Chapter 2

The ℓ-Hypergeometric function

2.1 Introduction

In this chapter, a function representing a *rapidly convergent* power series is proposed which extends the well-known confluent hypergeometric function ${}_1F_1[z]$ (1.5) as well as the integral function $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!^n}$ (1.8) considered by P. C. Sikkema [49]. With the aid of newly defined differential operators, an infinite order differential equations is obtained, for which these new special functions are the eigen functions.

At first place, some properties are established, as the order zero of these entire (integral) functions, integral representations, differential equations involving a kind of hyper-Bessel type operators of infinite order.

In the following a new class of special functions is defined which is suggested by the power series representations (1.5) and (1.8).

Definition 2.1. For $z \in \mathbb{C}$, define the ℓ-Hypergeometric function as

$$ {}_1H_1^1 \left[\begin{array}{lll} a; & & z \\ b; & (c:\ell); & \end{array} \right] = {}_1H_1^1(\ell : z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n \; (c)_n^{\ell n}} \frac{z^n}{n!}, \tag{2.1} $$

where $(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$, $a, \ell \in \mathbb{C}$ with $\Re(\ell) \geq 0$, and $b, c \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$.

Here 'a' is the numerator parameter,'b' is denominator parameter as usual and 'c' is regarded as the "ℓ-denominator" parameter.

The function defined here will be referred to as ℓ-H function: ${}_1H_1^1(\ell : z)$ throughout the work.

Remark 2.2. It is easy to see that

$$ {}_1H_1^1\left[\begin{array}{lll} a; & & z \\ b; & (c:0); & \end{array}\right] = {}_1F_1\left[\begin{array}{ll} a; & z \\ b; & \end{array}\right]. $$

Throughout the chapter,

$$ \frac{(a)_n}{(b)_n \ (c)_n^{\ell n} \ n!} = \varphi_n, $$

so that

$$ {}_1H_1^1\left[\begin{array}{lll} a; & & z \\ b; & (c:\ell); & \end{array}\right] = \sum_{n=0}^{\infty} \varphi_n \ z^n. \tag{2.2} $$

2.2 Main Results

2.2.1 Convergence

The series in (2.1) is convergent for all $z \in \mathbb{C}$ which is proved in the following theorem.

Theorem 2.3. *If* $\Re(\ell) \geq 0$ *and* $\Re\left(c\ell - \frac{\ell}{2} + 1\right) > 0$, *then the* ℓ-*H function* ${}_1H_1^1(\ell : z)$ *is an entire function of* z.

Proof. Using the Cauchy-Hadamard formula:

$$ \frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|\varphi_n|}, $$

one gets

$$ \begin{aligned} \frac{1}{R} &= \limsup_{n\to\infty} \left| \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n \ (c)_n^{\ell n} \ n!} \right|^{\frac{1}{n}} \\ &= \limsup_{n\to\infty} \left|\frac{\Gamma(b)}{\Gamma(a)}\right|^{\frac{1}{n}} \left|\frac{\Gamma(a+n)}{\Gamma(b+n)}\right|^{\frac{1}{n}} \times \left|\frac{\Gamma(c)}{\Gamma(c+n)}\right|^{\Re(\ell)} \frac{1}{\Gamma^{\frac{1}{n}}(n+1)}. \end{aligned} $$

Now applying the Stirling's asymptotic formula [18] for large n , given by

$$ \Gamma(\alpha+n) \sim \sqrt{2\pi} e^{-(\alpha+n)} \ (\alpha+n)^{(\alpha+n-1/2)}, \tag{2.3} $$

with $\alpha = a, b, c, 1$ in turn, one obtains

$$\begin{aligned}
\frac{1}{R} &\sim \left|\frac{\Gamma(c)}{\sqrt{2\pi}}\right|^{\Re(\ell)} \limsup_{n\to\infty} \left|\frac{\Gamma(b)}{\Gamma(a)}\right|^{\frac{1}{n}} \left|\frac{\sqrt{2\pi}\ e^{-(a+n)}\ (a+n)^{a+n-1/2}}{\sqrt{2\pi}\ e^{-(b+n)}\ (b+n)^{b+n-1/2}}\right|^{\frac{1}{n}} \\
&\quad \times \frac{1}{|e^{-(c+n)}\ (c+n)^{c+n-1/2}|^{\Re(\ell)}\ \left|\sqrt{2\pi}\ e^{-(n+1)}\ (n+1)^{n+1-1/2}\right|^{\frac{1}{n}}} \\
&= \left|\frac{\Gamma(c)}{\sqrt{2\pi}}\right|^{\Re(\ell)} \limsup_{n\to\infty} \left|\frac{1}{n^{c\ell-\frac{\ell}{2}+1}} \left(\frac{e}{n}\right)^{\ell n}\right| \\
&= 0,
\end{aligned}$$

provided that $\Re(\ell) \geq 0$ and $\Re\left(c\ell - \frac{\ell}{2} + 1\right) > 0$. □

