Chapter 3

The q-ℓ-Ψ hypergeometric functions

3.1 Introduction

This chapter deals with a rapidly convergent power series which generalizes the basic hypergeometric functions ${}_1\phi_1[z]$ and ${}_2\phi_1[z]$ together with the q-analogue of Sikkema's function (1.15). For this generalized function, the difference equations, eigen function properties and the contiguous functions relations are derived.

In the definition of the generalized basic hypergeometric series (1.11) if $r = s = 1$ then the reduced function takes the form

$$ {}_1\phi_1\left[\begin{array}{lll} a; & q, & z \\ b; & & \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(b;q)_n} (-1)^n \, q^{\binom{n}{2}} \, \frac{z^n}{(q;q)_n}, \tag{3.1} $$

which is an entire function of z where $b \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$.
Now as defined in Chapter 1, the q-form of the Sikkema's function (1.15) suggest an extension of (3.1) as follows.

Definition 3.1. For $0 < q < 1$, $\Re(\ell) \geq 0$, $a, z \in \mathbb{C}$, and $b, c \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, define the function Ψ by

$$ {}_1\Psi_1^1\left[\begin{array}{llll} a; & & q; & z \\ b; & (c:\ell); & & \end{array}\right] = {}_1\Psi_1^1(\ell : z) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(b;q)_n} \frac{(-1)^n \, q^{\binom{n}{2}}}{(c;q)_n^{\ell n}} \frac{z^n}{(q;q)_n}. \tag{3.2} $$

This function will be referred to as q-ℓ-Ψ hypergeometric function and in brief, the q-ℓ-Ψ function ${}_1\Psi_1^1(\ell : z)$. Also 'a' and 'b' as numerator and denominator

parameters respectively while 'c' is an ℓ-denominator parameter.
As $q \to 1$, this q-ℓ-Ψ function ${}_1\Psi_1^1(\ell : z)$ approaches to *ordinary* analogue, namely the ℓ-Hypergeometric function (2.1):

$$ {}_1H_1^1\left[\begin{array}{lll} a; & & z \\ b; & (c:\ell); & \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n\ (c)_n^{\ell n}} \frac{z^n}{n!}. $$

3.2 Main Results

In this section, certain properties of q-ℓ-Ψ function ${}_1\Psi_1^1(\ell : z)$ (3.2) are obtained namely, a difference equation of infinite order, an eigen function property and the contiguous function relations. But first the convergence behavior of the series. For the sake of brevity, throughout this chapter the coefficient of z^n in (3.2) will be abbreviated as ξ_n, that is,

$$ \frac{(a;q)_n\ \ (-1)^n\ q^{\binom{n}{2}}}{(b;q)_n\ (c;q)_n^{\ell n}\ (q;q)_n} = \xi_n. $$

3.2.1 Convergence

The absolute convergence of series in (3.2) is however evident due to the presence of the factor $q^{\binom{n}{2}}$, yet it will be shown that the series indeed represents the entire function.

Theorem 3.2. *If* $0 < q < 1,\ \Re(\ell) \geq 0$ *then* q-ℓ-Ψ *function* ${}_1\Psi_1^1(\ell : z)$*is an entire function of* z*.*

Proof. In view of the formula

$$ (a;q)_n = \frac{\Gamma_q(a+n)}{\Gamma_q(a)}\ (1-q)^n, \tag{3.3} $$

the n^{th} root of the coefficient ξ_n is given by

$$ \begin{aligned} |\xi_n|^{\frac{1}{n}} &= \left|\frac{(a;q)_n\ q^{n(n-1)/2}}{(b;q)_n\ (c;q)_n^{\ell n}\ (q;q)_n}\right|^{\frac{1}{n}} \\ &= \left|\frac{\Gamma_q(b)}{\Gamma_q(a)}\right|^{\frac{1}{n}} \left|\frac{\Gamma_q(a+n)}{\Gamma_q(b+n)}\right|^{\frac{1}{n}} \left|\frac{\Gamma_q(c)\ (1-q)^{-n}}{\Gamma_q(c+n)}\right|^{\Re(\ell)} \frac{q^{(n-1)/2}\ (1-q)^{-1}}{\left|\Gamma_q^{\frac{1}{n}}(n+1)\right|}. \end{aligned} $$

Here applying the Stirling's formula of q-Gamma function [41, Eq.(2.25), p.482]:

$$\Gamma_q(z) \sim (1+q)^{\frac{1}{2}}\ \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-z}\ e^{\frac{\theta q^z}{1-q-q^z}},\ \text{for large } |z|,\quad 0<\theta<1, \tag{3.4}$$

one finds

$$\begin{aligned} |\xi_n|^{\frac{1}{n}} \quad \sim \quad & \left|\frac{\Gamma_q(b)}{\Gamma_q(a)}\right|^{\frac{1}{n}} \left|\frac{(1+q)^{\frac{1}{2}}\ \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-(a+n)}\ e^{\frac{\theta\ q^{a+n}}{1-q-q^{a+n}}}}{(1+q)^{\frac{1}{2}}\ \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-(b+n)}\ e^{\frac{\theta\ q^{b+n}}{1-q-q^{b+n}}}}\right|^{\frac{1}{n}} \\ & \times \frac{\left|\Gamma_q^{\ell}(c)\ (1-q)^{-\ell n}\right|}{\left|(1+q)^{\frac{1}{2}}\ \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-(c+n)}\ e^{\frac{\theta\ q^{c+n}}{1-q-q^{c+n}}}\right|^{\Re(\ell)}} \\ & \times \frac{q^{(n-1)/2}\ (1-q)^{-1}}{\left|(1+q)^{\frac{1}{2}}\ \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-(n+1)}\ e^{\frac{\theta\ q^{n+1}}{1-q-q^{n+1}}}\right|^{\frac{1}{n}}} \end{aligned}$$

for large n. Finally, using the Cauchy-Hadamard formula, one arrives at

$$\begin{aligned} \frac{1}{R} &= \lim_{n\to\infty} \sup \sqrt[n]{|\xi_n|} \\ &= \lim_{n\to\infty} \left|\frac{\Gamma_q^{\ell}(c)\ (1-q)^{c\ell-\frac{\ell}{2}}}{(1+q)^{\frac{\ell}{2}}\ \Gamma_{q^2}^{\ell}\left(\frac{1}{2}\right)}\right| q^{(n-1)/2} \\ &= 0, \end{aligned}$$

provided $\Re(\ell) \geq 0$ and $0<q<1$. □

Remark 3.3. If $\ell = 0$ then the q-ℓ-Ψ function ${}_1\Psi_1^1(\ell : z)$ reduces to the basic hypergeometric function ${}_1\phi_1(a;b;q;z), z \in \mathbb{C}$.

3.2.2 q-Difference Equation

The differential equation of the q-ℓ-Ψ function ${}_1\Psi_1^1(\ell : z)$ occurs for $\ell \in \mathbb{N}\cup\{0\}$, which is obtained by using the following operator.

Definition 3.4. Let $f(z) = \sum\limits_{n=1}^{\infty} a_{n,q}\ z^n$, $0 \neq z \in \mathbb{C}$, $p \in \mathbb{N}\cup\{0\}$ and $\alpha \in \mathbb{C}$. Define

$${}_p\Delta_{\alpha}^{\theta_q} f(z) = \begin{cases} \sum\limits_{n=1}^{\infty} a_{n,q}\ (\alpha;q)_{n-1}^{p}\ (q^{\alpha-1}\ \theta_q - q^{\alpha-1}+1)^{pn}\ z^n, & \text{if } p\in\mathbb{N} \\ f(z), & \text{if } p=0 \end{cases}, \tag{3.5}$$

where the difference operator θ_q is defined by

$$\theta_q f(z) = f(z) - f(zq). \tag{3.6}$$

Note 3.5. *It may be noted that the operator* $(q^{\alpha-1}\ \theta_q - q^{\alpha-1} + 1)^{pn}$ *is a q-form of hyper-Bessel differential operators given in (2.8).*

The operator defined in (3.5) indeed helps in constructing an infinite order difference operator as given below.

Definition 3.6. Let $f(z) = \sum_{n=1}^{\infty} a_{n,q}\ z^n,\ 0 \neq z \in \mathbb{C},\ p \in \mathbb{N} \cup \{0\}$ and $\alpha, \beta \in \mathbb{C}$. Define the operator

$${}_{\beta}\Lambda^{\theta_q}_{(\alpha,p)} f(z) = \left[\left\{ {}_{p}\Delta^{\theta_q}_{\alpha} \right\} \left\{ q^{\beta-1}\theta_q - q^{\beta-1} + 1 \right\} \theta_q \right]\ (-q)\ f\left(\frac{z}{q}\right), \tag{3.7}$$

where ${}_{p}\Delta^{\theta_q}_{\alpha}$ is as defined in (3.5).

Note 3.7. *The operators* ${}_{p}\Delta^{\theta_q}_{\alpha}$ *and* θ_q *do not commute.*

By means of this operator, the difference equation of the q-ℓ-Ψ function ${}_1\Psi^1_1(\ell : z)$ is obtained as stated in the following theorem.

