambda,\mu)}(y^k;s,r|q)\, w^{ms}.$$

Comparing the coefficients of w^{ms} in above inequality, one arrives at (7.6.2). □

7.6.1 Special cases

(i) From (7.6.1) and (7.6.2) , one gets finite summation formulas for $s = 1$:

$$B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k;1,r|q) = (q^{\beta+1};q)_{km} \left(\frac{x}{y}\right)^{\frac{km}{\delta}} \sum_{j=0}^{m} \frac{(-1)^j\, q^{kj(j-1)/2}}{(q^k;q^k)_j} \left(\left(\frac{y}{x}\right)^{\frac{k}{\delta}} q^{-kj};q^k\right)_j$$

$$\times \frac{B_{(m-j)^*}^{(k,\beta,\lambda,\mu)}(y^k;1,r|q)}{(q^{\beta+1};q)_{k(m-j)}} \tag{7.6.11}$$

and

$$b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q) \;=\; \left(\frac{x}{y}\right)^{\frac{km}{\delta}} \sum_{j=0}^{m} \frac{(-1)^j q^{kj(j+1)/2-kmj}\;(q^{\beta+1};q)_{km}}{(q^{\beta+1};q)_{k(m-j)}\;(q^k;q^k)_j} \left(\frac{x}{k}\right)^{\frac{kj}{\delta}} \times \left(\left(\frac{y}{x}\right)^{\frac{k}{\delta}};q^k\right)_j \; b_{(m-j)^*}^{(\alpha,\beta,\lambda,\mu)}(y^k;1,r|q). \tag{7.6.12}$$

From (7.6.11), with $r=0$, the following summation formula involving the generalized Laguerre polynomial (6.2.2) occurs.

$$L_{m^*}^{(k,\beta)}(x^k|q) \;=\; (q^{\beta+1};q)_{km} \left(\frac{x}{y}\right)^{\frac{km}{\delta}} \sum_{j=0}^{m} \frac{(-1)^j\; q^{kj(j-1)/2}}{(q^k;q^k)_j} \left(\left(\frac{y}{x}\right)^{\frac{k}{\delta}} q^{-kj};q^k\right)_j \times \frac{L_{(m-j)^*}^{(k,\beta)}(y^k|q)}{(q^{\beta+1};q)_{k(m-j)}}.$$

Further, $\delta=1$ in (7.6.13) provides

$$Z_m^{\beta}(x;k|q) \;=\; (q^{\beta+1};q)_{km} \left(\frac{x}{y}\right)^{km} \sum_{j=0}^{m} \frac{(-1)^j\; q^{kj(j-1)/2}}{(q^k;q^k)_j} \left(\left(\frac{y}{x}\right)^{k} q^{-kj};q^k\right)_j \times \frac{Z_{(m-j)}^{\beta}(y;k|q)}{(q^{\beta+1};q)_{k(m-j)}}.$$

The Laguerre polynomial case follows immediately with $k=1$ in the form:

$$L_m^{(\beta)}(x|q) \;=\; (q^{\beta+1};q)_m \left(\frac{x}{y}\right)^{m} \sum_{j=0}^{m} \frac{(-1)^j\; q^{j(j-1)/2}}{(q;q)_j} \left(\left(\frac{y}{x}\right) q^{-j};q\right)_j \times \frac{L_{(m-j)}^{(\beta)}(y|q)}{(q^{\beta+1};q)_{(m-j)}}.$$

7.7 Mixed relations

Theorem 7.7.1. *For $\beta,\lambda>0$, $m,\delta,\mu,k,s\in\mathbb{N}$, $r\in\mathbb{N}\cup\{0\}$ there hold the mixed relations:*

$$(1-q^{\beta})\;B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k;s,r|q) + (1-q)\;q^{\beta}\;x\;D_qB_{m^*}^{(k,\beta,\lambda,\mu)}(x^k;s,r|q) \;=\; (1-q^{\beta+km})B_{m^*}^{(k,\beta-1,\lambda,\mu)}((xq^{\delta})^k;s,r|q), \tag{7.7.1}$$

$$(1-q^{\beta})\;b_{m^*}^{(k,\beta,\lambda,\mu)}(x^k;s,r|q) + (1-q)\;q^{\beta}\;x\;D_qb_{m^*}^{(k,\beta,\lambda,\mu)}(x^k;s,r|q)$$

$$= \quad (1-q^{\beta+km})b_{m^*}^{(k,\beta-1,\lambda,\mu)}(x^k;s,r|q), \qquad (7.7.2)$$

where $D_q f(x) = \dfrac{f(x)-f(xq)}{x-xq}$.

Proof.

Here

$$\begin{aligned}
l.h.s. \quad &= \quad (1-q^\beta)\ B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k;s,r|q) + (1-q)\ q^\beta\ x\ D_q B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k;s,r|q)\\
&= \quad (1-q^\beta)\frac{(q^{\beta+1};q)_{km}}{[(q^k;q^k)_m]^s}\sum_{n=0}^{m^*}\frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(k(\beta+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{kn}\ [(q^\lambda;q)_{\mu n}]^r}\\
&\quad \times\frac{[(q^{-mk};q^k)_{\delta n}]^s\ x^{kn}}{(q^k;q^k)_n} + (1-q)\ q^\beta\ x\ D_q\frac{(q^{\beta+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s}\\
&\quad \times\sum_{n=0}^{m^*}\frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{kn}\ [(q^\lambda;q)_{\mu n}]^r}\frac{[(q^{-mk};q^k)_{\delta n}]^s\ x^{kn}}{(q^k;q^k)_n}\\
&= \quad (1-q^\beta)\frac{(q^{\beta+1};q)_{km}}{[(q^k;q^k)_m]^s}\sum_{n=0}^{m^*}\frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(k(\beta+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{kn}\ [(q^\lambda;q)_{\mu n}]^r}\\
&\quad \times\frac{[(q^{-mk};q^k)_{\delta n}]^s\ x^{kn}}{(q^k;q^k)_n} + (1-q)\ q^\beta x\frac{(q^{\beta+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s}\\
&\quad \times\sum_{n=0}^{m^*}\frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{kn}\ [(q^\lambda;q)_{\mu n}]^r}\frac{[(q^{-mk};q^k)_{\delta n}]^s}{(q^k;q^k)_n}D_q(x^{kn})\\
&= \quad (1-q^\beta)\frac{(q^{\beta+1};q)_{km}}{[(q^k;q^k)_m]^s}\sum_{n=0}^{m^*}\frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(k(\beta+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{kn}\ [(q^\lambda;q)_{\mu n}]^r}\\
&\quad \times\frac{[(q^{-mk};q^k)_{\delta n}]^s\ x^{kn}}{(q^k;q^k)_n} + (1-q)\ q^\beta x\frac{(q^{\beta+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s}\\
&\quad \times\sum_{n=0}^{m^*}\frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{kn}\ [(q^\lambda;q)_{\mu n}]^r}\frac{[(q^{-mk};q^k)_{\delta n}]^s}{(q^k;q^k)_n}\frac{(1-q^{kn})}{x(1-q)}\\
&= \quad (1-q^\beta)\frac{(q^{\beta+1};q)_{km}}{[(q^k;q^k)_m]^s}\sum_{n=0}^{m^*}\frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(k(\beta+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{kn}\ [(q^\lambda;q)_{\mu n}]^r}\\
&\quad \times\frac{[(q^{-mk};q^k)_{\delta n}]^s\ x^{kn}}{(q^k;q^k)_n} + (q^\beta - q^{kn+\beta})\frac{(q^{\beta+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s}\\
&\quad \times\sum_{n=0}^{m^*}\frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{kn}\ [(q^\lambda;q)_{\mu n}]^r}\frac{[(q^{-mk};q^k)_{\delta n}]^s}{(q^k;q^k)_n}\\
&\quad \frac{(1-q^{\beta+km})(q^\beta;q)_{km}}{[(q^k;q^k)_m]^s}\sum_{n=0}^{m^*}\frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(k(\beta+1)+r\mu\lambda)}}{[q^\beta]_{kn}\ [(q^\lambda;q)_{\mu n}]^r}\\
&\quad \times\frac{[(q^{-mk};q^k)_{\delta n}]^s\ x^{kn}}{(q^k;q^k)_n}.
\end{aligned}$$