Remark 2.4. 1. The series $\sum \varphi_n\ z^n$ thus converges uniformly, in any compact subset of $\mathbb{C}$.

2. It may be seen that the proposed new class of functions (2.1) preserves the entire function property of the confluent hypergeometric function (1.5).

2.2.2 Order of ℓ-H function ${}_1H_1^1(\ell : z)$

Theorem 2.5. *If the conditions stated in Theorem 2.3 hold, then the ℓ-H function* ${}_1H_1^1(\ell : z)$ *is an entire function of order zero.*

Proof. It is known that [8, 40] if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function then the order $\rho(f)$ of f is given by

$$\rho(f) = \limsup_{n\to\infty} \frac{n \ln n}{\ln |\varphi_n|^{-1}}. \tag{2.4}$$

Now from (2.2),

$$|\varphi_n|^{-1} = \left|\frac{\Gamma(a)}{\Gamma(b)} \frac{\Gamma(b+n)\ \Gamma^{\ell n}(c+n)\ \Gamma(n+1)}{\Gamma(a+n)\ \Gamma^{\ell n}(c)}\right|.$$

Hence,

$$\begin{aligned}
\ln |\varphi_n|^{-1} &= |\ln \Gamma(a) - \ln \Gamma(b) + \ln(\Gamma(b+n)) + \ell n\ \ln(\Gamma(c+n)) \\
&\quad + \ln(\Gamma(n+1)) - \ln(\Gamma(a+n)) - \ell n\ \ln(\Gamma(c))|.
\end{aligned}$$

Since

$$\ln\Gamma(r) \sim \left(r-\frac{1}{2}\right)\ln r - r + \frac{1}{2}\ln\sqrt{2\pi}$$

for large r,

$$\begin{aligned}\ln|\varphi_n|^{-1} \leq\ & |\ln\Gamma(a) - \ln\Gamma(b)| \\ & + \left|\left(b+n-\frac{1}{2}\right)\ln(b+n) - (b+n) + \frac{1}{2}\ln\sqrt{2\pi}\right| \\ & + \left|\ell n\left[\left(c+n-\frac{1}{2}\right)\ln(c+n) - (c+n) + \frac{1}{2}\ln\sqrt{2\pi}\right]\right| \\ & + \left|\left(n+1-\frac{1}{2}\right)\ln(n+1) - (n+1) + \frac{1}{2}\ln\sqrt{2\pi}\right| + |\ell n(\ln\Gamma(c))| \\ & + \left|\left(a+n-\frac{1}{2}\right)\ln(a+n) + (a+n) - \frac{1}{2}\ln\sqrt{2\pi}\right|. \qquad (2.5)\end{aligned}$$

But since

$$\lim_{n\to\infty}\frac{\ln|\varphi_n|^{-1}}{n\ln n}$$

is unbounded as n increases without bound, it follows from (2.4) and (2.5) that

$$\rho\left({}_1H_1^1\left[\begin{matrix} a; & & z \\ b; & (c:\ell); & \end{matrix}\right]\right) = \lim_{n\to\infty}\frac{n\ln n}{\ln|\varphi_n|^{-1}} = 0.$$

□

Remark 2.6. It is proved that [4, Theorem 1.1] "If f is entire and $\rho(f)$ is finite and is not equal to a positive integer, then f has infinitely many zeros or it is a polynomial." From this it follows that the ℓ-H function ${}_1H_1^1(\ell : z)$ has infinitely many zeros.

2.2.3 Integral Representation

The integral representation occurs by routine calculations [46, Ch. 7, Eq. 9, p.124]; which is as obtained below.

