Chapter 1

Introduction

1.1 The Mittag-Leffler function

Gösta Magnus Mittag-Leffler was born on March 16, 1846, in Stockholm, Sweden. His father, John Olof Leffler, was a school teacher, and was also elected as a member of the Swedish Parliament. At his birth Gösta was given the name Leffler and later (when he was a student) he added his mother's name "Mittag" as a tribute to this family, which was very important in Sweden in the nineteenth century. Both sides of his family were of German origin.

He studied at the University of Uppsala, matriculated in 1865 and completed his Ph.D. in 1872.

He founded *Acta Mathematica*, an international mathematical journal in 1882 and served as the Editor-in-Chief of the journal for 45 years.

In 1903, Mittag-Leffler [47] proposed a function $E_\alpha(z)$ defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)},$$

where z is a complex variable and $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$. Later on this function was referred to as Mittag-Leffler function. The Mittag-Leffler function is direct generalization of the exponential function to which it reduces for $\alpha = 1$. This function has some interesting properties which later became essential for the description of many problems arising in applications. Nowadays the Mittag-Leffler function and its numerous generalizations have acquired a new life. The recent notable increased interest in the study of their relevant properties is due to the close connection to the Fractional Calculus and its application to the study of Differential and Integral Equations. Many modern models of fractional type have recently been proposed in Probability Theory, Mechanics, Mathematical Physics, Chemistry, Biology, Mathematical Economics, Engineering and Applied Sciences etc. There are many applications of Mittag-leffler function and its generalizations in Astrophysics problems (see [24]). One application of Mittag-Leffler function is described below.

In a reaction-diffusion process if $N(t)$ is the number density at a time t and if the production rate is proportional to original number, then

$$\frac{d}{dt}N(t) \;=\; \lambda N(t), \quad \lambda > 0 \tag{1.1.1}$$

where λ is the rate of production. If the consumption or destruction rate is also proportional to the original number then

$$\frac{d}{dt}N(t) \;=\; -\mu N(t), \quad \mu > 0 \tag{1.1.2}$$

where μ is the destruction rate. Then the residual part is given by

$$\frac{d}{dt}N(t) \;=\; -cN(t), \quad c = \mu - \lambda. \tag{1.1.3}$$

If c is free of t then the solution is exponential model

$$N(t) \;=\; N_0 e^{-ct}, \quad N_0 = N(t) \text{ at } t = t_0 \tag{1.1.4}$$

where t_0 is the starting time. Instead of total derivative in (1.1.1) to (1.1.3) if the fractional derivative or fractional nature of reactions is considered, that is, an equation of the form

$$N(t) - N_0 \;=\; -c^v \, {}_0D_t^{-v} \, N(t) \tag{1.1.5}$$

is considered where ${}_0D_t^{-v}$ is the standard Riemann-Liouville fractional integral operator, then the solution

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{(-1)^k \; (ct)^{vk}}{\Gamma(vk+1)} = N_0 \; E_v(-(ct)^v), \tag{1.1.6}$$

involves $E_v(.)$ which is nothing but the Mittag-Leffler function.

1.2 Definitions and formulas

1.2.1 Ordinary forms

The following definitions and formulas will be used in the work.

Definition 1.2.1. *The Pochhammer symbol is defined as ([63],[78])*

$$(\lambda)_n = \begin{cases} (\lambda)(\lambda+1)(\lambda+2)\cdots(\lambda+n-1) & \text{if } n \in \mathbb{N}, \\ 1 & \text{if } n = 0. \end{cases} \tag{1.2.1}$$

Here, $(\lambda)_n$ *is also called the factorial function. If* $\lambda = 1$ *then it reduces to* $n!$, *thus* $(1)_n = n!$.

Definition 1.2.2. *The binomial coefficient is given by*

$$\binom{\lambda}{n} = \frac{(\lambda)(\lambda-1)(\lambda-2)\cdots(\lambda-n+1)}{n!} = \frac{(-1)^n\,(-\lambda)_n}{n!}, \tag{1.2.2}$$

or equivalently [63],

$$\binom{\lambda}{n} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-n+1)\,n!}. \tag{1.2.3}$$

Remark 1.2.1. *For* $0 \le k \le n$,

$$(\lambda)_{n-k} = \frac{(-1)^k\,(\lambda)_n}{(1-\lambda-n)_k}. \tag{1.2.4}$$

For $\lambda = 1$, *it reduces to*

$$(-n)_k = \frac{(-1)^k\,n!}{(n-k)!}, \quad 0 \le k \le n. \tag{1.2.5}$$

Definition 1.2.3. *The Gamma function is defined as [63]:*

$$\Gamma(z) = \int_0^\infty e^{-t}\,t^{z-1}\,\mathrm{d}t; \quad \Re(z) > 0. \tag{1.2.6}$$

