# Chapter 4

# The generalized $\ell$-Hypergeometric function

## 4.1 Introduction

By introducing a finite number of numerator, denominator and $\ell$-denominator parameters, a generalization of the $\ell$-H functions

$$ {}_1H_1^1\left[\begin{array}{lll} a; & & z \\ b; & (c:\ell); & \end{array}\right] = \sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n\ (c)_n^{\ell n}}\frac{z^n}{n!}, $$

and

$$ {}_2H_1^2\left[\begin{array}{lll} a_1,\ a_2; & & z \\ b; & (c_1,c_2:\ell); & \end{array}\right] = \sum_{n=0}^{\infty}\frac{(a_1)_n\ (a_2)_n}{(b)_n\ (c_1)_n^{\ell n}\ (c_2)_n^{\ell n}}\frac{z^n}{n!}, $$

of Chapter 2 is studied in this chapter. First the series convergence and the order of convergence of the series of proposed function are taken up. This is followed by the derivation of the differential equation with the help of the hyper-Bessel type operators defined in Chapter 2. The eigen function property and certain contiguous function relations are also obtained.

As the special cases, the $\ell$-extensions of exponential, trigonometric and hyperbolic functions are illustrated together with their graphical representations by means of the *Maple software*.

Finally, the Ramanujan's theorem and Kummer's first formula are extended by means of this theory.

The proposed generalization is defined as follows:

**Definition 4.1.** For $p, r, s \in \mathbb{N} \cup \{0\}$, $a_i, z \in \mathbb{C}, \forall i = 1, 2, \dots, r$,and $b_j, c_k \in \mathbb{C} \setminus \{0, -1, -2, \dots, \} \, \forall j = 1, 2, \dots, s$, $\forall k = 1, 2, \dots, p$, the generalized $\ell$-Hypergeometric function is defined as

$$\begin{aligned} {}_rH_s^p(\ell; z) &= {}_rH_s^p \left[ \begin{array}{ccccccccc} a_1, & a_2, & \dots, & a_r; & & & & & z \\ b_1, & b_2, & \dots, & b_s; & (c_1, & c_2, & \dots, & c_p : \ell); & \end{array} \right] \\ &= \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_r)_n}{(b_1)_n (b_2)_n \cdots (b_s)_n \; (c_1)_n^{\ell n} \; (c_2)_n^{\ell n} \cdots (c_p)_n^{\ell n}} \frac{z^n}{n!}, \end{aligned} \tag{4.1}$$

where $\ell \in \mathbb{C}$ with $\Re(\ell) \geq 0$ and $\Re\left((c_1 + c_2 + \cdots + c_p)\ell - \frac{\ell p}{2} + s - r + 1\right) > 0$.

Now onward the function defined by (4.1) will be referred to as the *generalized ℓ-H function.* Also, throughout the work, the range of values of $i, j, k$ to be $i = 1, 2, \dots, r$, $j = 1, 2, \dots, s$, and $k = 1, 2, \dots, p$ will be kept fixed.

*Remark* 4.2. The numbers $r$, $s$, $p$ can all be zero simultaneously. The absence of parameters is emphasized by a dash. As an example,

$${}_0H_0^1 \left[ \begin{array}{ccc} -; & & z \\ -; & (c : \ell); & \end{array} \right] = \sum_{n=0}^{\infty} \frac{z^n}{(c)_n^{\ell n} \; n!}. \tag{4.2}$$

In Section 4.3 this function in (4.2) will be seen to be an $\ell$-H exponential function.

*Remark* 4.3. The function ${}_rH_s^p(\ell; z)$ reduces to the generalized hypergeometric function ${}_rF_s[z]$ when $\ell = 0$.

## 4.2 Main Results

The coefficient of $z^n$ in the series (4.1) will be symbolized as $\varphi_n$, that is,

$$\frac{(a_1)_n (a_2)_n \cdots (a_r)_n}{(b_1)_n (b_2)_n \cdots (b_s)_n \; (c_1)_n^{\ell n} \; (c_2)_n^{\ell n} \cdots (c_p)_n^{\ell n} \; n!} = \varphi_n.$$

### 4.2.1 Convergence

In the following theorem, it is shown that the function in (4.1) exists.