Theorem 2.7. *With* $\Re(b) > \Re(a) > 0, c, b \neq 0, -1, \ldots,$ *and* $\Re\left(c\ell - \frac{\ell}{2} + 1\right) > 0$,

$${}_1H_1^1\left[\begin{matrix} a; & & z \\ b; & (c:\ell); & \end{matrix}\right] = \frac{\Gamma(b)}{\Gamma(a)\,\Gamma(b-a)}\int_0^1 t^{a-1}(1-t)^{b-a-1}$$

$$\times\ {}_0H_0^1\left[\begin{matrix} -; & & zt \\ -; & (c:\ell); & \end{matrix}\right] dt. \tag{2.6}$$

Proof. Consider

$$\begin{aligned}
\frac{(a)_n}{(b)_n} &= \frac{\Gamma(a+n)}{\Gamma(a)}\frac{\Gamma(b)}{\Gamma(b+n)}\frac{\Gamma(b-a)}{\Gamma(b-a)} \\
&= \frac{\Gamma(b)}{\Gamma(b-a)\ \Gamma(a)}\frac{\Gamma(a+n)\ \Gamma(b-a)}{\Gamma(b+n)} \\
&= \frac{\Gamma(b)}{\Gamma(b-a)\ \Gamma(a)}B(a+n,b-a) \\
&= \frac{\Gamma(b)}{\Gamma(b-a)\ \Gamma(a)}\int_0^1 t^{a+n-1}\ (1-t)^{b-a-1}dt.
\end{aligned}$$

So that

$$\begin{aligned}
&{}_1H_1^1\left[\begin{matrix} a; & & z \\ b; & (c:\ell); & \end{matrix}\right] \\
&= \sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n\ (c)_n^{\ell n}}\frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty}\frac{\Gamma(b)}{\Gamma(b-a)\ \Gamma(a)}\int_0^1 t^{a+n-1}\ (1-t)^{b-a-1}\frac{z^n}{(c)_n^{\ell n}\ n!}dt \\
&= \sum_{n=0}^{\infty}\frac{\Gamma(b)}{\Gamma(b-a)\ \Gamma(a)}\int_0^1 t^{a-1}\ (1-t)^{b-a-1}\frac{(zt)^n}{(c)_n^{\ell n}\ n!}dt \\
&= \frac{\Gamma(b)}{\Gamma(a)\ \Gamma(b-a)}\int_0^1 t^{a-1}(1-t)^{b-a-1}\ {}_0H_0^1\left[\begin{matrix} -; & & zt \\ -; & (c:\ell); & \end{matrix}\right] dt.
\end{aligned}$$

□

Remark 2.8. When $\ell = 0$, the theorem reduced to the integral representation of the confluent hypergeometric function [46, Ch.4, p.47] as stated in Theorem 1.6.

2.2.4 Differential Equation

The differential equation of the ℓ-H function ${}_1H_1^1(\ell : z)$ is derived with the aid of the following operator.

Definition 2.9. Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $0 \neq z \in \mathbb{C}$, $p \in \mathbb{N} \cup \{0\}$ and $\alpha \in \mathbb{C}$. Define

$$ {}_p\Delta_\alpha^\theta(f(z)) = \begin{cases} \sum_{n=1}^{\infty} a_n (\alpha)_{n-1}^p (\theta + \alpha - 1)^{pn} z^n, & \text{if} \quad p \in \mathbb{N} \\ f(z), & \text{if} \quad p = 0 \end{cases}, \tag{2.7} $$

where θ is the Euler differential operator $z\frac{d}{dz}$ and

$$ (\theta + \alpha)^r = \underbrace{(\theta + \alpha)(\theta + \alpha) \ldots (\theta + \alpha)}_{r \ times} $$

is a special case of the hyper-Bessel differential operators B which has the following form (see e.g. [33, 34, 36]).

$$ B = x^{\alpha_0} D x^{\alpha_1} D \cdots x^{\alpha_{m-1}} D x^{\alpha_m}, \quad m - (\alpha_o + \alpha_1 + \cdots + \alpha_m) > 0, \tag{2.8} $$

where $m \in \mathbb{N}$, $0 < x < \infty$ and $\alpha_0, \alpha_1, \ldots, \alpha_m \in \mathbb{R}$.

In this notation, the following theorem proves the differential equation.

