Chapter 2

Generalized Mittag-Leffler function

2.1 Introduction

The well known Mittag-Leffler function [47]

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \tag{2.1.1}$$

where z is a complex variable and $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ was generalized by Wiman [94] in 1905 in the form:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \Re(\alpha, \beta) > 0, \tag{2.1.2}$$

which is known as Wiman's function or generalized Mittag-Leffler function.

Note 2.1.1. $E_{\alpha,1}(z) = E_\alpha(z)$.

In 1971, Prabhakar [56] introduced its extension:

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \tag{2.1.3}$$

wherein $\Re(\alpha, \beta, \gamma) > 0$.

Note 2.1.2. $E^{1}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z)$.

In 2007, Shukla and Prajapati [73] introduced the function:

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}, \tag{2.1.4}$$

in which $\alpha, \beta, \gamma \in \mathbb{C}$; $\Re(\alpha,\beta,\gamma) > 0$ and $q \in (0,1) \cup \mathbb{N}$.

Note 2.1.3. $E^{\gamma,1}_{\alpha,\beta}(z) = E^{\gamma}_{\alpha,\beta}(z)$.

In the present work, the following function is defined and studied [59]:

Definition 2.1.1. *For* $\alpha,\beta,\gamma,\lambda \in \mathbb{C}$, $\Re(\alpha,\beta,\gamma,\lambda) > 0$, $\delta,\mu > 0, r \in \{-1,0\} \cup \mathbb{N}$, $s \in \mathbb{N} \cup \{0\}$

$$E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r) = \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s}{\Gamma(\alpha n+\beta)\ [(\lambda)_{\mu n}]^r\ n!} z^n. \tag{2.1.5}$$

Note 2.1.4. $E^{\gamma,q}_{\alpha,\beta,\lambda,\mu}(z;1,0) = E^{\gamma,q}_{\alpha,\beta}(z)$.

The objective of constructing this function is to
(i) include certain existing generalizations of Mittag-Leffler function,
(ii) also include the functions such as Bessel Maitland function, Dotsenko function, Bessel function, generalized Bessel Maitland function, Lommel function etc. especially by means of parameters r, γ, λ (Table-1 below)
(iii) obtain inverse inequality relations and some other inequalities by means of the integer $'s'$.

Definition 2.1.2. *Bessel-Maitland function [24, Eq.(1.7.8), p.19] :*

$$J^{\mu}_{\nu}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\nu+n\mu+1)} \frac{z^n}{n!},$$

Definition 2.1.3. *Dotsenko function [24, Eq.(1.8.9), p.24] :*

$${}_2R_1(a,b;c,\omega;\nu;z) = \frac{\Gamma(c)}{\Gamma(a)\ \Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\ \Gamma(b+n\frac{\omega}{\nu})}{\Gamma(c+n\frac{\omega}{\nu})} \frac{z^n}{n!},$$

A particular form $(m = 2)$ of extension of Mittag-Leffler function due to Saxena and Nishimoto [69] given by

Definition 2.1.4.

$$E_{\gamma,K}[(\alpha_j,\beta_j)_{1,2};z] = \sum_{n=0}^{\infty} \frac{(\gamma)_{Kn}}{\Gamma(\alpha_1 n+\beta_1)\ \Gamma(\alpha_2 n+\beta_2)} \frac{z^n}{n!},$$

where $z, \gamma, \alpha_j, \beta_j \in \mathbb{C}, \Re(\alpha_1 + \alpha_2) > \Re(K) - 1, \Re(K) > 0.$

Definition 2.1.5. *The Elliptic function [41, Eq.(1), p.211] :*

$$K(k) = \frac{\pi}{2}\ {}_2F_1\left(\begin{array}{ll} \frac{1}{2}, & \frac{1}{2}; \quad k^2 \\ 1; & \end{array}\right).$$

All these functions are tabulated below as particular cases of (2.1.5).

Table-1

Function	**r**	**s**	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\gamma}$	$\boldsymbol{\delta}$	$\boldsymbol{\lambda}$	$\boldsymbol{\mu}$
Mittag-Leffler	0	1	α	1	1	1	-	-
Wiman	0	1	α	β	1	1	-	-
Prabhakar	0	1	α	β	γ	1	-	-
Shukla and Prajapati	0	1	α	β	γ	q	-	-
Bessel-Maitland	0	0	μ	$\nu+1$	-	-	-	-
Dotsenko	-1	1	ω/ν	c	a	1	b	ω/ν
Saxena-Nishimoto	1	1	α_1	β_1	γ	K	β_2	α_2
Elliptic	-1	1	1	1	$\frac{1}{2}$	1	$\frac{1}{2}$	1

The purpose of consideration of the parameters $'r'$ and $'s'$ is now clear from the above table (Table-1). Also, the parameter $'s'$ plays a vital roll in deriving certain inequalities (Chapter-6 and Chapter-7).

2.2 Main results

For the function (2.1.5), the absolute convergence test is first taken up. The subsequent properties include order of the function (2.1.5) and its type, asymptotic estimate, differential equation, Eigen function property and Mellin-Barnes contour integral representation. Certain mixed recurrence type relations are also derived; and the results involving integral transforms namely, Euler-Beta transform, Mellin-Barnes transform, Laplace transform and Whittaker transform are recorded. Alternative representation of (2.1.5) as generalized hypergeometric function, Fox H-function, Wright function are also illustrated.

2.2.1 Convergence

Theorem 2.2.1. *The series represented by the function* $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r)$ *is convergent for* $\Re(\alpha)+r\mu-s\delta+1>0$.

Proof. Take

$$u_n = \frac{[(\gamma)_{\delta n}]^s}{\Gamma(\alpha n+\beta)\ [(\lambda)_{\mu n}]^r\ n!} = \frac{[\Gamma(\gamma+\delta n)]^s\ [\Gamma(\lambda)]^r}{[\Gamma(\gamma)]^s\ \Gamma(\alpha n+\beta)\ [\Gamma(\lambda+\mu n)]^r\ \Gamma(n+1)}. \tag{2.2.1}$$

Now by using the Stirling's asymptotic formula (1.2.7) of the Gamma-function for large $|z|$, one gets

$$\begin{aligned} u_n &\sim \frac{\left(\sqrt{2\pi}\ e^{-(\gamma+\delta n)}(\gamma+\delta n)^{\gamma+\delta n-1/2}\right)^s}{\left(\sqrt{2\pi}e^{-\gamma}\gamma^{\gamma-1/2}\right)^s\left(\sqrt{2\pi}e^{-(\alpha n+\beta)}(\alpha n+\beta)^{\alpha n+\beta-1/2}\right)} \\ &\quad\times\frac{\left(\sqrt{2\pi}\ e^{-\lambda}\lambda^{\lambda-1/2}\right)^r}{\left(\sqrt{2\pi}\ e^{-(\lambda+\mu n)}(\lambda+\mu n)^{\lambda+\mu n-1/2}\right)^r\ \left(\sqrt{2\pi}e^{-(n+1)}(n+1)^{n+1/2}\right)} \\ &= \frac{e^{-\delta ns}\ (\delta n)^{s(\gamma+\delta n-1/2)}\left(1+\frac{\gamma}{\delta n}\right)^{s(\gamma+\delta n-1/2)}}{2\pi\ \gamma^{s(\gamma-1/2)}\ e^{-(\alpha n+\beta)}\ (\alpha n)^{\alpha n+\beta-1/2}\left(1+\frac{\beta}{\alpha n}\right)^{\alpha n+\beta-1/2}} \\ &\quad\times\frac{\lambda^{r(\lambda-1/2)}\ \left(1+\frac{1}{n}\right)^{-(n+1/2)}}{e^{-\mu nr}\ (\mu n)^{r(\lambda+\mu n-1/2)}\ \left(1+\frac{\lambda}{\mu n}\right)^{r(\lambda+\mu n-1/2)}\ e^{-n}\ n^{n+1/2}}. \end{aligned}$$

Hence

$$\begin{aligned} |u_n|^{\frac{1}{n}} &= \left|\frac{e^{-\delta ns}\ (\delta n)^{s(\gamma+\delta n-1/2)}\left(1+\frac{\gamma}{\delta n}\right)^{s(\gamma+\delta n-1/2)}}{2\pi\ \gamma^{s(\gamma-1/2)}\ e^{-(\alpha n+\beta)}(\alpha n)^{\alpha n+\beta-1/2}\left(1+\frac{\beta}{\alpha n}\right)^{\alpha n+\beta-1/2}}\right. \\ &\quad\left.\times\frac{\lambda^{r(\lambda-1/2)}\left(1+\frac{1}{n}\right)^{-(n+1/2)}}{e^{-\mu nr}\ (\mu n)^{r(\lambda+\mu n-1/2)}\left(1+\frac{\lambda}{\mu n}\right)^{r(\lambda+\mu n-1/2)}\ e^{-n}\ n^{n+1/2}}\right|^{\frac{1}{n}}. \\ &= \left|\frac{e^{-s\delta}\ (\delta n)^{\frac{s}{n}(\gamma+\delta n-1/2)}\left(1+\frac{\gamma}{\delta n}\right)^{\frac{s}{n}(\gamma+\delta n-1/2)}}{(2\pi)^{\frac{1}{n}}\ \gamma^{\frac{s}{n}(\gamma-1/2)}\ e^{-\frac{1}{n}(\alpha n+\beta)}\ (\alpha n)^{\frac{1}{n}(\alpha n+\beta-1/2)}\left(1+\frac{\beta}{\alpha n}\right)^{\frac{1}{n}(\alpha n+\beta-1/2)}}\right. \\ &\quad\left.\times\frac{\lambda^{\frac{r}{n}(\lambda-1/2)}\left(1+\frac{1}{n}\right)^{-\frac{1}{n}(n+1/2)}}{e^{-r\mu}\ (\mu n)^{\frac{r}{n}(\lambda+\mu n-1/2)}\left(1+\frac{\lambda}{\mu n}\right)^{\frac{r}{n}(\lambda+\mu n-1/2)}\ e^{-1}\ n^{\frac{1}{n}(n+1/2)}}\right|. \end{aligned}$$

Now making limit as $n \to \infty$, this gives

$$\begin{aligned}\lim_{n\to\infty} |u_n|^{\frac{1}{n}} &= \left|\frac{e^{\alpha+r\mu-s\delta+1}\delta^{s\delta}}{\alpha^{\alpha}\mu^{r\mu}}\right| \lim_{n\to\infty} \left|n^{s\delta-\alpha-r\mu-1}\right| \\ &= \frac{e^{\Re(\alpha)+r\mu-s\delta+1}\delta^{s\delta}}{\{\Re(\alpha)\}^{\Re(\alpha)}\mu^{r\mu}} \lim_{n\to\infty} n^{s\delta-\Re(\alpha)-r\mu-1}.\end{aligned}$$

Hence the series of (2.1.5) converges absolutely when $\Re(\alpha)+r\mu+1 > s\delta$.

□

Theorem 2.2.2. *Let* $\Re(\alpha,\beta,\gamma,\lambda) > 0, \Re(\alpha)+r\mu-s\delta+1 > 0,\ \delta,\mu > 0,\ r \in \{-1,0\} \cup \mathbb{N},\ s \in \mathbb{N}\cup\{0\}$*. Then* $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r)$ *is an entire function of order* $\varrho = \frac{1}{\Re(\alpha)+r\mu-s\delta+1}$ *and type* $\sigma = \frac{1}{\varrho}\left(\frac{\delta^{s\delta}}{\{\Re(\alpha)\}^{\Re(\alpha)}\mu^{r\mu}}\right)^{\varrho}$*.*

Proof. The Stirling's asymptotic formula (1.2.7) of the Gamma-function for large $|z|$ when applied to the general term

$$\begin{aligned}u_n &= \frac{[(\gamma)_{\delta n}]^s}{\Gamma(\alpha n+\beta)\ [(\lambda)_{\mu n}]^r\ n!}. \\ &= \frac{[\Gamma(\gamma+\delta n)]^s\ [\Gamma(\lambda)]^r}{[\Gamma(\gamma)]^s\ \Gamma(\alpha n+\beta)\ [\Gamma(\lambda+\mu n)]^r\ \Gamma(n+1)},\end{aligned}$$

gives

$$\begin{aligned}u_n &\sim \frac{\left(\sqrt{2\pi}\ e^{-(\gamma+\delta n)}\ (\gamma+\delta n)^{\gamma+\delta n-1/2}\right)^s}{\left(\sqrt{2\pi}\ e^{-\gamma}\ \gamma^{\gamma-1/2}\right)^s\ \left(\sqrt{2\pi}\ e^{-(\alpha n+\beta)}\ (\alpha n+\beta)^{\alpha n+\beta-1/2}\right)} \\ &\quad\times\frac{\left(\sqrt{2\pi}\ e^{-\lambda}\ \lambda^{\lambda-1/2}\right)^r}{\left(\sqrt{2\pi}\ e^{-(\lambda+\mu n)}\ (\lambda+\mu n)^{\lambda+\mu n-1/2}\right)^r\ \left(\sqrt{2\pi}\ e^{-(n+1)}\ (n+1)^{n+1/2}\right)} \\ &= \frac{e^{-\delta n s}\ (\delta n)^{s(\gamma+\delta n-1/2)}\ \left(1+\frac{\gamma}{\delta n}\right)^{s(\gamma+\delta n-1/2)}}{\gamma^{s(\gamma-1/2)}\sqrt{2\pi}\ e^{-(\alpha n+\beta)}\ (\alpha n)^{\alpha n+\beta-1/2}\ \left(1+\frac{\beta}{\alpha n}\right)^{\alpha n+\beta-1/2}} \\ &\quad\times\frac{\lambda^{r(\lambda-1/2)}\ \left(1+\frac{\lambda}{\mu n}\right)^{-r(\lambda+\mu n-1/2)}}{e^{-\mu n r}\ (\mu n)^{r(\lambda+\mu n-1/2)}\ \sqrt{2\pi}e^{-n}\ n^{n+1/2}\ \left(1+\frac{1}{n}\right)^{n+1/2}}.\end{aligned}$$

If R is the radius of convergence of the series of $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r)$, then with the use of Cauchy-Hadamard formula:

$$\frac{1}{R} = \lim_{n\to\infty}\sup \sqrt[n]{|u_n|}.$$

one obtains

$$\begin{aligned}\frac{1}{R} &= \lim_{n\to\infty}\sup\left|\frac{e^{\alpha+r\mu-s\delta+1}\delta^{s\delta}}{\alpha^{\alpha}\mu^{r\mu}}\right|\left|n^{s\delta-\alpha-r\mu-1}\right|\\ &= \frac{e^{\Re(\alpha)+r\mu-s\delta+1}\,\delta^{s\delta}}{\{\Re(\alpha)\}^{\Re(\alpha)}\,\mu^{r\mu}}\lim_{n\to\infty} n^{s\delta-\Re(\alpha)-r\mu-1}\\ &= 0\end{aligned}$$

when $\Re(\alpha)+r\mu-s\delta+1>0$. Therefore, the above function (2.1.5) turns out to be an *entire* function.