The Stirling's asymptotic formula [16] for the Gamma function is

$$\Gamma(z) \sim \sqrt{2\pi}\,e^{-z}\,z^{z-\frac{1}{2}}, \quad \text{for large } |z|. \tag{1.2.7}$$

Note 1.2.1. *This function is related with the factorial function by means of the formula [63]:*

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}. \tag{1.2.8}$$

Definition 1.2.4. *The Beta function denoted by $\mathfrak{B}(a,b)$, is defined as [63]:*

$$\mathfrak{B}(a,b) \quad = \quad \int_0^1 z^{a-1}\,(1-z)^{b-1}\,\mathrm{d}z; \quad \Re(a,b) > 0. \tag{1.2.9}$$

Note 1.2.2. *Its relation with Gamma function is given by [63]*

$$\mathfrak{B}(a,b) \quad = \quad \frac{\Gamma(a)\;\Gamma(b)}{\Gamma(a+b)}; \quad \Re(a,b) > 0. \tag{1.2.10}$$

Definition 1.2.5. *The Wright generalized hypergeometric function is denoted and defined as [44]*

$${}_p\psi_q\left[\begin{array}{ll}(a_1,A_1),\cdots,(a_p,A_p); & z\\ (b_1,B_1),\cdots,(b_q,B_q); & \end{array}\right] = \sum_{r=0}^{\infty}\frac{\prod\limits_{j=1}^{p}\Gamma(a_j+rA_j)}{\prod\limits_{j=1}^{q}\Gamma(b_j+rB_j)}\frac{z^r}{r!} \tag{1.2.11}$$

$$= H^{1,p}_{p,q+1}\left[-z\left|\begin{array}{c}(1-a_1,A_1),\;\cdots,\;(1-a_p,A_p)\\ (0,1),\;(1-b_1,B_1),\;\cdots,\;(1-b_q,B_q)\end{array}\right.\right], \tag{1.2.12}$$

where $H^{m,n}_{p,q}\left[-z\left|\begin{array}{ccc}(a_1,A_1), & \cdots, & (a_p,A_p)\\ (b_1,B_1), & \cdots, & (b_q,B_q)\end{array}\right.\right]$ *denotes the Fox $H-$function in which $a_i, b_j \in \mathbb{C}$, $A_i, B_j \in \mathbb{R}$, $(i = 1,2,...p\,; j = 1,2...q)$, and* $1+\sum\limits_{j=1}^{q}B_j - \sum\limits_{i=1}^{p}A_i > 0$. *In particular [40],*

$${}_1\psi_1\left[\;\alpha;\quad\beta;\quad z\;\right] = \sum_{r=0}^{\infty}\frac{z^r}{\Gamma(\alpha r+\beta)\;r!}, \quad \alpha > -1.$$

Theorem 1.2.1. *If $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ is an entire function then the order $\varrho(f)$ of f is given by [7, Eq.(1.2)]*

$$\varrho(f) = \lim_{n\to\infty}\sup\frac{n\;\log n}{\log(1/|a_n|)}. \tag{1.2.13}$$

and the type of the function σ is given by [33]

$$e\varrho\sigma = \lim_{n\to\infty} \sup \left(n \left| a_n \right|^{\varrho/n} \right). \tag{1.2.14}$$

For every positive ϵ, the asymptotic estimate [33, Eq.(16)]

$$|f(z)| < exp\left((\sigma+\epsilon)\ |z|^{\varrho}\right),\ \ |z| \geq r_0 > 0 \tag{1.2.15}$$

holds with ϱ, σ as in (1.2.13), (1.2.14) for $|z| \geq r_0(\epsilon),\ r_0(\epsilon)$ sufficiently large.

Definition 1.2.6. *The space $L(a,b)$ of (real or complex valued) Lebesgue measurable functions is given by ([67], [31])*

$$L(a,b) = \left\{ f:\ \|f\|_1 = \int_a^b |f(t)|\, \mathrm{d}t < \infty \right\}. \tag{1.2.16}$$

The following double series identities will also be used [63].