Theorem 3.8. *For* $\ell \in \mathbb{N} \cup \{0\}, a, z \in \mathbb{C},$ *and* $b, c \in \mathbb{C} \setminus \{0, -1, -2, \ldots\},$ *the function* $w = {}_1\Psi^1_1 \left[\begin{matrix} a; & & q; & z \\ b; & (c:\ell); & & \end{matrix} \right]$ *satisfies the q-difference equation*

$$\left[{}_{b}\Lambda^{\theta_q}_{(c,\ell)} - z(q^a\theta_q - q^a + 1) \right] w = 0, \tag{3.8}$$

where the operator ${}_{b}\Lambda^{\theta_q}_{(\alpha,p)}$ *is as defined in (3.7).*

Unlike the finite order operators that act on operand in straightforward manner, here the operator (3.5) acts on w subject to a condition which is stated and proved here as

Lemma 3.9. *If* $\ell \in \mathbb{N} \cup \{0\}, a, z \in \mathbb{C},\ b, c \in \mathbb{C} \setminus \{0, -1, -2, \ldots\},$ *and*

$$w = {}_1\Psi^1_1 \left[\begin{matrix} a; & & q; & z \\ b; & (c:\ell); & & \end{matrix} \right] = \sum_{n=0}^{\infty} \xi_n\ z^n$$

then the operator ${}_b\Lambda^{\theta_q}_{(c:\ell)}\ w := \sum_{n=0}^{\infty} f_{n,q}(a,b,(c:\ell);z)$ *is applicable to the* q-ℓ-Ψ *function* ${}_1\Psi_1^1(\ell : z)$ *provided that the series*

$$\sum_{n=0}^{\infty} \xi_n\ f_{n,q}(a,b,(c:\ell);z)$$

converges (cf. [49, Definition 11, p.20]).

Proof. Put

$$\frac{(a;q)_n\ (-1)^{n+1} q^{(n-1)(n-2)/2}}{(b;q)_n\ (c;q)_n^{\ell n}} = A_n, \tag{3.9}$$

then

$$\begin{aligned}
{}_b\Lambda^{\theta_q}_{(c,\ell)} w &= \left[\left\{{}_\ell\Delta_c^{\theta_q}\right\}\left\{q^{b-1}\theta_q - q^{b-1}+1\right\}\theta_q\right]\sum_{n=0}^{\infty}\xi_n\ (-q)\left(\frac{z}{q}\right)^n \\
&= \left[\left\{{}_\ell\Delta_c^{\theta_q}\right\}\left\{q^{b-1}\theta_q - q^{b-1}+1\right\}\theta_q\right]\sum_{n=0}^{\infty}A_n\ \frac{z^n}{(q;q)_n} \\
&= \left[\left\{{}_\ell\Delta_c^{\theta_q}\right\}\left\{q^{b-1}\theta_q - q^{b-1}+1\right\}\right]\sum_{n=1}^{\infty}A_n\ \frac{(1-q^n)}{(q;q)_n} z^n \\
&= \left[\left\{{}_\ell\Delta_c^{\theta_q}\right\}\left\{q^{b-1}\theta_q - q^{b-1}+1\right\}\right]\sum_{n=1}^{\infty}A_n\ \frac{z^n}{(q;q)_{n-1}} \\
&= \left\{{}_\ell\Delta_c^{\theta_q}\right\}\sum_{n=1}^{\infty}\frac{A_n}{(q;q)_{n-1}}\left(q^{b-1}(z^n - z^n q^n) - z^n(q^{b-1}-1)\right) \\
&= \left\{{}_\ell\Delta_c^{\theta_q}\right\}\sum_{n=1}^{\infty}\frac{A_n}{(q;q)_{n-1}}\ (1-q^{b+n-1})\ z^n \\
&= \sum_{n=1}^{\infty}\frac{A_n}{(q;q)_{n-1}}\ (1-q^{b+n-1})\ (c;q)_{n-1}^{\ell}\ \left(q^{c-1}\theta_q - q^{c-1}+1\right)^{\ell n}\ z^n.
\end{aligned}$$

Now observing that for n=1,

$$\begin{aligned}
&\left(q^{c-1}\theta_q - q^{c-1}+1\right)^{\ell}\ z \\
&\quad = \left(q^{c-1}\theta_q - q^{c-1}+1\right)^{\ell-1}\ (q^{c-1}(z-zq)+(1-q^{c-1})z) \\
&\quad = \left(q^{c-1}\theta_q - q^{c-1}+1\right)^{\ell-1}\ (1-q^{c+1-1})\ z \\
&\quad = \cdots = (1-q^{c+1-1})^{\ell}\ z, \\
&\left(q^{c-1}\theta_q - q^{c-1}+1\right)^{2\ell}\ z^2 \\
&\quad = \left(q^{c-1}\theta_q - q^{c-1}+1\right)^{2\ell-1}\ (q^{c-1}(z^2 - z^2q^2)+z^2(1-q^{c-1}))
\end{aligned}$$

$$\begin{aligned} &= \left(q^{c-1}\theta_q - q^{c-1}+1\right)^{2\ell-1}\ (1-q^{2+c-1})\ z^2 \\ &= \cdots = (1-q^{2+c-1})^{2\ell} z^2, \end{aligned}$$

and in general for $n \in \mathbb{N}$,

$$\left(q^{c-1}\theta_q - q^{c-1}+1\right)^{\ell n}\ z^n = (1-q^{n+c-1})^{\ell n}\ z^n, \tag{3.10}$$

one further gets

$$\begin{aligned} {}_b\Lambda^{\theta_q}_{(c,\ell)} w &= \sum_{n=1}^{\infty} \frac{A_n\ (c;q)^{\ell}_{n-1}}{(q;q)_{n-1}}\ (1-q^{b+n-1})(1-q^{n+c-1})^{\ell n}\ z^n \\ &= \sum_{n=1}^{\infty} \frac{(a;q)_n\ (-1)^{n+1} q^{(n-1)(n-2)/2}\ z^n}{(b;q)_{n-1}\ (c;q)^{\ell n-\ell}_{n-1}\ (q;q)_{n-1}} \quad (3.11) \\ &= \sum_{n=0}^{\infty} \xi_n\ (1-aq^n)\ z^{n+1} \quad (3.12) \\ &= \sum_{n=0}^{\infty} f_{n,q}(a,b,(c:\ell);z)\ \text{(say)}. \end{aligned}$$

To complete the proof of lemma, it suffices to show that the series

$$\sum_{n=0}^{\infty} \xi_n\ f_{n,q}(a,b,(c:\ell);z) = \sum_{n=0}^{\infty} \frac{(a;q)^2_n\ (1-aq^n)\ (-1)^n q^{n(n-1)/2}\ z^{n+1}}{(b;q)^2_n\ (c;q)^{2\ell n}_n\ (q;q)^2_n}$$

is convergent. For that, take $\xi_n\ f_{n,q}(a,b,(c:\ell);z) = \mu_n$ then

$$\begin{aligned} |\mu_n| &= |\xi_n\ f_{n,q}(a,b,(c:\ell);z)| \\ &= \left| \frac{(a;q)^2_n\ (1-aq^n)\ (-1)^n q^{n(n-1)/2}\ z^{n+1}}{(b;q)^2_n\ (c;q)^{2\ell n}_n\ (q;q)^2_n} \right|. \end{aligned}$$

Hence in view of (3.3),

$$\begin{aligned} |\mu_n| &= \left| \left(\frac{\Gamma_q(a+n)}{\Gamma_q(a)\ (1-q)^n}\right)^2 \left(\frac{\Gamma_q(b)\ (1-q)^n}{\Gamma_q(b+n)}\right)^2 \left(\frac{\Gamma_q(c)\ (1-q)^{-n}}{\Gamma_q(c+n)}\right)^{2\ell n} \right| \\ &\quad \times \left| \left(\frac{q^{n(n-1)/2}\ (1-q)^{-1}}{\Gamma_q(n+1)}\right)^2\ (1-aq^n)\ z^{n+1} \right| \\ &\sim \left|\frac{\Gamma_q(b)}{\Gamma_q(a)}\right|^2\ |\Gamma^{2\ell n}_q(c)\ z^{n+1}|\ \left| \frac{(1+q)^{\frac{1}{2}}\ \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-a-n}\ e^{\theta \frac{q^{a+n}}{1-q-q^{a+n}}}}{(1+q)^{\frac{1}{2}}\ \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-b-n}\ e^{\theta \frac{q^{b+n}}{1-q-q^{b+n}}}} \right|^2 \end{aligned}$$

$$\times \frac{q^{n(n-1)}\ (1-q)^{-2\ell n^2}}{\left|(1+q)^{\frac{1}{2}}\ \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-c-n}\ e^{\theta\frac{q^{c+n}}{1-q-q^{c+n}}}\right|^{2n\ell}}$$
$$\times \frac{|(1-aq^n)|\ (1-q)^{-2}}{\left|(1+q)^{\frac{1}{2}}\ \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-n-1}\ e^{\theta\frac{q^{n+1}}{1-q-q^{n+1}}}\right|^{2}}.$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty}|\mu_n|^{\frac{1}{n}} &\sim \lim_{n\to\infty}\frac{|\Gamma_q^{2\ell}(c)|\ |z|\ |(1-aq^n)|^{\frac{1}{n}}}{|(1+q)^{\ell}|\ \left|\Gamma_{q^2}^{2\ell}\left(\frac{1}{2}\right)\right|\ |(1-q)^{\ell-2\ell c}|}\ q^{n-1} \\ &= 0\end{aligned}$$

whenever $\ell \in \mathbb{N}\cup\{0\}$ as $0<q<1$. □

Proof. (of Theorem 3.8) From (3.12),

$$\begin{aligned}{}_b\Lambda_{(c,\ell)}^{\theta_q} w &= \sum_{n=0}^{\infty} \xi_n\ (1-aq^n)\ z^{n+1} \\ &= z\sum_{n=0}^{\infty} \xi_n\ (q^a z^n - q^a z^n q^n - q^a z^n + z^n) \\ &= z\sum_{n=0}^{\infty} \xi_n\ (q^a\theta_q - q^a + 1)\ z^n \\ &= z\ (q^a\theta_q - q^a + 1)\ w.\end{aligned}$$