In order to determine its order, one may use the result [7, Eq.(1.1)] which states that if $f(z)=\sum_{n=0}^{\infty}u_n z^n$ is an entire function then the order $\varrho(f)$ of f is given by [7, Eq.(1.2)]

$$\varrho(f)=\lim_{n\to\infty}\sup\frac{n\,\log n}{\log(1/|u_n|)}.$$

By choosing $f(z)=E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r)$, this particularizes to

$$\varrho(E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r))=\lim_{n\to\infty}\sup\frac{n\,\log n}{\log(1/|u_n|)},$$

Here,

$$\begin{aligned}\log\left(\frac{1}{|u_n|}\right) &= \log\left|\frac{[\Gamma(\gamma)]^{s}\,\Gamma(\alpha n+\beta)\,[\Gamma(\lambda+\mu n)]^{r}\,\Gamma(n+1)}{[\Gamma(\gamma+\delta n)]^{s}\,[\Gamma(\lambda)]^{r}}\right|\\ &\sim \log\left|\frac{\left(\sqrt{2\pi}\,e^{-(\alpha n+\beta)}\,(\alpha n+\beta)^{\alpha n+\beta-1/2}\right)\,\left(\sqrt{2\pi}\,e^{-\gamma}\gamma^{\gamma-1/2}\right)^{s}}{\left(\sqrt{2\pi}\,e^{-(\gamma+\delta n)}(\gamma+\delta n)^{\gamma+\delta n-1/2}\right)^{s}}\right.\\ &\qquad\left.\times\frac{\left(\sqrt{2\pi}\,e^{-(\lambda+\mu n)}\,(\lambda+\mu n)^{\lambda+\mu n-1/2}\right)^{r}\,\left(\sqrt{2\pi}\,e^{-(n+1)}\,(n+1)^{n+1/2}\right)}{\left(\sqrt{2\pi}\,e^{-\lambda}\,\lambda^{\lambda-1/2}\right)^{r}}\right|\\ &= \log\left|2\pi\,\gamma^{s(\gamma-1/2)}\,e^{s\delta n-\alpha n-\beta-r\mu n-n}\,(\alpha n+\beta)^{(\alpha n+\beta-1/2)}\right.\\ &\qquad\left.\times(\lambda+\mu n)^{r(\lambda+\mu n-1/2)}\,(n+1)^{n+1/2}\right|\\ &\qquad-\log\left|(\gamma+\delta n)^{s(\gamma+\delta n-1/2)}\lambda^{r(\lambda-1/2)}\right|\\ &= \log\left(2\pi\,|\gamma|^{s\Re(\gamma-1/2)}\,e^{\Re(s\delta n-\alpha n-\beta-r\mu n-n)}\,|\alpha n+\beta|^{\Re(\alpha n+\beta-1/2)}\right.\\ &\qquad\left.\times|\lambda+\mu n|^{r\Re(\lambda+\mu n-1/2)}\,(n+1)^{n+1/2}\right)\\ &\qquad-\log\left(|\gamma+\delta n|^{s\Re(\gamma+\delta n-1/2)}|\lambda|^{r\Re(\lambda-1/2)}\right)\\ &= \log(2\pi)+s\Re(\gamma-1/2)\log|\gamma|+\Re(s\delta n-\alpha n-\beta-r\mu n-n)\end{aligned}$$

$$+\Re(\alpha n+\beta-1/2)\log|\alpha n+\beta|+r\Re(\lambda+\mu n-1/2)\log|\lambda+\mu n|$$
$$+(n+1/2)\log(n+1)-s\Re(\gamma+\delta n-1/2)\log|\gamma+\delta n|$$
$$-r\Re(\lambda-1/2)\log|\lambda|. \tag{2.2.2}$$

Hence,

$$\frac{1}{\varrho}=\limsup_{n\to\infty}\left(\frac{\log(1/|u_n|)}{n\log n}\right)=\Re(\alpha)+r\mu-s\delta+1.$$

Thus the order of the function (2.1.5) is

$$\varrho=\frac{1}{\Re(\alpha)+r\mu-s\delta+1}. \tag{2.2.3}$$

The type σ of the function (2.1.5) is given by

$$\sigma(E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r))=\frac{1}{e\,\varrho}\limsup_{n\to\infty}\left(n\,|u_n|^{\varrho/n}\right). \tag{2.2.4}$$

The limit on the right hand side may be computed as follows.
Consider

$$\begin{aligned}|u_n| &= \left|\frac{[\Gamma(\gamma+\delta n)]^s\,[\Gamma(\lambda)]^r}{[\Gamma(\gamma)]^s\,\Gamma(\alpha n+\beta)\,[\Gamma(\lambda+\mu n)]^r\,\Gamma(n+1)}\right|\\ &\sim \left|\frac{\left(\sqrt{2\pi}e^{-(\gamma+\delta n)}(\gamma+\delta n)^{\gamma+\delta n-1/2}\right)^s\,\left(\sqrt{2\pi}e^{-\gamma}\gamma^{\gamma-1/2}\right)^{-s}}{\left(\sqrt{2\pi}e^{-(\alpha n+\beta)}(\alpha n+\beta)^{\alpha n+\beta-1/2}\right)}\right.\\ &\quad\left.\times\frac{\left(\sqrt{2\pi}e^{-\lambda}\lambda^{\lambda-1/2}\right)^r}{\left(\sqrt{2\pi}e^{-(\lambda+\mu n)}(\lambda+\mu n)^{\lambda+\mu n-1/2}\right)^r\,\left(\sqrt{2\pi}e^{-(n+1)}(n+1)^{n+1-1/2}\right)}\right|\\ &= \left|\frac{1}{2\pi}\,\frac{e^{\alpha n+\beta+r\mu n-s\delta n+n-1}\,(\delta n)^{s(\gamma+\delta n-1/2)}}{(\alpha n)^{\alpha n+\beta-1/2}\left(1+\frac{\beta}{\alpha n}\right)^{\alpha n+\beta-1/2}\gamma^{s(\gamma-1/2)}}\right.\\ &\quad\left.\times\frac{\lambda^{r(\lambda-1/2)}\,\left(1+\frac{\gamma}{\delta n}\right)^{s(\gamma+\delta n-1/2)}}{(\mu n)^{r(\lambda+\mu n-1/2)}\left(1+\frac{\lambda}{\mu n}\right)^{r(\lambda+\mu n-1/2)}(n+1)^{n+1/2}}\right|.\end{aligned}$$

On substituting this on the right hand side of (2.2.4) and using (2.2.3), one gets

$$\begin{aligned}\limsup_{n\to\infty}\left(n\;|u_n|^{\varrho/n}\right) &= \left(\frac{\delta^{s\delta}}{\{\Re(\alpha)\}^{\Re(\alpha)}\;\mu^{r\mu}}\right)^{\varrho}e^{\varrho(\Re(\alpha)+r\mu-s\delta+1)}\\ &\quad\times\lim_{n\to\infty}n^{\varrho(s\delta-\Re(\alpha)-r\mu-1)+1}.\end{aligned}$$

This gives

$$\sigma(E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r)) = \frac{1}{\varrho}\left(\frac{\delta^{s\delta}}{\{\Re(\alpha)\}^{\Re(\alpha)}\ \mu^{r\mu}}\right)^{\varrho}. \tag{2.2.5}$$

For every positive ϵ, from the equation (1.2.15) the asymptotic estimate is given by

$$\left|E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r)\right| < \exp\left((\sigma+\epsilon)\ |z|^{\varrho}\right),\ \ |z| \geq r_0 > 0 \tag{2.2.6}$$

with ϱ, σ as in (2.2.3), (2.2.5) and for $|z| \geq r_0(\epsilon)$, where $r_0(\epsilon)$ is sufficiently large.

□

2.2.2 Differential equation

By assuming the following notations for the indicated operator expressions:

$$\theta = zD,\quad D = \frac{d}{dz},\quad \frac{\delta^{s\delta}}{\alpha^{\alpha}\ \mu^{r\mu}} = P,\quad \Delta_j^{(a,b;m)} = \prod_{j=0}^{a-1}\left[\left(\theta + \frac{b+j}{a}\right)\right]^m,$$

$$\Upsilon_j^{(a,b;m)} = \prod_{j=0}^{a-1}\left[\left(\theta + \frac{b+j}{a} - 1\right)\right]^m, \tag{2.2.7}$$

the differential equation satisfied by the function (2.1.5) is derived below.

Theorem 2.2.3. *Let $\alpha, \mu, \delta \in \mathbb{N}$ then $y = E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r)$ satisfies the equation*

$$\left[\Upsilon_k^{(\mu,\lambda;r)}\ \Upsilon_j^{(\alpha,\beta;1)}\ \theta - z\frac{\delta^{s\delta}}{\alpha^{\alpha}\ \mu^{r\mu}}\ \Delta_m^{(\delta,\gamma;s)}\right] y = 0. \tag{2.2.8}$$

Proof. The function

$$\begin{aligned} y &= \sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s}{\Gamma(\alpha n+\beta)\ [(\lambda)_{\mu n}]^r}\frac{z^n}{n!} \\ &= \frac{1}{\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s}{(\beta)_{\alpha n}\ [(\lambda)_{\mu n}]^r}\frac{z^n}{n!} \\ &= \frac{1}{\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{\delta^{s\delta n}[(\frac{\gamma}{\delta})_n]^s\ [(\frac{\gamma+1}{\delta})_n]^s \ldots [(\frac{\gamma+\delta-1}{\delta})_n]^s}{\alpha^{\alpha n}(\frac{\beta}{\alpha})_n\ (\frac{\beta+1}{\alpha})_n \ldots (\frac{\beta+\alpha-1}{\alpha})_n} \end{aligned}$$

$$\times \frac{1}{\mu^{r\mu n}[(\frac{\lambda}{\mu})_n]^r\ [(\frac{\lambda+1}{\mu})_n]^r...[(\frac{\lambda+\mu-1}{\mu})_n]^r}\frac{z^n}{n!}$$

$$= \frac{1}{\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{\delta^{s\delta n}}{\alpha^{\alpha n}\ \mu^{r\mu n}}\frac{\left\{\prod_{m=0}^{\delta-1}[(\frac{\gamma+m}{\delta})_n]^s\right\}}{\left\{\prod_{j=0}^{\alpha-1}(\frac{\beta+j}{\alpha})_n\right\}\ \left\{\prod_{k=0}^{\mu-1}[(\frac{\lambda+k}{\mu})_n]^r\right\}}\frac{z^n}{n!}. \qquad (2.2.9)$$

is put in the form

$$y = \sum_{n=0}^{\infty}\frac{A_n\ P^n}{B_n\ C_n}\frac{z^n}{n!}$$

by taking

$$\frac{1}{\Gamma(\beta)}\prod_{m=0}^{\delta-1}\left[\left(\frac{\gamma+m}{\delta}\right)_n\right]^s = A_n,\ \prod_{j=0}^{\alpha-1}\left(\frac{\beta+j}{\alpha}\right)_n = B_n,\ \prod_{k=0}^{\mu-1}\left[\left(\frac{\lambda+k}{\mu}\right)_n\right]^r = C_n. \qquad (2.2.10)$$

Now,

$$\theta y = \sum_{n=0}^{\infty}\frac{A_n\ P^n}{B_n\ C_n\ n!}\ \theta\ z^n = \sum_{n=1}^{\infty}\frac{A_n\ P^n}{B_n\ C_n}\frac{z^n}{(n-1)!}.$$

Further,

$$\Upsilon_j^{(\alpha,\beta;1)}\ \theta y = \sum_{n=1}^{\infty}\frac{A_n\ P^n}{B_n\ C_n\ (n-1)!}\prod_{j=0}^{\alpha-1}\left(\theta+\frac{\beta+j}{\alpha}-1\right)\ z^n$$

$$= \sum_{n=1}^{\infty}\frac{A_n\ P^n}{B_n\ C_n\ (n-1)!}\prod_{j=0}^{\alpha-1}\left(n+\frac{\beta+j}{\alpha}-1\right)\ z^n$$

$$= \sum_{n=1}^{\infty}\frac{A_n\ P^n}{B_{n-1}\ C_n}\frac{z^n}{(n-1)!}.$$

Finally,

$$\Upsilon_k^{(\mu,\lambda;r)}\ \Upsilon_j^{(\alpha,\beta;1)}\ \theta y$$

$$= \sum_{n=1}^{\infty}\frac{A_n\ P^n}{B_{n-1}\ C_n\ (n-1)!}\ \prod_{k=0}^{\mu-1}\left[\left(\theta+\frac{\lambda+k}{\mu}-1\right)\right]^r\ z^n$$

$$= \sum_{n=1}^{\infty} \frac{A_n\ P^n}{B_{n-1}\ C_n\ (n-1)!} \prod_{k=0}^{\mu-1} \left[\left(n + \frac{\lambda+k}{\mu} - 1 \right) \right]^r\ z^n$$

$$= \sum_{n=1}^{\infty} \frac{A_n\ P^n}{B_{n-1}\ C_{n-1}} \frac{z^n}{(n-1)!}. \tag{2.2.11}$$

On the other hand,

$$\begin{aligned} \Delta_m^{(\delta,\gamma;s)}\ y &= \sum_{n=0}^{\infty} \frac{A_n\ P^n}{B_n\ C_n\ n!} \prod_{m=0}^{\delta-1} \left[\left(\theta + \frac{\gamma+m}{\delta} \right) \right]^s\ z^n \\ &= \sum_{n=0}^{\infty} \frac{A_n\ P^n}{B_n\ C_n\ n!} \prod_{m=0}^{\delta-1} \left[\left(n + \frac{\gamma+m}{\delta} \right) \right]^s \Bigg]\ z^n \\ &= \sum_{n=0}^{\infty} \frac{A_{n+1}\ P^n}{B_n\ C_n} \frac{z^n}{n!}, \end{aligned}$$

whence

$$P\ z\ \Delta_m^{(\delta,\gamma;s)}\ y = \sum_{n=0}^{\infty} \frac{A_{n+1}\ P^{n+1}}{B_n\ C_n} \frac{z^{n+1}}{n!}. \tag{2.2.12}$$

Now (3.2.17) follows on comparing (2.2.11) and (2.2.12). □

2.2.3 Eigen function property

In deriving the eigen function property for the function (2.1.5), the following operators are required.

$$\Theta_j^{(a,b;m)} = \prod_{j=0}^{a-1} \left[\left(-\theta + \frac{b+j}{a} - 1 \right) \right]^m, \tag{2.2.13}$$

and

$$\Omega_{\Theta;\Upsilon} = P^{-1}\ D\ \Theta_m^{(\delta,\gamma;-s)} \Upsilon_k^{(\mu,\lambda;r)} \Upsilon_j^{(\alpha,\beta;1)}. \tag{2.2.14}$$

Here the operators $\Theta_m^{(\delta,\gamma;-s)}$, $\Upsilon_k^{(\mu,\lambda;r)}$, $\Upsilon_j^{(\alpha,\beta;1)}$ in (2.2.14) are not commutative with the operator D.

Theorem 2.2.4. *Let $\alpha, \mu, \delta \in \mathbb{N}$ then $E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(z;s,r)$ is an eigen function with respect to the operator $\Omega_{\Theta;\Upsilon}$ as defined by (2.2.14).*

That is,

$$\Omega_{\Theta;\Upsilon}\left(E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\zeta z;s,r)\right)=\zeta\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\zeta z;s,r). \tag{2.2.15}$$

Proof. In view of (2.2.7), (2.2.13) and (2.2.14),

$$\begin{aligned} w &= E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\zeta z;s,r)\\ &= \sum_{n=0}^{\infty}\frac{A_n\ (\zeta P)^n}{B_n\ C_n}\frac{z^n}{n!}. \end{aligned}$$

Now

$$\begin{aligned} \Upsilon^{(\alpha,\beta;1)}_j\ w &= \sum_{n=0}^{\infty}\frac{A_n\ (\zeta P)^n}{B_n\ C_n\ n!}\prod_{j=0}^{\alpha-1}\left(\theta+\frac{\beta+j}{\alpha}-1\right)\ z^n\\ &= \sum_{n=0}^{\infty}\frac{A_n\ (\zeta P)^n}{B_n\ C_n\ n!}\prod_{j=0}^{\alpha-1}\left(n+\frac{\beta+j}{\alpha}-1\right)\ z^n\\ &= \sum_{n=0}^{\infty}\frac{A_n\ (\zeta P)^n}{B_{n-1}\ C_n}\frac{z^n}{n!}. \end{aligned}$$

Next

$$\begin{aligned} \Upsilon^{(\mu,\lambda;r)}_k\ \Upsilon^{(\alpha,\beta;1)}_j\ w &= \sum_{n=0}^{\infty}\frac{A_n\ (\zeta P)^n}{B_{n-1}\ C_n\ n!}\prod_{k=0}^{\mu-1}\left[\left(\theta+\frac{\lambda+k}{\mu}-1\right)\right]^r\ z^n\\ &= \sum_{n=0}^{\infty}\frac{A_n\ (\zeta P)^n}{B_{n-1}\ C_n\ n!}\prod_{k=0}^{\mu-1}\left[\left(n+\frac{\lambda+k}{\mu}-1\right)\right]^r\ z^n\\ &= \sum_{n=0}^{\infty}\frac{A_n\ (\zeta P)^n}{B_{n-1}\ C_{n-1}}\frac{z^n}{n!}. \end{aligned}$$