**Theorem 4.4.** *If* $\Re(\ell) \geq 0$ *and* $\Re\left((c_1 + c_2 + \cdots + c_p)\ell - \frac{\ell p}{2} + s - r + 1\right) > 0$ *then the generalized ℓ-H function is an entire function of* $z$.

*Proof.* With $\varphi_n$ as stated above and from the Cauchy-Hadamard formula,

$$\begin{aligned}
\frac{1}{R} &= \lim_{n\to\infty} \sup \sqrt[n]{|\varphi_n|} \\
&= \lim_{n\to\infty} \sup \left| \frac{(a_1)_n(a_2)_n\cdots(a_r)_n}{(b_1)_n(b_2)_n\cdots(b_s)_n\ (c_1)_n^{\ell n}(c_2)_n^{\ell n}\cdots(c_p)_n^{\ell n}} \frac{1}{n!}\right|^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \sup \left| \frac{\Gamma(b_1)\Gamma(b_2)\cdots\Gamma(b_s)}{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_r)}\right|^{\frac{1}{n}} \left| \frac{\Gamma(a_1+n)\Gamma(a_2+n)\cdots\Gamma(a_r+n)}{\Gamma(b_1+n)\Gamma(b_2+n)\cdots\Gamma(b_s+n)}\right|^{\frac{1}{n}} \\
&\quad \times \left| \frac{\Gamma(c_1)\Gamma(c_2)\cdots\Gamma(c_p)}{\Gamma(c_1+n)\Gamma(c_2+n)\cdots\Gamma(c_p+n)}\right|^{\Re(\ell)} \frac{1}{\Gamma^{\frac{1}{n}}(n+1)}.
\end{aligned}$$

Now applying of the Stirling's formula [18]:

$$\Gamma(\alpha+n) \sim \sqrt{2\pi}e^{-(\alpha+n)}\ (\alpha+n)^{(\alpha+n-1/2)} \tag{4.3}$$

for large $n$ and taking $\alpha = a_i, b_j, c_k$, in turn, one gets

$$\begin{aligned}
\frac{1}{R} &\sim \left| \frac{\prod_{k=1}^{p}\Gamma(c_k)}{(2\pi)^{\frac{p}{2}}}\right|^{\Re(\ell)} \lim_{n\to\infty}\sup \frac{\left|\prod_{k=1}^{p} e^{-(c_k+n)}\ (c_k+n)^{c_k+n-1/2}\right|^{-\Re(\ell)}}{\left|(2\pi)^{\frac{1}{2}}\ e^{-(n+1)}\ (n+1)^{n+1-1/2}\right|^{\frac{1}{n}}} \\
&\quad \times \left| \frac{\prod_{j=1}^{s}\Gamma(b_j)}{\prod_{i=1}^{r}\Gamma(a_i)}\right|^{\frac{1}{n}} \left| \frac{(2\pi)^{\frac{r}{2}}\ \prod_{i=1}^{r} e^{-(a_i+n)}\ (a_i+n)^{a_i+n-1/2}}{(2\pi)^{\frac{s}{2}}\ \prod_{j=1}^{s} e^{-(b_j+n)}\ (b_j+n)^{b_j+n-1/2}}\right|^{\frac{1}{n}} \\
&= \left| \frac{\prod_{k=1}^{p}\Gamma^{\ell}(c_k)}{(2\pi)^{\frac{p\ell}{2}}\ e^{r-s}}\right| \lim_{n\to\infty}\sup \left| \frac{1}{n^{s-r+1}n^{(c_1+c_2+\cdots+c_p)\ell-\frac{\ell p}{2}}} \left(\frac{e}{n}\right)^{pn\Re(\ell)}\right| \\
&= 0,
\end{aligned}$$

provided that $\Re(\ell) \geq 0$ and $\Re\left((c_1+c_2+\cdots+c_p)\ell - \frac{\ell p}{2} + s - r + 1\right) > 0$. □

*Remark* 4.5. The series $\sum \varphi_n\ z^n$ thus converges uniformly in any compact subset of $\mathbb{C}$.