Thus the equation (3.8) follows. □

3.2.3 Eigen function property

In order to obtain the q-ℓ-Ψ function ${}_1\Psi_1^1(\ell : z)$ as an eigen function, yet another operator is required to be defined. This is given as

Definition 3.10. Let $f(z) = \sum_{n=1}^{\infty} a_{n,q}\ z^n,\ |z| < R, z \neq 0, R > 0$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Define

$$I_q^{\alpha} f(z) = \frac{z^{-\alpha}}{1-q}\ I_q\left(z^{\alpha-1} f(z)\right), \tag{3.13}$$

where the q-integral of the function is given by [23, Eq.(1.11.1), p.19]

$$I_q f(z) = \int_0^z f(t) d_q t = z\ (1-q) \sum_{n=0}^{\infty} f(zq^n)\ q^n. \tag{3.14}$$

Definition 3.11. Let $f(z) = \sum_{n=0}^{\infty} a_{n,q}\ z^n,\ |z| < R, z \neq 0, R > 0$. Define

$$_{\alpha}\mathcal{E}_{\beta}^{(\gamma:p)} f(z) = \left[I_q^{\alpha}\ z^{-1}\ {}_{\beta}\Lambda_{(\gamma,p)}^{\theta_q} \right] f(z), \tag{3.15}$$

where ${}_{\beta}\Lambda_{(\gamma,p)}^{\theta_q}$ and I_q^{α} are as defined in (3.7) and (3.13) respectively.

The following theorem establishes the eigen function property with respect to the operator (3.15).

Theorem 3.12. *If $\ell \in \mathbb{N} \cup \{0\}$ then the q-ℓ-Ψ function ${}_1\Psi_1^1(\ell : z)$is an eigen function with respect to the operator ${}_a\mathcal{E}_b^{(c:\ell)}$ defined in (3.15). That is for $\lambda \in \mathbb{C}$,*

$$_{a}\mathcal{E}_{b}^{(c:\ell)} \left({}_1\Psi_1^1 \left[\begin{matrix} a; & & q; & \lambda z \\ b; & (c:\ell); & & \end{matrix} \right] \right) = \lambda\ {}_1\Psi_1^1 \left[\begin{matrix} a; & & q; & \lambda z \\ b; & (c:\ell); & & \end{matrix} \right]. \tag{3.16}$$

Proof. The applicability of this operator to the q-ℓ-Ψ function ${}_1\Psi_1^1(\ell : z)$ follows from Lemma 3.9.

Now for $z \neq 0$,

$$_{a}\mathcal{E}_{b}^{(c:\ell)} \left({}_1\Psi_1^1 \left[\begin{matrix} a; & & q; & \lambda z \\ b; & (c:\ell); & & \end{matrix} \right] \right) = \left[I_q^a\ z^{-1}\ {}_b\Lambda_{(c,\ell)}^{\theta_q} \left(\sum_{n=0}^{\infty} \lambda^n\ \xi_n\ z^n \right) \right].$$

Putting

$$\frac{\lambda^n\ (a;q)_n\ (-1)^{n+1}\ q^{(n-1)(n-2)/2}}{(b;q)_{n-1}\ (c;q)_{n-1}^{\ell n-\ell}\ (q;q)_{n-1}} = B_n$$

and using (3.11), one gets

$$\begin{aligned} &{}_{a}\mathcal{E}_{b}^{(c:\ell)} \left({}_1\Psi_1^1 \left[\begin{matrix} a; & & q; & \lambda z \\ b; & (c:\ell); & & \end{matrix} \right] \right) \\ = &\ I_q^a\ z^{-1} \left[\sum_{n=1}^{\infty} \frac{\lambda^n\ (a;q)_n\ (-1)^{n+1}\ q^{(n-1)(n-2)/2}}{(b;q)_{n-1}\ (c;q)_{n-1}^{\ell n-\ell}\ (q;q)_{n-1}}\ z^n \right] \\ = &\ I_q^a\ z^{-1} \left[\sum_{n=1}^{\infty} B_n\ z^n \right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{z^{-a}}{1-q} \; I_q \left[\sum_{n=1}^{\infty} B_n \; z^{a+n-2} \right] \\
&= \frac{z^{-a}}{1-q} \sum_{n=1}^{\infty} B_n \; z(1-q) \; \sum_{k=0}^{\infty} (zq^k)^{a+n-2} \; q^k \\
&= \frac{z^{-a}}{1-q} \sum_{n=1}^{\infty} B_n \; z(1-q) \; z^{a+n-2} \; \sum_{k=0}^{\infty} q^{k(a+n-1)} \\
&= \sum_{n=1}^{\infty} B_n \; \frac{z^{n-1}}{1-q^{a+n-1}} \\
&= \lambda \sum_{n=0}^{\infty} \frac{\lambda^n \; (a;q)_n \; (-1)^n \; q^{n(n-1)/2}}{(b;q)_n \; (c;q)_n^{\ell n} \; (q;q)_n} \; z^n \\
&= \lambda \; {}_1\Psi_1^1 \left[\begin{matrix} a; & & q; & \lambda z \\ b; & (c:\ell); & & \end{matrix} \right].
\end{aligned}
$$

□

3.2.4 Contiguous function relations

The contiguous function relations for the basic hypergeometric series have been derived by Swarttouw [52]. Here the attempt made in this direction led to the following two identities:

$$
(bq^{-1}-a)\Psi \;=\; bq^{-1}(1-a)\Psi(a+) - a(1-bq^{-1})\Psi(b-), \tag{3.17}
$$

$$
(1-b)\Psi \;=\; (1-a)\Psi(a+,b+) - (b-a)\Psi(b+,zq) \tag{3.18}
$$

in which

$$
\Psi \;=\; {}_1\Psi_1^1 \left[\begin{matrix} a; & & q; & z \\ b; & (c:\ell); & & \end{matrix} \right],
$$

and

$$
\Psi(a+) := {}_1\Psi_1^1 \left[\begin{matrix} aq; & & q; & z \\ b; & (c:\ell); & & \end{matrix} \right], \quad \Psi(a-) := {}_1\Psi_1^1 \left[\begin{matrix} aq^{-1}; & & q; & z \\ b; & (c:\ell); & & \end{matrix} \right],
$$

$$
\Psi(a+,b+) := {}_1\Psi_1^1 \left[\begin{matrix} aq; & & q; & z \\ bq; & (c:\ell); & & \end{matrix} \right], \Psi(b+,zq) := {}_1\Psi_1^1 \left[\begin{matrix} a; & & q; & zq \\ bq; & (c:\ell); & & \end{matrix} \right].
$$

The functions $\Psi(b+), \Psi(b-), \Psi(c+), \Psi(c-)$ etc. are defined similarly.
Now with

$$\xi_n \quad = \quad \frac{(a;q)_n\ (-1)^n\ q^{n(n-1)/2}}{(b;q)_n\ (c;q)_n^{\ell n}\ (q;q)_n}$$

as before, the function

$$\Psi \quad = \quad \sum_{n=0}^{\infty} \xi_n\ z^n$$

whence the following series representations follow.

$$\begin{aligned}\Psi(a+) &= \sum_{n=0}^{\infty} \tfrac{1-aq^n}{1-a}\ \xi_n\ z^n, &\quad \Psi(a-) &= \sum_{n=0}^{\infty} \tfrac{1-aq^{-1}}{1-aq^{n-1}}\ \xi_n\ z^n,\\ \Psi(b+) &= \sum_{n=0}^{\infty} \tfrac{1-b}{1-bq^n}\ \xi_n\ z^n, &\quad \Psi(b-) &= \sum_{n=0}^{\infty} \tfrac{1-bq^{n-1}}{1-bq^{-1}}\ \xi_n\ z^n,\\ \Psi(c+) &= \sum_{n=0}^{\infty} \tfrac{(1-c)^{\ell n}}{(1-cq^n)^{\ell n}}\ \xi_n\ z^n, &\quad \Psi(c-) &= \sum_{n=0}^{\infty} \tfrac{(1-cq^{n-1})^{\ell n}}{(1-cq^{-1})^{\ell n}}\ \xi_n\ z^n.\end{aligned} \tag{3.19}$$