Further using (2.2.13),

$$\begin{aligned} \Theta^{(\delta,\gamma;-s)}_m\ \Upsilon^{(\mu,\lambda;r)}_k\ \Upsilon^{(\alpha,\beta;1)}_j\ w &= \sum_{n=0}^{\infty}\frac{A_n\ (\zeta P)^n}{B_{n-1}\ C_{n-1}\ n!}\ \Theta^{(\delta,\gamma;-s)}_m\ z^n\\ &= \sum_{n=0}^{\infty}\frac{A_n\ (\zeta P)^n}{B_{n-1}\ C_{n-1}\ n!}\prod_{j=0}^{\delta-1}\left[\left(-\theta+\frac{\gamma+j}{\delta}-1\right)\right]^{-s}\ z^n\\ &= \sum_{n=0}^{\infty}\frac{A_n\ (\zeta P)^n}{B_{n-1}\ C_{n-1}\ n!}\prod_{j=0}^{\delta-1}\left[\left(n+\frac{\gamma+j}{\delta}-1\right)\right]^{-s}\ z^n\\ &= \sum_{n=0}^{\infty}\frac{A_{n-1}\ (\zeta P)^n}{B_{n-1}\ C_{n-1}}\frac{z^n}{n!}. \end{aligned}$$

Finally,

$$\begin{aligned}
\Omega_{\Theta;\Upsilon}^{(\delta,\gamma,s;\alpha,\beta,\mu,\lambda,r)} w &= P^{-1}\ D\ \Theta_m^{(\delta,\gamma;-s)}\ \Upsilon_k^{(\mu,\lambda;r)}\ \Upsilon_j^{(\alpha,\beta;1)}\ w \\
&= \sum_{n=0}^{\infty} \frac{A_{n-1}\ \zeta^n\ P^{n-1}}{B_{n-1}\ C_{n-1}\ n!}\ D\ z^n \\
&= \sum_{n=1}^{\infty} \frac{A_{n-1}\ \zeta^n\ P^{n-1}}{B_{n-1}\ C_{n-1}} \frac{z^{n-1}}{(n-1)!} \\
&= \sum_{n=0}^{\infty} \frac{A_n\ \zeta^{n+1}\ P^n}{B_n\ C_n} \frac{z^n}{n!} \\
&= \zeta \sum_{n=0}^{\infty} \frac{A_n\ \zeta^n\ P^n}{B_n\ C_n} \frac{z^n}{n!} \\
&= \zeta\ E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(\zeta z; s, r).
\end{aligned}$$

□

2.2.4 Mellin-Barnes integral representation

It is known that ([63, p.94], [78, p.109]) the integration of a group of Gamma functions of the form:

$$\frac{\Gamma(a+s)\ \Gamma(-s)}{\Gamma(b+s)} = g(s), \quad \text{say},$$

can be evaluated along Barnes path in s-plane which comes from $-i\infty$ and proceeds towards $+i\infty$, keeping the poles of $\Gamma(a+s)$ to its left side and the poles of $\Gamma(-s)$ to its right side. In the following theorem, the function (2.1.5) is expressed as integral over such path.

Theorem 2.2.5. *Let $\alpha, \delta, \mu > 0$; $\beta, \gamma, \lambda \in \mathbb{C}$, with $\Re(\beta, \gamma, \lambda) > 0$. Then the function $E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(z; s, r)$ is expressible as the Mellin - Barnes integral given by*

$$\begin{aligned}
E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(z; s, r) &= \frac{[\Gamma(\lambda)]^r}{2\pi i\ [\Gamma(\gamma)]^s} \int_L \frac{\Gamma(S)\ \Gamma(1-S)}{\Gamma(\beta-\alpha S)\ [\Gamma(\lambda-\mu S)]^r} \\
&\quad \times \frac{[\Gamma(\gamma-\delta S)]^s\ (-z)^{-S}}{\Gamma(1-S)} \mathrm{d}S, \qquad (2.2.16)
\end{aligned}$$

where $|argz| < \pi$; the contour L of integration begins from $-i\infty$ and proceeds towards $+i\infty$, and is indented to keep the poles at $S = -n$ of integrand to the left; and the poles at $S = (\gamma + n)/\delta$ to the right of the path, for all $n \in \mathbb{N}_0$.

Proof. The integral on the right hand side of (2.2.16) may be evaluated as the sum of the residues at the poles $S = 0, -1, -2, \ldots$.
In fact, in view of the definition of residue,

$$\begin{aligned}
I &= \frac{1}{2\pi i}\int_L \frac{\Gamma(S)\,\Gamma(1-S)\,[\Gamma(\gamma-\delta S)]^s\,(-z)^{-S}}{\Gamma(\beta-\alpha S)\,[\Gamma(\lambda-\mu S)]^r\;\Gamma(1-S)}\mathrm{d}S \\
&= \sum_{n=0}^{\infty}\underset{S=-n}{Res}\left[\frac{\Gamma(S)\,\Gamma(1-S)\,(-z)^{-S}}{\Gamma(\beta-\alpha S)\,[\Gamma(\lambda-\mu S)]^r\,[\Gamma(\gamma-\delta S)]^{-s}\,\Gamma(1-S)}\right] \\
&= \sum_{n=0}^{\infty}\lim_{S\to -n}\frac{\pi(S+n)}{sin\pi S}\frac{[\Gamma(\gamma-\delta S)]^s\,(-z)^{-S}}{\Gamma(\beta-\alpha S)\,[\Gamma(\lambda-\mu S)]^r\,\Gamma(1-S)} \\
&= \sum_{n=0}^{\infty}\frac{[\Gamma(\gamma+\delta n)]^s}{\Gamma(\beta+\alpha n)\,[\Gamma(\lambda+\mu n)]^r\,\Gamma(n+1)}\,z^n \\
&= \frac{[\Gamma(\gamma)]^s}{[\Gamma(\lambda)]^r}\,E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r).
\end{aligned}$$

□

2.2.5 Mixed relations

In the notations

$$\begin{aligned}
\dot{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r) &= \frac{d}{dz}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r), \\
\ddot{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r) &= \frac{d^2}{dz^2}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r),
\end{aligned}$$

there holds the following mixed relation.

Theorem 2.2.6. *For* $\alpha,\beta,\gamma,\lambda \in \mathbb{C},\;\; k \in \mathbb{N},\;\; \Re(\alpha,\beta,\gamma,\lambda) > 0, \delta,\mu > 0,$

$$\begin{aligned}
& E^{\gamma,\delta}_{\alpha,\beta+k,\lambda,\mu}(z;s,r) - E^{\gamma,\delta}_{\alpha,\beta+k+1,\lambda,\mu}(z;s,r) \\
=\;& \alpha^2 z^2\,\ddot{E}^{\gamma,\delta}_{\alpha,\beta+k+2,\lambda,\mu}(z;s,r) + \alpha z\,[\alpha+2(\beta+k)]\,\dot{E}^{\gamma,\delta}_{\alpha,\beta+k+2,\lambda,\mu}(z;s,r) \\
& +(\beta^2+2\beta k+k^2-1)E^{\gamma,\delta}_{\alpha,\beta+k+2,\lambda,\mu}(z;s,r). \qquad (2.2.17)
\end{aligned}$$

Proof. Here the change is taking place only in the parameter β, hence the relation will be obtained in the abbreviated notation:

$$E^{\gamma,\delta}_{\alpha,\,\beta+k,\,\lambda,\,\mu}(z;s,r) \equiv E_{\beta+k}.$$

Now putting

$$\frac{[(\gamma)_{\delta n}]^s\ z^n}{[(\lambda)_{\mu n}]^r\ n!} = d_n, \quad \alpha n + \beta = \sigma(n)$$

and applying the functional equation: $\Gamma(z+1) = z\,\Gamma(z)$, the function $E_{\beta+k}$ is expressible as

$$\begin{aligned} E_{\beta+k} &= \sum_{n=0}^{\infty} \frac{1}{\Gamma(\sigma(n)+k)}\ d_n \\ &= \sum_{n=0}^{\infty} \frac{d_n}{\Gamma(\sigma(n))}\ \left\{\prod_{i=0}^{k-1}(\sigma(n)+i)\right\}^{-1}. \end{aligned}$$

With $\sigma(n) + i = b_i$, this takes the form

$$E_{\beta+k} = \sum_{n=0}^{\infty} \frac{d_n}{\Gamma(\sigma(n))}\ \left\{\prod_{i=0}^{k-1} b_i\right\}^{-1}. \tag{2.2.18}$$

Similarly,

$$\begin{aligned} E_{\beta+k+1} &= \sum_{n=0}^{\infty} \frac{1}{\Gamma(\sigma(n)+k+1)}\ d_n \\ &= \sum_{n=0}^{\infty} \frac{d_n}{\Gamma(\sigma(n))}\ \left\{\prod_{i=0}^{k} b_i\right\}^{-1} \\ &= \sum_{n=0}^{\infty} \frac{d_n}{\Gamma(\sigma(n))}\left[\frac{1}{b_{k-1}} - \frac{1}{b_k}\right]\ \left\{\prod_{i=0}^{k-2} b_i\right\}^{-1} \\ &= \sum_{n=0}^{\infty} \frac{d_n}{\Gamma(\sigma(n))}\frac{1}{b_{k-1}}\ \left\{\prod_{i=0}^{k-2} b_i\right\}^{-1} - \sum_{n=0}^{\infty} \frac{d_n}{\Gamma(\sigma(n))}\frac{1}{b_k}\ \left\{\prod_{i=0}^{k-2} b_i\right\}^{-1} \\ &= \sum_{n=0}^{\infty} \frac{d_n}{\Gamma(\sigma(n))}\ \left\{\prod_{i=0}^{k-1} b_i\right\}^{-1} - \sum_{n=0}^{\infty} \frac{d_n}{b_k\ \Gamma(\sigma(n))}\ \left\{\prod_{i=0}^{k-2} b_i\right\}^{-1}. \end{aligned}$$

This in view of (2.2.18), gives

$$E_{\beta+k+1} = E_{\beta+k} - \sum_{n=0}^{\infty} \frac{d_n}{b_k\ \Gamma(\sigma(n))}\ \left\{\prod_{i=0}^{k-2} b_i\right\}^{-1}. \tag{2.2.19}$$

This may be considered as

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{d_n}{b_k\ \Gamma(\sigma(n))} \left\{\prod_{i=0}^{k-2} b_i\right\}^{-1} \\
=\ & E_{\beta+k} - E_{\beta+k+1} \qquad (2.2.20)\\
=\ & \sum_{n=0}^{\infty} \frac{d_n}{\Gamma(\sigma(n))} \left\{\prod_{i=0}^{k-1} b_i\right\}^{-1} - \sum_{n=0}^{\infty} \frac{d_n}{\Gamma(\sigma(n))} \left\{\prod_{i=0}^{k} b_i\right\}^{-1} \\
=\ & \sum_{n=0}^{\infty} \frac{d_n}{\Gamma(\sigma(n))} \left(1 - \frac{1}{b_k}\right) \left\{\prod_{i=0}^{k-1} b_i\right\}^{-1} \\
=\ & \sum_{n=0}^{\infty} \frac{d_n}{\Gamma(\sigma(n))} \frac{b_{k-1}}{b_k} \left\{\prod_{i=0}^{k-1} b_i\right\}^{-1} \\
=\ & \sum_{n=0}^{\infty} \frac{d_n}{b_k\ \Gamma(\sigma(n))} \left\{\prod_{i=0}^{k-2} b_i\right\}^{-1}. \qquad (2.2.21)
\end{aligned}$$

But since

$$\frac{1}{b_k\ \Gamma(\sigma(n))} = \frac{1}{b_{k+1}\ b_k\ \Gamma(\sigma(n))} + \frac{1}{b_{k+1}\ \Gamma(\sigma(n))},$$

hence

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{d_n}{b_k\ \Gamma(\sigma(n))} \left\{\prod_{i=0}^{k-2} b_i\right\}^{-1} \\
=\ & \sum_{n=0}^{\infty} \left[\frac{1}{b_{k+1}\ b_k\ \Gamma(\sigma(n))} + \frac{1}{b_{k+1}\ \Gamma(\sigma(n))}\right] d_n \left\{\prod_{i=0}^{k-2} b_i\right\}^{-1} \\
=\ & \sum_{n=0}^{\infty} \frac{d_n}{b_{k+1}\ b_k\ \Gamma(\sigma(n))} \left\{\prod_{i=0}^{k-2} b_i\right\}^{-1} + \sum_{n=0}^{\infty} \frac{d_n}{b_{k+1}\ \Gamma(\sigma(n))} \left\{\prod_{i=0}^{k-2} b_i\right\}^{-1} \\
=\ & \sum_{n=0}^{\infty} \frac{b_{k-1}}{\left\{\prod_{i=0}^{k+1} b_i\right\}\ \Gamma(\sigma(n))}\ d_n + \sum_{n=0}^{\infty} \frac{b_{k-1}\ b_k}{\left\{\prod_{i=0}^{k+1} b_i\right\}\ \Gamma(\sigma(n))}\ d_n \\
=\ & \sum_{n=0}^{\infty} \frac{b_{k-1}}{\Gamma(b_{k+2})}\ d_n + \sum_{n=0}^{\infty} \frac{b_{k-1}\ b_k}{\Gamma(b_{k+2})}\ d_n \\
=\ & \sum_{n=0}^{\infty} \frac{(\alpha n+\beta+k-1)}{\Gamma(\alpha n+\beta+k+2)} \frac{[(\gamma)_{\delta n}]^s\ z^n}{[(\lambda)_{\mu n}]^r\ n!} \\
& + \sum_{n=0}^{\infty} \frac{(\alpha n+\beta+k-1)\ (\alpha n+\beta+k)}{\Gamma(\alpha n+\beta+k+2)} \frac{[(\gamma)_{\delta n}]^s\ z^n}{[(\lambda)_{\mu n}]^r\ n!}
\end{aligned}$$

$$
\begin{aligned}
&= \alpha^2 \sum_{n=1}^{\infty} \frac{n\ [(\gamma)_{\delta n}]^s}{\Gamma(\alpha n+\beta+k+2)} \ \frac{z^n}{[(\lambda)_{\mu n}]^r\ (n-1)!} \\
&+2\alpha(\beta+k) \sum_{n=1}^{\infty} \frac{[(\gamma)_{\delta n}]^s}{\Gamma(\alpha n+\beta+k+2)} \frac{z^n}{[(\lambda)_{\mu n}]^r\ (n-1)!} \\
&+(\beta^2+2\beta k+k^2-1) \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s}{\Gamma(\alpha n+\beta+k+2)} \ \frac{z^n}{[(\lambda)_{\mu n}]^r\ n!}. \qquad (2.2.22)
\end{aligned}
$$

The right hand side of (2.2.22) will now be expressed as the derivatives of $E_{\beta+k+2}$. In fact,

$$
\begin{aligned}
& z^2\ \ddot{E}_{\beta+k+2} + 4z\ \dot{E}_{\beta+k+2} + 2E_{\beta+k+2} \\
&= z^2 \frac{d^2}{dz^2} E_{\beta+k+2} + 4z \frac{d}{dz} E_{\beta+k+2} + 2\ E_{\beta+k+2} \\
&= z^2 \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s}{\Gamma(\alpha n+\beta+k+2)\ [(\lambda)_{\mu n}]^r\ n!} \frac{d^2}{dz^2} z^n \\
&\quad +4z \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s}{\Gamma(\alpha n+\beta+k+2)\ [(\lambda)_{\mu n}]^r\ n!} \frac{d}{dz} z^n + 2\ E_{\beta+k+2} \\
&= z^2 \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\ n\ (n-1)\ z^{n-2}}{\Gamma(\alpha n+\beta+k+2)\ [(\lambda)_{\mu n}]^r\ n!} \\
&\quad +4z \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\ n\ z^{n-1}}{\Gamma(\alpha n+\beta+k+2)\ [(\lambda)_{\mu n}]^r\ n!} + 2\ E_{\beta+k+2} \\
&= \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\ n\ (n-1)\ z^{n}}{\Gamma(\alpha n+\beta+k+2)\ [(\lambda)_{\mu n}]^r\ n!} \\
&\quad +4 \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\ n\ z^{n}}{\Gamma(\alpha n+\beta+k+2)\ [(\lambda)_{\mu n}]^r\ n!} + 2\ E_{\beta+k+2} \\
&= \sum_{n=1}^{\infty} \frac{[(\gamma)_{\delta n}]^s\ n\ z^{n}}{\Gamma(\alpha n+\beta+k+2)\ [(\lambda)_{\mu n}]^r\ (n-1)!} \\
&\quad +3 \sum_{n=1}^{\infty} \frac{[(\gamma)_{\delta n}]^s\ z^{n}}{\Gamma(\alpha n+\beta+k+2)\ [(\lambda)_{\mu n}]^r\ (n-1)!} + 2\ E_{\beta+k+2}. \qquad (2.2.23)
\end{aligned}
$$