Theorem 2.10. *For $\ell \in \mathbb{N} \cup \{0\}, a, z \in \mathbb{C}$, and $b, c \in \mathbb{C}/\{0, -1, -2, \ldots\}$, the function $w = {}_1H_1^1 \left[\begin{array}{lll} a; & & z \\ b; & (c : \ell); & \end{array} \right]$ satisfies the differential equation*

$$ \left\{ \left\{ {}_\ell\Delta_c^\theta \right\} \{\theta + b - 1\} \theta - z(\theta + a) \right\} w = 0, \tag{2.9} $$

where the operator ${}_\ell\Delta_c^\theta$ is as defined in (2.7).

Since the operator defined by (2.7) involves the infinite series, its applicability is subject to the convergence of the series with the general term $\phi_n\ f_n(a, b, c, \ell; z)$, $n \geq 0$. This is proved in the following lemma.

Lemma 2.11. *If $\ell \in \mathbb{N} \cup \{0\}$ then $w = {}_1H_1^1 \left[\begin{array}{lll} a; & & z \\ b; & (c : \ell); & \end{array} \right] = \sum_{n=0}^{\infty} \varphi_n z^n$ and $\left(\left\{ {}_\ell\Delta_c^\theta \right\} \{\theta + b - 1\} \theta \right) w = \sum_{n=0}^{\infty} f_n(a, b, c, \ell; z)$, then $\left(\left\{ {}_\ell\Delta_c^\theta \right\} \{\theta + b - 1\} \theta \right)$ is applicable to the ℓ-H function ${}_1H_1^1(\ell : z)$ provided that*

$$ \sum_{n=0}^{\infty} \varphi_n\ f_n(a, b, c, \ell; z) $$

is convergent (cf. [49, Definition 11, p.20]).

Proof. Let

$$w = {}_1H_1^1\left[\begin{array}{lll} a; & & z \\ b; & (c:\ell); & \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n\ (c)_n^{\ell n}} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \varphi_n z^n,$$

then

$$\begin{aligned}
\left\{\left\{{}_\ell\Delta_c^\theta\right\}\{\theta+b-1\}\,\theta\right\} w &= \left\{{}_\ell\Delta_c^\theta\right\}\{\theta+b-1\}\,\theta\left(\sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n\ (c)_n^{\ell n}} \frac{z^n}{n!}\right) \\
&= \left\{{}_\ell\Delta_c^\theta\right\}\{\theta+b-1\}\left(\sum_{n=1}^{\infty} \frac{(a)_n}{(b)_n\ (c)_n^{\ell n}} \frac{z^n}{(n-1)!}\right) \\
&= \left\{{}_\ell\Delta_c^\theta\right\}\left(\sum_{n=1}^{\infty} \frac{(a)_n}{(b)_n\ (c)_n^{\ell n}} \frac{(n+b-1)\ z^n}{(n-1)!}\right) \\
&= \sum_{n=1}^{\infty} \frac{(a)_n}{(b)_{n-1}\ (c)_n^{\ell n}} \frac{(c)_{n-1}^{\ell}\ (\theta+c-1)^{\ell n}\ z^n}{(n-1)!}.
\end{aligned}$$

Now

$$\begin{aligned}
(\theta+c-1)^\ell\ z &= \underbrace{(\theta+c-1)(\theta+c-1)\dots(\theta+c-1)}_{\ell\ times} z \\
&= (1+c-1)^\ell\ z \\
(\theta+c-1)^{2\ell}\ z^2 &= \underbrace{(\theta+c-1)(\theta+c-1)\dots(\theta+c-1)}_{2\ell\ times} z^2 \\
&= \underbrace{(\theta+c-1)(\theta+c-1)\dots(\theta+c-1)}_{2\ell-1\ times}(\theta+c-1)\ z^2 \\
&= (2+c-1)\underbrace{(\theta+c-1)(\theta+c-1)\dots(\theta+c-1)}_{2\ell-1\ times} z^2 \\
&= \dots = (2+c-1)^{2\ell}\ z^2.
\end{aligned}$$

In general,

$$(\theta+c-1)^{\ell n}\ z^n = (n+c-1)^{\ell n}\ z^n,\ n \in \mathbb{N}. \tag{2.10}$$