$$\begin{aligned}
&= (1-q^{\beta+km})B_{m^*}^{(k,\beta-1,\lambda,\mu)}((xq^{\delta})^k;s,r|q) \\
&= r.h.s.
\end{aligned}$$

In similar manner, one can obtain the other relation (7.7.2). Hence proof is omitted. □

7.8 Integral representations

Theorem 7.8.1. *If* $\alpha,\beta,\lambda>0,\ m,\delta,\mu,k,s\in\mathbb{N},\ r\in\mathbb{N}\cup\{0\}$, *and* $\sigma\in\mathbb{C}$ *with* $\Re(\sigma)>-1$ *then*

$$\begin{aligned}
B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(tq^{\delta\alpha(\sigma-\beta)};s,r|q) &= \frac{(q^{\beta+1};q)_\infty\ (q^{\beta-\sigma};q)_\infty\ (q^{\beta+1};q)_{\alpha m}}{(1-q)\ (q;q)_\infty\ (q^{\beta+1};q)_\infty\ (q^{\sigma+1};q)_{\alpha m}} \\
&\quad\times\int_0^x (x-uq)_{\beta-\sigma-1}\ u^{\sigma}\ B_{m^*}^{(\alpha,\sigma,\lambda,\mu)}(u^{\alpha};s,r|q)\ \mathrm{d}_q u, \quad (7.8.1)
\end{aligned}$$

$$\begin{aligned}
b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(t;s,r|q) &= \frac{(q^{\beta+1};q)_\infty\ (q^{\beta-\sigma};q)_\infty\ (q^{\beta+1};q)_{\alpha m}}{(1-q)\ (q;q)_\infty\ (q^{\beta+1};q)_\infty\ (q^{\sigma+1};q)_{\alpha m}} \\
&\quad\times\int_0^x (x-uq)_{\beta-\sigma-1}\ u^{\sigma}\ B_{m^*}^{(\alpha,\sigma,\lambda,\mu)}(u^{\alpha};s,r|q)\ \mathrm{d}_q u. \qquad (7.8.2)
\end{aligned}$$

Proof. Consider

$$\begin{aligned}
&\int_0^x (x-uq)_{\beta-\sigma-1}\ u^{\sigma}\ B_{m^*}^{(\alpha,\sigma,\lambda,\mu)}(u^{\alpha};s,r|q)\ \mathrm{d}_q u \\
&= \int_0^x (x-uq)_{\beta-\sigma-1}\ u^{\sigma}\frac{(q^{\sigma+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s}\sum_{n=0}^{m^*}\frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\sigma+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r} \\
&\quad\times\frac{[(q^{-mk};q^k)_{\delta n}]^s\ u^{\alpha n}}{(q^k;q^k)_n}\ \mathrm{d}_q u \\
&= \frac{(q^{\sigma+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s}\sum_{n=0}^{m^*}\frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\sigma+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r} \\
&\quad\times\frac{[(q^{-mk};q^k)_{\delta n}]^s}{(q^k;q^k)_n}\int_0^x (x-uq)_{\beta-\sigma-1}\ u^{\alpha n+\sigma}\ \mathrm{d}_q u.
\end{aligned}$$

By taking $u = xt$, this gives

$$\begin{aligned}
&\int_0^x (x-uq)_{\beta-\sigma-1}\ u^{\sigma}\ B_{m^*}^{(\alpha,\sigma,\lambda,\mu)}(u^{\alpha};s,r|q)\ \mathrm{d}_q u \\
=\ & \frac{(q^{\sigma+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s} \sum_{n=0}^{m^*} \frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\sigma+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r} \\
&\times \frac{[(q^{-mk};q^k)_{\delta n}]^s\ x^{\alpha n+\beta}}{(q^k;q^k)_n} \int_0^1 (1-tq)_{\beta-\sigma-1}\ t^{\alpha n+\sigma}\ \mathrm{d}_q t \\
=\ & \frac{(q^{\sigma+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s} \sum_{n=0}^{m^*} \frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\sigma+1)+r\mu\lambda)}}{(q^{\sigma+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r} \\
&\times \frac{[(q^{-mk};q^k)_{\delta n}]^s\ x^{\alpha n+\beta}}{(q^k;q^k)_n} \int_0^1 t^{\alpha n+\sigma} \frac{(tq;q)_{\infty}}{(tq^{\beta-\sigma};q)_{\infty}}\ \mathrm{d}_q t \\
=\ & \frac{(q^{\sigma+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s} \sum_{n=0}^{m^*} \frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\sigma+1)+r\mu\lambda)}}{(q^{\sigma+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r} \frac{[(q^{-mk};q^k)_{\delta n}]^s\ t^n}{(q^k;q^k)_n} \\
&\times \mathcal{B}_q(\alpha n+\sigma+1,\beta-\sigma) \\
=\ & \frac{(q^{\sigma+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s} \sum_{n=0}^{m^*} \frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\sigma+1)+r\mu\lambda)}}{(q^{\sigma+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r} \frac{[(q^{-mk};q^k)_{\delta n}]^s\ t^n}{(q^k;q^k)_n} \\
&\times \frac{(1-q)\ (q;q)_{\infty}\ (q^{\alpha n+\beta+1};q)_{\infty}}{(q^{\alpha n+\sigma+1};q)_{\infty}\ (q^{\beta-\sigma};q)_{\infty}} \\
=\ & \sum_{n=0}^{m^*} \frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\sigma+1)+r\mu\lambda)}}{(q^{\sigma+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r} \frac{[(q^{-mk};q^k)_{\delta n}]^s\ t^n}{(q^k;q^k)_n} \\
&\times \frac{(1-q)\ (q;q)_{\infty}\ (q^{\beta+1};q)_{\infty}\ (q^{\sigma+1};q)_{\alpha n}}{(q^{\beta+1};q)_{\alpha n}(q^{\sigma+1};q)_{\infty}\ (q^{\beta-\sigma};q)_{\infty}} \\
=\ & \frac{(1-q)\ (q;q)_{\infty}\ (q^{\beta+1};q)_{\infty}}{(q^{\sigma+1};q)_{\infty}\ (q^{\beta-\sigma};q)_{\infty}} \frac{(q^{\sigma+1};q)_{\alpha m}\ (q^{\beta+1};q)_{\alpha m}}{(q^{\beta+1};q)_{\alpha m}\ [(q^k;q^k)_m]^s} \\
&\times \sum_{n=0}^{m^*} \frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\sigma+1)+r\mu\lambda)}}{(q^{\beta+a+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r} \frac{[(q^{-mk};q^k)_{\delta n}]^s\ t^n}{(q^k;q^k)_n} \\
=\ & \frac{(1-q)\ (q;q)_{\infty}\ (q^{\beta+1};q)_{\infty}\ (q^{\sigma+1};q)_{\alpha m}}{(q^{\beta+1};q)_{\infty}\ (q^{\beta-\sigma};q)_{\infty}\ (q^{\beta+1};q)_{\alpha m}}\ B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(tq^{\delta\alpha(\sigma-\beta)};s,r|q).
\end{aligned}$$