If $\Theta_q = zD_q$, where $D_q = \frac{f(z)-f(zq)}{z(1-q)}$ then,

$$\Theta_q\Psi \quad = \quad \sum_{n=0}^{\infty} \frac{(1-q^n)}{(1-q)}\xi_n\ z^n, \tag{3.20}$$

and consequently,

$$\begin{aligned}\left(a\Theta_q + \frac{1-a}{1-q}\right)\Psi \quad &= \quad a\sum_{n=0}^{\infty} \frac{(1-q^n)}{(1-q)}\xi_n z^n + \frac{1-a}{1-q}\sum_{n=0}^{\infty}\xi_n z^n\\ &= \quad \sum_{n=0}^{\infty} \frac{\xi_n}{1-q}\ [a - aq^n + 1 - a]\ z^n\\ &= \quad \frac{1-a}{1-q}\ \Psi(a+).\end{aligned}$$

This implies that

$$\left(abq^{-1}\Theta_q + bq^{-1}\frac{1-a}{1-q}\right)\Psi \quad = \quad bq^{-1}\frac{1-a}{1-q}\Psi(a+). \tag{3.21}$$

Also,

$$\left(bq^{-1}\Theta_q + \frac{1-bq^{-1}}{1-q}\right)\Psi \quad = \quad bq^{-1}\sum_{n=0}^{\infty} \frac{(1-q^n)}{(1-q)}\xi_n z^n + \frac{1-bq^{-1}}{1-q}\sum_{n=0}^{\infty}\xi_n z^n$$

$$= \sum_{n=0}^{\infty} \frac{\xi_n z^n}{1-q}[bq^{-1} - bq^{n-1} + 1 - bq^{-1}]$$
$$= \frac{1-bq^{-1}}{1-q}\Psi(b-).$$

Hence

$$\left(abq^{-1}\Theta_q + a\frac{1-bq^{-1}}{1-q}\right)\Psi = a\frac{1-bq^{-1}}{1-q}\Psi(b-). \tag{3.22}$$

Elimination of Θ_q from (3.21) and (3.22) yields

$$(bq^{-1}-a)\Psi = bq^{-1}(1-a)\Psi(a+) - a(1-bq^{-1})\Psi(b-)$$

which is (3.17).
Next, using (3.9),

$$\begin{aligned}
\{\,{}_\ell\Delta_c^{\theta_q}\}\theta_q\,(-q)\ \Psi(z/q) &= \left\{{}_\ell\Delta_c^{\theta_q}\right\}\theta_q\left(\sum_{n=0}^{\infty} A_n\,\frac{z^n}{(q;q)_n}\right)\\
&= \left\{{}_\ell\Delta_c^{\theta_q}\right\}\left(\sum_{n=1}^{\infty} A_n\,\frac{1-q^n}{(q;q)_n}\,z^n\right)\\
&= \left\{{}_\ell\Delta_c^{\theta_q}\right\}\left(\sum_{n=1}^{\infty} A_n\,\frac{z^n}{(q;q)_{n-1}}\right)\\
&= \sum_{n=1}^{\infty}\frac{A_n}{(q;q)_{n-1}}(c;q)_{n-1}^{\ell}\left(q^{c-1}\theta_q - q^{c-1}+1\right)^{\ell n} z^n.
\end{aligned}$$

This in view of (3.10), simplifies to

$$\begin{aligned}
\{\,{}_\ell\Delta_c^{\theta_q}\}\theta_q\,(-q)\ \Psi(z/q) &= \sum_{n=1}^{\infty}\frac{A_n\,(c;q)_{n-1}^{\ell}}{(q;q)_{n-1}}(1-q^{n+c-1})^{\ell n} z^n.\\
&= \sum_{n=1}^{\infty}\frac{(a;q)_n\,(-1)^{n+1}q^{(n-1)(n-2)/2}\ z^n}{(b;q)_n\,(c;q)_{n-1}^{\ell n-\ell}\,(q;q)_{n-1}}\\
&= \sum_{n=0}^{\infty}\frac{(a;q)_{n+1}\,(-1)^n q^{n(n-1)/2}\ z^{n+1}}{(b;q)_{n+1}\,(c;q)_n^{\ell n}\,(q;q)_n}\\
&= z\sum_{n=0}^{\infty}\frac{1-aq^n}{1-bq^n}\,\xi_n\,z^n \qquad (3.23)\\
&= z\,\frac{1-a}{1-b}\,\Psi(a+,b+). \qquad (3.24)
\end{aligned}$$

Next in (3.23) putting

$$\frac{1-aq^n}{1-bq^n} = 1+\frac{bq^n-aq^n}{1-bq^n},$$

one finds that

$$\begin{aligned}\{\,_\ell\Delta_c^{\theta_q}\}\theta_q\,(-q)\;\Psi(z/q) &= z\sum_{n=0}^{\infty}\left(1+\frac{bq^n-aq^n}{1-bq^n}\right)\xi_n z^n \\ &= z\left[\sum_{n=0}^{\infty}\xi_n z^n+\sum_{n=0}^{\infty}\frac{b-a}{1-b}\frac{1-b}{1-bq^n}\;\xi_n\;(zq)^n\right] \\ &= z\left[\Psi+\frac{b-a}{1-b}\Psi(b,zq)\right]. \qquad (3.25)\end{aligned}$$

The identity (3.18) thus follows from (3.24) and (3.25), that is,

$$(1-b)\Psi=(1-a)\Psi(a+,b+)-(b-a)\Psi(b+,zq).$$

3.3 The q-ℓ-Ψ function ${}_2\Psi_1^2(\ell : z)$

In this section, the q-ℓ-Ψ function ${}_2\Psi_1^2(\ell : z)$ is considered by adding one numerator and one ℓ-denominator parameter and dropping the factor $(-1)^n\;q^{\binom{n}{2}}$. This proposed function is an extension of the basic hypergeometric function ${}_2\phi_1[z]$ which is valid for $|z|<1$.
This extension is defined as follows.

Definition 3.13. For $\Re(\ell)\geq 0, b, c_1, c_2 \in \mathbb{C}/\{0,-1,-2,\ldots\}$ and $a_1, a_2, z \in \mathbb{C}$, define the function

$${}_2\Psi_1^2\left[\begin{array}{lll} a_1, a_2; & & q;\; z \\ \quad b; & (c_1, c_2 : \ell); & \end{array}\right] = \sum_{n=0}^{\infty}\frac{(a_1;q)_n\;(a_2;q)_n}{(b;q)_n\;(c_1;q)_n^{\ell n}\;(c_2;q)_n^{\ell n}}\frac{z^n}{(q;q)_n}. \qquad (3.26)$$

This function will be abbreviated as the q-ℓ-Ψ function.
The properties namely series convergence, a difference equation, eigen function property and contiguous function relations will be derived in the subsequent subsections.

3.3.1 Convergence

Theorem 3.14. *If* $0 < q < 1,\ \ \Re(\ell) > 0$ *then the* q-ℓ-Ψ *function* ${}_2\Psi_1^2(\ell : z)$ *is an analytic function of* z.

Proof. Put

$$\frac{(a_1;q)_n\ (a_2;q)_n}{(b;q)_n\ (c_1;q)_n^{\ell n}\ (c_2;q)_n^{\ell n}\ (q;q)_n} = \zeta_n$$

then in view of the formula

$$(a;q)_n = \frac{\Gamma_q(a+n)}{\Gamma_q(a)}\ (1-q)^n,$$

one gets

$$\begin{aligned} |\zeta_n|^{\frac{1}{n}} &= \left|\frac{(a_1;q)_n\ (a_2;q)_n}{(b;q)_n\ (c_1;q)_n^{\ell n}\ (c_2;q)_n^{\ell n}\ (q;q)_n}\right|^{\frac{1}{n}} \\ &= \left|\frac{\Gamma_q(b)}{\Gamma_q(a_1)\ \Gamma_q(a_2)}\right|^{\frac{1}{n}} \left|\frac{\Gamma_q(a_1+n)\ \Gamma_q(a_2+n)}{\Gamma_q(b+n)\ \Gamma_q(n+1)}\right|^{\frac{1}{n}} \\ &\quad \times \left|\frac{\Gamma_q(c_1)\ \Gamma_q(c_2)}{\Gamma_q(c_1+n)\ \Gamma_q(c_2+n)\ (1-q)^{2n}}\right|^{\Re(\ell)}. \end{aligned}$$