This may be written as

$$
\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n\ [(\gamma)_{\delta n}]^s}{\Gamma(\alpha n+\beta+k+2)\ [(\lambda)_{\mu n}]^r} \frac{z^n}{(n-1)!} \\
&= z^2\ \ddot{E}_{\beta+k+2} + 4z\ \dot{E}_{\beta+k+2} - 3 \sum_{n=1}^{\infty} \frac{[(\gamma)_{\delta n}]^s\ z^n}{\Gamma(\alpha n+\beta+k+2)\ [(\lambda)_{\mu n}]^r\ (n-1)!}. \qquad (2.2.24)
\end{aligned}
$$

Also,

$$\begin{aligned}
& z\ \dot{E}_{\beta+k+2} + E_{\beta+k+2} \\
= & \ z\ \frac{d}{dz} E_{\beta+k+2} + E_{\beta+k+2} \\
= & \ z \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s}{\Gamma(\alpha n+\beta+k+2)\ [(\lambda)_{\mu n}]^r\ n!} \frac{d}{dz} z^n + E_{\beta+k+2} \\
= & \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\ n\ z^n}{\Gamma(\alpha n+\beta+k+2)\ [(\lambda)_{\mu n}]^r\ n!} + E_{\beta+k+2} \\
= & \sum_{n=1}^{\infty} \frac{[(\gamma)_{\delta n}]^s\ z^n}{\Gamma(\alpha n+\beta+k+2)\ [(\lambda)_{\mu n}]^r\ (n-1)!} + E_{\beta+k+2}.
\end{aligned} \tag{2.2.25}$$

That is,

$$\sum_{n=1}^{\infty} \frac{[(\gamma)_{\delta n}]^s\ z^n}{\Gamma(\alpha n+\beta+k+2)\ [(\lambda)_{\mu n}]^r\ (n-1)!} = z\ \dot{E}_{\beta+k+2}. \tag{2.2.26}$$

Here using this in (2.2.24), one gets

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{[(\gamma)_{\delta n}]^s\ n\ z^n}{\Gamma(\alpha n+\beta+k+2)\ [(\lambda)_{\mu n}]^r\ (n-1)!} \\
= & \ z^2\ \ddot{E}_{\beta+k+2}\ + 4z\ \dot{E}_{\beta+k+2} - 3z\dot{E}_{\beta+k+2} \\
= & \ z^2\ \ddot{E}_{\beta+k+2}\ + z\ \dot{E}_{\beta+k+2}.
\end{aligned} \tag{2.2.27}$$

Finally from (2.2.22), (2.2.26) and (2.2.27), it follows that

$$\begin{aligned}
& E_{\beta+k} - E_{\beta+k+1} \\
= & \ \alpha^2 z^2\ \ddot{E}_{\beta+k+2}\ + \alpha^2 z\ \dot{E}_{\beta+k+2} + 2\alpha(\beta+k)z\ \dot{E}_{\beta+k+2} \\
& +(\beta^2+2\beta k+k^2-1)E_{\beta+k+2} \\
= & \ \alpha^2 z^2\ \ddot{E}_{\beta+k+2}\ + \alpha z\left[\alpha+2(\beta+k)\right]\dot{E}_{\beta+k+2} \\
& +(\beta^2+2\beta k+k^2-1)E_{\beta+k+2}.
\end{aligned} \tag{2.2.28}$$

□

Theorem 2.2.7. *For $\alpha, \beta, \gamma, \lambda \in \mathbb{C}; \Re(\alpha, \beta, \gamma, \lambda) > 0$ and $\delta, \mu > 0$, the differential recurrence relation*

$$\beta\ E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu}(z;s,r) + \alpha z\ \frac{d}{dz}\ E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu}(z;s,r) = E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r)$$

holds.

Proof. Here

$$\begin{aligned} l.h.s. &= \beta\ E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu}(z;s,r)\ +\ \alpha z\ \frac{d}{dz}\ E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu}(z;s,r) \\ &= \beta\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s\ z^n}{\Gamma(\alpha n+\beta+1)\ [(\lambda)_{\mu n}]^r\ n!}+\alpha z\frac{d}{dz}\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s(z)^n}{\Gamma(\alpha n+\beta+1)\ [(\lambda)_{\mu n}]^r\ n!} \\ &= \beta\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s\ z^n}{\Gamma(\alpha n+\beta+1)\ [(\lambda)_{\mu n}]^r\ n!}+\alpha z\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s\ n\ z^{n-1}}{\Gamma(\alpha n+\beta+1)\ [(\lambda)_{\mu n}]^r\ n!} \\ &= \sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s(\alpha n+\beta)z^n}{\Gamma(\alpha n+\beta+1)\ [(\lambda)_{\mu n}]^r\ n!} \\ &= \sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s z^n}{\Gamma(\alpha n+\beta)\ [(\lambda)_{\mu n}]^r\ n!} \\ &= E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r) \\ &= r.h.s. \end{aligned}$$

□

2.2.5.1 Particular cases

1. Setting $r=0, s=1, \delta=q, k=1$, (2.2.17), one gets mixed relation for $E^{\gamma,q}_{\alpha,\beta}(z)$ due to Shukla and Prajapati [76] in the form:

$$\begin{aligned} E^{\gamma,q}_{\alpha,\beta+1}(z)-E^{\gamma,q}_{\alpha,\beta+2}(z) &= \alpha^2\ z^2\ddot{E}^{\gamma,q}_{\alpha,\beta+3}(z)+\alpha z\big[\alpha+2(\beta+1)\big]\dot{E}^{\gamma,q}_{\alpha,\beta+3}(z) \\ &\quad +\beta\ (\beta+2)\ E^{\gamma,q}_{\alpha,\beta+3}(z). \end{aligned} \tag{2.2.29}$$

2. With the substitutions $r=0, \gamma=\delta=s=k=1, \beta=m\in\mathbb{N}$, (2.2.17) reduces to a known recurrence relation of $E_{\alpha,\beta}(z)$ due to Gupta and Debnath[20]:

$$\begin{aligned} E_{\alpha,m+1}(z) &= \alpha^2 z^2\ddot{E}_{\alpha,m+3}(z)\ +\ \alpha(\alpha+2m+2)\ z\ \dot{E}_{\alpha,m+3}(z) \\ &\quad +m(m+2)\ E_{\alpha,m+3}(z)\ +\ E_{\alpha,m+2}(z). \end{aligned} \tag{2.2.30}$$

2.2.6 Integral formula

Theorem 2.2.8. *For* $\alpha,\beta,\gamma,\lambda\in\mathbb{C}, \Re(\alpha,\beta,\gamma,\lambda)>0, \delta,\mu>0, r\in\mathbb{N}\cup\{-1,0\}, s\in\mathbb{N}\cup\{0\}$,

$$\int_0^1 u^{\beta+k-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(u^\alpha;s,r)\,du = E^{\gamma,\delta}_{\alpha,\beta+k,\lambda,\mu}(1;s,r)\ -\ E^{\gamma,\delta}_{\alpha,\beta+k+1,\lambda,\mu}(1;s,r). \tag{2.2.31}$$

Proof. Putting $z = 1$ in (2.2.20), one gets

$$\sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s}{(\alpha n+\beta+k)\ \left\{\prod\limits_{i=0}^{k-2}(\alpha n+\beta+i)\right\}\ \Gamma(\alpha n+\beta)\ [(\lambda)_{\mu n}]^r\ n!}$$
$$= E^{\gamma,\delta}_{\alpha,\beta+k,\lambda,\mu}(1;s,r) - E^{\gamma,\delta}_{\alpha,\beta+k+1,\lambda,\mu}(1;s,r). \tag{2.2.32}$$

Now

$$\int_0^z u^{\beta+k-1}\ E^{\gamma,\delta}_{\alpha,\beta+k-1,\lambda,\mu}(u^\alpha;s,r)\,\mathrm{d}u$$
$$= \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s}{\Gamma(\alpha n+\beta+k-1)\ [(\lambda)_{\mu n}]^r\ n!} \int_0^z u^{\alpha n+\beta+k-1}\,\mathrm{d}u$$
$$= \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s}{\left\{\prod\limits_{i=0}^{k-2}(\alpha n+\beta+i)\right\}\ \Gamma(\alpha n+\beta)\ [(\lambda)_{\mu n}]^r\ n!} \int_0^z u^{\alpha n+\beta+k-1}\,\mathrm{d}u$$
$$= \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\ z^{\alpha n+\beta+1}}{(\alpha n+\beta+k)\ \left\{\prod\limits_{i=0}^{k-2}(\alpha n+\beta+i)\right\}\ \Gamma(\alpha n+\beta)\ [(\lambda)_{\mu n}]^r\ n!}. \tag{2.2.33}$$

Taking $z = 1$ in (2.2.33), one gets

$$\int_0^1 u^{\beta+k-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(u^\alpha;s,r)\,\mathrm{d}u$$
$$= \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s}{(\alpha n+\beta+k)\ \left\{\prod\limits_{i=0}^{k-2}(\alpha n+\beta+i)\right\}\ \Gamma(\alpha n+\beta)\ [(\lambda)_{\mu n}]^r\ n!}. \tag{2.2.34}$$

On comparing (2.2.32) and (2.2.34), the result follows. □

2.2.6.1 Particular cases

1. Taking $r = 0, s = 1, \delta = q$ in (2.2.31), one gets (Shukla and Prajapati [76]):

$$\int_0^1 u^\beta\ E^{\gamma,q}_{\alpha,\beta}(u^\alpha)\,\mathrm{d}u \ = \ E^{\gamma,q}_{\alpha,\beta+1}(1) \ - \ E^{\gamma,\ q}_{\alpha,\beta+2}(1). \tag{2.2.35}$$

2. The substitutions $r = 0$ and $\alpha = \beta = \delta = s = 1$ in (2.2.35), yields the formula

$$\int_0^1 u\, e^u \, \mathrm{d}u \;=\; E_{1,2}(1) \;-\; E_{1,3}(1)$$

for $\gamma = 1$, whereas for $\gamma = 2$ it furnishes

$$\int_0^1 u\, E_{1,1}^{2,1}(1)\, \mathrm{d}u \;=\; E_{1,2}^{2,1}(1) \;-\; E_{1,3}^{2,1}(1).$$

2.2.7 Double series representation

As in Theorem 2.2.2, take

$$u_n \;=\; \frac{[(\gamma)_{\delta n}]^s}{\Gamma(\alpha n + \beta)\; [(\lambda)_{\mu n}]^r \; n!}.$$

and put

$$^*E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta,\rho}(z; s, r) = \sum_{n=0}^{\infty} u_n \, (\rho)_n \, z^n$$

which is valid under the condition $\Re(\alpha) + r\mu - s\delta > 0$.

Theorem 2.2.9. *The function*

$$^*E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta,\rho}(z; s, r) \;=\; \sum_{i,j=0}^{\infty} \frac{1}{(\rho)_{i+j}} \frac{(-1)^i}{i!\; j!} \; {}^*E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta,\rho+i+j}(z; s, r). \qquad (2.2.36)$$

Proof. Here introducing the function (2.1.5) in the integrand of the integral (1.2.6) and applying this integral to the identity $e^{-y}e^{y} = 1$, one gets the integral

$$\int_0^\infty e^{-t}\, t^{\rho-1} e^{-xt}\; e^{xt} E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(zt; s, r)\, \mathrm{d}t \;=\; \int_0^\infty e^{-t}\, t^{\rho-1} E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(zt; s, r)\, \mathrm{d}t. \qquad (2.2.37)$$

Here substituting the corresponding series representations for e^{-t} and e^{t} on the left hand side, one gets

$$l.h.s. \;=\; \int_0^\infty e^{-t}\; t^{\rho-1}\; e^{-xt}\; e^{xt}\; E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(zt; s, r)\, \mathrm{d}t$$

$$= \int_0^\infty e^{-t}\ t^{\rho-1} \left\{ \sum_{i=0}^\infty \frac{(-1)^i x^i t^i}{i!} \right\} \left\{ \sum_{j=0}^\infty \frac{x^j t^j}{j!} \right\}\ E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(zt; s, r)\,\mathrm{d}t$$

$$= \sum_{i=0}^\infty \sum_{j=0}^\infty \frac{(-1)^i}{i!\ j!} x^{i+j} \int_0^\infty e^{-t}\ t^{\rho+i+j-1} E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(zt; s, r)\,\mathrm{d}t$$

$$= \sum_{i=0}^\infty \sum_{j=0}^\infty \frac{(-1)^i}{i!\ j!} x^{i+j} \int_0^\infty e^{-t}\ t^{\rho+i+j-1} \left\{ \sum_{n=0}^\infty \frac{[(\gamma)_{\delta n}]^s}{\Gamma(\alpha n+\beta)\ [(\lambda)_{\mu n}]^r} \frac{(zt)^n}{n!} \right\} \mathrm{d}t$$

$$= \sum_{i=0}^\infty \sum_{j=0}^\infty \frac{(-1)^i}{i!\ j!} x^{i+j} \sum_{n=0}^\infty u_n\ z^n \int_0^\infty e^{-t}\ t^{\rho+i+j+n-1}\,\mathrm{d}t$$

$$= \sum_{i=0}^\infty \sum_{j=0}^\infty \frac{(-1)^i}{i!\ j!} x^{i+j} \sum_{n=0}^\infty u_n\ z^n\ \Gamma(\rho+i+j+n)$$

$$= \sum_{i=0}^\infty \sum_{j=0}^\infty \frac{(-1)^i}{i!\ j!} x^{i+j} \sum_{n=0}^\infty u_n\ z^n \frac{(\rho+i+j)_n}{\Gamma(\rho+i+j)}$$

$$= \sum_{i=0}^\infty \sum_{j=0}^\infty \frac{1}{\Gamma(\rho+i+j)} \frac{(-1)^i}{i!\ j!} x^{i+j}\ {}^*E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta,\rho+i+j}(z; s, r). \tag{2.2.38}$$

Likewise,

$$r.h.s. = \int_0^\infty e^{-t}\ t^{\rho-1} \left\{ E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(zt; s, r) \right\} \mathrm{d}t$$

$$= \sum_{n=0}^\infty u_n\ z^n \int_0^\infty e^{-t}\ t^{\rho+n-1}\,\mathrm{d}t$$

$$= \sum_{n=0}^\infty u_n\ \Gamma(\rho+n)\ z^n$$

$$= \sum_{n=0}^\infty u_n \frac{(\rho)_n}{\Gamma(\rho)}\ z^n$$

$$= \frac{1}{\Gamma(\rho)}\ {}^*E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta,\rho}(z; s, r). \tag{2.2.39}$$

Consequently, from (2.2.38) and (2.2.39), (2.2.37) yields

$${}^*E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta,\rho}(z; s, r) = \sum_{i,j=0}^\infty \frac{\Gamma(\rho)}{\Gamma(\rho+i+j)} \frac{(-1)^i}{i!\ j!}\ {}^*E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta,\rho+i+j}(z; s, r)$$

$$= \sum_{i,j=0}^\infty \frac{1}{(\rho)_{i+j}} \frac{(-1)^i}{i!\ j!}\ {}^*E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta,\rho+i+j}(z; s, r). \tag{2.2.40}$$

□

2.3 Other results

2.3.1 Differentiation

As the series given in (2.1.5) converges absolutely in any compact subset of $\mathbb{C}$, the term by term differentiation is permitted and it leads to the following theorem.

Theorem 2.3.1. *If* $m \in \mathbb{N}, \alpha,\ \beta,\ \gamma,\ \lambda,\ \in \mathbb{C},\ \Re(\alpha,\beta,\gamma,\lambda) > 0$ *and* $\delta,\ \mu > 0$ *then*

$$\left(\frac{d}{dz}\right)^m E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\big(z;s,r\big) = \frac{[(\gamma)_{\delta m}]^s}{[(\lambda)_{\mu m}]^r} E^{\gamma+\delta m,\delta}_{\alpha,\beta+\alpha m,\lambda+\mu m,\mu}\big(z;s,r\big), \tag{2.3.1}$$

$$\left(\frac{d}{dz}\right)^m \left[z^{\beta-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\big(\omega z^{\alpha};s,r\big)\right] = z^{\beta-m-1}\ E^{\gamma,\delta}_{\alpha,\beta-m,\lambda,\mu}\big(\omega\ z^{\alpha};s,r\big), \tag{2.3.2}$$

for $\Re(\beta - m) > 0$.