One can easily prove the remaining integral form (7.8.2). Hence proof is omitted.

□

7.8.1 Special cases

Here putting $\delta = s = 1, r = 0, \alpha = k \in \mathbb{N}$ in (7.8.1), one gets

$$Z_m^{\beta}(t^{1/k}q^{\beta-\sigma}; k|q) = \frac{(q^{\beta+1};q)_\infty\ (q^{\beta-\sigma};q)_\infty\ (q^{\beta+1};q)_{km}}{(1-q)\ (q;q)_\infty\ (q^{\beta+1};q)_\infty\ (q^{\sigma+1};q)_{km}} \times \int_0^x (x-uq)_{\beta-\sigma-1}\ u^{\sigma}\ Z_m^{\beta}(u;k|q)\ \mathrm{d}_q u.$$

Further, taking $k = 1$, this reduces to

$$L_m^{(\beta)}(tq^{\beta-\sigma}|q) = \frac{(q^{\beta+1};q)_\infty\ (q^{\beta-\sigma};q)_\infty\ (q^{\beta+1};q)_m}{(1-q)\ (q;q)_\infty\ (q^{\beta+1};q)_\infty\ (q^{\sigma+1};q)_m} \times \int_0^x (x-uq)_{\beta-\sigma-1}\ u^{\sigma}\ L_m^{(\beta)}(u;k|q)\ \mathrm{d}_q u.$$

7.9 Fractional q-operators

The fractional operators are applied on q-GKP here and the following results are obtained.

7.9.1 Fractional q-integral operator

First the fractional q-integral operator is used.

Theorem 7.9.1. *If* $\alpha, \beta, \lambda > 0,\ m, \delta, \mu, k, s \in \mathbb{N},\ r \in \mathbb{N} \cup \{0\},\ \nu \in \mathbb{C}$, *then*

$${}_qI_{0+}^{\nu}\left[t^{\beta} B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(t^{\alpha}; s, r|q)\right] = \frac{(1-q)^{\nu}\ (q^{\beta+\nu+1};q)_\infty\ (q^{\beta+1};q)_{\alpha m}\ x^{\beta+\nu}}{(q^{\beta+1};q)_\infty\ (q^{\beta+\nu+1};q)_{\alpha m}} \times B_{m^*}^{(\alpha,\beta+\nu,\lambda,\mu)}(q^{-\delta\alpha\nu}x^{\alpha}; s, r|q). \tag{7.9.1}$$

$${}_qI_{0+}^{\nu}\left[t^{\beta} b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(t^{\alpha}; s, r|q)\right] = \frac{(1-q)^{\nu}\ (q^{\beta+\nu+1};q)_\infty\ (q^{\beta+1};q)_{\alpha m}\ x^{\beta+\nu}}{(q^{\beta+1};q)_\infty\ (q^{\beta+\nu+1};q)_{\alpha m}} \times b_{m^*}^{(\alpha,\beta+\nu,\lambda,\mu)}(x^{\alpha}; s, r|q). \tag{7.9.2}$$

Proof. Beginning with left hand side of (7.9.2),

$${}_qI_{0+}^{\nu}\left[t^{\beta} B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(t^{\alpha}; s, r|q)\right]$$

$$= \frac{(q^{\beta+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s} \sum_{n=0}^{m^*} \frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r}\ {}_qI^\nu_{0+}\left(t^{\alpha n+\beta}\right)$$

$$= \frac{(q^{\beta+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s} \sum_{n=0}^{m^*} \frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\ [(q^{-mk};q^k)_{\delta n}]^s}{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r (q^k;q^k)_n}$$

$$\times \frac{1}{\Gamma_q(\nu)} \int_0^x (x-tq)_{\nu-1} t^{\alpha n+\beta} \mathrm{d}_q t.$$

Now taking $t = xu$, this gives

$$ {}_qI^\nu_{0+}\left[t^\beta B^{(\alpha,\beta,\lambda,\mu)}_{m^*}(t^\alpha;s,r|q)\right]$$

$$= \frac{(q^{\beta+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s} \sum_{n=0}^{m^*} \frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\ [(q^{-mk};q^k)_{\delta n}]^s}{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r (q^k;q^k)_n}$$

$$\times \frac{x^{\alpha n+\beta+\nu}}{\Gamma_q(\nu)} \int_0^1 (1-uq)_{\nu-1}\ u^{\alpha n+\beta} \mathrm{d}_q u$$

$$= \frac{(q^{\beta+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s} \sum_{n=0}^{m^*} \frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r}\ \frac{[(q^{-mk};q^k)_{\delta n}]^s}{(q^k;q^k)_n}$$

$$\times \frac{x^{\alpha n+\beta+\nu}}{\Gamma_q(\nu)} \int_0^1 u^{\alpha n+\beta+1-1} \frac{(uq;q)_\infty}{(uq^\nu;q)_\infty} \mathrm{d}_q u$$

$$= \frac{(q^{\beta+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s} \sum_{n=0}^{m^*} \frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r}$$