Here using Stirling's formula of q-Gamma function [41, Eq.(2.25), p.482]:

$$\Gamma_q(z) \sim (1+q)^{\frac{1}{2}}\ \Gamma_{q^2}\left(\frac{1}{2}\right) (1-q)^{\frac{1}{2}-z}\ e^{\frac{\theta q^z}{1-q-q^z}}, \tag{3.27}$$

for large $|z|$ and $0 < \theta < 1$, this further gives

$$\begin{aligned} |\zeta_n|^{\frac{1}{n}} \sim\ & \left|\frac{\Gamma_q(b)}{\Gamma_q(a_1)\ \Gamma_q(a_2)}\right|^{\frac{1}{n}} \left|\frac{(1+q)^{\frac{1}{2}}\ \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-(a_1+n)}\ e^{\frac{\theta\ q^{a_1+n}}{1-q-q^{a_1+n}}}}{(1+q)^{\frac{1}{2}}\ \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-(b+n)}\ e^{\frac{\theta\ q^{b+n}}{1-q-q^{b+n}}}}\right|^{\frac{1}{n}} \\ & \times \left|\frac{(1+q)^{\frac{1}{2}}\ \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-(a_2+n)}\ e^{\frac{\theta\ q^{a_2+n}}{1-q-q^{a_2+n}}}}{(1+q)^{\frac{1}{2}}\ \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-(n+1)}\ e^{\frac{\theta\ q^{n+1}}{1-q-q^{n+1}}}}\right|^{\frac{1}{n}} \\ & \times \frac{\left|\Gamma_q^{\ell}(c_1)\ (1-q)^{-\ell n}\right|}{\left|(1+q)^{\frac{1}{2}}\ \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-(c_1+n)}\ e^{\frac{\theta\ q^{c_1+n}}{1-q-q^{c_1+n}}}\right|^{\Re(\ell)}} \\ & \times \frac{\left|\Gamma_q^{\ell}(c_2)\ (1-q)^{-\ell n}\right|}{\left|(1+q)^{\frac{1}{2}}\ \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-(c_2+n)}\ e^{\frac{\theta\ q^{c_2+n}}{1-q-q^{c_2+n}}}\right|^{\Re(\ell)}} \end{aligned}$$

for large n. Now from the Cauchy-Hadamard formula:

$$\begin{aligned}\frac{1}{R} &= \limsup_{n\to\infty} \sqrt[n]{|\zeta_n|} \\ &= \left|\frac{\Gamma_q(c_1)\ \Gamma_q(c_2)\ (1-q)^{c_1+c_2}}{\Gamma^2_{q^2}\left(\frac{1}{2}\right)\ (1-q^2)}\right|^{\Re(\ell)},\end{aligned}$$

hence it follows that the radius of convergence

$$R = \left|\frac{\Gamma^2_{q^2}\left(\frac{1}{2}\right)\ (1-q^2)}{\Gamma_q(c_1)\ \Gamma_q(c_2)\ (1-q)^{c_1+c_2}}\right|^{\Re(\ell)}. \tag{3.28}$$

□

Remark 3.15. From (3.28), it is evident that if $\ell = 0$ then the q-ℓ-Ψ function ${}_2\Psi^2_1(\ell : z)$ reduces to the basic hypergeometric function ${}_2\phi_1(a_1, a_2; b; q, z)$ whose series has radius of convergence unity.

3.3.2 q-Difference Equation

The difference equation of the q-ℓ-Ψ function ${}_2\Psi^2_1[z]$ function occurs for $\ell \in \mathbb{N}\cup\{0\}$, which is obtained by means of already defined operator (3.5). This is given as

Theorem 3.16. *For $\ell \in \mathbb{N}\cup\{0\}, a_1, a_2, z \in \mathbb{C}$, and $b, c_1, c_2 \in \mathbb{C}/\{0, -1, -2, \ldots\}$, the function $w = {}_2\Psi^2_1\left[\begin{matrix} a_1, a_2; & & q; & z \\ b; & (c_1, c_2 : \ell); & & \end{matrix}\right]$ satisfies the difference equation*

$$\left[\ {}_\ell\Delta^{\theta_q}_{c_2}\ {}_\ell\Delta^{\theta_q}_{c_1} \left\{q^{b-1}\theta_q - q^{b-1} + 1\right\} \theta_q - z(q^{a_1}\theta_q - q^{a_1} + 1)(q^{a_2}\theta_q - q^{a_2} + 1)\right] w = 0, \tag{3.29}$$

where the operator ${}_\ell\Delta^{\theta_q}_{c_i}$, $i = 1, 2$ is as defined in (3.5).

The proof needs the following lemma which actually permits to apply the operator $\left\{{}_\ell\Delta^{\theta_q}_{c_2}\ {}_\ell\Delta^{\theta_q}_{c_1}\right\}\left\{q^{b-1}\theta_q - q^{b-1} + 1\right\}\theta_q$ on the operand w.
For the sake of brevity, put

$$\left\{{}_\ell\Delta^{\theta_q}_{c_2}\ {}_\ell\Delta^{\theta_q}_{c_1}\right\}\left\{q^{b-1}\theta_q - q^{b-1} + 1\right\}\theta_q = {}_b\Lambda^{\theta_q}_{(c_1,c_2:\ell)} \tag{3.30}$$

and as mentioned earlier, $\alpha \equiv q^\alpha$. In this notation, there holds the

Lemma 3.17. *If $\ell \in \mathbb{N} \cup \{0\}, a_1, a_2, z \in \mathbb{C},\ b, c_1, c_2 \in \mathbb{C}/\{0, -1, -2, \ldots\}$,*

$$w = {}_2\Psi_1^2\left[\begin{array}{ccc} a_1, a_2; & & q; \quad z \\ b; & (c_1, c_2 : \ell); & \end{array}\right] = \sum_{n=0}^{\infty} \zeta_n\ z^n$$

and ${}_b\Lambda^{\theta_q}_{(c_1,c_2:\ell)} w = \sum_{n=0}^{\infty} f_{n,q}(a_1, a_2; b, (c_1, c_2 : \ell); z)$ *then the operator* ${}_b\Lambda^{\theta_q}_{(c_1,c_2:\ell)}$ *is applicable to the q-ℓ-Ψ function* ${}_2\Psi_1^2(\ell : z)$ *provided that the series*

$$\sum_{n=0}^{\infty} \zeta_n\ f_{n,q}(a_1, a_2; b, (c_1, c_2 : \ell); z)$$

converges (cf. [49, Definition 11, p.20]).

Proof. Indeed

$$\begin{aligned}
&{}_b\Lambda^{\theta_q}_{(c_1,c_2:\ell)} w \\
&= \left[\left\{{}_\ell\Delta^{\theta_q}_{c_2}\ {}_\ell\Delta^{\theta_q}_{c_1}\right\}\left\{q^{b-1}\theta_q - q^{b-1} + 1\right\}\ \theta_q\right]\left(\sum_{n=0}^{\infty} \frac{(a_1;q)_n\ (a_2;q)_n}{(b;q)_n\ (c_1;q)_n^{\ell n}\ (c_2;q)_n^{\ell n}} \frac{z^n}{(q;q)_n}\right) \\
&= \left[\left\{{}_\ell\Delta^{\theta_q}_{c_2}\ {}_\ell\Delta^{\theta_q}_{c_1}\right\}\left\{q^{b-1}\theta_q - q^{b-1} + 1\right\}\right]\left(\sum_{n=1}^{\infty} \frac{(a_1;q)_n\ (a_2;q)_n}{(b;q)_n\ (c_1;q)_n^{\ell n}\ (c_2;q)_n^{\ell n}} \frac{z^n - z^n q^n}{(q;q)_{n-1}}\right) \\
&= \left\{{}_\ell\Delta^{\theta_q}_{c_2}\ {}_\ell\Delta^{\theta_q}_{c_1}\right\}\left(\sum_{n=1}^{\infty} \frac{(a_1;q)_n\ (a_2;q)_n}{(b;q)_n\ (c_1;q)_n^{\ell n}\ (c_2;q)_n^{\ell n}\ (q;q)_{n-1}}\right. \\
&\qquad \left.\times\left(q^{b-1}(z^n - z^n q^n) - z^n(q^{b-1} - 1)\right)\right) \\
&= \left\{{}_\ell\Delta^{\theta_q}_{c_2}\ {}_\ell\Delta^{\theta_q}_{c_1}\right\}\left(\sum_{n=1}^{\infty} \frac{(a_1;q)_n\ (a_2;q)_n}{(b;q)_n\ (c_1;q)_n^{\ell n}\ (c_2;q)_n^{\ell n}\ (q;q)_{n-1}} z^n\ (1 - q^{b+n-1})\right) \\
&= \left\{{}_\ell\Delta^{\theta_q}_{c_2}\ {}_\ell\Delta^{\theta_q}_{c_1}\right\}\left(\sum_{n=1}^{\infty} \frac{(a_1;q)_n\ (a_2;q)_n}{(b;q)_{n-1}\ (c_1;q)_n^{\ell n}\ (c_2;q)_n^{\ell n}\ (q;q)_{n-1}} z^n\right) \\
&= {}_\ell\Delta^{\theta_q}_{c_2} \sum_{n=1}^{\infty} \frac{(a_1;q)_n\ (a_2;q)_n}{(b;q)_{n-1}\ (c_1;q)_n^{\ell n}\ (c_2;q)_n^{\ell n}\ (q;q)_{n-1}} \\
&\qquad \times (c_1;q)_{n-1}^{\ell}\ \left(q^{c_1-1}\theta_q - q^{c_1-1} + 1\right)^{\ell n} z^n.
\end{aligned}$$