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} f(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} f(k,n-k). \tag{1.2.17}$$

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} f(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} f(k,n+k). \tag{1.2.18}$$

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} f(k,n) \quad = \quad \sum_{n=0}^{\infty} \sum_{k=0}^{[\frac{n}{2}]} f(k,n-2k) \tag{1.2.19}$$

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[\frac{n}{2}]} f(k,n) \quad = \quad \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} f(k,n+2k) \tag{1.2.20}$$

$$\sum_{i=0}^{mn} \sum_{j=0}^{[\frac{i}{m}]} f(i,j) = \sum_{j=0}^{n} \sum_{i=0}^{mn-mj} f(i+mj,j). \tag{1.2.21}$$

$$\sum_{k=0}^{n}\sum_{j=0}^{k} f(k,j) = \sum_{j=0}^{n}\sum_{k=j}^{n} f(k,j). \tag{1.2.22}$$

$$\sum_{k=0}^{n}\sum_{j=0}^{k} A(k,j) = \sum_{j=0}^{n}\sum_{k=0}^{n-j} A(k+j,j) \tag{1.2.23}$$

The binomial series is given by [63]

$$\sum_{n=0}^{\infty}(a)_n \frac{z^n}{n!} = (1-z)^{-a}, \quad |z| < 1. \tag{1.2.24}$$

Lemma 1.2.1. *For* $0 < t < 1,\ a > 0,$ *where* $s \in \mathbb{N}$,

$$\sum_{n=0}^{\infty}\left((a)_n \frac{t^n}{n!}\right)^s \leq \left(\sum_{n=0}^{\infty}(a)_n \frac{t^n}{n!}\right)^s. \tag{1.2.25}$$

On comparing the coefficients of like powers of t, the above inequality follows.

1.2.2 q-Analogues

A q-analogue of a non zero number ‘ a’ denoted by $[a]$, is defined by [27]

$$[a] = \frac{1-q^a}{1-q}$$

with the convention that $[a] \to a$ when $q \to 1$. Based on this notion, the q-theory has been extensively developed by a large number of eminent researchers in varied directions such as Special Functions, Number theory, Distribution theory etc. ([18], [5], [17]).

Definition 1.2.7. *For* $a \in \mathbb{C}$, *and* $0 < |q| < 1$, *the* q*-shifted factorial is defined by [18, Eq.(1.2.15), p.3 and Eq.(1.2.30), p.6]*

$$(a;q)_n = \begin{cases} 1 & if \quad n = 0 \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & if \quad n \in \mathbb{N} \\ \dfrac{(q;q)_\infty}{(aq^n;q)_\infty} & if \quad n \in \mathbb{C}, \end{cases} \tag{1.2.26}$$

where

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)\ , \quad |q|<1.$$

A well-known extension of the q-shifted factorial is given by [12]

$$[t-|a]_n = (t-a)(t-aq)(t-aq^2)\cdots(t-aq^{n-1}). \tag{1.2.27}$$

A finite series-product identity is [12]

$$\sum_{k=0}^{n} q^{k(k-1)/2}\begin{bmatrix} n \\ k \end{bmatrix} x^k = \prod_{k=1}^{n}(1+xq^{k-1}). \tag{1.2.28}$$

Definition 1.2.8. *The basic binomial coefficient with base q^r is (cf. [18, Ex.(1.2), p.20] with r=1):*

$$\begin{bmatrix} n \\ m \end{bmatrix}_r = \frac{(q^r;q^r)_n}{(q^r;q^r)_{n-m}\ (q^r;q^r)_m}, r \neq 0. \tag{1.2.29}$$

Note 1.2.3.

$$(q^{-k};q^{-k})_n \quad = \quad (q^k;q^k)_n\ (-q^{-k})\ q^{-kn(n-1)/2}. \tag{1.2.30}$$

Definition 1.2.9. *A q-Gamma function is defined as [23]:*

$$\Gamma_q(\alpha) \quad = \quad \frac{(q;q)_\infty\ (1-q)^{1-\alpha}}{(q^\alpha;q)_\infty}, \tag{1.2.31}$$

where $\alpha \neq 0,-1,-2,...,$ *and* $0<q<1.$

The q-analogue of Stirling's asymptotic formula [42, Eq.(2.25), p.482] for the q-Gamma function is

$$\Gamma_q(x) \sim (1+q)^{\frac{1}{2}}\ \Gamma_{q^2}\left(\frac{1}{2}\right)\ (1-q)^{\frac{1}{2}-x}\ e^{\mu_q(x)}, \tag{1.2.32}$$

where $\mu_q(x) = \dfrac{\theta\ q^x}{1-q-q^x},\ 0<\theta<1.$

Definition 1.2.10. *A q-Beta function $\mathfrak{B}_q(x,y)$ is expressible in different ways [18].*

$$\mathfrak{B}_q(x,y) = \int_0^1 t^{x-1}(tq)_{y-1}\ \mathrm{d}_q t, \tag{1.2.33}$$

$$\mathfrak{B}_q(x,y) = \frac{(1-q)\,(q)_\infty\,(q^{x+y})_\infty}{(q^x)_\infty\,(q^y)_\infty}, \tag{1.2.34}$$

and

$$\mathfrak{B}_q(x,y) = \int_0^1 t^{x-1}\frac{(tq;q)_\infty}{(tq^y;q)_\infty}\,\mathrm{d}_q t \tag{1.2.35}$$

in which $y \neq 0,-1,-2,...,\ \Re(x) > 0$.