Proof. Consider the left hand side of (2.3.1), that is,

$$\begin{aligned}
&\left(\frac{d}{dz}\right)^m E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r)\\
=&\left(\frac{d}{dz}\right)^m \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s z^n}{[\Gamma(\alpha n+\beta)][(\lambda)_{\mu n}]^r\ n!}\\
=&\sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s}{[\Gamma(\alpha n+\beta)][(\lambda)_{\mu n}]^r n!}\left(\frac{d}{dz}\right)^m (z^n)\\
=&\sum_{n=m}^{\infty} \frac{[(\gamma)_{\delta n}]^s\ \ z^{n-m}}{[\Gamma(\alpha\ n+\beta)][(\lambda)_{\mu n}]^r (n-m)!}\\
=&\sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta(n+m)}]^s\ z^n}{[\Gamma(\alpha(n+m)+\beta)][(\lambda)_{\mu(n+m)}]^r n!}\\
=&\sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta m}]^s\ [(\gamma+\delta m)_{\delta n}]^s z^n}{[\Gamma(\alpha\ n+\beta+\alpha\ m)]\ [(\lambda)_{\mu m}]^r\ [(\lambda+\mu m)_{\mu n}]^r n!}\\
=&\frac{[(\gamma)_{\delta m}]^s}{[(\lambda)_{\mu m}]^r}\sum_{n=0}^{\infty} \frac{[(\gamma+\delta m)_{\delta n}]^s\ z^n}{[(\lambda+\mu m)_{\mu n}]^r\ [\Gamma(\alpha n+\beta+\alpha m)]\ n!}\\
=&\frac{[(\gamma)_{\delta m}]^s}{[(\lambda)_{\mu m}]^r}\ E^{\gamma+\delta m,\delta}_{\alpha,\beta+\alpha m,\lambda+\mu m,\mu}\big(z;s,r\big).
\end{aligned}$$

Likewise

$$
\begin{aligned}
& \left(\frac{d}{dz}\right)^m \left[z^{\beta-1}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\big(\omega\, z^{\alpha}; s, r\big)\right] \\
= & \left(\frac{d}{dz}\right)^m \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\, z^{\alpha n}\, z^{\beta-1}\, \omega^n}{[\Gamma(\alpha\, n+\beta)]\, [(\lambda)_{\mu n}]^r\, n!} \\
= & \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\, \omega^n}{[\Gamma(\alpha\, n+\beta)]\, [(\lambda)_{\mu n}]^r\, n!} \left(\frac{d}{dz}\right)^m \left(z^{\alpha n+\beta-1}\right) \\
= & \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\, (\omega\, z^{\alpha})^n\, z^{\beta-m-1}}{[\Gamma(\alpha\, n+\beta-m)]\, [(\lambda)_{\mu n}]^r\, n!} \\
= & z^{\beta-m-1}\, E^{\gamma,\delta}_{\alpha,\beta-m,\lambda,\mu}\left(\omega z^{\alpha}; s, r\right).
\end{aligned}
$$

□

2.3.2 Integral representations

Taking $f(z) = E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\big(zu^{\alpha}; s, r\big)$, in the Euler (Beta) transform (1.3), one finds the following.

Theorem 2.3.2. *If* $\alpha, \beta, \gamma, \lambda, \sigma, \eta, \nu \in \mathbb{C},\ \Re(\alpha, \beta, \gamma, \lambda, \sigma, \eta, \nu) > 0$ *and* $\delta,\ \mu > 0$ *then*

$$\frac{1}{\Gamma(\eta)} \int_0^1 u^{\beta-1}\, (1-u)^{\eta-1}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\big(zu^{\alpha}; s, r\big)\, \mathrm{d}u = E^{\gamma,\delta}_{\alpha,\beta+\eta,\lambda,\mu}(z; s, r), \tag{2.3.3}$$

$$
\begin{aligned}
& \frac{1}{\Gamma(\eta)} \int_t^x (x-t)^{\eta-1}\, (s-t)^{\beta-1} E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\, \left(\nu(s-t)^{\alpha}; s, r\right)\, \mathrm{d}s \\
& \quad = (x-t)^{\eta+\beta-1}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\nu(x-t)^{\alpha}; s, r),
\end{aligned} \tag{2.3.4}
$$

$$\int_0^z t^{\beta-1}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\, \left(\omega t^{\alpha}; s, r\right)\, \mathrm{d}t = z^b\, E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu}\, \left(\omega t^{\alpha}; s, r\right), \tag{2.3.5}$$

$$\frac{1}{\Gamma(\sigma)} \int_0^1 z^{\sigma-1}(1-z)^{\beta-1} E^{\gamma,\delta}_{\alpha,\ \beta,\lambda,\mu}(x(1-z)^{\alpha}; s, r)\, \mathrm{d}z = E^{\gamma,\delta}_{\alpha,\beta+\sigma,\lambda,\mu}(x; s, r). \tag{2.3.6}$$

Proof. For proving (2.3.3), one may begin as follows.

$$\begin{aligned}
L.H.S. &= \frac{1}{\Gamma(\eta)}\int_0^1 u^{\beta-1}\ (1-u)^{\eta-1}\ E^{\gamma,\ \delta}_{\alpha,\ \beta,\ \lambda,\ \mu}\big(zu^{\alpha}\ s,\ r\big)\,\mathrm{d}u\\
&= \frac{1}{\Gamma(\eta)}\int_0^1 u^{\beta-1}\ (1-u)^{\eta-1}\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s\ u^{\alpha n}z^n}{\Gamma(\alpha n+\beta)\ [(\lambda)_{\mu n}]^r\ n!}\,\mathrm{d}u\\
&= \sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s\ z^n}{[\Gamma(\alpha n+\beta)]\ [(\lambda)_{\mu n}]^r\ n!\Gamma(\eta)}\int_0^1 u^{\alpha n+\beta-1}(1-u)^{\eta-1}\,\mathrm{d}u\\
&= \sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s\ z^n}{[\Gamma(\alpha n+\beta)]\ [(\lambda)_{\mu n}]^r\ n!\ \Gamma(\eta)}\ \frac{\Gamma(\eta)\ [\Gamma(\alpha n+\beta)]}{[\Gamma(\alpha n+\beta)+\eta]}\\
&= \sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s\ z^n}{\Gamma(\alpha n+\beta+\eta)\ [(\lambda)_{\mu n}]^r\ n!}\\
&= E^{\gamma,\delta}_{\alpha,\beta+\eta,\lambda,\mu}(z;s,r)\\
&= R.H.S.
\end{aligned}$$

Now, denoting the L. H. S. of (2.3.4) by I, one gets

$$\begin{aligned}
I &= \frac{1}{\Gamma(\eta)}\int_t^x (x-t)^{\eta-1}(s-t)^{\beta-1}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\left(\nu\ (s-t)^{\alpha};s,r\right)\mathrm{d}s\\
&= \frac{1}{\Gamma(\eta)}\int_t^x (x-t)^{\eta-1}(s-t)^{\beta-1}\left\{\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s(s-t)^{\alpha n}\nu^n}{\Gamma(\alpha n+\beta)[(\lambda)_{\mu n}]^r\ n!}\right\}\mathrm{d}s\\
&= \sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s\nu^n}{\Gamma(\alpha n+\beta)[(\lambda)_{\mu n}]^r n!\Gamma(\eta)}\int_t^x (x-t)^{\eta-1}(s-t)^{\alpha n+\beta-1}\mathrm{d}s.
\end{aligned}$$

In this, introducing u as a new variable of integration by means of the substitution

$$u=\frac{s-t}{x-t},$$

one gets,

$$\begin{aligned}
I &= \sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s\ \nu^n(x-t)^{\alpha n+\beta+\eta-1}}{\Gamma(\alpha n+\beta)[(\lambda)_{\mu n}]^r n!\Gamma(\eta)}\int_0^1 (1-u)^{\eta-1}\ u^{\alpha n+\beta-1}\mathrm{d}u\\
&= \sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s\nu^n(x-t)^{\alpha n+\beta+\eta-1}}{[\Gamma(\alpha n+\beta)][(\lambda)_{\mu n}]^r n!\Gamma(\eta)}\ \frac{\Gamma(\eta)\ [\Gamma(\alpha n+\beta)]}{[\Gamma(\alpha n+\beta+\eta)]}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^{s}\ \nu^{n}(x-t)^{\alpha n+\beta+\eta-1}}{[\Gamma(\alpha n+\beta+\eta)]\ [(\lambda)_{\mu n}]^{r} n!}$$
$$= (x-t)^{\eta+\beta-1}\ E^{\gamma,\delta}_{\alpha,\beta+\eta,\lambda,\mu}(\nu(x-t)^{\alpha}; s, r)$$

as desired.

In order to prove (2.3.5), beginning with the integral on the left hand side, one gets

$$\int_0^z t^{\beta-1} E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega t^{\alpha}; s, r)\mathrm{d}t$$
$$= \int_0^z t^{\beta-1} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^{s}\ t^{\alpha n}\omega^{n}}{[\Gamma(\alpha n+\beta)]\ [(\lambda)_{\mu n}]^{r} n!}\ \mathrm{d}t$$
$$= \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^{s}\ \omega^{n}}{[\Gamma(\alpha n+\beta)]\ [(\lambda)_{\mu n}]^{r}\ n!} \int_0^z t^{\alpha n+\beta-1}\ \mathrm{d}t$$
$$= \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^{s}\ z^{\alpha n+\beta}(\omega)^{n}}{[\Gamma(\alpha n+\beta)]\ [(\lambda)_{\mu n}]^{r}\ n!(\alpha n+\beta)}$$
$$= \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^{s}\ z^{\alpha n+\beta}\omega^{n}}{[\Gamma(\alpha n+\beta+1)]\ [(\lambda)_{\mu n}]^{r}\ n!}$$
$$= z^{\beta}\ E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu}(\omega z^{\alpha}; s, r).$$

Hence the result.

Next, consider

$$\frac{1}{\Gamma(\sigma)} \int_0^1 z^{\sigma-1}(1-z)^{\beta-1} E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(x\ (1-z)^{\alpha}; s, r)\ \mathrm{d}z$$
$$= \frac{1}{\Gamma(\sigma)} \int_0^1 z^{\sigma-1}\ (1-z)^{\beta-1} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^{s}(1-z)^{\alpha n}x^{n}}{[\Gamma(\alpha n+\beta)][(\lambda)_{\mu n}]^{r} n!}\ \mathrm{d}z$$
$$= \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^{s}x^{n}}{[\Gamma(\alpha n+\beta)][(\lambda)_{\mu n}]^{r} n!\Gamma(\sigma)} \int_0^1 (1-z)^{\alpha n+\beta-1} z^{\sigma-1}\ \mathrm{d}z$$
$$= \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^{s}x^{n}}{[\Gamma(\alpha n+\beta)][(\lambda)_{\mu n}]^{r} n!\Gamma(\sigma)} \frac{\Gamma(\sigma)[\Gamma(\alpha n+\beta)]}{[\Gamma(\alpha n+\beta)+\sigma]},$$

simplification of above series yields (2.3.6). □

2.4 Integral transforms

In this section, Euler transform, Laplace transform, Whittaker transform and Mellin transform are applied on the function (2.1.5).
For convenience, the notation $\frac{[\Gamma(\beta+\alpha n)]^r}{[\Gamma(\beta)]^r}$ will be used for $(\beta,\alpha)^r$.
The evaluation will be expressed as the Wright generalized hypergeometric function [44]:

$$ {}_p\psi_q\left[\begin{array}{ll}(a_1,A_1),\ldots,\ (a_p,A_p); & z\\ (b_1,B_1),\ldots,\ (b_q,B_q); & \end{array}\right]=\sum_{r=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+rA_j)}{\prod_{j=1}^{q}\Gamma(b_j+rB_j)}\frac{z^r}{r!}.$$

2.4.1 The Euler (Beta) transform

Theorem 2.4.1. *The Euler (Beta) transform of the function $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(xz^\sigma;s,r)$ is given by*

$$\int_0^1 z^{a-1}(1-z)^{b-1}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(xz^\sigma;s,r)\ \mathrm{d}z$$
$$=\frac{[\Gamma(\lambda)]^r\ \Gamma(b)}{[\Gamma(\gamma)]^s}\ {}_{s+1}\Psi_{r+2}\left[\begin{array}{llll}(\gamma,\delta)^s, & (a,\sigma); & & x\\ (\beta,\alpha), & (\lambda,\mu)^r, & (a+b,\sigma); & \end{array}\right].$$

where $\alpha,\beta,\gamma,\lambda,\sigma,a,b\in\mathbb{C},\ \Re(\alpha,\beta,\gamma,\lambda,\sigma,a,b)>0$ and $\delta,\ \mu>0$.

Proof. Here

$$\int_0^1 z^{a-1}(1-z)^{b-1}E^{\gamma,\delta}_{\alpha,\ \beta,\ \lambda,\ \mu}(xz^\sigma;s,r)\ \mathrm{d}z$$
$$=\int_0^1 z^{a-1}\ (1-z)^{b-1}\left\{\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s\ x^n z^{\sigma n}}{\Gamma(\alpha n+\beta)\ [(\lambda)_{\mu\ n}]^r\ n!}\right\}\ \mathrm{d}z$$
$$=\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s\ x^n}{\Gamma(\alpha n+\beta)\ [(\lambda)_{\mu n}]^r\ n!}\int_0^1 z^{\sigma n+a-1}(1-z)^{b-1}\ \mathrm{d}z$$
$$=\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s\ x^n}{\Gamma(\alpha n+\beta)\ [(\lambda)_{\mu n}]^r\ n!}\mathfrak{B}(\sigma n+a,b)$$
$$=\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s\ x^n}{\Gamma(\alpha n+\beta)\ [(\lambda)_{\mu n}]^r\ n!}\frac{\Gamma(\sigma n+a)\Gamma(b)}{\Gamma(\sigma n+a+b)}$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma[(\gamma+\delta n)]^s \ [\Gamma(\lambda)]^r \ \Gamma(b) \ \Gamma(\sigma n + a) \ x^n}{[\Gamma(\gamma)]^s \ \Gamma(\alpha n+\beta) \ [\Gamma(\lambda+\mu n)]^r \Gamma(\sigma n + a + b) n!}$$

$$= \frac{[\Gamma(\lambda)]^r \ \Gamma(b)}{[\Gamma(\gamma)]^s} \ {}_{s+1}\Psi_{r+2}\left[\begin{array}{llll} (\gamma,\delta)^s, & (a,\sigma); & & x \\ (\beta,\alpha), & (\lambda,\mu)^r, & (a+b,\sigma); & \end{array} \right].$$

□

2.4.2 The Laplace tansform

Theorem 2.4.2. *The Laplace transform of the function (2.1.5) is:*

$$\int_0^{\infty} z^{a-1} \ e^{-Sz} E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(xz^{\sigma}; s, r) \ \mathrm{d}z \ = \ \frac{[\Gamma(\lambda)]^r}{[\Gamma(\gamma)]^s \ S^a} \ {}_{s+1}\Psi_{r+1}\left[\begin{array}{lll} (\gamma,\delta)^s, & (a,\sigma); & \frac{x}{S^{\sigma}} \\ (\beta,\alpha), & (\lambda,\mu)^r; & \end{array} \right],$$

where $\alpha, \beta, \gamma, \lambda, \sigma, a, b \in \mathbb{C}, \ \Re(\alpha, \beta, \gamma, \lambda, \sigma, a, b) > 0$ *and* $\delta, \ \mu > 0.$

Proof. Beginning with

$$I = \int_0^{\infty} z^{a-1} \ e^{-Sz} \ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(xz^{\sigma}; s, r) \ \mathrm{d}z$$

$$= \int_0^{\infty} z^{a-1} e^{-Sz} \left\{ \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s x^n z^{\sigma n}}{\Gamma(\alpha n+\beta)[(\lambda)_{\mu n}]^r \ n!} \right\} \mathrm{d}z$$

$$= \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s x^n}{\Gamma(\alpha n+\beta)[(\lambda)_{\mu n}]^r n!} \int_0^{\infty} e^{-Sz} \ z^{\sigma n + a - 1} \ \mathrm{d}z,$$

and using the substitution $t = Sz$, one finds

$$I = \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s \ x^n}{\Gamma(\alpha n+\beta)[(\lambda)_{\mu n}]^r n!} \int_0^{\infty} e^{-t} \ (t/S)^{\sigma n + a - 1} \ (1/S) \ \mathrm{d}t$$

$$= \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s \ x^n}{\Gamma(\alpha n+\beta)[(\lambda)_{\mu n}]^r \ n!} \frac{\Gamma(\sigma n + a)}{S^{\sigma n + a}}$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma[(\gamma+\delta n)]^s \ [\Gamma(\lambda)]^r \ x^n \ S^{-\sigma n - a} \ \Gamma(\sigma n + a)}{[\Gamma(\gamma)]^s \ \Gamma(\alpha n+\beta) \ [\Gamma(\lambda+\mu n)]^r \ n!}$$

$$= \frac{[\Gamma(\lambda)]^r}{[\Gamma(\gamma)]^s \ S^a} \ {}_{s+1}\Psi_{r+1}\left[\begin{array}{lll} (\gamma,\delta)^s, & (a,\sigma); & \frac{x}{S^{\sigma}} \\ (\beta,\alpha), & (\lambda,\mu)^r; & \end{array} \right].$$

□

2.4.3 The Whittaker transform

In proving the following theorem, the integral formula involving the Whittaker function:

$$\int_0^\infty t^{\nu-1}\, e^{-t/2}\, W_{\lambda,\mu}(t)\, \mathrm{d}t = \frac{\Gamma(1/2+\mu+\nu)\Gamma(1/2-\mu+\nu)}{\Gamma(1-\lambda+\nu)},\ \Re(\nu\pm\mu) > -1/2$$

will be used.