### 4.2.2 Order of ${}_rH_s^p(\ell; z)$ function

**Theorem 4.6.** *If the conditions stated in Theorem 4.4 hold then the generalized ℓ-H function is an entire function of order zero.*

*Proof.* If the function $f(z)=\sum\limits_{n=0}^{\infty} a_n z^n$ is an entire function then the order $\rho(f)$ of $f$ is given by [8, 40]

$$\rho(f)=\lim_{n\to\infty}\sup\ \frac{n\ln n}{\ln|a_n|^{-1}}. \tag{4.4}$$

For $f(z)= {}_rH_s^p(\ell;z)=\sum\varphi_n\ z^n$,

$$\rho({}_rH_s^p(\ell;z))=\lim_{n\to\infty}\sup\ \frac{n\ln n}{\ln|\varphi_n|^{-1}}.$$

Here,

$$\begin{aligned}|\varphi_n|^{-1} &= \left|\frac{\left\{\prod\limits_{i=1}^{r}\Gamma(a_i)\right\}\ \left\{\prod\limits_{j=1}^{s}\Gamma(b_j+n)\right\}}{\left\{\prod\limits_{j=1}^{s}\Gamma(b_j)\right\}\ \left\{\prod\limits_{i=1}^{r}\Gamma(a_i+n)\right\}}\right| \\ &\quad\times\left|\frac{\left\{\prod\limits_{k=1}^{p}\Gamma^{\ell n}(c_k+n)\right\}}{\left\{\prod\limits_{k=1}^{p}\Gamma^{\ell n}(c_k)\right\}}\right|\ \Gamma(n+1).\end{aligned}$$

Using asymptotic expansion

$$\ln\Gamma(r)\sim\left(r-\frac{1}{2}\right)\ln r-r+\frac{1}{2}\ln\sqrt{2\pi},$$

for large $r$, one further gets

$$\begin{aligned}\ln|\varphi_n|^{-1} &\sim \left|\sum_{i=1}^{r}\ln\Gamma(a_i)\right|-\left|\sum_{j=1}^{s}\ln\Gamma(b_j)\right| \\ &+\sum_{j=1}^{s}\left|\left(b_j+n-\frac{1}{2}\right)\ln(b_j+n)-(b_j+n)+\frac{1}{2}\ln\sqrt{2\pi}\right| \\ &-\sum_{i=1}^{r}\left|\left(a_i+n-\frac{1}{2}\right)\ln(a_i+n)+(a_i+n)-\frac{1}{2}\ln\sqrt{2\pi}\right| \\ &+\left|\ell n\sum_{k=1}^{p}\left[\left(c_k+n-\frac{1}{2}\right)\ln(c_k+n)-(c_k+n)+\frac{1}{2}\ln\sqrt{2\pi}\right]\right| \\ &+\left|\left(n+1-\frac{1}{2}\right)\ln(n+1)-(n+1)+\frac{1}{2}\ln\sqrt{2\pi}\right| \\ &-\left|\ell n\ \sum_{k=1}^{p}\ln\Gamma(c_k)\right|.\end{aligned} \tag{4.5}$$

From this, one finds that

$$\lim_{n\to\infty} \frac{\ln|\varphi_n|^{-1}}{n\ln n}$$

is unbounded, consequently, from (4.4) and (4.5),

$$\rho\left({}_rH_s^p(\ell;z)\right) = \lim_{n\to\infty} \sup \frac{n\ln n}{\ln|\varphi_n|^{-1}} = 0.$$

□

*Remark* 4.7. It is known that [4, Theorem 1.1] "If $f$ is entire and $\rho(f)$ is finite and is not equal to a positive integer, then $f$ has infinitely many zeros or it is a polynomial." Thus, the generalized $\ell$-H function has infinitely many zeros.