Definition 1.2.11. *The basic exponential functions are defined by (cf. W. Hahn [22], Gasper and Rahman [18, p.236]):*

$$e_q(x) = {}_1\phi_0(0;-;q,x) = \sum_{n=0}^{\infty}\frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty}, \quad |x|<1 \tag{1.2.36}$$

and

$$E_q(x) = {}_0\phi_0(-;-;q,x) = \sum_{n=0}^{\infty} q^{n(n-1)/2}\frac{x^n}{(q;q)_n} = (-x;q)_\infty, \quad |x|<\infty. \tag{1.2.37}$$

Definition 1.2.12. *A q-derivative of a function $f(x)$ is defined by [18, Ex.1.12, p.22]*

$$D_q f(x) = \frac{f(x)-f(xq)}{x(1-q)} \tag{1.2.38}$$

and [34]

$$\Delta_q f(x) \;=\; \frac{f(x)-f(xq^{-1})}{x-xq^{-1}}. \tag{1.2.39}$$

Definition 1.2.13. *A q-derivative of product of two functions is given by [9].*

$$D_q\left(f(x).g(x)\right) \;=\; g(qx)D_q f(x) + f(x)D_q(g(x)). \tag{1.2.40}$$

$$\Delta_q\left(f(x).g(x)\right) \;=\; g(q^{-1}x)\Delta_q f(x) + f(x)\Delta_q(g(x)). \tag{1.2.41}$$

Definition 1.2.14. *The q-integrals are defined by [29]*

$$\int_0^x f(t)\,\mathrm{d}_q t \;=\; x(1-q)\sum_{k=0}^{\infty} q^k\, f(xq^k), \tag{1.2.42}$$

and

$$\int_{x}^{\infty} f(t)\, \mathrm{d}_q t \;=\; x(1-q)\sum_{k=1}^{\infty} q^{-k}\, f(xq^{-k}). \tag{1.2.43}$$

Definition 1.2.15. *The q-Beta integral due to W. Hahn [23] is*

$$\int_0^1 t^{\lambda-1} E_q(tq)\, \mathrm{d}_q t \;=\; (1-q)\frac{(q;q)_\infty}{(q^\lambda;q)_\infty}, \quad \lambda > 0. \tag{1.2.44}$$

1.3 Generalized hypergeometric function and basic hypergeometric function

The hypergeometric function and its associated series are given by [63]:

$$ {}_2F_2\left[\begin{array}{ccc} a, & b, & x \\ c, & & \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n}\frac{x^n}{n!}, \tag{1.3.1}$$

where c is neither zero nor negative integer and $|x| < 1$.
The series is convergent for $|x| < 1$. For $|x| = 1$, the series converges if $\Re(c-a-b) > 0$. E. Heine ([25], [26]) introduced an interesting extension of this series in the form:

$$1 + \frac{(1-q^a)\,(1-q^b)}{(1-q^c)\,(1-q)}x + \frac{(1-q^a)\,(1-q^{a+1})\,(1-q^b)\,(1-q^{b+1})}{(1-q^c)\,(1-q^{c+1})\,(1-q)\,(1-q^2)}x^2 + \ldots \tag{1.3.2}$$
$$(c \neq 0, -1, -2, \ldots;\; |x| < 1,\; |q| < 1).$$

It readily follows that as $q \to 1$, the series in (1.3.2) appoaches to the Gauss series (1.3.1).

$$1 + \frac{a\;b}{c}x + \frac{a\;(a+1)\;b\;(b+1)}{c\;(c+1)\;1\;2}x^2 + \ldots$$
$$(c \neq 0, -1, -2, \ldots;\; |x| < 1).$$