$$\times \frac{[(q^{-mk};q^k)_{\delta n}]^s}{(q^k;q^k)_n}\ \frac{x^{\alpha n+\beta+\nu}}{\Gamma_q(\nu)} \mathfrak{B}_q(\alpha n+\beta+1,\nu)$$

$$= \frac{(q^{\beta+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s} \sum_{n=0}^{m^*} \frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r} \frac{[(q^{-mk};q^k)_{\delta n}]^s\ \ x^{\alpha n}}{(q^k;q^k)_n}$$

$$\times \frac{(1-q)^\nu\ (q^{\alpha n+\beta+\nu+1};q)_\infty\ x^{\beta+\nu}}{(q^{\alpha n+\beta+1};q)_\infty}$$

$$= \sum_{n=0}^{m^*} \frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r} \frac{[(q^{-mk};q^k)_{\delta n}]^s\ x^{\alpha n}}{(q^k;q^k)_n}$$

$$\times \frac{(1-q)^\nu\ (q^{\beta+\nu+1};q)_\infty\ (q^{\beta+1};q)_{\alpha n}\ x^{\beta+\nu}}{[q^{\beta+\nu+1}]_{\alpha n}(q^{\beta+1};q)_\infty}$$

$$= \frac{(1-q)^\nu\ (q^{\beta+\nu+1};q)_\infty}{(q^{\beta+1};q)_\infty} \frac{(q^{\beta+1};q)_{\alpha m}\ (q^{\beta+\nu+1};q)_{\alpha m}\ x^{\beta+\nu}}{[q^{\beta+\nu+1}]_{\alpha m}\ [(q^k;q^k)_m]^s}$$

$$\times \sum_{n=0}^{m^*} \frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+\nu+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r} \frac{[(q^{-mk};q^k)_{\delta n}]^s\ x^{\alpha n}}{(q^k;q^k)_n}$$

$$= \frac{(1-q)^{\nu}\ (q^{\beta+\nu+1};q)_{\infty}\ (q^{\beta+1};q)_{\alpha m}\ x^{\beta+\nu}}{(q^{\beta+1};q)_{\infty}\ (q^{\beta+\nu+1};q)_{\alpha m}}\ B_{m^*}^{(\alpha,\beta+\nu,\lambda,\mu)}(q^{-\delta\alpha\nu}x^{\alpha};s,r|q).$$

The second expression (7.9.2), proceeding similarly. Hence proof is omitted.

$$\begin{aligned}
&{}_qI_{0+}^{\nu}\left[t^{\beta}b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(t^{\alpha};s,r|q)\right]\\
&= \frac{(q^{\beta+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s}\sum_{n=0}^{m^*}\frac{[(q^{-mk};q^k)_{\delta n}]^s}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r\ (q^k;q^k)_n}\ {}_qI_{0+}^{\nu}\left(t^{\alpha n+\beta}\right)\\
&= \frac{(q^{\beta+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s}\sum_{n=0}^{m^*}\frac{[(q^{-mk};q^k)_{\delta n}]^s}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r(q^k;q^k)_n}\frac{1}{\Gamma_q(\nu)}\int_0^x (x-tq)_{\nu-1}\ t^{\alpha n+\beta}\ \mathrm{d_q t}.
\end{aligned}$$

The substitution $t = xu$ simplifies this as follows.

$$\begin{aligned}
&{}_qI_{0+}^{\nu}\left[t^{\beta}b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(t^{\alpha};s,r|q)\right]\\
&= \frac{(q^{\beta+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s}\sum_{n=0}^{m^*}\frac{[(q^{-mk};q^k)_{\delta n}]^s}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r(q^k;q^k)_n}\frac{x^{\alpha n+\beta+\nu}}{\Gamma_q(\nu)}\\
&\quad\times\int_0^1 (1-uq)_{\nu-1}\ u^{\alpha n+\beta}\ \mathrm{d_q u}\\
&= \frac{(q^{\beta+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s}\sum_{n=0}^{m^*}\frac{[(q^{-mk};q^k)_{\delta n}]^s}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r\ (q^k;q^k)_n}\frac{x^{\alpha n+\beta+\nu}}{\Gamma_q(\nu)}\\
&\quad\times\int_0^1 u^{\alpha n+\beta+1-1}\frac{(uq;q)_{\infty}}{(uq^{\nu};q)_{\infty}}\ \mathrm{d_q u}\\
&= \frac{(q^{\beta+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s}\sum_{n=0}^{m^*}\frac{[(q^{-mk};q^k)_{\delta n}]^s}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r\ (q^k;q^k)_n}\frac{x^{\alpha n+\beta+\nu}}{\Gamma_q(\nu)}\mathfrak{B}_q(\alpha n+\beta+1,\nu)\\
&= \frac{(q^{\beta+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s}\sum_{n=0}^{m^*}\frac{[(q^{-mk};q^k)_{\delta n}]^s\ x^{\alpha n}}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r\ (q^k;q^k)_n}\\
&\quad\times\frac{(1-q)^{\nu}\ (q^{\alpha n+\beta+\nu+1};q)_{\infty}\ x^{\beta+\nu}}{(q^{\alpha n+\beta+1};q)_{\infty}}\\
&= \sum_{n=0}^{m^*}\frac{[(q^{-mk};q^k)_{\delta n}]^s\ x^{\alpha n}}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r\ (q^k;q^k)_n}\frac{(1-q)^{\nu}\ (q^{\beta+\nu+1};q)_{\infty}\ (q^{\beta+1};q)_{\alpha n}\ x^{\beta+\nu}}{[q^{\beta+\nu+1}]_{\alpha n}(q^{\beta+1};q)_{\infty}}\\
&= \frac{(1-q)^{\nu}\ (q^{\beta+\nu+1};q)_{\infty}\ (q^{\beta+1};q)_{\alpha m}\ (q^{\beta+\nu+1};q)_{\alpha m}\ x^{\beta+\nu}}{(q^{\beta+1};q)_{\infty}\ (q^{\beta+\nu+1};q)_{\alpha m}\ [(q^k;q^k)_m]^s}\\
&\quad\times\sum_{n=0}^{m^*}\frac{[(q^{-mk};q^k)_{\delta n}]^s\ [(q^{\lambda};q)_{\mu n}]^{-r}\ x^{\alpha n}}{(q^{\beta+\nu+1};q)_{\alpha n}\ (q^k;q^k)_n}\\
&= \frac{(1-q)^{\nu}\ (q^{\beta+\nu+1};q)_{\infty}\ (q^{\beta+1};q)_{\alpha m}\ x^{\beta+\nu}}{(q^{\beta+1};q)_{\infty}\ [q^{\beta+\nu+1}]_{\alpha m}}\ b_{m^*}^{(\alpha,\beta+\nu,\lambda,\mu)}(x^{\alpha};s,r|q).
\end{aligned}$$