Theorem 2.4.3. *In the above notation, the Whittaker transform of the function is:*

$$\begin{aligned}&\int_0^\infty e^{-qt/2}t^{\psi-1}\, W_{\eta,\nu}(qt)\, E^{\gamma,\delta}_{\alpha,\ \beta,\ \lambda,\ \mu}(\omega t^\sigma; s, r)\, \mathrm{d}t\\ =\ &\frac{[\Gamma(\lambda)]^r}{[\Gamma(\gamma)]^s\, q^\psi}\ {}_{s+2}\Psi_{r+2}\left[\begin{matrix}(\gamma,\delta)^s, & (\psi+\nu+1/2,\sigma), & (\psi-\nu+1/2,\sigma); & \frac{\omega}{q^\sigma}\\ (\beta,\alpha), & (\lambda,\mu)^r, & (\psi-\eta+1,\sigma); & \end{matrix}\right],\end{aligned}$$

where $\alpha,\beta,\gamma,\lambda,\sigma,a,b\in\mathbb{C},\ \Re(\alpha,\beta,\gamma,\lambda,\sigma,a,b)>0$ *and* $\delta,\ \mu>0$.

Proof. Let

$$\begin{aligned}I &= \int_0^\infty e^{-qt/2}\, t^{\psi-1}\, W_{\eta,\nu}(qt)\, E^{\gamma,\delta}_{\alpha,\ \beta,\ \lambda,\ \mu}(\omega t^\sigma; s, r)\, \mathrm{d}t\\ &= \int_0^\infty e^{-qt/2}\, t^{\psi-1}\, W_{\eta,\nu}(qt)\left\{\sum_{n=0}^\infty \frac{[(\gamma)_{\delta n}]^s\, \omega^{pn+\rho-1}\, t^{\sigma n}}{\Gamma(\alpha n+\beta)\, [(\lambda)_{\mu n}]^r\, n!}\right\}\mathrm{d}t\\ &= \sum_{n=0}^\infty \frac{[(\gamma)_{\delta n}]^s\, \omega^{pn+\rho-1}}{\Gamma(\alpha n+\beta)\, [(\lambda)_{\mu n}]^r\, n!}\int_0^\infty e^{-qt/2}\, t^{\sigma n+\psi-1}\, W_{\eta,\nu}(qt)\, \mathrm{d}t,\end{aligned}$$

then using the substitution $\xi = qt$, this gives

$$\begin{aligned}I &= \sum_{n=0}^\infty \frac{[(\gamma)_{\delta n}]^s\, \omega^n\, q^{-\sigma n-\psi}}{\Gamma(\alpha n+\beta)\, [(\lambda)_{\mu n}]^r\, n!}\int_0^\infty e^{-\xi/2}\, \xi^{\sigma n+\psi-1}\, W_{\eta,\nu}(\xi)\, \mathrm{d}\xi\\ &= \sum_{n=0}^\infty \frac{\Gamma[(\gamma+\delta n)]^s\, [\Gamma(\lambda)]^r\, \omega^n q^{-\sigma n-\psi}}{[\Gamma(\gamma)]^s\, \Gamma(\alpha n+\beta)\, [\Gamma(\lambda+\mu n)]^r\, n!}\\ &\quad\times\frac{\Gamma(\sigma n+\psi+\nu+1/2)\, \Gamma(\sigma n+\psi-\nu+1/2)}{\Gamma(\sigma n+\psi-\eta+1)}\end{aligned}$$

$$= \frac{[\Gamma(\lambda)]^r}{[\Gamma(\gamma)]^s\ q^{\psi}}\ {}_{s+2}\Psi_{r+2}\left[\begin{array}{llll}(\gamma,\delta)^s, & (\psi+\nu+1/2,\sigma), & (\psi-\nu+1/2,\sigma); & \frac{\omega}{q^{\sigma}}\\ (\beta,\alpha), & (\lambda,\mu)^r, & (\psi-\eta+1,\sigma); & \end{array}\right].$$

□

2.4.4 The Mellin transform

Theorem 2.4.4. *The Mellin transform of the function is:*

$$\int_0^{\infty} t^{S-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(-\omega t; s,r)\ \mathrm{d}t = \frac{\Gamma(S)\ [\Gamma(\gamma-\delta S)]^s}{\omega^S\ \Gamma(\beta+\alpha-\alpha-\alpha S)\ [\Gamma(\lambda-\mu S)]^r}, \quad (2.4.1)$$

where $\alpha,\beta,\gamma,\lambda,S\in\mathbb{C},\ \Re(\alpha,\beta,\gamma,\lambda,S)>0,\ \delta,\ \mu>0.$

Proof. Putting $z=-\omega t$ in (2.2.16) gives

$$\begin{aligned} E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(-\omega t; s,r) &= \frac{[\Gamma(\lambda)]^r}{2\pi i\ [\Gamma(\gamma)]^s}\int_L \frac{\Gamma(S)\ [\Gamma(\gamma-\delta S)]^s\ (\omega t)^{-S}}{\Gamma(\beta-\alpha S)\ [\Gamma(\lambda-\mu S)]^r}\ \mathrm{d}S \\ &= \frac{[\Gamma(\lambda)]^r}{2\pi i\ [\Gamma(\gamma)]^s}\int_L f^*(S)\ t^{-S}\ \mathrm{d}S, \quad (2.4.2)\end{aligned}$$

in which

$$f^*(S) = \frac{\Gamma(S)\ [\Gamma(\gamma-\delta S)]^s\ (\omega)^{-S}}{\Gamma(\beta-\alpha S)\ [\Gamma(\lambda-\mu S)]^r}$$

using (1.4.8) and (1.4.9) in (2.4.2), one arrives at (2.4.1). □

2.5 Relationship with certain known special functions

In the following it is shown that the proposed unified function may be reduced appropriately to get some special functions such as generalized Konhauser polynomial, Fox H- function, Wright hypergeometric function etc.

2.5.1 Generalized hypergeometric function

If $\alpha, \delta, \mu \in \mathbb{N}$, then (2.1.5) gives

$$\begin{aligned}
& E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r) \\
&= \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s \; z^n}{\Gamma(\alpha n+\beta)\; [(\lambda)_{\mu n}]^r \; n!} \\
&= \frac{1}{\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s \; z^n}{(\beta)_{\alpha n}\; [(\lambda)_{\mu n}]^r \; n!} \\
&= \frac{1}{\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\left[\prod_{i=1}^{\delta}\left(\frac{\gamma+i-1}{\delta}\right)_n\right]^s \left[\prod_{k=1}^{\mu}\left(\frac{\lambda+k-1}{\mu}\right)_n\right]^{-r}}{\prod_{j=1}^{\alpha}\left(\frac{\beta+j-1}{\alpha}\right)_n} \frac{\frac{\delta^{s\delta n} \; z^n}{\mu^{r\mu n} \; (\alpha)^{\alpha n}}}{n!} \\
&= \frac{1}{\Gamma(\beta)} \; {}_{s\delta}F_{\alpha+r\mu}\left[\begin{array}{ll} \Delta(\delta,\gamma)^s; & \frac{\delta^{s\delta} \; z}{\mu^{r\mu} \; \alpha^{\alpha}} \\ \Delta(\alpha,\beta), \quad \Delta(\mu,\lambda)^r; & \end{array}\right],
\end{aligned}$$

where $\Delta(n;\alpha)$ stands for the set of n parameters $\frac{\alpha}{n}, \frac{\alpha+1}{n}, ..., \frac{\alpha+n-1}{n}$. And $\Delta(n;\alpha)^s$ represents n parameters $\left(\frac{\alpha}{n}\right)^s, \left(\frac{\alpha+1}{n}\right)^s, ..., \left(\frac{\alpha+n-1}{n}\right)^s$.

2.5.2 Generalized Konhauser polynomial

Putting $\alpha = k,\ \beta = \nu+1,\ \gamma = -m,\ r = 0,\ s = 1, \delta = j$, where $j, k, m \in \mathbb{N}$, and replacing z by z^k in (2.1.5), one gets

$$\begin{aligned}
E^{-m,j}_{k,\nu+1,\lambda,\mu,1,1}(z^k;1,0) &= \sum_{n=0}^{[\frac{m}{j}]} \frac{(-m)_{jn}}{\Gamma(kn+\nu+1)} \frac{z^{kn}}{n!} \\
&= \sum_{n=0}^{[\frac{m}{j}]} \frac{(-1)^{jn}\; m!}{(m-jn)!} \frac{1}{\Gamma(kn+\nu+1)} \frac{z^{kn}}{n!} \\
&= \frac{\Gamma(m+1)}{\Gamma(km+\nu+1)} \sum_{n=0}^{[\frac{m}{j}]} \frac{(-1)^{jn}}{(m-jn)!} \frac{\Gamma(km+\nu+1)}{\Gamma(kn+\nu+1)} \frac{z^{kn}}{n!} \\
&= \frac{\Gamma(m+1)}{\Gamma(km+\nu+1)} Z^{(\nu)}_{[\frac{m}{j}]}(z,k), \qquad (2.5.1)
\end{aligned}$$

where $Z^{(\nu)}_{[\frac{m}{j}]}(z,k)$ is a generalized Konhauser polynomial of degree $[\frac{m}{j}]$ in z^k.

In particular, $Z_m^{(\nu)}(z,1) = L_m^{(\nu)}(z)$ the Laguerre polynomial, that is,

$$E_{k,\nu+1,\lambda,\mu,1,1}^{-m,1}(z;1,0) = \frac{\Gamma(m+1)}{\Gamma(km+\nu+1)} L_m^{(\nu)}(z). \tag{2.5.2}$$

2.5.3 Fox H- function

In view of the contour integral representation (2.2.16),

$$E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(z;s,r) = \frac{[\Gamma(\lambda)]^r}{2\pi i\ [\Gamma(\gamma)]^s} \int_L \frac{\Gamma(S)\,[\Gamma(\gamma-\delta S)]^s\ (-z)^{-S}}{\Gamma(\beta-\alpha S)\ [\Gamma(\lambda-\mu S)]^r} \mathrm{d}S,$$

$$= \frac{[\Gamma(\lambda)]^r}{[\Gamma(\gamma)]^s} H_{s,r+3}^{1,s}\left[-z \left| \begin{matrix} [(1-\gamma,\delta)]^s & & \\ (0,1), & (1-\beta,\alpha), & [(1-\lambda,\mu)]^r \end{matrix} \right. \right].$$

2.5.4 Wright function

In the notation of the Wright function, the function (2.1.5) assumes the representation as follows.

$$E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(z;s,r) = \frac{[\Gamma(\lambda)]^r}{[\Gamma(\gamma)]^s}\ {}_s\Psi_{r+1}\left[\begin{matrix} [(\gamma,\delta)]^s; & & z \\ (\beta,\alpha), & [(\lambda,\mu)]^r; & \end{matrix} \right].$$

2.6 Special cases

Each of the above obtained properties will be illustrated by taking one special case from Table-1, Section 2.1.

- **Contour integral** - Elliptic function:

$$K(\sqrt{z}) = \frac{1}{4\pi i} \int_L \frac{\Gamma(S)[\Gamma(\frac{1}{2}-S)]^2\,(-z)^{-S}}{\Gamma(1-S)} \mathrm{d}S.$$

- **Differential equation** - Shukla and Prajapati's function:

$$\left[\Upsilon_j^{(\alpha,\beta;1)}\ \theta - z\frac{q^q}{\alpha^\alpha} \Delta_m^{(q,\gamma;1)} \right] E_{\alpha,\beta}^{\gamma,q}(z) = 0.$$

- **Eigen function property** - Dotsenko function:

$$\left[D\ \Theta_m^{(1,a;-1)}\Upsilon_k^{(\frac{\omega}{\nu},b;-1)}\Upsilon_j^{(\frac{\omega}{\nu},c;1)}\right]\ {}_2R_1(a,b;c,\omega;\nu;\zeta z)=\zeta\ {}_2R_1(a,b;c,\omega;\nu;\zeta z).$$

- **Mixed relations** - Bessel-Maitland function:

1. $J^{\mu}_{\nu+k}(z)-J^{\mu}_{\nu+k+1}(z)$
   $=\mu^2z^2\ \ddot{J}^{\mu}_{\nu+k+2}(z)+\mu z\left[\mu+2(\nu+k+1)\right]\dot{J}^{\mu}_{\nu+k+2}(z)$
   $+((\nu+1)^2+2(\nu+1)k+k^2-1)J^{\mu}_{\nu+k+2}(z).$
2. $(\nu+1)\ J^{\mu}_{\nu+1}(z)+\mu z\ \frac{d}{dz}\ J^{\mu}_{\nu+1}(z)=J^{\mu}_{\nu}(z).$

- **Double series relation** - Saxena and Nishimoto function:

$${}^*E_{\gamma,K,\rho,1}[(\alpha_m,\beta_m)_{1,2};z]\ =\ \sum_{i,j=0}^{\infty}\frac{1}{(\rho)_{i+j}}\frac{(-1)^i}{i!\ j!}\ {}^*E_{\gamma,K,\rho+i+j,1}[(\alpha_m,\beta_m)_{1,2};z].$$

2.7 Bessel function family

In (2.1.5) replacing z by $-\frac{z^2}{4}$ and multiplying the series by $\left(\frac{z}{2}\right)^{\xi}$, one gets

$$\left(\frac{z}{2}\right)^{\xi}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\left(-\frac{z^2}{4};s,r\right)=\left(\frac{z}{2}\right)^{\xi}\ \sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s}{\Gamma(\alpha n+\beta)\ [(\lambda)_{\mu n}]^r\ n!}\left(-\frac{z^2}{4}\right),\quad(2.7.1)$$

in which $\alpha,\beta,\gamma,\lambda\in\mathbb{C},\ \Re(\alpha,\beta,\gamma,\lambda)>0,\ \xi,\delta,\mu>0,\ r,s\in\mathbb{N}\cup\{0\}$.
For suitable choices of ξ, this gives Bessel function, generalized Bessel Maitland function, Lommel function and Struve function.
(I) The Bessel function of first kind [24, Eq.(1.2.19), p.9]:
$(r=0,\ s=0,\ \alpha=1,\ \beta=\nu+1$ and $\xi=\nu)$

$$J_\nu(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\ \Gamma(1+\nu+n)}\left(\frac{z}{2}\right)^{\nu+2n}.$$

(II) The generalized Bessel-Maitland function [24, Eq.(1.7.9), p.19] :
$(s=0,\ r=1,\ \alpha=\sigma,\ \beta=\nu+\eta+1,\ \mu=1,\ \lambda=\eta+1$ and $\xi=\nu+2\eta)$

$$J^{\sigma}_{\nu,\eta}(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(\nu+n\sigma+\eta+1)\ \Gamma(n+\eta+1)}\left(\frac{z}{2}\right)^{\nu+2\eta+2n}.$$

(III) Lommel function [41, Eq.(1), p.217] :
$(s=1,\ r=1,\ \alpha=1,\ \beta=\frac{1}{2}(\eta-\nu+3),\ \mu=1,\ \lambda=\frac{1}{2}(\eta+\nu+3),\ \gamma=1,\ \delta=1$

and $\xi = \eta + 1)$

$$S_{\eta,\nu}(z) = \frac{z^{\eta+1}}{(\eta-\nu+1)(\eta+\nu+1)} \times {}_1F_2\left(\begin{array}{ccc} & 1; & -\frac{z^2}{4} \\ \frac{1}{2}(\eta-\nu+3), & \frac{1}{2}(\eta+\nu+3); & \end{array}\right).$$

(IV) Struve function [41, Eq.(3), p.217] :
$(s = 1,\ r = 1,\ \alpha = 1,\ \beta = 3/2,\ \mu = 1,\ \lambda = 3/2 + \nu,\ \gamma = 1,\ \delta = 1$ and $\xi = \nu + 1)$

$$H_\nu(z) = \frac{(z/2)^{\nu+1}}{\Gamma(3/2)\Gamma(3/2+\nu)}\ {}_1F_2\left(\begin{array}{ccc} & 1; & -\frac{z^2}{4} \\ 3/2, & 3/2+\nu; & \end{array}\right).$$

All these functions are tabulated below as particular cases of (2.7.1).