Thus, Heine's series defines a basic analogue (or a q-analogue) of the Gauss series; and for this reason the Heine's series is called a basic hypergeometric series or q-hypergeometric series.
Just as it happened with the Gauss series that it was known in other particular forms before its introduction, this q-series (1.3.2) was also known in special forms

prior to its introduction. For example, the identity

$$1+\sum_{n=1}^{\infty}(-1)^n\left\{q^{n(3n-1)/2}+q^{n(3n+1)/2}\right\}=\prod_{n=1}^{\infty}(1-q^n)$$

due to Leonhard Euler is dated back to 1748 A. D. The triple product identity

$$\prod_{n=0}^{\infty}\left\{(1-xq^n)\left(1-q^{n+1}x^{-1}\right)\left(1-q^{n+1}\right)\right\}=\sum_{n=-\infty}^{n=\infty}(-1)^n\,q^{n^2/2}x^n,$$

and the four Theta functions : $\theta_i(z,q),\ i=1,2,3,4,$ were given by C.G.J. Jacobi also dates back to in 1829 A.D.
The series in (1.3.2), was denoted by Heine by using the notation $\phi(a,b;c;q,x)$. Alternatively, the other notations, namely

$${}_2\phi_1(a,b;c;q,x),\qquad {}_2\phi_1\left[\begin{array}{cccc} a, & b; & q, & x \\ & c; & & \end{array}\right]$$

are often occurs in the literature. In this notation, the series (1.3.2) is representable in the form

$${}_2\phi_1(a,b;c;q,x) \;=\; \sum_{n=0}^{\infty}\frac{(a;q)_n\,(b;q)_n}{(c;q)_n\,(q;q)_n}x^n, \tag{1.3.3}$$

where $|x|<1,\ |q|<1$ *and* $(a;q)_n$ is as given in (1.2.26).
A generalization of (1.3.3) which is a basic analogue of (1.3.1) is ${}_r\phi_s$ function defined by ([8], [18]):

$$\begin{aligned}{}_r\phi_s\left[\begin{array}{cccccc} a_1, & a_2, & \cdots, & a_r; & q, & x \\ b_1, & b_2, & \cdots, & b_s; & & \end{array}\right] \;=\; & \sum_{n=0}^{\infty}\frac{[a_1]_n\,[a_2]_n\cdots[a_r]_n\;x^n}{[b_1]_n\,[b_2]_n\cdots[b_s]_n\;(q;q)_n}\\ & \times\left\{(-1)^n\;q^{n(n-1)/2}\right\}^{s-r+1}. \end{aligned}\tag{1.3.4}$$

This infinite basic series converges for all x if $r\le s,\ 0<|q|<1$. If $r=s+1$ then it converges for $|x|<1$. The various specializations of this ${}_r\phi_s[x]$ function include the basic exponential functions (1.2.36) and (1.2.37), and basic Bessel functions [18, p.25])

$$J_\nu^{(1)}(x;q) \;=\; \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\,(x/2)^\nu\;{}_2\phi_1\left(0,0;q^{\nu+1};q,-x^2/4\right),$$

and

$$J_\nu^{(2)}(x;q) \;=\; \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\,(x/2)^\nu\;{}_0\phi_1\left(-;q^{\nu+1};q,-\frac{x^2\,q^{\nu+1}}{4}\right),$$

1.4 Integral transforms

Some well-known integral transforms are given as follows ([14], [63], [79], [80]).

Definition 1.4.1. *Beta (or Euler) transform.*

$$\mathfrak{B}\{f(x);a,b\} = \int_0^1 x^{a-1}\,(1-x)^{b-1} f(z)\,\mathrm{d}z; \quad \Re(a,b)>0. \tag{1.4.1}$$

Definition 1.4.2. *Finite Laplace transform.*

$$\mathcal{L}_T\{f(x)\} = \int_0^T e^{-St} f(t)\,\mathrm{d}t; \ \Re(S)>0, \tag{1.4.2}$$

where T is a positive number.

Definition 1.4.3. *Laplace transform.*

$$\mathcal{L}\{f(x)\} = \int_0^\infty e^{-St} f(t)\,\mathrm{d}t; \quad \Re(S)>0. \tag{1.4.3}$$

Definition 1.4.4. *Convolution formula.*
If $\mathcal{L}(f_1(t)) = F_1(S)$ and $\mathcal{L}(f_2(t)) = F_2(S)$ then

$$\mathcal{L}\left\{\int_0^t f_1(x)\, f_2(x-t)\,\mathrm{d}x\right\} = F_1(S)\,F_2(S). \tag{1.4.4}$$

Definition 1.4.5. *Derivative rule.*

$$\frac{\partial^n}{\partial S^n}\left([\mathcal{L}\, y(x)]\,(S)\right) = (-1)^n\,\mathcal{L}\,[x^n\; y(x)]\,(S), \quad n\in\mathbb{N}_0. \tag{1.4.5}$$

Definition 1.4.6. *Laguerre transform.*

$$L\{f(x)\} = \int_0^\infty e^{-x}\; x^\mu\; L_n^{(\mu)}(x)\; f(x)\,\mathrm{d}x; \tag{1.4.6}$$

where $L_n^{(\mu)}(x)$, $\mu>-1$, is the Laguerre polynomial of degree $n\geq 0$.