□

7.9.1.1 Special cases

Taking $r = 0, s = 1$ and $\delta \in \mathbb{N}$ in (7.9.1), it gives

$$ {}_qI^{\nu}_{0+}\left[t^{\beta} L^{(\alpha,\beta)}_{m^*}(t^{\alpha}|q)\right] = \frac{(1-q)^{\nu}\ (q^{\beta+\nu+1};q)_{\infty}\ (q^{\beta+1};q)_{\alpha m}\ x^{\beta+\nu}}{(q^{\beta+1};q)_{\infty}\ (q^{\beta+\nu+1};q)_{\alpha m}}\ L^{(\alpha,\beta+\nu)}_{m^*}(q^{-\delta\alpha\nu}x^{\alpha}|q). $$

For $\alpha = k \in \mathbb{N}, and\delta = 1$, this reduces to

$$ {}_qI^{\nu}_{0+}\left[t^{\beta} Z^{\beta}_{m}(t;k|q)\right] = \frac{(1-q)^{\nu}\ (q^{\beta+\nu+1};q)_{\infty}\ (q^{\beta+1};q)_{km}\ x^{\beta+\nu}}{(q^{\beta+1};q)_{\infty}\ (q^{\beta+\nu+1};q)_{km}}\ Z^{\beta+\nu}_{m}(q^{-\nu}x;k|q). $$

The case $k = 1$ is

$$ {}_qI^{\nu}_{0+}\left[t^{\beta} L^{(\beta)}_{m}(t|q)\right] = \frac{(1-q)^{\nu}\ (q^{\beta+\nu+1};q)_{\infty}\ (q^{\beta+1};q)_{m}\ x^{\beta+\nu}}{(q^{\beta+1};q)_{\infty}\ (q^{\beta+\nu+1};q)_{m}}\ L^{(\beta+\nu)}_{m}(q^{-\nu}x|q). $$

7.9.2 Fractional q-differential operators

Next the fractional q-differential operator is applied on q-GKP.

Theorem 7.9.2. *If* $\alpha, \beta, \lambda > 0,\ m, \delta, \mu, l, s \in \mathbb{N},\ r \in \mathbb{N} \cup \{0\},\ \nu \in \mathbb{C}$ *and* $\delta \in \mathbb{N}$, *then*

$$ {}_qD^{\nu}_{0+}\left[t^{\beta} B^{(\alpha,\beta,\lambda,\mu)}_{m^*}(t^{\alpha};s,r|q)\right] = \frac{(1-q)^{-\nu}\ (q^{\beta+l-\nu+1};q)_{\infty}\ (q^{\beta+1};q)_{\alpha m}\ x^{\alpha n+\beta-\nu}}{(q^{\beta};q)_{\infty}\ (q^{\beta+l-\nu+1};q)_{\alpha m}\ (q^{\beta-\nu+1};q)_{\alpha m}} \times B^{(\alpha,\beta-\nu,\lambda,\mu)}_{m^*}(q^{\delta\alpha\nu}x^{\alpha};s,r|q). \tag{7.9.3} $$

$$ {}_qD^{\nu}_{0+}\left[t^{\beta} b^{(\alpha,\beta,\lambda,\mu)}_{m^*}(t^{\alpha};s,r|q)\right] = \frac{(1-q)^{-\nu}\ (q^{\beta+l-\nu+1};q)_{\infty}\ (q^{\beta+1};q)_{\alpha m}\ x^{\alpha n+\beta-\nu}}{(q^{\beta};q)_{\infty}\ (q^{\beta+l-\nu+1};q)_{\alpha m}\ (q^{\beta-\nu+1};q)_{\alpha m}} \times b^{(\alpha,\beta-\nu,\lambda,\mu)}_{m^*}(x^{\alpha};s,r|q). \tag{7.9.4} $$

Proof. Use of (1.6.4) and Theorem 7.9.1 on the left hand side of (7.9.3) gives

$$ \begin{aligned} & {}_qD^{\nu}_{0+}\left[t^{\beta} B^{(\alpha,\beta,\lambda,\mu)}_{m^*}(t^{\alpha};s,r|q)\right] \\ = & \left(\frac{d_q}{d_q x}\right)^{l} \left({}_qI^{l-\nu}_{0+}\left[t^{\beta} B^{(\alpha,\beta,\lambda,\mu)}_{m^*}(t^{\alpha};s,r|q)\right]\right) \\ = & \left(\frac{d_q}{d_q x}\right)^{l} \frac{(1-q)^{l-\nu}\ (q^{\beta+l-\nu+1};q)_{\infty}\ (q^{\beta+1};q)_{\alpha m}\ x^{\beta+l-\nu}}{(q^{\beta};q)_{\infty}\ (q^{\beta+l-\nu+1};q)_{\alpha m}} \end{aligned} $$

$$\times B_{m^*}^{(\alpha,\beta+l-\nu,\lambda,\mu)}(q^{-\delta\alpha(l-\nu)}x^{\alpha};s,r|q)$$

$$= \frac{(1-q)^{l-\nu}\ (q^{\beta+l-\nu+1};q)_{\infty}\ (q^{\beta+1};q)_{\alpha m}}{(q^{\beta};q)_{\infty}\ (q^{\beta+l-\nu+1};q)_{\alpha m}}\frac{(q^{\beta+l-\nu+1};q)_{\alpha m}}{[(q^{\alpha};q^{\alpha})_m]^s}$$

$$\times\sum_{n=0}^{m^*}\frac{q^{s(\alpha\delta n(\alpha\delta n-1)/2+\alpha\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+l-\nu+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r}\frac{[(q^{-m\alpha};q^{\alpha})_{\delta n}]^s}{(q^{\alpha};q^{\alpha})_n}\left(\frac{d_q}{d_qx}\right)^l(x^{\alpha n+\beta+l-\nu})$$

$$= \frac{(1-q)^{l-\nu}\ (q^{\beta+l-\nu+1};q)_{\infty}\ (q^{\beta+1};q)_{\alpha m}}{(q^{\beta};q)_{\infty}\ (q^{\beta+l-\nu+1};q)_{\alpha m}}\frac{(q^{\beta+l-\nu+1};q)_{\alpha m}}{[(q^{\alpha};q^{\alpha})_m]^s}$$

$$\times\sum_{n=0}^{m^*}\frac{q^{s(\alpha\delta n(\alpha\delta n-1)/2+\alpha\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+l-\nu+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r}\frac{[(q^{-m\alpha};q^{\alpha})_{\delta n}]^s}{(q^{\alpha};q^{\alpha})_n}$$

$$\times\frac{1}{x^l\ (q-1)^l\ q^{l(l-1)/2}}\sum_{j=0}^{l}(-1)^j\ q^{j(j-1)/2}\begin{bmatrix}l\\j\end{bmatrix}_q\ \left(xq^{l-j}\right)^{\alpha n+\beta+l-\nu}$$