Table-2

Function	**r**	**s**	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\gamma}$	$\boldsymbol{\delta}$	$\boldsymbol{\lambda}$	$\boldsymbol{\mu}$	$\boldsymbol{\xi}$
Bessel	0	0	1	$\nu+1$	-	-	-	-	ν
Generalized Bessel-Maitland	1	0	σ	$\nu+\eta+1$	-	-	$\eta+1$	1	$\nu+2\eta$
Lommel	1	1	1	$\frac{\eta-\nu+3}{2}$	1	1	$\frac{\eta+\nu+3}{2}$	1	$\eta+1$
Struve	1	1	1	$3/2$	1	1	$3/2+\nu$	1	$\nu+1$

2.7.1 Convergence

Theorem 2.7.1. *The series represented by the function* $\left(\frac{z}{2}\right)^\xi E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\left(-\frac{z^2}{4}; s, r\right)$ *converges for* $\Re(\alpha) + r\mu - s\delta + 1 > 0$.

The proof of the above theorem is simillar to the proof of the Theorem 2.2.1 wherein one has to take

$$u_{n,\xi} = \frac{(-1)^n\ [(\gamma)_{\delta n}]^s\ z^\xi}{\Gamma(\alpha n+\beta)\ [(\lambda)_{\mu n}]^r\ 2^{\xi+2n}\ n!} \tag{2.7.2}$$

instead of u_n as given in (2.2.1). Hence the proof omitted.

Theorem 2.7.2. *Let* $\Re(\alpha,\beta,\gamma,\lambda) > 0, \Re(\alpha)+r\mu-s\delta+1 > 0,\ \delta,\mu > 0,\ r,s \in \mathbb{N}\cup\{0\}$. *Then* $\left(\frac{z}{2}\right)^\xi E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\left(-\frac{z^2}{4}; s, r\right)$ *is an entire function of order* $\varrho = \frac{1}{\Re(\alpha)+r\mu-s\delta+1}$ *and type* $\sigma = \frac{1}{\varrho}\left(\frac{\delta^{s\delta}}{4\ \{\Re(\alpha)\}^{\Re(\alpha)}\mu^{r\mu}}\right)^\varrho$.

Proof. It can be easily shown that the function given by (2.7.1) is an entire function of order $\varrho = \frac{1}{\Re(\alpha)+r\mu-s\delta+1}$ just as it is shown in Theorem 2.2.2 with the use of (2.7.2).

The proof runs quite parallel hence not repeated.

Now, the type σ of the function (2.7.1) is given by

$$\sigma\left(\left(\frac{z}{2}\right)^{\xi} E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\left(-\frac{z^2}{4};s,r\right)\right) = \frac{1}{e\ \varrho}\ \limsup_{n\to\infty}\left(n\,|u_{n,\xi}|^{\varrho/n}\right). \tag{2.7.3}$$

Here

$$\begin{aligned}
|u_{n,\xi}| &= \left|\frac{(-1)^n\ [\Gamma(\gamma+\delta n)]^s\ [\Gamma(\lambda)]^r\ z^{\xi}}{[\Gamma(\gamma)]^s\ \Gamma(\alpha n+\beta)\ [\Gamma(\lambda+\mu n)]^r\ 2^{\xi+2n}\ \Gamma(n+1)}\right| \\
&\sim \left|\frac{(-1)^n\ \left(\sqrt{2\pi}e^{-(\gamma+\delta n)}(\gamma+\delta n)^{\gamma+\delta n-1/2}\right)^s\ \left(\sqrt{2\pi}e^{-\gamma}\gamma^{\gamma-1/2}\right)^{-s}}{\left(\sqrt{2\pi}e^{-(\alpha n+\beta)}(\alpha n+\beta)^{\alpha n+\beta-1/2}\right)}\right. \\
&\quad \left.\times\frac{\left(\sqrt{2\pi}e^{-\lambda}\lambda^{\lambda-1/2}\right)^r\ z^{\xi}}{\left(\sqrt{2\pi}e^{-(\lambda+\mu n)}(\lambda+\mu n)^{\lambda+\mu n-1/2}\right)^r\ 2^{\xi+2n}\ \left(\sqrt{2\pi}e^{-(n+1)}(n+1)^{n+1-1/2}\right)}\right| \\
&= \left|\frac{1}{2\pi}\ \frac{(-1)^n\ e^{\alpha n+\beta+r\mu n-s\delta n+n-1}\ (\delta n)^{s(\gamma+\delta n-1/2)}}{(\alpha n)^{\alpha n+\beta-1/2}\left(1+\frac{\beta}{\alpha n}\right)^{\alpha n+\beta-1/2}\gamma^{s(\gamma-1/2)}}\right. \\
&\quad \left.\times\frac{\lambda^{r(\lambda-1/2)}\ \left(1+\frac{\gamma}{\delta n}\right)^{s(\gamma+\delta n-1/2)}\ z^{\xi}}{(\mu n)^{r(\lambda+\mu n-1/2)}\left(1+\frac{\lambda}{\mu n}\right)^{r(\lambda+\mu n-1/2)}\ 2^{\xi+2n}\ (n+1)^{n+1/2}}\right|.
\end{aligned}$$

On substituting this on the right hand side of (2.2.4) and using (2.2.3), one gets

$$\begin{aligned}
\limsup_{n\to\infty}\left(n\ |u_{n,\xi}|^{\varrho/n}\right) &= \left(\frac{\delta^{s\delta}}{4\ \{\varrho(\Re(\alpha)\}^{\Re(\alpha)}\ \mu^{r\mu}}\right)^{\varrho} e^{\Re(\alpha)+r\mu-s\delta+1)} \\
&\quad\times\lim_{n\to\infty} n^{\varrho(s\delta-\Re(\alpha)-r\mu-1)+1}.
\end{aligned}$$

This gives

$$\sigma\left(\left(\frac{z}{2}\right)^{\xi} E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\left(-\frac{z^2}{4};s,r\right)\right) = \frac{1}{\varrho}\left(\frac{\delta^{s\delta}}{4\ \{\Re(\alpha)\}^{\Re(\alpha)}\ \mu^{r\mu}}\right)^{\varrho}. \tag{2.7.4}$$

For every positive ϵ, from the equation (1.2.15) the asymptotic estimate is given by

$$\left|\left(\frac{z}{2}\right)^{\xi} E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\left(-\frac{z^2}{4};s,r\right)\right| < exp\left((\sigma+\epsilon)\ |z|^{2\varrho}\right),\ \ |z|\geq r_0>0. \tag{2.7.5}$$

□

2.7.2 Differential equation

The differential equation of (2.7.1) is derived with the help of the following operators.

$$\theta = zD,\; D^i = \frac{d^i}{dz^i},\; \frac{\delta^{s\delta}}{\alpha^{\alpha}\; \mu^{r\mu}} = P,$$

$$\theta_\xi f(z) = z^{\xi/2} f(z^{1/2})\; \theta \left(z^{-\xi/2} f(z^{1/2}) \right),\quad \mathfrak{D}^i_\xi f(z) = z^{\xi/2-i} f(z^{1/2})\; D^i \left(z^{-\xi/2} f(z^{1/2}) \right),$$

$$\Upsilon^{(a,b;m)}_{\xi,j} = \prod_{j=0}^{a-1} \left[\left(\theta_\xi + \frac{b+j}{a} - 1 \right) \right]^m,\; \Delta^{(a,b;m)}_{\xi,j} = \prod_{j=0}^{a-1} \left[\left(\theta_\xi + \frac{b+j}{a} \right) \right]^m,$$

$$\Theta^{(a,b;m)}_{\xi,j} = \prod_{j=0}^{a-1} \left[\left(-\theta_{-\xi} + \frac{b+j}{a} - 1 \right) \right]^m \tag{2.7.6}$$

and

$$\Omega^{\Upsilon}_{\xi,\Theta} = -4P^{-1}\; \mathfrak{D}_\xi\; \Theta^{(\delta,\gamma;-s)}_{\xi,m}\; \Upsilon^{(\mu,\lambda;r)}_{\xi,k}\; \Upsilon^{(\alpha,\beta;1)}_{\xi,j}. \tag{2.7.7}$$

It may be noted that the operators $\Theta^{(\delta,\gamma;-s)}_{\xi,m}$, $\Upsilon^{(\mu,\lambda;r)}_{\xi,k}$, $\Upsilon^{(\alpha,\beta;1)}_{\xi,j}$ in (2.7.7) are not commutative with the operator $\mathfrak{D}_\xi$.

Theorem 2.7.3. *Let $\alpha, \mu, \delta \in \mathbb{N}$ then $y_\xi = \left(\frac{z}{2}\right)^\xi\; E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\left(-\frac{z^2}{4}; s, r\right)$ satisfies the equation*

$$\left[\Upsilon^{(\mu,\lambda;r)}_{\xi,k}\; \Upsilon^{(\alpha,\beta;1)}_{\xi,j}\; \theta_\xi + \frac{z^2}{4}\; P\; \Delta^{(\delta,\gamma;s)}_{\xi,m} \right] y_\xi = 0. \tag{2.7.8}$$

Proof. Here

$$\begin{aligned} y_\xi &= \sum_{n=0}^{\infty} \frac{(-1)^n\; [(\gamma)_{\delta n}]^s\; z^{2n+\xi}}{\Gamma(\alpha n+\beta)\; [(\lambda)_{\mu n}]^r\; 2^{2n+\xi}\; n!} \\ &= \frac{1}{\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{(-1)^n\; [(\gamma)_{\delta n}]^s\; z^{2n+\xi}}{(\beta)_{\alpha n}\; [(\lambda)_{\mu n}]^r\; 2^{2n+\xi}\; n!} \\ &= \frac{1}{\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{(-1)^n\; \delta^{s\delta n} [(\frac{\gamma}{\delta})_n]^s\; [(\frac{\gamma+1}{\delta})_n]^s \ldots [(\frac{\gamma+\delta-1}{\delta})_n]^s}{\alpha^{\alpha n} (\frac{\beta}{\alpha})_n\; (\frac{\beta+1}{\alpha})_n \ldots (\frac{\beta+\alpha-1}{\alpha})_n} \end{aligned}$$

$$\times \frac{z^{2n+\xi}}{2^{2n+\xi}\ \mu^{r\mu n}[(\frac{\lambda}{\mu})_n]^r\ [(\frac{\lambda+1}{\mu})_n]^r ... [(\frac{\lambda+\mu-1}{\mu})_n]^r\ n!}$$

$$= \frac{1}{\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{(-1)^n\ P^n}{2^{2n+\xi}} \frac{\prod_{m=0}^{\delta-1} [(\frac{\gamma+m}{\delta})_n]^s\ z^{2n+\xi}}{\left\{\prod_{j=0}^{\alpha-1} (\frac{\beta+j}{\alpha})_n\right\} \left\{\prod_{k=0}^{\mu-1} [(\frac{\lambda+k}{\mu})_n]^r\right\}\ n!}. \tag{2.7.9}$$

By using (2.2.10), the equation (2.7.1) takes the form

$$y_\xi = \sum_{n=0}^{\infty} \frac{(-1)^n\ A_n\ P^n\ z^{2n+\xi}}{2^{2n+\xi}\ B_n\ C_n\ n!}.$$

Now,

$$\begin{aligned} \theta_\xi\ y_\xi &= \sum_{n=0}^{\infty} \frac{(-1)^n\ A_n\ P^n}{2^{2n+\xi}\ B_n\ C_n\ n!}\ \theta_\xi\ z^{2n+\xi} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n\ A_n\ P^n}{2^{2n+\xi}\ B_n\ C_n\ n!}\ z^{\xi/2} z^{n+\xi/2} \theta \left(z^{-\xi/2} z^{n+\xi/2}\right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n\ A_n\ P^n}{2^{2n+\xi}\ B_n\ C_n\ n!}\ z^{\xi/2} z^{n+\xi/2} \theta z^n \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n\ A_n\ P^n\ z^{2n+\xi}}{2^{2n+\xi}\ B_n\ C_n\ (n-1)!}. \end{aligned}$$

Further,

$$\begin{aligned} \Upsilon_{\xi,j}^{(\alpha,\beta;1)}\ \theta_\xi\ y_\xi &= \sum_{n=1}^{\infty} \frac{(-1)^n\ A_n\ P^n}{2^{2n+\xi}\ B_n\ C_n\ (n-1)!} \prod_{j=0}^{\alpha-1} \left(\theta_\xi + \frac{\beta+j}{\alpha} - 1\right)\ z^{2n+\xi} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n\ A_n\ P^n}{2^{2n+\xi}\ B_n\ C_n\ (n-1)!} \prod_{j=0}^{\alpha-1} \left(n + \frac{\beta+j}{\alpha} - 1\right)\ z^{2n+\xi} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n\ A_n\ P^n}{2^{2n+\xi}\ B_{n-1}\ C_n} \frac{z^{2n+\xi}}{(n-1)!}. \end{aligned}$$

Finally,

$$\begin{aligned} &\Upsilon_{\xi,k}^{(\mu,\lambda;r)}\ \Upsilon_{\xi,j}^{(\alpha,\beta;1)}\ \theta_\xi\ y_\xi \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n\ A_n\ P^n}{2^{2n+\xi}\ B_{n-1}\ C_n\ (n-1)!} \prod_{k=0}^{\mu-1} \left[\left(\theta_\xi + \frac{\lambda+k}{\mu} - 1\right)\right]^r\ z^{2n+\xi} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n\ A_n\ P^n}{2^{2n+\xi}\ B_{n-1}\ C_n\ (n-1)!} \prod_{k=0}^{\mu-1} \left[\left(n + \frac{\lambda+k}{\mu} - 1\right)\right]^r\ z^{2n+\xi} \end{aligned}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n \; A_n \; P^n}{2^{2n+\xi} \; B_{n-1} \; C_{n-1}} \frac{z^{2n+\xi}}{(n-1)!}.$$

Thus,

$$\Upsilon_{\xi,k}^{(\mu,\lambda;r)} \; \Upsilon_{\xi,j}^{(\alpha,\beta;1)} \; \theta_\xi \; y_\xi = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} A_{n+1} \; P^{n+1} \; z^{2(n+1)+\xi}}{2^{2(n+1)+\xi} \; B_n \; C_n \; n!}. \tag{2.7.10}$$

On the other hand,

$$\begin{aligned}
\Delta_{\xi,m}^{(\delta,\gamma;s)} \; y_\xi &= \sum_{n=0}^{\infty} \frac{(-1)^n \; A_n \; P^n}{2^{2n+\xi} \; B_n \; C_n \; n!} \prod_{m=0}^{\delta-1} \left[\left(\theta_\xi + \frac{\gamma+m}{\delta}\right)\right]^s \; z^{2n+\xi} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n \; A_n \; P^n}{2^{2n+\xi} \; B_n \; C_n \; n!} \left[\prod_{m=0}^{\delta-1} \left[\left(n + \frac{\gamma+m}{\delta}\right)\right]^s\right] \; z^{2n+\xi} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n \; A_{n+1} \; P^n \; z^{2n+\xi}}{2^{2n+\xi} \; B_n \; C_n \; n!},
\end{aligned}$$

hence

$$\frac{-z^2}{4} \; P \; \Delta_{\xi,m}^{(\delta,\gamma;s)} \; y_\xi = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \; A_{n+1} \; P^{n+1} \; z^{2(n+1)+\xi}}{2^{2(n+1)+\xi} \; B_n \; C_n \; n!}. \tag{2.7.11}$$