Definition 1.4.7. *Generalized Stieltjes transform.*

$$S_g\{f(x)\} = \int \frac{f(x)}{(x+z)^\rho}\,\mathrm{d}x, \tag{1.4.7}$$

where z is a complex variable, $|arg(z)| < \pi, \ \rho \geq 1$.

Definition 1.4.8. *Mellin-Barnes transform.*

$$M[f(z); s] = \int_0^\infty z^{s-1} f(z)\,\mathrm{d}z = f^*(s), \quad \Re(s) > 0, \tag{1.4.8}$$

then

$$f(z) = M^{-1}[f^*(s); x] = \frac{1}{2\pi i} \int f^*(s)\, x^{-s}\,\mathrm{d}s. \tag{1.4.9}$$

Definition 1.4.9. *Whittaker transform.*

$$W(f(x); \nu, \lambda, \mu) \ = \ \int_0^\infty e^{-\frac{t}{2}}\, t^{\nu-1}\, W_{\lambda,\mu}(t)\, f(t)\,\mathrm{d}t, \quad \Re(\nu) > 0, \tag{1.4.10}$$

in which $W_{\lambda,\mu}(t)$ *is a Whittaker function [93] defined as*

$$W_{\lambda,\mu}(z) \ = \ \frac{e^{-\frac{z}{2}}\, z^{\lambda}}{\Gamma\left(\frac{1}{2} - \lambda + \mu\right)} \int_0^\infty t^{-\lambda-\frac{1}{2}+\mu} \left(1 + \frac{t}{z}\right)^{\lambda-\frac{1}{2}+\mu} e^{-t}\,\mathrm{d}t, \tag{1.4.11}$$

with $\Re\left(\lambda - \frac{1}{2} - \mu\right) \leq 0$ *and* $\lambda - \frac{1}{2} - \mu$ *is not an integer.*

1.5 q-Integral transforms

Definition 1.5.1. *q-Laplace transform.*
Hahn [23] defined the q-analogues of the well known Laplace transform:

$$F(S) \ = \ \phi(S) \ = \ \int_0^\infty e^{-St}\, f(t)\,\mathrm{d}t,$$

by means of the following two integral equations.

$$\mathcal{L}_q\{f(t)\} \ = \ \frac{1}{(1-q)} \int_0^{S^{-1}} E_q(qSt)\, f(t)\,\mathrm{d}_q t, \tag{1.5.1}$$

and

$$\mathcal{L}_q\{f(t)\} = \frac{1}{(1-q)}\int_0^{\infty} e_q(-St)\, f(t)\, \mathrm{d}_q t, \tag{1.5.2}$$

where $\Re(S) > 0$.
A q-Laplace transform of integration is given by [34]

$$\mathcal{L}_q\left[\int_0^x f(t)\, \mathrm{d_q t}\right] = \frac{1}{S}\, F_q(S), \tag{1.5.3}$$

whereas the formula for q-Laplace transform of differentiation is ([34])

$$\mathcal{L}_q\left[D_q\ f(t)\right] = S\, F_q(S) - f(0). \tag{1.5.4}$$

$$\mathcal{L}_q\left[x f(x)\right] = -\frac{1}{q}\ \Delta_q F_q(S), \tag{1.5.5}$$

in which $F_q(S) = \mathcal{L}_q(f(x))(S)$.

Definition 1.5.2. *q-Convolution formula [34].*

$$\mathcal{L}_q\left[\int_0^x f_1(t)\ f_2(x-tq)\, \mathrm{d}_q t\right] = F_{1_q}(S)\ F_{2_q}(S), \tag{1.5.6}$$

whenever $F_{1_q}(S), F_{2_q}(S)$ *exist.*
Here $F_{1_q}(S) = \mathcal{L}_q(f_1(x))(S)$ *and* $F_{2_q}(S) = \mathcal{L}_q(f_2(x))(S)$.

Definition 1.5.3. *The q-Euler (Beta) transform is [18]:*

$$\mathfrak{B}\{f(z) : a,\ b|q\} = \int_0^1 u^{\beta-1}\frac{(uq;q)_\infty}{(uq^\eta;q)_\infty} f(z)\, \mathrm{d}_q u. \tag{1.5.7}$$

1.6 Fractional integrals and derivatives

In a letter dated September 30^{th}, 1695 Guillaume De L'Hospital(1661-1704) wrote to Leibnitz asking him about a particular notation he had used in his publications for the nth-derivative $\frac{D^n}{Dx^n}x$ of the linear function $f(x) = x$. L'Hospital posed the question to Leibnitz, what would the result be if $n = \frac{1}{2}$? Leibnitz's responded by saying: "An apparent paradox, from which one day useful consequences will be drawn'. By these words, the fractional calculus was born.