Hence

$$\begin{aligned}
\left\{\left\{{}_\ell\Delta_c^\theta\right\}\{\theta+b-1\}\,\theta\right\} w &= \sum_{n=1}^{\infty} \frac{(a)_n}{(b)_{n-1}\ (c)_n^{\ell n}} \frac{(c)_{n-1}^{\ell}\ (n+c-1)^{\ell n}\ z^n}{(n-1)!} \\
&= \sum_{n=1}^{\infty} \frac{(a)_n}{(b)_{n-1}\ (c)_{n-1}^{\ell n}} \frac{(c)_{n-1}^{\ell}\ z^n}{(n-1)!}
\end{aligned}$$

$$= \sum_{n=1}^{\infty} \frac{(a)_n}{(b)_{n-1}\ (c)_{n-1}^{\ell n-\ell}} \frac{z^n}{(n-1)!} \tag{2.11}$$

$$= \sum_{n=0}^{\infty} \frac{(a)_{n+1}}{(b)_n\ (c)_n^{\ell n}} \frac{z^{n+1}}{n!} \tag{2.12}$$

$$= \sum_{n=0}^{\infty} f_n(a, b, c, \ell; z).$$

To complete the proof of the lemma it suffices to show that

$$\sum_{n=0}^{\infty} \varphi_n\ f_n(a, b, c, \ell; z) = \sum_{n=0}^{\infty} \frac{(a)_n (a)_{n+1}}{(b)_n^2\ (c)_n^{2\ell n}} \frac{z^{n+1}}{(n!)^2}$$

is convergent.

Put

$$\xi_n = \frac{(a)_n^2\ (a+n)\ z^{n+1}}{(b)_n^2\ (c)_n^{2\ell n}\ (n!)^2} = \frac{\Gamma^2(b)}{\Gamma^2(a)}\ \frac{\Gamma^2(a+n)\ (a+n)\ \Gamma^{2\ell n}(c)\ z^{n+1}}{\Gamma^2(b+n)\ \Gamma^{2\ell n}(c+n)\ \Gamma^2(n+1)}.$$

Now applying the Stirling's asymptotic formula for large n given in (2.3), one gets

$$|\xi_n|^{\frac{1}{n}} \sim \left|\frac{\Gamma^2(b)}{\Gamma^2(a)}\right|^{\frac{1}{n}} \frac{\left|e^{-(a+n)}(a+n)^{a+n-\frac{1}{2}}\sqrt{2\pi}\right|^{\frac{2}{n}}}{\left|e^{-(b+n)}(b+n)^{b+n-\frac{1}{2}}\sqrt{2\pi}\right|^{\frac{2}{n}}}$$

$$\times \frac{|a+n|^{\frac{1}{n}}\ \left|\Gamma^{2\ell}(c)\right|\ |z|^{1+\frac{1}{n}}}{\left|e^{-(c+n)}(c+n)^{c+n-\frac{1}{2}}\sqrt{2\pi}\right|^{2\Re(\ell)}\ \left|e^{-(n+1)}(n+1)^{n+1-\frac{1}{2}}\sqrt{2\pi}\right|^{\frac{2}{n}}}.$$

Hence,

$$\begin{aligned}\lim_{n\to\infty} \sup |\xi_n|^{\frac{1}{n}} &\sim \lim_{n\to\infty} \sup \left|\frac{\Gamma(c)}{\sqrt{2\pi}}\right|^{2\ell}\ |z|\ \frac{1}{|n^{2c\ell-\ell+2}|}\ \left|\frac{e}{n}\right|^{2n\ell} \\ &= 0,\end{aligned}$$

when $\Re\,(2c\ell - \ell + 2) > 0$. □

Proof. (of Theorem 2.10) From (2.12),

$$\begin{aligned}\left\{\left\{{}_\ell\Delta_c^\theta\right\}\left\{(\theta+b-1)\right\}\theta\right\} w &= \sum_{n=0}^{\infty} \frac{(a)_{n+1}}{(b)_n\ (c)_n^{\ell n}} \frac{z^{n+1}}{n!} \\ &= z\sum_{n=0}^{\infty} \frac{(a)_n(a+n)}{(b)_n\ (c)_n^{\ell n}} \frac{z^n}{n!} \\ &= z(\theta+a)w\end{aligned}$$

which gives (2.9). □

Remark 2.12. It is noteworthy that when $\ell = 0$, the differential equation (2.9) reduces to the form

$$[(\theta + b - 1)\,\theta - z(\theta + a)]\,w = 0$$

which is the differential equation satisfied by $w = {}_1F_1[z]$, as stated in Theorem 1.4.