Now from (3.10),

$${}_b\Lambda^{\theta_q}_{(c_1,c_2:\ell)} w = {}_\ell\Delta^{\theta_q}_{c_2} \sum_{n=1}^{\infty} \frac{(a_1;q)_n\ (a_2;q)_n\ (c_1;q)_{n-1}^{\ell}}{(b;q)_{n-1}\ (c_1;q)_n^{\ell n}\ (c_2;q)_n^{\ell n}\ (q;q)_{n-1}} (1 - q^{n+c_1-1})^{\ell n} z^n$$

$$= \ {}_{\ell}\Delta_{c_2}^{\theta_q} \sum_{n=1}^{\infty} \frac{(a_1;q)_n \ (a_2;q)_n \ z^n}{(b;q)_{n-1} \ (c_1;q)_{n-1}^{\ell n-\ell} \ (c_2;q)_n^{\ell n} \ (q;q)_{n-1}}.$$

Further applying ${}_{\ell}\Delta_{c_2}^{\theta_q}$ operator, one finds

$$ {}_{b}\Lambda_{(c_1,c_2:\ell)}^{\theta_q} w = \sum_{n=1}^{\infty} \frac{(a_1;q)_n \ (a_2;q)_n \ z^n}{(b;q)_{n-1} \ (c_1;q)_{n-1}^{\ell n-\ell} \ (c_2;q)_{n-1}^{\ell n-\ell} \ (q;q)_{n-1}} \tag{3.31}$$

$$= \sum_{n=0}^{\infty} \frac{(a_1;q)_{n+1} \ (a_2;q)_{n+1} \ z^{n+1}}{(b;q)_n \ (c_1;q)_n^{\ell n} \ (c_2;q)_n^{\ell n} \ (q;q)_n} \tag{3.32}$$

$$= \sum_{n=0}^{\infty} f_{n,q}(a_1,a_2;b,(c_1,c_2:\ell);z).$$

To complete the proof of lemma, it remains to show that

$$\sum_{n=0}^{\infty} \zeta_n \ f_{n,q}(a_1,a_2;b,(c_1,c_2:\ell);z) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n^2 \ (a_2;q)_n^2 (1-a_1q^n)(1-a_2q^n) z^{n+1}}{(b;q)_n^2 \ (c_1;q)_n^{2\ell n} \ (c_2;q)_n^{2\ell n} \ (q;q)_n^2}$$

is convergent. For that, take

$$\begin{aligned} |\mu_n| &= |\zeta_n \ f_{n,q}(a_1,a_2;b,(c_1,c_2:\ell);z)| \\ &= \left| \frac{(a_1;q)_n^2 \ (a_2;q)_n^2 \ (1-a_1q^n) \ (1-a_2q^n) \ z^{n+1}}{(b;q)_n^2 \ (c_1;q)_n^{2\ell n} \ (c_2;q)_n^{2\ell n} \ (q;q)_n^2} \right| \\ &= \left| \frac{\Gamma_q(a_1+n) \ \Gamma_q(a_2+n) \ \Gamma_q(b)}{\Gamma_q(a_1) \ \Gamma_q(a_2) \ \Gamma_q(b+n) \ \Gamma_q(n+1)} \right|^2 \ |(1-a_1q^n) \ (1-a_2q^n)| \\ &\quad \times \left| \frac{\Gamma_q(c_1) \ \Gamma_q(c_2)}{\Gamma_q(c_1+n) \ \Gamma_q(c_2+n) \ (1-q)^{2n}} \right|^{2n\Re(\ell)} \ |z|^{n+1}. \end{aligned}$$

Once again applying the Stirling's formula of q-Gamma function:

$$\Gamma_q(\alpha+n) \sim (1+q)^{\frac{1}{2}} \ \Gamma_{q^2}\left(\frac{1}{2}\right) (1-q)^{\frac{1}{2}-\alpha-n} \ e^{\theta \frac{q^{\alpha+n}}{1-q-q^{\alpha+n}}}, \qquad (0<\theta<1)$$

with α replaced by a_1, a_2, b, c_1 and c_2 in turn, and proceeding as in Theorem 3.14, one obtains

$$\lim_{n\to\infty} \sup |\mu_n|^{\frac{1}{n}} = \left| \frac{\Gamma_q^2(c_1) \ \Gamma_q^2(c_2) \ (1-q)^{2(c_1+c_2)}}{(1-q^2) \ \Gamma_{q^2}^2\left(\frac{1}{2}\right)} \right|^{\ell} \ |z|.$$

Hence the lemma follows under the condition

$$|z| < \left| \frac{(1-q^2) \ \Gamma_{q^2}^2\left(\frac{1}{2}\right)}{\Gamma_q^2(c_1) \ \Gamma_q^2(c_2) \ (1-q)^{2(c_1+c_2)}} \right|^{\ell}.$$

□

The theorem can now be proved.

Proof. (of Theorem 3.8) From (3.32),

$$\begin{aligned}
&{}_b\Lambda^{\theta_q}_{(c_1,c_2\ell)} w \\
&= \sum_{n=0}^{\infty} \frac{(a_1;q)_{n+1}\ (a_2;q)_{n+1}\ z^{n+1}}{(b;q)_n\ (c_1;q)_n^{\ell n}\ (c_2;q)_n^{\ell n}\ (q;q)_n} \\
&= z \sum_{n=0}^{\infty} \frac{(a_1;q)_n (q^{a_1} z^n - q^{a_1} z^n q^n - q^{a_1} z^n + z^n)\ (a_2;q)_{n+1}}{(b;q)_n\ (c_1;q)_n^{\ell n}\ (c_2;q)_n^{\ell n}\ (q;q)_n} \\
&= z\ (q^{a_1}\theta_q - q^{a_1} + 1) \sum_{n=0}^{\infty} \frac{(a_1;q)_n\ (a_2;q)_{n+1}\ z^n}{(b;q)_n\ (c_1;q)_n^{\ell n}\ (c_2;q)_n^{\ell n}\ (q;q)_n} \\
&= z\ (q^{a_1}\theta_q - q^{a_1} + 1) \sum_{n=0}^{\infty} \frac{(a_1;q)_n\ (b;q)_n (q^{a_2} z^n - q^{a_2} z^n q^n - q^{a_2} z^n + z^n)}{(b;q)_n\ (c_1;q)_n^{\ell n}\ (c_2;q)_n^{\ell n}\ (q;q)_n} \\
&= z\ (q^{a_1}\theta_q - q^{a_1} + 1)\ (q^{a_2}\theta_q - q^{a_2} + 1)\ w.
\end{aligned}$$

Thus the equation (3.29) holds. □

3.3.3 Eigen function property

To derive the eigen function property of the q-ℓ-Ψ function ${}_2\Psi_1^2(\ell : z)$, the following operator needs to be constructed and defined.

Definition 3.18. Let $f(z) = \sum_{n=0}^{\infty} a_{n,q}\ z^n,\ 0 \neq |z| < R, R > 0,\ p \in \mathbb{N} \cup \{0\}$ and $\alpha, \beta \in \mathbb{C}$ with $\Re(\alpha,\beta) \geq 0$. Define

$$ {}^{\delta}_{\alpha}\mathcal{E}^{(\gamma_1,\gamma_2:p)}_{\beta} f(z) = \left[I_q^{\beta} I_q^{\alpha}\ z^{-1}\ {}_{\delta}\Lambda^{\theta_q}_{(\gamma_1,\gamma_2:p)} \right] f(z), \tag{3.33}$$

where ${}_{\delta}\Lambda^{\theta_q}_{(\gamma_1,\gamma_2:p)}$ and I_q^{α} are as defined in (3.30) and (3.13) respectively.

In these notations, there holds

Theorem 3.19. *If* $\ell \in \mathbb{N}\cup\{0\}$ *and* $\Re(a_1,a_2) \geq 0$ *then the* q-ℓ-Ψ *function* ${}_2\Psi_1^2(\ell : z)$ *is an eigen function with respect to the operator* ${}^{b}_{a_1}\mathcal{E}^{(c_1,c_2:\ell)}_{a_2}$ *defined in (3.33). That is, for* $\lambda \in \mathbb{C}$,

$$ {}^{b}_{a_1}\mathcal{E}^{(c_1,c_2:\ell)}_{a_2} \left({}_2\Psi_1^2 \left[\begin{matrix} a_1, a_2; & & q;\ \lambda z \\ & b;\ \ (c_1,c_2:\ell); & \end{matrix} \right] \right) = \lambda\ {}_2\Psi_1^2 \left[\begin{matrix} a_1, a_2; & & q;\ \lambda z \\ & b;\ \ (c_1,c_2:\ell); & \end{matrix} \right].$$

Proof. The applicability of this operator to the q-ℓ-Ψ function ${}_2\Psi_1^2(\ell : z)$ follows from Lemma 3.17.