Definition 1.6.1. *Riemann-Liouville fractional integral operators.*

For $f(x) \in$ L(a,b), $\mu \in \mathbb{C}$, *and* $\Re(\mu) > 0$, *the Riemann-Liouville (R-L) fractional integrals of order* μ *[67] are defined as follows.*

The left-sided R-L fractional integral operator of order μ *is defined as*

$$ {}_xI_a^{\mu} f(x) = I_{a+}^{\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_a^x \frac{f(t)}{(x-t)^{1-\mu}} \mathrm{d}t, \qquad x > a, \tag{1.6.1}$$

whereas the right-sided R-L fractional integral operator of order μ *is defined as*

$$ {}_xI_b^{\mu} f(x) := I_{b-}^{\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_x^b \frac{f(t)}{(x-t)^{1-\mu}} \mathrm{d}t, \qquad x < b, \tag{1.6.2}$$

Further, if $\mu,\ \beta \in \mathbb{C},\ \Re(\mu, \beta) > 0$, *then ([46],[67])*

$$I_{a+}^{\mu}[(t-a)^{\beta-1}](x) = \frac{\Gamma(\beta)}{\Gamma(\mu+\beta)}(x-a)^{\mu+\beta-1}. \tag{1.6.3}$$

Definition 1.6.2. *Riemann-Liouville fractional derivative.*

For $\mu \in \mathbb{C}, \Re(\mu) > 0;\ n = [\Re(\mu)] + 1$, *the R-L fractional derivative [67] is*

$$(D_{a+}^{\alpha}\ f)(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n (I_{a+}^{n-\alpha}\ f)(x). \tag{1.6.4}$$

Definition 1.6.3. *Generalized Riemann-Liouville fractional integral operator.*

The fractional integral operator investigated by Erdélyi and Kober is defined as

$$I_{0+}^{\eta;\nu}\{f(x)\} = \frac{x^{-\eta-\nu+1}}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)\,\mathrm{d}t,\ \ \Re(\nu) > 0,\ \eta > 0. \tag{1.6.5}$$

This is a generalization of the Riemann-Liouville fractional integral operator (1.6.2).

Definition 1.6.4. *Generalized Riemann-Liouville fractional derivative operator.*

Hilfer ([15], [28]) generalized the Riemann-Liouville fractional derivative operator D^{μ}_{a+} *in (1.6.4) by introducing a right-sided fractional derivative operator* $D^{\mu,\nu}_{a+}$ *of order* $0 < \mu < 1$ *and type* $0 \leq \nu \leq 1$ *with respect to* x *as follows.*

$$(D^{\mu,\nu}_{a+} f)(x) = \left(I^{\nu(1-\mu)}_{a+} \frac{\mathrm{d}}{\mathrm{d}x} (I^{(1-\nu)(1-\mu)}_{a+} f) \right)(x). \tag{1.6.6}$$

Remark 1.6.1. *The Laplace transforms of generalized the Riemann-Liouville fractional derivative operator is given by ([15], [28]):*

$$\mathcal{L}[D^{\mu,\nu}_{0+} f(x)](S) = S^{\mu}\, \mathcal{L}[f(x)](S) - S^{\nu(1-\mu)}(I^{(1-\nu)(1-\mu)}_{0+} f)(0+), \tag{1.6.7}$$

where $0 < \mu < 1$, *and the initial-value term:* $(I^{(1-\nu)(1-\mu)}_{0} f)(0+)$ *involves the Riemann-Liouville fractional integral operator of order* $(1-\nu)(1-\mu)$ *evaluated by taking limit* $t \to 0+$. *Here, as usual*

$$\mathcal{L}[f(x)](S) = \int_{0}^{\infty} e^{-Sx}\, f(x)\, \mathrm{d}x, \tag{1.6.8}$$

provided that the integral exists.

1.7 Fractional q-integrals and q-derivatives

Definition 1.7.1. *Riemann-Liouville fractional q-integral operator.*

A q-analogue of Riemann-Liouville fractional integral operator [1] is given by

$${}_{q}I^{\mu}_{a+} f(x) \;=\; \frac{1}{\Gamma_q(\mu)} \int_{a}^{x} (x - |yq)_{\mu-1}\, f(y)\, \mathrm{d}_q y, \tag{1.7.1}$$

where μ *is an arbitrary order of integration with* $Re(\mu) > 0$.
In particular, for $f(x) = x^{\nu-1}$, *the equation (1.7.1) reduces to*

$${}_{q}I^{\mu}_{0+} f(x)[x^{\nu-1}] \;=\; \frac{\Gamma_q(\nu)}{\Gamma_q(\nu+\mu)} x^{\nu+\mu-1}. \tag{1.7.2}$$