### 4.2.3 Integral Representation

A generalized form of the integral representation of Theorem 2.7 is obtained in

**Theorem 4.8.** *If* $a_i, b_j, c_k \in \mathbb{C}$ *with* $\Re(b_1) > \Re(a_1) > 0$, $b_j, c_k \neq 0, -1, -2, \ldots,$ *and* $\Re\left(\sum_{k=1}^{p} c_k\ell - \frac{\ell p}{2} + s - r + 1\right) > 0$ *then*

$$\begin{aligned}
&{}_rH_s^p\left[\begin{array}{llllllll} a_1, & a_2, & \ldots, & a_r; & & & & z \\ b_1, & b_2, & \ldots, & b_s; & (c_1, & c_2, & \ldots, & c_p:\ell); \end{array}\right] \\
&= \frac{\Gamma(b_1)}{\Gamma(a_1)\,\Gamma(b_1-a_1)} \int_0^1 t^{a_1-1}(1-t)^{b_1-a_1-1} \\
&\quad \times {}_{r-1}H_{s-1}^p\left[\begin{array}{lllllll} a_2, & \ldots, & a_r; & & & & zt \\ b_2, & \ldots, & b_s; & (c_1, & c_2, & \ldots, & c_p:\ell) \end{array}\right] \mathrm{dt}.
\end{aligned}$$

*Proof.* Beginning with

$$\begin{aligned}
\frac{(a_1)_n}{(b_1)_n} &= \frac{\Gamma(a_1+n)}{\Gamma(a_1)} \frac{\Gamma(b_1)}{\Gamma(b_1+n)} \frac{\Gamma(b_1-a_1)}{\Gamma(b_1-a_1)} \\
&= \frac{\Gamma(b_1)}{\Gamma(b_1-a_1)\,\Gamma(a_1)} \frac{\Gamma(a_1+n)\,\Gamma(b_1-a_1)}{\Gamma(b_1+n)} \\
&= \frac{\Gamma(b_1)}{\Gamma(b_1-a_1)\,\Gamma(a_1)} B(a_1+n, b_1-a_1) \\
&= \frac{\Gamma(b_1)}{\Gamma(b_1-a_1)\,\Gamma(a_1)} \int_0^1 t^{a_1+n-1}\,(1-t)^{b_1-a_1-1}\,\mathrm{dt},
\end{aligned}$$

Under the convergence conditions permitting to interchange the series and the integral, one obtains

$$
\begin{aligned}
&{}_rH_s^p\left[\begin{matrix} a_1, & a_2, & \dots, & a_r; & & & & & z\\ b_1, & b_2, & \dots, & b_s; & (c_1, & c_2, & \dots, & c_p:\ell); & \end{matrix}\right]\\
&= \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n\dots(a_r)_n}{(b_1)_n(b_2)_n\dots(b_s)_n\ (c_1)_n^{\ell n}(c_2)_n^{\ell n}\dots(c_p)_n^{\ell n}} \frac{z^n}{n!}\\
&= \sum_{n=0}^{\infty} \frac{\Gamma(b_1)}{\Gamma(b_1-a_1)\ \Gamma(a_1)} \int_0^1 t^{a_1+n-1}\ (1-t)^{b_1-a_1-1}\\
&\quad \times \frac{(a_2)_n(a_3)_n\dots(a_r)_n}{(b_2)_n(b_3)_n\dots(b_s)_n\ (c_1)_n^{\ell n}(c_2)_n^{\ell n}\dots(c_p)_n^{\ell n}} \frac{z^n}{n!}\mathrm{dt}\\
&= \frac{\Gamma(b_1)}{\Gamma(b_1-a_1)\ \Gamma(a_1)} \sum_{n=0}^{\infty} \frac{(a_2)_n(a_3)_n\dots(a_r)_n}{(b_2)_n(b_3)_n\dots(b_s)_n\ (c_1)_n^{\ell n}(c_2)_n^{\ell n}\dots(c_p)_n^{\ell n}}\\
&\quad \times \int_0^1 t^{a_1-1}\ (1-t)^{b_1-a_1-1} \frac{(zt)^n}{n!}\ \mathrm{dt}\\
&= \frac{\Gamma(b_1)}{\Gamma(a_1)\ \Gamma(b_1-a_1)} \int_0^1 t^{a_1-1}(1-t)^{b_1-a_1-1}\\
&\quad \times {}_{r-1}H_{s-1}^p\left[\begin{matrix} a_2, & \dots, & a_r; & & & & & zt\\ b_2, & \dots, & b_s; & (c_1, & c_2, & \dots, & c_p:\ell); & \end{matrix}\right]\mathrm{dt}.
\end{aligned}
$$