2.2.5 Eigen Function Property

To obtain the ℓ-H function ${}_1H_1^1(\ell : z)$ as the eigen function, one more operator is needed. This is defined below.

Definition 2.13. Let $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$, $|z| < R, z \neq 0, R > 0$ and $\Re(\alpha) > 0$. Define

$$ {}_\alpha\mathcal{H}_\beta^{(\gamma:p)}(f(z)) = \left[I_\alpha\left(z^{-1}\ \left\{{}_p\Delta_\gamma^\theta\right\}\{\theta+\beta-1\}\ \theta\right)\right](f(z)), \tag{2.13}$$

where the operator ${}_p\Delta_\gamma^\theta$ is as defined in (2.7) and

$$I_\alpha(f(z)) = z^{-\alpha}\int_0^z t^{\alpha-1} f(t)dt. \tag{2.14}$$

Theorem 2.14. *If $\ell \in \mathbb{N}\cup\{0\}$ and $\Re(a) > 0$, then the ℓ-H function ${}_1H_1^1(\ell : z)$ is the eigen function with respect to the operator ${}_a\mathcal{H}_b^{(c:\ell)}$ defined in (2.13). That is,*

$$ {}_a\mathcal{H}_b^{(c:\ell)}\left({}_1H_1^1\left[\begin{matrix} a; & & \lambda z\\ b; & (c:\ell); & \end{matrix}\right]\right) = \lambda\ {}_1H_1^1\left[\begin{matrix} a; & & z\\ b; & (c:\ell); & \end{matrix}\right],\quad \lambda\in\mathbb{C}. \tag{2.15}$$

Proof. The applicability of this operator to the ℓ-H function ${}_1H_1^1(\ell : z)$ follows from the Lemma 2.11.
It may be noted that for $z \neq 0$,

$$\begin{aligned} &{}_a\mathcal{H}_b^{(c:\ell)}\left({}_1H_1^1\left[\begin{matrix} a; & & \lambda z\\ b; & (c:\ell); & \end{matrix}\right]\right)\\ &= \left[I_a\left\{z^{-1}\left(\left\{{}_\ell\Delta_c^\theta\right\}\{(\theta+b-1)\}\,\theta\ \sum_{n=0}^{\infty}\frac{\lambda^n\ (a)_n}{(b)_n\ (c)_n^{\ell n}\ n!}z^n\right)\right\}\right]. \end{aligned}$$

In view of (2.11),

$$
\begin{aligned}
&{}_a\mathcal{H}_b^{(c:\ell)}\left({}_1H_1^1\left[\begin{matrix} a; & & \lambda z \\ b; & (c:\ell); & \end{matrix}\right]\right) \\
&= I_a\left\{ z^{-1}\left[\sum_{n=1}^{\infty}\frac{\lambda^n\,(a)_n}{(b)_{n-1}\,(c)_{n-1}^{\ell n-\ell}}\frac{z^n}{(n-1)!}\right]\right\} \\
&= I_a\left\{\sum_{n=1}^{\infty}\frac{\lambda^n\,(a)_n}{(b)_{n-1}\,(c)_{n-1}^{\ell n-\ell}\,(n-1)!}\frac{z^{n-1}}{(n-1)!}\right\} \\
&= \sum_{n=1}^{\infty}\frac{\lambda^n\,(a)_n}{(b)_{n-1}\,(c)_{n-1}^{\ell n-\ell}\,(n-1)!} z^{-a}\int_0^z t^{a-1}t^{n-1}dt \\
&= \sum_{n=1}^{\infty}\frac{\lambda^n\,(a)_n\,z^{-a}}{(b)_{n-1}\,(c)_{n-1}^{\ell n-\ell}\,(n-1)!}\left[\frac{z^{a+n-1}}{a+n-1}\right] \\
&= \sum_{n=1}^{\infty}\frac{\lambda^n\,(a)_{n-1}\,z^{n-1}}{(b)_{n-1}\,(c)_{n-1}^{\ell n-\ell}\,(n-1)!} \\
&= \sum_{n=0}^{\infty}\frac{\lambda^{n+1}\,(a)_n\,z^n}{(b)_n\,(c)_n^{\ell n}\,n!}.
\end{aligned}
$$

□

Remark 2.15. If z is zero then λ must be zero.