Definition 1.7.2. *Generalized Riemann-Liouville fractional q-integral operator.*

A basic analogue of the Kober fractional integral operator is given by [1],

$$ {}_qI_{0+}^{\eta;\mu} f(t) \;=\; \frac{t^{-\eta-\mu}}{\Gamma_q(\mu)} \int_0^t (t-|xq)_{\mu-1}\, x^{\eta}\, f(x)\, \mathrm{d}_q x, \tag{1.7.3}$$

where μ is an arbitrary order of integration with $\Re(\mu) > 0$ and type η is in general, complex.

Definition 1.7.3. *Riemann-Liouville fractional q-derivative.*

The fractional q-differential operator of arbitrary order α, is defined as [3] :

$$\left({}_qD_{0+}^{\alpha} f\right)(x) \;=\; \frac{1}{\Gamma_q(-\alpha)} \int_0^x (x-|yq)_{-\alpha-1}\, f(y)\, \mathrm{d}_q y, \tag{1.7.4}$$

in which $\Re(\alpha) < 0,\ 0 < |q| < 1$.

It is to be noted that $\left({}_qD_{0+}^{\alpha} f\right)(x) = D_{x,q}^{\alpha} f(x)$. In this context,

$$\left({}_qD_{a+}^{\alpha} f\right)(x) = \left(\frac{\mathrm{d}_q}{\mathrm{d}_q x}\right)^{n} \left({}_qI_{a+}^{n-\alpha} f\right)(x). \tag{1.7.5}$$

If $f(x) = x^{\mu-1}$, then (1.7.4) reduces to

$${}_qD_{0+}^{\alpha}[x^{\mu-1}] \;=\; \frac{\Gamma_q(\mu)}{\Gamma_q(\mu-\alpha)} x^{\mu-\alpha-1}. \tag{1.7.6}$$

Note 1.7.1. *Hilfer ([15], [28]) generalized the Riemann-Liouville fractional derivative operator D_{a+}^{μ} in (1.6.4) by introducing a right-sided fractional derivative operator $D_{a+}^{\mu,\nu}$ of order $0 < \mu < 1$ and type $0 \le \nu \le 1$ with respect to x given by (1.6.6). Its q-analogue is defined here in the form:*

$$({}_qD_{a+}^{\mu,\nu} f)(x) = \left({}_qI_{a+}^{\nu(1-\mu)} \frac{\mathrm{d}}{\mathrm{d}x} \left({}_qI_{a+}^{(1-\nu)(1-\mu)} f\right)\right)(x), \tag{1.7.7}$$

in which ${}_qI_{a+}^{(1-\nu)(1-\mu)}$ denotes a q-analogue of the Kober fractional integral operator (1.6.5).

A q-analogue of the formula (1.6.7) is given by

$$\mathcal{L}_q[\,{}_qD_{0+}^{\mu,\nu} f(x)](S) = S^{\mu}\, \mathcal{L}_q[f(x)](S) - S^{\nu(1-\mu)}(\,{}_qI_{0+}^{(1-\nu)(1-\mu)} f)(0+), \tag{1.7.8}$$

where $0 < \mu < 1$, *and the initial-value term:* $\left({}_qI_{0+}^{(1-\nu)(1-\mu)}f\right)(0+)$ *involves the Riemann-Liouville fractional q-integral operator of order* $(1-\nu)(1-\mu)$ *evaluated when the limit* $t \to 0+$.

1.8 Inverse series relation

A series is said to be the inverse series of a given series if one of the series when substituted into the other, simplifies to the expression involving the Kronecker delta :

$$\delta_{nk} = \begin{cases} 0, & if \quad k \neq n \\ 1, & if \quad k = n \end{cases} .$$

To illustrate this, consider the inverse pair

$$a_n = \sum_{k=0}^{n} \binom{n}{k} b_k, \quad b_n = \sum_{k=0}^{n} (-1)^{k+n} \binom{n}{k} a_k.$$

Here, if second series is substituted into the first series then the inner sum simplifies to the form

$$\sum_{k=0}^{n} (-1)^{k+j} \binom{n}{j} \binom{j}{k} = \delta_{nk},$$

thus proving one side of inverse relation. The poof of the converse part is similar. Now in order to illustrate a q-analogue of inverse pair, consider the series

$$a_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} b_k.$$

Then its inverse series is given by

$$b_n = \sum_{k=0}^{n} (-1)^{n+k} q^{k(k-2n+1)/2} \begin{bmatrix} n \\ k \end{bmatrix} a_k$$

and vise versa.