Now for $z \neq 0$,

$$
\begin{aligned}
&{}^{\;b}_{a_1}\mathcal{E}^{(c_1,c_2:\ell)}_{a_2}\left({}_2\Psi_1^2\left[\begin{matrix} a_1, a_2; & & q; & \lambda z \\ b; & (c_1, c_2 : \ell); & & \end{matrix}\right]\right) \\
&= \left[I_q^{a_2} I_q^{a_1}\ z^{-1}\ {}_c\Lambda^{\theta_q}_{(c_1,c_2:\ell)}\left(\sum_{n=0}^{\infty}\frac{\lambda^n\ (a_1;q)_n\ (a_2;q)_n\ z^n}{(b;q)_n\ (c_1;q)_n^{\ell n}\ (c_2;q)_n^{\ell n}\ (q;q)_n}\right)\right].
\end{aligned}
$$

In view of (3.31),

$$
\begin{aligned}
&{}^{\;b}_{a_1}\mathcal{E}^{(c_1,c_2:\ell)}_{a_2}\left({}_2\Psi_1^2\left[\begin{matrix} a_1, a_2; & & q; & \lambda z \\ b; & (c_1, c_2 : \ell); & & \end{matrix}\right]\right) \\
&= I_q^{a_2}\ I_q^{a_1}\left[\sum_{n=1}^{\infty}\frac{\lambda^n\ (a_1;q)_n\ (a_2;q)_n}{(b;q)_{n-1}\ (c_1;q)_{n-1}^{\ell n-\ell}\ (c_2;q)_{n-1}^{\ell n-\ell}}\frac{z^{n-1}}{(q;q)_{n-1}}\right] \\
&= I_q^{a_2}\left\{\frac{z^{-a_1}}{1-q}\ I_q\left[\sum_{n=1}^{\infty}\frac{\lambda^n\ (a_1;q)_n\ (a_2;q)_n}{(b;q)_{n-1}\ (c_1;q)_{n-1}^{\ell n-\ell}\ (c_2;q)_{n-1}^{\ell n-\ell}}\frac{z^{a_1+n-2}}{(q;q)_{n-1}}\right]\right\} \\
&= I_q^{a_2}\left\{\frac{z^{-a_1}}{1-q}\ \sum_{n=1}^{\infty}\frac{\lambda^n\ (a_1;q)_n\ (a_2;q)_n}{(b;q)_{n-1}\ (c_1;q)_{n-1}^{\ell n-\ell}\ (c_2;q)_{n-1}^{\ell n-\ell}\ (q;q)_{n-1}}\right. \\
&\qquad \left.\times z(1-q)\ \sum_{k=0}^{\infty}(zq^k)^{a_1+n-2}\ q^k\right\} \\
&= I_q^{a_2}\left\{\frac{z^{-a_1}}{1-q}\ \sum_{n=1}^{\infty}\frac{\lambda^n\ (a_1;q)_n\ (a_2;q)_n}{(b;q)_{n-1}\ (c_1;q)_{n-1}^{\ell n-\ell}\ (c_2;q)_{n-1}^{\ell n-\ell}\ (q;q)_{n-1}}\right. \\
&\qquad \left.\times z(1-q)\ z^{a_1+n-2}\ \sum_{k=0}^{\infty}q^{k(a_1+n-1)}\right\} \\
&= I_q^{a_2}\left\{\sum_{n=1}^{\infty}\frac{\lambda^n\ (a_1;q)_n\ (a_2;q)_n}{(b;q)_{n-1}\ (c_1;q)_{n-1}^{\ell n-\ell}\ (c_2;q)_{n-1}^{\ell n-\ell}\ (q;q)_{n-1}}\frac{z^{n-1}}{1-q^{a_1+n-1}}\right\} \\
&= I_q^{a_2}\left\{\sum_{n=1}^{\infty}\frac{\lambda^n\ (a_1;q)_{n-1}\ (a_2;q)_n\ z^{n-1}}{(b;q)_{n-1}\ (c_1;q)_{n-1}^{\ell n-\ell}\ (c_2;q)_{n-1}^{\ell n-\ell}\ (q;q)_{n-1}}\right\} \\
&= \frac{z^{-a_2}}{1-q}\ I_q\left[\sum_{n=1}^{\infty}\frac{\lambda^n\ (a_1;q)_{n-1}\ (a_2;q)_n}{(b;q)_{n-1}\ (c_1;q)_{n-1}^{\ell n-\ell}\ (c_2;q)_{n-1}^{\ell n-\ell}}\frac{z^{a_2+n-2}}{(q;q)_{n-1}}\right] \\
&= \frac{z^{-a_2}}{1-q}\ \sum_{n=1}^{\infty}\frac{\lambda^n\ (a_1;q)_{n-1}\ (a_2;q)_n}{(b;q)_{n-1}\ (c_1;q)_{n-1}^{\ell n-\ell}\ (c_2;q)_{n-1}^{\ell n-\ell}\ (q;q)_{n-1}} \\
&\qquad \times z(1-q)\ \sum_{k=0}^{\infty}(zq^k)^{a_2+n-2}\ q^k
\end{aligned}
$$

$$= \frac{z^{-a_2}}{1-q} \sum_{n=1}^{\infty} \frac{\lambda^n\ (a_1;q)_{n-1}\ (a_2;q)_n}{(b;q)_{n-1}\ (c_1;q)_{n-1}^{\ell n-\ell}\ (c_2;q)_{n-1}^{\ell n-\ell}\ (q;q)_{n-1}}$$

$$\times z(1-q)\ z^{a_2+n-2} \sum_{k=0}^{\infty} q^{k(a_2+n-1)}$$

$$= \sum_{n=1}^{\infty} \frac{\lambda^n\ (a_1;q)_{n-1}\ (a_2;q)_n}{(b;q)_{n-1}\ (c_1;q)_{n-1}^{\ell n-\ell}\ (c_2;q)_{n-1}^{\ell n-\ell}\ (q;q)_{n-1}} \frac{z^{n-1}}{1-q^{a_2+n-1}}$$

$$= \sum_{n=1}^{\infty} \frac{\lambda^n\ (a_1;q)_{n-1}\ (a_2;q)_{n-1}\ z^{n-1}}{(b;q)_{n-1}\ (c_1;q)_{n-1}^{\ell n-\ell}\ (c_2;q)_{n-1}^{\ell n-\ell}\ (q;q)_{n-1}}$$

$$= \lambda \sum_{n=0}^{\infty} \frac{\lambda^n\ (a_1;q)_n\ (a_2;q)_n}{(b;q)_n\ (c_1;q)_n^{\ell n}\ (c_2;q)_n^{\ell n}} \frac{z^n}{(q;q)_n}.$$

□

3.3.4 Contiguous function relations

The contiguous function relations involving the q-ℓ-Ψ function ${}_2\Psi_1^2(\ell:z)$ derived here by proceeding in the manner exactly as in the case of the q-ℓ-Ψ function ${}_1\Psi_1^1(\ell:z)$ of Subsection 3.2.4. The following are the identities obtained as contiguous function relations for the q-ℓ-Ψ function ${}_2\Psi_1^2(\ell:z)$.

$$(bq^{-1}-a_1)\Psi = bq^{-1}(1-a_1)\Psi(a_1+) - a_1(1-bq^{-1})\Psi(b-), \quad (3.34)$$

$$(1-b)\ \Psi = (1-a_1)\ \Psi(a_1+,b+) - (b-a_1)\ \Psi(b+,zq), \quad (3.35)$$

$$(bq^{-1}-a_2)\Psi = bq^{-1}(1-a_2)\Psi(a_2+) - a_2(1-bq^{-1})\Psi(b-), \quad (3.36)$$

and

$$(1-b)\ \Psi = (1-a_2)\ \Psi(a_2+,b+) - (b-a_2)\ \Psi(b+,zq). \quad (3.37)$$

Here the notation used are:

$$\Psi = {}_2\Psi_1^2\left[\begin{matrix} a_1,a_2; & & q; & z \\ b; & (c_1,c_2:\ell); & & \end{matrix}\right],$$

$$\Psi(a_1+) := {}_2\Psi_1^2\left[\begin{matrix} a_1q,\ a_2; & & q; & z \\ b; & (c_1,c_2:\ell); & & \end{matrix}\right],$$

$$\Psi(a_1-) := {}_2\Psi_1^2\left[\begin{matrix} a_1q^{-1},\ a_2; & & q; & z \\ b; & (c_1,c_2:\ell); & & \end{matrix}\right],$$

$$\Psi(a_1+,b+) \;:=\; {}_2\Psi_1^2\left[\begin{array}{ccc} a_1q,\ a_2; & q; & z \\ bq; & (c_1,c_2:\ell); & \end{array}\right],$$

and

$$\Psi(b+,zq) \;:=\; {}_2\Psi_1^2\left[\begin{array}{ccc} a_1,a_2; & & q;\ \ zq \\ bq; & (c_1,c_2:\ell); & \end{array}\right].$$

The functions $\Psi(a_2+), \Psi(a_2-), \Psi(b+), \Psi(b-)$ are defined similarly.
Now with

$$\zeta_n \;=\; \frac{(a_1;q)_n\ (a_2;q)_n}{(b;q)_n\ (c_1;q)_n^{\ell n}\ (c_2;q)_n^{\ell n}\ (q;q)_n},$$

the function

$$\Psi \;=\; \sum_{n=0}^{\infty} \zeta_n z^n$$

whence the following series representations follow immediately.