$$= \frac{(1-q)^{l-\nu}\ (q^{\beta+l-\nu+1};q)_{\infty}\ (q^{\beta+1};q)_{\alpha m}}{(q^{\beta};q)_{\infty}\ (q^{\beta+l-\nu+1};q)_{\alpha m}}\frac{(q^{\beta+l-\nu+1};q)_{\alpha m}}{[(q^{\alpha};q^{\alpha})_m]^s}$$

$$\times\sum_{n=0}^{m^*}\frac{q^{s(\alpha\delta n(\alpha\delta n-1)/2+\alpha\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+l-\nu+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r}\frac{[(q^{-m\alpha};q^{\alpha})_{\delta n}]^s}{(q^{\alpha};q^{\alpha})_n}$$

$$\times(-1)^l\ (1-q)^{-l}\ x^{\alpha n+\beta-\nu}q^{l(\alpha n+\beta+l-\nu-l/2+1/2)}\prod_{j=1}^{l}(1-q^{-(\alpha n+\beta+l-\nu)}q^{j-1})$$

$$= \frac{(-1)^l\ (1-q)^{-\nu}\ (q^{\beta+l-\nu+1};q)_{\infty}\ (q^{\beta+1};q)_{\alpha m}}{(q^{\beta};q)_{\infty}\ (q^{\beta+l-\nu+1};q)_{\alpha m}}$$

$$\times\frac{[q^{\beta+l-\nu+1}]_{\alpha m}}{[(q^{\alpha};q^{\alpha})_m]^s}\sum_{n=0}^{m^*}\frac{q^{s(\alpha\delta n(\alpha\delta n-1)/2+\alpha\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+l-\nu+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r}$$

$$\times\frac{[(q^{-m\alpha};q^{\alpha})_{\delta n}]^s}{(q^{\alpha};q^{\alpha})_n}\ x^{\alpha n+\beta-\nu}q^{l(\alpha n+\beta+l-\nu-l/2+1/2)}\ (q^{-(\alpha n+\beta+l-\nu)};q)_l$$

$$= \frac{(-1)^l\ (1-q)^{-\nu}\ (q^{\beta+l-\nu+1};q)_{\infty}\ (q^{\beta+1};q)_{\alpha m}}{(q^{\beta};q)_{\infty}\ (q^{\beta+l-\nu+1};q)_{\alpha m}}$$

$$\times\frac{[q^{\beta+l-\nu+1}]_{\alpha m}}{[(q^{\alpha};q^{\alpha})_m]^s}\sum_{n=0}^{m^*}\frac{q^{s(\alpha\delta n(\alpha\delta n-1)/2+\alpha\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+l-\nu+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r}\frac{[(q^{-m\alpha};q^{\alpha})_{\delta n}]^s}{(q^{\alpha};q^{\alpha})_n}$$

$$\times x^{\alpha n+\beta-\nu}q^{l(\alpha n+\beta+l-\nu-l/2+1/2)}\frac{(q;q)_{\alpha n+\beta+l-\nu}}{(q;q)_{\alpha n+\beta-\nu}}(-1)^l q^{l(l-1)/2-(\alpha n+\beta+l-\nu)l}$$

$$= \frac{(1-q)^{l-\nu}\ (q^{\beta+l-\nu+1};q)_{\infty}\ (q^{\beta+1};q)_{\alpha m}}{(q^{\beta};q)_{\infty}\ (q^{\beta+l-\nu+1};q)_{\alpha m}}\frac{(q^{\beta+l-\nu+1};q)_{\alpha m}}{[(q^{\alpha};q^{\alpha})_m]^s}$$

$$\times\sum_{n=0}^{m^*}\frac{q^{s(\alpha\delta n(\alpha\delta n-1)/2+\alpha\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+l-\nu+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r}\frac{[(q^{-m\alpha};q^{\alpha})_{\delta n}]^s}{(q^{\alpha};q^{\alpha})_n}$$

$$\times(1-q)^{-l}\ x^{\alpha n+\beta-\nu}\ \frac{(q^{\alpha n+\beta-\nu+1};q)_{\infty}}{(q^{\alpha n+\beta+l-\nu+1};q)_{\infty}}$$

$$= \frac{(1-q)^{-\nu}\ (q^{\beta+l-\nu+1};q)_{\infty}\ (q^{\beta+1};q)_{\alpha m}}{(q^{\beta};q)_{\infty}\ (q^{\beta+l-\nu+1};q)_{\alpha m}}$$

$$\times \frac{(q^{\beta+l-\nu+1};q)_{\alpha m}}{[(q^{\alpha};q^{\alpha})_m]^s} \sum_{n=0}^{m^*} \frac{q^{s(\alpha\delta n(\alpha\delta n-1)/2+\alpha\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+l-\nu+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r}$$
$$\times \frac{[(q^{-m\alpha};q^{\alpha})_{\delta n}]^s\ x^{\alpha n+\beta-\nu}}{(q^l;q^l)_n}\ \frac{(q^{\beta+l-\nu+1};q)_{\alpha n}\ (q^{\beta-\nu+1};q)_{\infty}}{(q^{\beta+l-\nu+1};q)_{\infty}\ (q^{\beta-\nu+1};q)_{\alpha n}}$$
$$= \frac{(1-q)^{-\nu}\ (q^{\beta+l-\nu+1};q)_{\infty}\ (q^{\beta+1};q)_{\alpha m}}{(q^{\beta};q)_{\infty}\ (q^{\beta+l-\nu+1};q)_{\alpha m}\ (q^{\beta-\nu+1};q)_{\alpha m}}\ \frac{(q^{\beta-\nu+1};q)_{\alpha m}}{[(q^l;q^l)_m]^s}$$
$$\times \sum_{n=0}^{m^*} \frac{q^{s(\alpha\delta n(\alpha\delta n-1)/2+\alpha\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+l-\nu+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r}\ \frac{[(q^{-m\alpha};q^{\alpha})_{\delta n}]^s\ x^{\alpha n+\beta-\nu}}{(q^l;q^l)_n}$$
$$= \frac{(1-q)^{-\nu}\ (q^{\beta+l-\nu+1};q)_{\infty}\ (q^{\beta+1};q)_{\alpha m}\ x^{\alpha n+\beta-\nu}}{(q^{\beta};q)_{\infty}\ (q^{\beta+l-\nu+1};q)_{\alpha m}\ (q^{\beta-\nu+1};q)_{\alpha m}} B_{m^*}^{(\alpha,\beta-\nu,\lambda,\mu)}(q^{\delta\alpha\nu}x^{\alpha};s,r|q).$$