□

*Remark* 4.9. For $\ell = 0$, the theorem reduced to the a simple integral form of ${}_pF_q[z]$ [46, Ch.5, p.85].

### 4.2.4 Differential Equation

While deriving the differential equation of the ℓ-H function ${}_1H_1^1(\ell : z)$ (2.1) in Chapter 2, the following differential operator was defined.
Let $f(z) = \sum\limits_{n=1}^{\infty} a_n z^n$, $0 \neq z \in \mathbb{C},\ p \in \mathbb{N} \cup \{0\}$ and $\alpha \in \mathbb{C}$. Then

$$
{}_p\Delta_\alpha^\theta(f(z)) = \begin{cases} \sum\limits_{n=1}^{\infty} a_n(\alpha)_{n-1}^p(\theta+\alpha-1)^{pn}z^n, & \textit{if} \quad p \in \mathbb{N}\\ f(z), & \textit{if} \quad p = 0 \end{cases}, \tag{4.6}
$$

where $\theta$ is the Euler differential operator $z\frac{d}{dz}$ and

$$(\theta+\alpha)^r = \underbrace{(\theta+\alpha)(\theta+\alpha)\ldots(\theta+\alpha)}_{r\ times}$$

is a special case of the hyper-Bessel differential operators (2.8) defined in Chapter 2. With the aid of this operator it was shown in Theorem 2.10 that the $\ell$-H function ${}_1H_1^1(\ell : z)$ (2.1) is a solution of the differential equation

$$\left\{\left\{{}_\ell\Delta_c^\theta\right\}\{\theta+b-1\}\,\theta - z(\theta+a)\right\}w = 0.$$

Using the same operator (4.6), an infinite order differential equation for the generalized $\ell$-H function can be obtained as in the following theorem.

**Theorem 4.10.** *For $\ell, p, r, s \in \mathbb{N}\cup\{0\}$, $a_i, b_j, c_k \in \mathbb{C}$ with $b_j, c_k \neq 0, -1, -2, \ldots,$ the function*

$$w = {}_rH_s^p\left[\begin{matrix} a_1, & a_2, & \ldots, & a_r; & & & & & z \\ b_1, & b_2, & \ldots, & b_s; & (c_1, & c_2, & \ldots, & c_p:\ell); & \end{matrix}\right]$$

*satisfies the equation*

$$\left[\left\{\prod_{k=1}^{p}{}_\ell\Delta_{c_k}^\theta\right\}\left\{\prod_{j=1}^{s}(\theta+b_j-1)\right\}\theta - z\prod_{i=1}^{r}(\theta+a_i)\right]w = 0, \tag{4.7}$$

*where ${}_\ell\Delta_{c_k}^\theta$ is as defined in (4.6).*

In parallel to the Lemma 2.11 which was required to prove Theorem 2.10, here also the following lemma is proved which enables one to apply the operator (4.6) to the operand $w$. It uses the notation

$$\left[\left\{\prod_{k=1}^{p}{}_\ell\Delta_{c_k}^\theta\right\}\left\{\prod_{j=1}^{s}(\theta+b_j-1)\right\}\theta\right] = {}_{(b)}\Lambda_{(c:\ell)}^\theta.$$