$$\left.\begin{array}{ll} \Psi(a_1+) = \sum\limits_{n=0}^{\infty} \frac{1-a_1q^n}{1-a_1}\ \zeta_n\ z^n, & \Psi(a_1-) = \sum\limits_{n=0}^{\infty} \frac{1-a_1q^{-1}}{1-a_1q^{n-1}}\ \zeta_n\ z^n, \\ \Psi(a_2+) = \sum\limits_{n=0}^{\infty} \frac{1-a_2q^n}{1-a_2}\ \zeta_n\ z^n, & \Psi(a_2-) = \sum\limits_{n=0}^{\infty} \frac{1-a_2q^{-1}}{1-a_2q^{n-1}}\ \zeta_n\ z^n, \\ \Psi(b+) = \sum\limits_{n=0}^{\infty} \frac{1-b}{1-cbq^n}\ \zeta_n\ z^n, & \Psi(b-) = \sum\limits_{n=0}^{\infty} \frac{1-bq^{n-1}}{1-bq^{-1}}\ \zeta_n\ z^n, \\ \Psi(c_1+) = \sum\limits_{n=0}^{\infty} \frac{(1-c_1)^{\ell n}}{(1-c_1q^n)^{\ell n}}\ \zeta_n\ z^n, & \Psi(c_1-) = \sum\limits_{n=0}^{\infty} \frac{(1-c_1q^{n-1})^{\ell n}}{(1-c_1q^{-1})^{\ell n}}\ \zeta_n\ z^n \\ \Psi(c_2+) = \sum\limits_{n=0}^{\infty} \frac{(1-c_2)^{\ell n}}{(1-c_2q^n)^{\ell n}}\ \zeta_n\ z^n, & \Psi(c_2-) = \sum\limits_{n=0}^{\infty} \frac{(1-c_2q^{n-1})^{\ell n}}{(1-c_2q^{-1})^{\ell n}}\ \zeta_n\ z^n. \end{array}\right\} \tag{3.38}$$

From (3.20),

$$\Theta_q\Psi \;=\; \sum_{n=0}^{\infty} \frac{(1-q^n)}{(1-q)}\ \zeta_n\ z^n. \tag{3.39}$$

Now following the procedure of obtaining (3.21) and (3.22), one finds

$$\left(a\Theta_q + \frac{1-a_1}{1-q}\right)\Psi \;=\; \frac{1-a_1}{1-q}\ \Psi(a_1+) \tag{3.40}$$

and

$$\left(bq^{-1}\Theta_q + \frac{1-bq^{-1}}{1-q}\right)\Psi \;=\; \frac{1-bq^{-1}}{1-q}\ \Psi(b-), \tag{3.41}$$

respectively.

Eliminating Θ_q from (3.40) and (3.41), yields (3.34).

Next, for $z \neq 0$, using the technique adopted in the proof of Lemma 3.17 and Theorem 3.19, one gets

$$\begin{aligned}
&\left\{I_q^{a_2}\ z^{-1}\ {}_\ell\Delta_{c_2}^{\theta_q}\ {}_\ell\Delta_{c_1}^{\theta_q}\right\}(\theta_q\Psi)\\
&= \left\{I_q^{a_2}\ z^{-1}\ {}_\ell\Delta_{c_2}^{\theta_q}\ {}_\ell\Delta_{c_1}^{\theta_q}\right\}\left(\sum_{n=1}^{\infty}\frac{(a_1;q)_n\ (a_2;q)_n}{(b;q)_n\ (c_1;q)_n^{\ell n}\ (c_2;q)_n^{\ell n}}\frac{z^n - z^n q^n}{(q;q)_n}\right)\\
&= \left\{I_q^{a_2}\ z^{-1}\ {}_\ell\Delta_{c_2}^{\theta_q}\ {}_\ell\Delta_{c_1}^{\theta_q}\right\}\left(\sum_{n=1}^{\infty}\frac{(a_1;q)_n\ (a_2;q)_n}{(b;q)_n\ (c_1;q)_n^{\ell n}\ (c_2;q)_n^{\ell n}}\frac{z^n}{(q;q)_{n-1}}\right)\\
&= I_q^{a_2}\ z^{-1}\ {}_\ell\Delta_{c_2}^{\theta_q}\sum_{n=1}^{\infty}\frac{(a_1;q)_n\ (a_2;q)_n}{(b;q)_n\ (c_1;q)_n^{\ell n}\ (c_2;q)_n^{\ell n}\ (q;q)_{n-1}}\\
&\quad\times(c_1;q)_{n-1}\ \left(q^{c_1-1}\theta_q - q^{c_1-1}+1\right)^{\ell n}\ z^n.
\end{aligned}$$

Hence with the aid of (3.10),

$$\begin{aligned}
&\left\{I_q^{a_2}\ z^{-1}\ {}_\ell\Delta_{c_2}^{\theta_q}\ {}_\ell\Delta_{c_1}^{\theta_q}\right\}(\theta_q\Psi)\\
&= I_q^{a_2}\ z^{-1}\ {}_\ell\Delta_{c_2}^{\theta_q}\sum_{n=1}^{\infty}\frac{(a_1;q)_n\ (a_2;q)_n\ (c_1;q)_{n-1}^{\ell}}{(b;q)_n\ (c_1;q)_n^{\ell n}\ (c_2;q)_n^{\ell n}\ (q;q)_{n-1}}(1-q^{n+c_1-1})^{n\ell}z^n\\
&= I_q^{a_2}\ z^{-1}\ {}_\ell\Delta_{c_2}^{\theta_q}\sum_{n=1}^{\infty}\frac{(a_1;q)_n\ (a_2;q)_n\ z^n}{(b;q)_n\ (c_1;q)_{n-1}^{\ell n-\ell}\ (c_2;q)_n^{\ell n}\ (q;q)_{n-1}}.
\end{aligned}$$

Similarly applying ${}_\ell\Delta_{c_2}^{\theta_q}$ operator, one finds

$$\begin{aligned}
&\left\{I_q^{a_2}\ z^{-1}\ {}_\ell\Delta_{c_2}^{\theta_q}\ {}_\ell\Delta_{c_1}^{\theta_q}\right\}(\theta_q\Psi)\\
&= I_q^{a_2}\ z^{-1}\sum_{n=1}^{\infty}\frac{(a_1;q)_n\ (a_2;q)_n\ z^{n-1}}{(b;q)_n\ (c_1;q)_{n-1}^{\ell n-\ell}\ (c_2;q)_{n-1}^{\ell n-\ell}\ (q;q)_{n-1}}\\
&= \frac{z^{-a_2}}{1-q}\ I_q\left[\sum_{n=1}^{\infty}\frac{(a_1;q)_n\ (a_2;q)_n\ z^{a_2+n-2}}{(b;q)_n\ (c_1;q)_{n-1}^{\ell n-\ell}\ (c_2;q)_{n-1}^{\ell n-\ell}\ (q;q)_{n-1}}\right]\\
&= \frac{z^{-a_2}}{1-q}\left[\sum_{n=1}^{\infty}\frac{(a_1;q)_n\ (a_2;q)_n}{(b;q)_n\ (c_1;q)_{n-1}^{\ell n-\ell}\ (c_2;q)_{n-1}^{\ell n-\ell}\ (q;q)_{n-1}}\right]\\
&\quad\times z\ (1-q)\ \sum_{k=0}^{\infty}(zq^k)^{a_2+n-2}\ q^k\\
&= \sum_{n=1}^{\infty}\frac{(a_1;q)_n\ (a_2;q)_{n-1}\ z^{n-1}}{(b;q)_n\ (c_1;q)_{n-1}^{\ell n-\ell}\ (c_2;q)_{n-1}^{\ell n-\ell}\ (q;q)_{n-1}}\\
&= \sum_{n=0}^{\infty}\frac{(a_1;q)_{n+1}\ (a_2;q)_n\ z^n}{(b;q)_{n+1}\ (c_1;q)_n^{\ell n}\ (c_2;q)_n^{\ell n}\ (q;q)_n}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{1-a_1q^n}{1-bq^n}\, \zeta_n\, z^n \tag{3.42}$$

$$= \frac{1-a_1}{1-b}\, \Psi(a_1+, b+). \tag{3.43}$$

Here in (3.42), putting

$$\frac{1-a_1q^n}{1-bq^n} \quad = \quad 1 + \frac{bq^n - a_1q^n}{1-bq^n}$$

one obtains

$$\begin{aligned}
&\left\{ I_q^{a_2}\ z^{-1}\ {}_\ell\Delta_{c_2}^{\theta_q}\ {}_\ell\Delta_{c_1}^{\theta_q} \right\} (\theta_q \Psi) \\
&= \sum_{n=0}^{\infty} \left(1 + \frac{bq^n - a_1q^n}{1-bq^n} \right) \zeta_n z^n \\
&= \sum_{n=0}^{\infty} \zeta_n z^n + \sum_{n=0}^{\infty} \frac{b-a_1}{1-b} \frac{1-b}{1-bq^n}\, \zeta_n\, (zq)^n \\
&= \Psi + \frac{b-a_1}{1-b} \Psi(b+, zq). \qquad (3.44)
\end{aligned}$$

The relation (3.35) now follows from (3.43) and (3.44).
Since the q-ℓ-Ψ function ${}_2\Psi_1^2(\ell : z)$ given by (3.26) is symmetric in its numerator parameters a_1 and a_2, the identities (3.36) and (3.37) follow immediately from (3.34) and (3.35) respectively.