The proof of (7.9.4) runs similar, hence omitted. □

7.9.2.1 Special cases

Taking $r=0, s=1$ and $\delta \in \mathbb{N}$ in (7.9.3), gives

$${}_qD_{0+}^{\nu}\left[t^{\beta}L_{m^*}^{(\alpha,\beta)}(t^{\alpha}|q)\right] = \frac{(1-q)^{-\nu}\ (q^{\beta+k-\nu+1};q)_{\infty}\ (q^{\beta+1};q)_{\alpha m}\ x^{\alpha n+\beta-\nu}}{(q^{\beta};q)_{\infty}\ (q^{\beta+k-\nu+1};q)_{\alpha m}\ (q^{\beta-\nu+1};q)_{\alpha m}} \times L_{m^*}^{(\alpha,\beta-\nu)}(q^{\delta\alpha\nu}x^{\alpha}|q).$$

If we put $\alpha = k \in \mathbb{N}, \delta = 1$, then reduces to

$${}_qD_{0+}^{\nu}\left[t^{\beta}Z_m^{\beta}(t;k|q)\right] = \frac{(1-q)^{-\nu}(q^{\beta+k-\nu+1};q)_{\infty}\ (q^{\beta+1};q)_{km}x^{kn+\beta-\nu}}{(q^{\beta};q)_{\infty}(q^{\beta+k-\nu+1};q)_{km}\ (q^{\beta-\nu+1};q)_{km}} Z_m^{\beta-\nu}(q^{k\nu}x;k|q).$$

Putting $k=1$, it further gives

$${}_qD_{0+}^{\nu}\left[t^{\beta}L_m^{\beta}(t|q)\right] = \frac{(1-q)^{-\nu}\ (q^{\beta-\nu+2};q)_{\infty}\ (q^{\beta+1};q)_m\ x^{kn+\beta-\nu}}{(q^{\beta};q)_{\infty}\ (q^{\beta+k-\nu+1};q)_{km}\ (q^{\beta-\nu+1};q)_{km}} L_{m^*}^{(\beta-\nu)}(q^{\nu}x|q).$$

7.10 q-Integral transforms

7.10.1 Euler(Beta) transform:

Theorem 7.10.1. *If* $\alpha,\beta,\lambda >,\ m,\delta,\mu,k,s \in \mathbb{N},\ r \in \mathbb{N}\cup\{0\},\ a,b \in \mathbb{C},$ *then*

$$\mathfrak{B}_q\left\{B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(tx^{\alpha}q^{\delta\alpha a};s,r|q) : \beta+1, a\right\}$$

$$= \frac{(1-q)\ (q;q)_\infty\ (q^{\beta+a+1};q)_\infty\ (q^{\beta+1};q)_{\alpha m}}{(q^{\beta+1};q)_\infty\ (q^a;q)_\infty\ (q^{\beta+a+1};q)_{\alpha m}}\ B_{m^*}^{(\alpha,\beta+a,\lambda,\mu)}(t;s,r|q). \quad (7.10.1)$$

$$\mathfrak{B}_q\left\{b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(tx^\alpha;s,r|q):\beta+1,a\right\}$$
$$= \frac{(1-q)\ (q;q)_\infty\ (q^{\beta+a+1};q)_\infty\ (q^{\beta+1};q)_{\alpha m}}{(q^{\beta+1};q)_\infty\ (q^a;q)_\infty\ (q^{\beta+a+1};q)_{\alpha m}}\ b_{m^*}^{(\alpha,\beta+a,\lambda,\mu)}(t;s,r|q). \quad (7.10.2)$$

Proof. The left hand expression

$$\mathfrak{B}_q\left\{B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(tx^\alpha q^{\delta\alpha a};s,r|q):\beta+1,a\right\}$$
$$= \int_0^1 x^{\beta+1-1}\frac{(xq;q)_\infty}{(xq^a;q)_\infty}\ B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(tx^\alpha q^{\delta\alpha a};s,r|q)\ d_qx$$
$$= \int_0^1 x^{\beta+1-1}\frac{(xq;q)_\infty}{(xq^a;q)_\infty}\ \frac{(q^{\beta+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s}\sum_{n=0}^{m^*}\frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\beta+a+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r}$$
$$\times\frac{[(q^{-mk};q^k)_{\delta n}]^s\ t^n\ x^{\alpha n}}{(q^k;q^k)_n}\ d_qx$$
$$= \frac{(q^{\beta+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s}\sum_{n=0}^{m^*}\frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\beta+a+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r}\frac{[(q^{-mk};q^k)_{\delta n}]^s\ t^n}{(q^k;q^k)_n}$$
$$\times\int_0^1 x^{\alpha n+\beta+1-1}\frac{(xq;q)_\infty}{(xq^a;q)_\infty}\ d_qx$$
$$= \frac{(q^{\beta+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s}\sum_{n=0}^{m^*}\frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\beta+a+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r}\frac{[(q^{-mk};q^k)_{\delta n}]^s\ t^n}{(q^k;q^k)_n}$$
$$\times\mathfrak{B}_q(\alpha n+\beta+1,a)$$
$$= \frac{(q^{\beta+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s}\sum_{n=0}^{m^*}\frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\beta+a+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r}\frac{[(q^{-mk};q^k)_{\delta n}]^s\ t^n}{(q^k;q^k)_n}$$
$$\times\frac{(1-q)\ (q;q)_\infty\ (q^{\alpha n+\beta+1+a};q)_\infty}{(q^{\alpha n+\beta+1};q)_\infty\ (q^a;q)_\infty}$$
$$= \frac{(1-q)\ (q;q)_\infty\ (q^{\beta+a+1};q)_\infty}{(q^{\beta+1};q)_\infty\ (q^a;q)_\infty}\frac{(q^{\beta+1};q)_{\alpha m}\ (q^{\beta+a+1};q)_{\alpha m}}{[q^{\beta+a+1}]_{\alpha m}\ [(q^k;q^k)_m]^s}$$
$$\times\sum_{n=0}^{m^*}\frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\beta+a+1)+r\mu\lambda)}}{(q^{\beta+a+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r}\frac{[(q^{-mk};q^k)_{\delta n}]^s\ t^n}{(q^k;q^k)_n}$$
$$= \frac{(1-q)\ (q;q)_\infty\ (q^{\beta+a+1};q)_\infty\ (q^{\beta+1};q)_{\alpha m}}{(q^{\beta+1};q)_\infty\ (q^a;q)_\infty\ (q^{\beta+a+1};q)_{\alpha m}}\ B_{m^*}^{(\alpha,\beta+a,\lambda,\mu)}(t;s,r|q).$$

Similarly, one can obtain (7.10.2), hence proof is omitted. □

7.10.1.1 Special cases

(i) Taking $r = 0, s = 1$ and $\delta \in \mathbb{N}$ in (7.10.1), it gives

$$\mathfrak{B}_q\left\{L_{m^*}^{(\alpha,\beta)}(tx^\alpha q^{\delta\alpha a}|q) : \beta+1, a\right\} = \frac{(1-q)\ (q;q)_\infty\ (q^{\beta+a+1};q)_\infty\ (q^{\beta+1};q)_{\alpha m}}{(q^{\beta+1};q)_\infty\ (q^a;q)_\infty\ (q^{\beta+a+1};q)_{\alpha m}} \times L_{m^*}^{(\alpha,\beta+a)}(t|q).$$