On comparing (2.7.10) and (2.7.11), one finds (2.7.8). □

Note 2.7.1. *The equation (2.7.8) with* $\xi = \nu, \nu + 2\eta, \eta + 1, \nu + 1$ *reduces to the particular differential equation for the Bessel, Bessel Maitland, Lommel and Struve functions, respectively.*

2.7.3 Eigen function property

Theorem 2.7.4. *For* $\alpha, \mu, \delta \in \mathbb{N}$, $\left(\frac{z}{2}\right)^\xi \; E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}\left(-\frac{z^2}{4}; s, r\right)$ *is an eigen function with respect to the operator* $\Omega_{\xi,\Theta}^{\Upsilon}$ *as defined in (2.7.7).*
That is,

$$\Omega_{\xi,\Theta}^{\Upsilon}\left(\left(\frac{z}{2}\right)^\xi \; E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}\left(-\zeta\frac{z^2}{4}; s, r\right)\right) = \zeta \; \left(\frac{z}{2}\right)^\xi \; E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}\left(-\zeta\frac{z^2}{4}; s, r\right). \tag{2.7.12}$$

Proof. Take

$$w_\xi \;\; = \;\; \left(\frac{z}{2}\right)^\xi \; E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}\left(-\zeta\frac{z^2}{4}; s, r\right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\ A_n\ (\zeta P)^n\ z^{2n+\xi}}{2^{2n+\xi}\ B_n\ C_n\ n!},$$

Now,

$$\begin{aligned}
\Upsilon_{\xi,j}^{(\alpha,\beta;1)}\ w_\xi &= \sum_{n=0}^{\infty} \frac{(-1)^n\ A_n\ (\zeta P)^n}{2^{2n+\xi}\ B_n\ C_n\ n!} \prod_{j=0}^{\alpha-1} \left(\theta_\xi + \frac{\beta+j}{\alpha} - 1\right)\ z^{2n+\xi} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n\ A_n\ (\zeta P)^n}{2^{2n+\xi}\ B_n\ C_n\ n!} \prod_{j=0}^{\alpha-1} \left(n + \frac{\beta+j}{\alpha} - 1\right) z^{2n+\xi} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n\ A_n\ (\zeta P)^n\ z^{2n+\xi}}{2^{2n+\xi}\ B_{n-1}\ C_n\ n!},
\end{aligned}$$

and

$$\begin{aligned}
\Upsilon_{\xi,k}^{(\mu,\lambda;r)}\ \Upsilon_{\xi,j}^{(\alpha,\beta;1)}\ w_\xi &= \sum_{n=0}^{\infty} \frac{(-1)^n\ A_n\ (\zeta P)^n}{2^{2n+\xi}\ B_{n-1}\ C_n\ n!} \prod_{k=0}^{\mu-1} \left[\left(\theta_\xi + \frac{\lambda+k}{\mu} - 1\right)\right]^r\ z^{2n+\xi} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n\ A_n\ (\zeta P)^n}{2^{2n+\xi}\ B_{n-1}\ C_n\ n!} \prod_{k=0}^{\mu-1} \left[\left(n + \frac{\lambda+k}{\mu} - 1\right)\right]^r\ z^{2n+\xi} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n\ A_n\ (\zeta P)^n\ z^{2n+\xi}}{2^{2n+\xi}\ B_{n-1}\ C_{n-1}\ n!}.
\end{aligned}$$

Now using (2.7.6),

$$\begin{aligned}
\Theta_{\xi,m}^{(\delta,\gamma;-s)}\ \Upsilon_{\xi,k}^{(\mu,\lambda;r)}\ \Upsilon_{\xi,j}^{(\alpha,\beta;1)}\ w_\xi &= \sum_{n=0}^{\infty} \frac{(-1)^n\ A_n\ (\zeta P)^n}{2^{2n+\xi}\ B_{n-1}\ C_{n-1}\ n!}\ \Theta_{\xi,m}^{(\delta,\gamma;-s)}\ z^{2n+\xi} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n\ A_n\ (\zeta P)^n}{2^{2n+\xi}\ B_{n-1}\ C_{n-1}\ n!} \\
&\quad \times \prod_{m=0}^{\delta-1} \left[\left(-\theta_{-\xi} + \frac{\gamma+m}{\delta} - 1\right)\right]^{-s}\ z^{2n+\xi} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n\ A_n\ (\zeta P)^n}{2^{2n+\xi}\ B_{n-1}\ C_{n-1}\ n!} \\
&\quad \times \prod_{m=0}^{\delta-1} \left[\left(n + \frac{\gamma+m}{\delta} - 1\right)\right]^{-s}\ z^{2n+\xi} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n\ A_{n-1}\ (\zeta P)^n\ z^{2n+\xi}}{2^{2n+\xi}\ B_{n-1}\ C_{n-1}\ n!}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\Omega^{\Upsilon}_{\xi,\Theta}\ w_\xi &= -4\ P^{-1}\ \mathfrak{D}_\xi\ \Theta^{(\delta,\gamma;-s)}_{\xi,m}\ \Upsilon^{(\mu,\lambda;r)}_{\xi,k}\ \Upsilon^{(\alpha,\beta;1)}_{\xi,j}\ w_\xi \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n-1}\ A_{n-1}\ \zeta^n\ P^{n-1}}{2^{2(n-1)+\xi}\ B_{n-1}\ C_{n-1}\ n!}\ \mathfrak{D}^1_\xi\ z^{2n+\xi} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\ A_{n-1}\ \zeta^n\ P^{n-1}\ z^{2(n-1)+\xi}}{2^{2(n-1)+\xi}\ B_{n-1}\ C_{n-1}\ (n-1)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n\ A_n\ \zeta^{n+1}\ P^n\ z^{2n+\xi}}{2^{2n+\xi}\ B_n\ C_n\ n!} \\
&= \zeta \sum_{n=0}^{\infty} \frac{(-1)^n\ A_n\ \zeta^n\ P^n\ z^{2n+\xi}}{2^{2n+\xi}\ B_n\ C_n\ n!} \\
&= \zeta\ \left(\frac{z}{2}\right)^\xi\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\left(-\zeta\frac{z^2}{4};s,r\right).
\end{aligned}$$

□

2.7.4 Mellin - Barnes integral representation

Theorem 2.7.5. *Let $\alpha,\delta,\mu>0;\beta,\gamma,\lambda\in\mathbb{C}$, with $\Re(\beta,\gamma,\lambda)>0$. Then the function $\left(\frac{z}{2}\right)^\xi\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\left(-\frac{z^2}{4};s,r\right)$ is expressible as the Mellin - Barnes integral given by*

$$\left(\frac{z}{2}\right)^\xi\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\left(-\frac{z^2}{4};s,r\right) = \frac{[\Gamma(\lambda)]^r\ \left(\frac{z}{2}\right)^\xi}{2\pi i\ [\Gamma(\gamma)]^s}\int_L \frac{\Gamma(S)\ \Gamma(1-S)}{\Gamma(\beta-\alpha S)\ [\Gamma(\lambda-\mu S)]^r} \times\frac{[\Gamma(\gamma-\delta S)]^s\ z^{-2S}}{2^{-2S}\ \Gamma(1-S)}\mathrm{d}S, \tag{2.7.13}$$

where $|argz|<\pi$; the contour L of integration begins from $-i\infty$ and proceeds towards $+i\infty$, and is indented to keep the poles of integrand at $S=-n$ to the left; and the poles at $S=(\gamma+n)/\delta$ to the right of the path, for all $n\in\mathbb{N}_0$.

The proof of the Theorem 2.7.5 is similar to that of the Theorem 2.2.5, hence it is omitted.

2.7.5 Mixed relations

Theorem 2.7.6. *For* $\alpha,\ \beta,\ \gamma,\ \lambda\ \in \mathbb{C},\ k \in \mathbb{N},\ \Re(\alpha,\beta,\gamma,\lambda) > 0, \delta,\ \mu > 0,$ *we get*

$$\begin{aligned} &\left(\frac{z}{2}\right)^{\xi} \left(E^{\gamma,\delta}_{\alpha,\beta+k,\lambda,\mu}\left(-\frac{z^2}{4};s,r\right) - E^{\gamma,\delta}_{\alpha,\beta+k+1,\lambda,\mu}\left(-\frac{z^2}{4};s,r\right)\right) \\ = \ & \alpha^2 z^4\ \mathfrak{D}^2_{\xi}\left(\left(\frac{z}{2}\right)^{\xi}\ E^{\gamma,\delta}_{\alpha,\beta+k+2,\lambda,\mu}\left(-\frac{z^2}{4};s,r\right)\right) \\ &+\alpha z^2\left[\alpha+2(\beta+k)\right]\ \mathfrak{D}_{\xi}\left(\left(\frac{z}{2}\right)^{\xi}\ E^{\gamma,\delta}_{\alpha,\beta+k+2,\lambda,\mu}\left(-\frac{z^2}{4};s,r\right)\right) \\ &+\left(\beta^2+2\beta k+k^2-1\right)\ \left(\frac{z}{2}\right)^{\xi}\ E^{\gamma,\delta}_{\alpha,\beta+k+2,\lambda,\mu}\left(-\frac{z^2}{4};s,r\right). \qquad (2.7.14) \end{aligned}$$

The proof of the Theorem 2.7.6 just differing in operator, runs parallel to that of Theorem 2.2.6, hence it is omitted.

Theorem 2.7.7. *For* $\alpha,\beta,\gamma,\lambda \in \mathbb{C}; \Re(\alpha,\beta,\gamma,\lambda) > 0$ *and* $\delta,\mu > 0$, *the differential recurrence relation*

$$\begin{aligned} &\beta\ \left(\frac{z}{2}\right)^{\xi}\ E^{\gamma,\delta}_{\alpha,\,\beta+1,\,\lambda,\,\mu}\left(-\frac{z^2}{4};s,r\right)\ +\ \alpha z^2\ \ {}_{\xi}\mathfrak{D}^1\ \left(\frac{z}{2}\right)^{\xi}\ E^{\gamma,\delta}_{\alpha,\,\beta+1,\,\lambda,\,\mu}\left(-\frac{z^2}{4};s,r\right) \\ = \ &\left(\frac{z}{2}\right)^{\xi}\ E^{\gamma,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu}\left(-\frac{z^2}{4};s,r\right) \end{aligned}$$

holds.

The proof of the Theorem 2.7.7 again differing just in operator, hence not given.

2.7.6 Double series representation

The double series representation of (2.7.1) with

$$u_{n,\xi}\ =\ \frac{(-1)^n\ \left[(\gamma)_{\delta n}\right]^s}{2^{2n+\xi}\ \Gamma(\alpha n+\beta)\ \left[(\lambda)_{\mu n}\right]^r\ n!}.$$

and

$$\left(\frac{z}{2}\right)^{\xi}\ {}^{*}E^{\gamma,\delta,\rho}_{\alpha,\beta,\lambda,\mu}\left(-\frac{z^2}{4};s,r\right) = \sum_{n=0}^{\infty} u_{n,\xi}\ (\rho)_n\ z^{2n+\xi}, \quad \Re(\alpha)+r\mu-s\delta > 0$$

is stated as

Theorem 2.7.8. *The function*

$$\left(\frac{z}{2}\right)^{\xi} \; {}^{*}E^{\gamma,\delta,\rho}_{\alpha,\beta,\lambda,\mu}\left(-\frac{z^2}{4}; s, r\right) = \sum_{i,j=0}^{\infty} \frac{1}{(i+j)_\rho} \frac{(-1)^i}{i!\; j!} \left(\frac{z}{2}\right)^{\xi} \; {}^{*}E^{\gamma,\delta,\rho+i+j}_{\alpha,\beta,\lambda,\mu}\left(-\frac{z^2}{4}; s, r\right).$$

The proof runs parallel to the proof of Theorem 2.2.9.

2.8 Special cases

The special cases of above obtained properties are illustrated by taking one special case from Table-2, section 2.7 for each property.

- **Contour integral** - Generalized Bessel-Maitland function:

$$J^{\sigma}_{\nu,\eta}(z) = \frac{\Gamma(\eta+1)\; \left(\frac{z}{2}\right)^{\nu+2\eta}}{2\pi i} \int_L \frac{\Gamma(S)\; \Gamma(1-S)}{\Gamma(\nu+\eta+1-\sigma S)\; \Gamma(\eta+1-S)} \times \frac{z^{-2S}}{2^{-2S}\; \Gamma(1-S)}\; \mathrm{d}S.$$

- **Differential equation** - Bessel function:

$$\left[\Upsilon^{(1,\nu+1;1)}_{\nu,j}\; \theta_\nu + \frac{z^2}{4}\right] J_\nu(z) = 0.$$

- **Eigen function Property** - Bessel Function:

$$-4\, \mathfrak{D}_\nu\; \Upsilon^{(1,\nu+1;1)}_{\nu,j} J_\nu(\zeta z) = \zeta\; J_\nu(\zeta z).$$

- **Mixed relations** - Lommel function:

1. $$\left(\tfrac{z}{2}\right)^{\eta+1}\Big(E^{1,1}_{1,\frac{\eta-\nu+3}{2}+k,\frac{\eta+\nu+3}{2},1}\left(-\tfrac{z^2}{4};1,1\right) - E^{1,1}_{1,\frac{\eta-\nu+3}{2}+k+1,\frac{\eta+\nu+3}{2},1}\left(-\tfrac{z^2}{4};1,1\right)\Big)$$
$$= z^4\, \mathfrak{D}^2_{\eta+1}\left(\left(\tfrac{z}{2}\right)^{\eta+1}\; E^{1,1}_{1,\frac{\eta-\nu+3}{2}+k+2,\frac{\eta+\nu+3}{2},1}\left(-\tfrac{z^2}{4};1,1\right)\right)$$
$$+z^2\left[1+2\left(\tfrac{\eta-\nu+3}{2}+k\right)\right]$$
$$\times\, \mathfrak{D}_{\eta+1}\;\left(\left(\tfrac{z}{2}\right)^{\eta+1}\; E^{1,1}_{1,\frac{\eta-\nu+3}{2}+k+2,\frac{\eta+\nu+3}{2},1}\left(-\tfrac{z^2}{4};1,1\right)\right)$$
$$+\;\left(\left(\tfrac{\eta-\nu+3}{2}\right)^2 + 2\tfrac{\eta-\nu+3}{2}k + k^2 - 1\right)$$

$$\times \left(\tfrac{z}{2}\right)^{\eta+1} \; E^{1,1}_{1,\frac{\eta-\nu+3}{2}+k+2,\frac{\eta+\nu+3}{2},1}\left(-\tfrac{z^2}{4};1,1\right).$$

2. $$\tfrac{\eta-\nu+3}{2} \; \left(\tfrac{z}{2}\right)^{\eta+1} \; E^{1,1}_{1,\frac{\eta-\nu+3}{2}+1,\frac{\eta+\nu+3}{2},1}\left(-\tfrac{z^2}{4};1,1\right)$$
$$+ \alpha z^2 \; \mathfrak{D}_\nu \; \left(\tfrac{z}{2}\right)^{\eta+1} \; E^{1,1}_{1,\frac{\eta-\nu+3}{2}+1,\frac{\eta+\nu+3}{2},1}\left(-\tfrac{z^2}{4};1,1\right)$$
$$= \left(\tfrac{z}{2}\right)^{\eta+1} \; E^{1,1}_{1,\frac{\eta-\nu+3}{2},\frac{\eta+\nu+3}{2},1}\left(-\tfrac{z^2}{4};1,1\right).$$

- **Double Series relation** - Sturve function:

$$\left(\frac{z}{2}\right)^{\nu+1} \; {}^{*}E^{1,1,\rho}_{1,3/2,3/2+\nu,1}\left(-\frac{z^2}{4};1,1\right)$$
$$= \sum_{i,j=0}^{\infty} \frac{1}{(\rho)_{i+j}} \frac{(-1)^i}{i! \; j!} \; \left(\frac{z}{2}\right)^{\nu+1} \; {}^{*}E^{1,1,\rho+i+j}_{1,3/2,3/2+\nu,1}\left(-\frac{z^2}{4};1,1\right).$$

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