2.3 The ℓ-H function ${}_2H_1^2(\ell : z)$

By considering one more numerator parameter and one more ℓ-denominator parameter, the function ${}_1H_1^1(\ell : z)$ admits a mild extension in the form:

$$
{}_2H_1^2\left[\begin{matrix} a_1,\ a_2; & & z \\ b; & (c_1, c_2 : \ell); & \end{matrix}\right] = {}_2H_1^2(\ell : z) = \sum_{n=0}^{\infty}\frac{(a_1)_n\,(a_2)_n}{(b)_n\,(c_1)_n^{\ell n}\,(c_2)_n^{\ell n}}\frac{z^n}{n!}, \tag{2.16}
$$

with $\Re(\ell) \geq 0$, $z, a_1, a_2 \in \mathbb{C}$ and $b, c_1, c_2 \in \mathbb{C}\setminus\{0,-1,-2,\ldots\}$.
Its properties occur in a straight forward manner which are as listed below.

Theorem 2.16. *If $\Re(\ell) \geq 0$ and $\Re(c_1+c_2-1)\,\ell > 0$, then the ℓ-H function (2.16) is an entire function of z.*

Theorem 2.17. *If the conditions stated in Theorem 2.16 hold, then the ℓ-H function (2.16) is an entire function of order zero.*

Hence from Remark 2.6, the function ${}_2H_1^2(\ell : z)$ has infinitely many zeros.

Theorem 2.18. *With* $\Re(b) > \Re(a_1) > 0, \quad b, c_1, c_2 \neq 0, -1, \ldots, \quad \Re(\ell) \geq 0$ *and* $\Re\left((c_1 + c_2 - 1)\ell\right) > 0$,

$$ {}_2H_1^2\left[\begin{array}{lll} a_1, \ a_2; & & z \\ \quad b; & (c_1, c_2 : \ell); & \end{array}\right] = \frac{\Gamma(b)}{\Gamma(a_1)\ \Gamma(b - a_1)} \int_0^1 t^{a_1 - 1}(1 - t)^{b - a_1 - 1} $$
$$ \times\ {}_1H_0^2\left[\begin{array}{lll} a_2; & & zt \\ -; & (c_1, c_2 : \ell); & \end{array}\right] dt. $$

Theorem 2.19. *For* $\ell \in \mathbb{N} \cup \{0\}, a_1, a_2, z \in \mathbb{C},$ *and* $b, c_1, c_2 \in \mathbb{C}/\{0, -1, -2, \ldots\}$, *the function* $w = {}_2H_1^2\left[\begin{array}{lll} a_1, \ a_2; & & z \\ \quad b; & (c_1, c_2 : \ell); & \end{array}\right]$ *satisfies the differential equation*

$$ \left\{\left\{{}_\ell\Delta_{c_2}^\theta\ {}_\ell\Delta_{c_1}^\theta\right\}\{\theta + b - 1\}\,\theta - z(\theta + a_1)(\theta + a_1)\right\} w = 0, $$

where the operator ${}_\ell\Delta_c^\theta$ *is as defined in (2.7).*

Theorem 2.20. *If* $\ell \in \mathbb{N} \cup \{0\}$ *and* $\Re(a_1, a_2) > 0$, *then the function* ${}_2H_1^2(\ell : z)$ *is the eigen function with respect to the operator* ${}_{(a_1,a_2)}\mathcal{H}_b^{(c_1,c_2:\ell)}$ *where*

$$ {}_{(a_1,a_2)}\mathcal{H}_b^{(c_1,c_2:\ell)} = \left[I_{a_2}\ I_{a_1}\left(z^{-1}\ \left\{{}_\ell\Delta_{c_2}^\theta\ {}_\ell\Delta_{c_1}^\theta\right\}\{\theta + b - 1\}\ \theta\right)\right]. \tag{2.17} $$

That is, for $\lambda \in \mathbb{C}$,

$$ {}_{(a_1,a_2)}\mathcal{H}_b^{(c_1,c_2:\ell)}\left({}_2H_1^2\left[\begin{array}{lll} a_1, \ a_2; & & \lambda z \\ \quad b; & (c_1, c_2 : \ell); & \end{array}\right]\right) = \lambda\ {}_2H_1^2\left[\begin{array}{lll} a_1, \ a_2; & & z \\ \quad b; & (c_1, c_2 : \ell); & \end{array}\right]. $$