For $\alpha = k \in \mathbb{N}$, and $\delta = 1$, this reduces to

$$\mathfrak{B}_q\left\{Z_m^{(k,\beta)}(t^{1/k}xq^a;k|q) : \beta+1, a\right\} = \frac{(1-q)\ (q;q)_\infty\ (q^{\beta+a+1};q)_\infty\ (q^{\beta+1};q)_{km}}{(q^{\beta+1};q)_\infty\ (q^a;q)_\infty\ (q^{\beta+a+1};q)_{km}} \times Z_m^{(k,\beta+a)}(t^{1/k};k|q).$$

The case $k = 1$ is

$$\mathfrak{B}_q\left\{L_m^{(\beta)}(txq^a|q) : \beta+1, a\right\} = \frac{(1-q)\ (q;q)_\infty\ (q^{\beta+a+1};q)_\infty\ (q^{\beta+1};q)_m}{(q^{\beta+1};q)_\infty\ (q^a;q)_\infty\ (q^{\beta+a+1};q)_m} \times Z_m^{(k,\beta+a)}(t;k|q).$$

7.10.2 Laplace transform:

Theorem 7.10.2. *If* $\alpha, \lambda, \sigma, \beta, \nu > 0$, *then*

$$\mathfrak{L}_q\left\{t^\nu B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(xt^\sigma;s,r|q)\right\} = \frac{(q^{\beta+1};q)_{\alpha m}}{(1-q)\ [(q^k;q^k)_m]^s}\sum_{n=0}^{m^*}\frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}}{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r} \times \frac{q^{\delta n(\alpha(\beta+a+1)+r\mu\lambda)}\ q^{-(\sigma n+\nu+1)(\sigma n+\nu)/2}}{S^{\sigma n+\nu+1}\ (q^{\sigma n+\nu+1};q)_\infty} \times \frac{(1-q)\ (q;q)_\infty\ [(q^{-mk};q^k)_{\delta n}]^s\ x^n}{(q^k;q^k)_n}. \quad (7.10.3)$$

$$\mathfrak{L}_q\left\{t^\nu b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(xt^\sigma;s,r|q)\right\} = \frac{(q^{\beta+1};q)_{\alpha m}}{(1-q)\ [(q^k;q^k)_m]^s}\sum_{n=0}^{m^*}\frac{[(q^{-mk};q^k)_{\delta n}]^s\ x^n}{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r} \times \frac{(1-q)\ (q;q)_\infty\ q^{-(\sigma n+\nu+1)(\sigma n+\nu)/2}}{S^{\sigma n+\nu+1}\ (q^{\sigma n+\nu+1};q)_\infty\ (q^k;q^k)_n}. \quad (7.10.4)$$

Proof.

$$\mathfrak{L}_q\left\{t^\nu B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(xt^\sigma;s,r|q)\right\}$$

$$= \frac{1}{1-q}\int_0^\infty e_q^{-St}\ t^\nu B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(xt^\sigma;s,r|q)\ d_qt$$

$$= \frac{1}{1-q}\int_0^\infty e_q^{-St}\ t^\nu \frac{(q^{\beta+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s}\sum_{n=0}^{m^*}\frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\beta+a+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r}$$
$$\times\frac{[(q^{-mk};q^k)_{\delta n}]^s\ x^n\ t^{\sigma n}}{(q^k;q^k)_n}d_qt$$

$$= \frac{(q^{\beta+1};q)_{\alpha m}}{(1-q)\ [(q^k;q^k)_m]^s}\sum_{n=0}^{m^*}\frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}q^{\delta n(\alpha(\beta+a+1)+r\mu\lambda)}[(q^{-mk};q^k)_{\delta n}]^s\ x^n}{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r\ (q^k;q^k)_n}$$
$$\times\int_0^\infty e_q^{-St}\ t^{\sigma n+\nu}\ d_qt$$

$$= \frac{(q^{\beta+1};q)_{\alpha m}}{(1-q)\ [(q^k;q^k)_m]^s}\sum_{n=0}^{m^*}\frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}q^{\delta n(\alpha(\beta+a+1)+r\mu\lambda)}[(q^{-mk};q^k)_{\delta n}]^s\ x^n}{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r\ (q^k;q^k)_n}$$
$$\times\frac{(1-q)\ (q;q)_\infty\ q^{-(\sigma n+\nu+1)(\sigma n+\nu)/2}}{S^{\sigma n+\nu+1}\ (q^{\sigma n+\nu+1};q)_\infty}.$$

Similarly, one can obtain (7.10.4), hence proof is omitted. □

7.10.2.1 Special cases

(i) Taking $r=0, s=1$ and $\delta\in\mathbb{N}$ in (7.10.3), it gives

$$\mathfrak{L}_q\left\{t^\nu L_{m^*}^{(\alpha,\beta)}(xt^\sigma;1,0|q)\right\} = \frac{(q^{\beta+1};q)_{\alpha m}}{(1-q)\ (q^k;q^k)_m}\sum_{n=0}^{m^*}\frac{q^{(k\delta n(k\delta n-1)/2+k\delta nm)}}{(q^{\beta+1};q)_{\alpha n}}$$
$$\times\frac{q^{\delta n(\alpha(\beta+a+1)+r\mu\lambda)}\ q^{-(\sigma n+\nu+1)(\sigma n+\nu)/2}}{S^{\sigma n+\nu+1}\ (q^{\sigma n+\nu+1};q)_\infty}$$
$$\times\frac{(1-q)\ (q;q)_\infty\ (q^{-mk};q^k)_{\delta n}\ x^n}{(q^k;q^k)_n}.$$

For $\alpha=k\in\mathbb{N}$, and $\delta=1$, this reduces to

$$\mathfrak{L}_q\left\{Z_m^{(k,\beta)}(t^{1/k}xq^a;k|q):\beta+1,a\right\} = \frac{(1-q)\ (q;q)_\infty\ (q^{\beta+a+1};q)_\infty\ (q^{\beta+1};q)_{km}}{(q^{\beta+1};q)_\infty\ (q^a;q)_\infty\ (q^{\beta+a+1};q)_{km}}$$
$$\times Z_m^{(k,\beta+a)}(t^{1/k};k|q).$$

The case $k=1$ is

$$\mathfrak{L}_q\left\{L_m^{(\beta)}(txq^a|q):\beta+1,a\right\} = \frac{(1-q)\ (q;q)_\infty\ (q^{\beta+a+1};q)_\infty\ (q^{\beta+1};q)_m}{(q^{\beta+1};q)_\infty\ (q^a;q)_\infty\ (q^{\beta+a+1};q)_m}$$
$$\times Z_m^{(k,\beta+a)}(t;k|q).$$