**Lemma 4.11.** *If $\ell \in \mathbb{N}\cup\{0\}$, $w = \sum\limits_{n=0}^{\infty}\varphi_n z^n$ and*

$${}_{(b)}\Lambda_{(c:\ell)}^\theta w = \sum_{n=0}^{\infty} f_n((a,r),(b,s),(c,p:\ell);z),$$

*then the operator* ${}_{(b)}\Lambda^{\theta}_{(c:\ell)}$ *is applicable to the generalized* $\ell$*-H function provided that the series*

$$\sum_{n=0}^{\infty} \varphi_n \ f_n((a,r),(b,s),(c,p:\ell);z)$$

*converges (cf. [49, Definition 11, p.20]).*

*Proof.* With $w = \sum\limits_{n=0}^{\infty} \varphi_n z^n$, and $n!\ \varphi_n = A_n$,

$$\begin{aligned}
&{}_{(b)}\Lambda^{\theta}_{(c:\ell)} w \\
&= \left\{\prod_{k=1}^{p} {}_{\ell}\Delta^{\theta}_{c_k}\right\}\{(\theta+b_1-1)(\theta+b_2-1)\cdots(\theta+b_s-1)\}\sum_{n=0}^{\infty} A_n\ \frac{\theta z^n}{n!} \\
&= \left\{\prod_{k=1}^{p} {}_{\ell}\Delta^{\theta}_{c_k}\right\}\ (\theta+b_1-1)(\theta+b_2-1)\cdots(\theta+b_s-1)\ \sum_{n=1}^{\infty} A_n \frac{z^n}{(n-1)!} \\
&= \left\{\prod_{k=1}^{p} {}_{\ell}\Delta^{\theta}_{c_k}\right\}\{(\theta+b_1-1)(\theta+b_2-1)\cdots(\theta+b_{s-1}-1)\} \\
&\quad \times \sum_{n=1}^{\infty} A_n\ \frac{(\theta+b_s-1)z^n}{(n-1)!} \\
&= \left\{\prod_{k=1}^{p} {}_{\ell}\Delta^{\theta}_{c_k}\right\}\{(\theta+b_1-1)(\theta+b_2-1)\cdots(\theta+b_{s-1}-1)\} \\
&\quad \times \sum_{n=1}^{\infty} A_n\ \frac{(n+b_s-1)z^n}{(n-1)!}.
\end{aligned}$$

By applying the operator $(\theta + b_j - 1)$ for $j = 1, 2, \dots, s-1$, and proceeding as above, one obtains

$$\begin{aligned}
&{}_{(b)}\Lambda^{\theta}_{(c:\ell)} w \\
&= \left\{\prod_{k=1}^{p} {}_{\ell}\Delta^{\theta}_{c_k}\right\}\sum_{n=1}^{\infty}\left\{\prod_{j=1}^{s}(b_j+n-1)\right\}\ A_n\ \frac{z^n}{(n-1)!} \\
&= \left\{\prod_{k=1}^{p-1} {}_{\ell}\Delta^{\theta}_{c_k}\right\}\ {}_{\ell}\Delta^{\theta}_{c_p}\left(\sum_{n=1}^{\infty}\left\{\prod_{j=1}^{s}(b_j+n-1)\right\}\ A_n\ \frac{z^n}{(n-1)!}\right) \\
&= \left\{\prod_{k=1}^{p-1} {}_{\ell}\Delta^{\theta}_{c_k}\right\}\sum_{n=1}^{\infty}\left\{\prod_{j=1}^{s}(b_j+n-1)\right\}\ A_n\ \frac{(c_p)^{\ell}_{n-1}}{(n-1)!} \\
&\quad \times\ (\theta+c_p-1)^{\ell n}\ z^n.
\end{aligned} \tag{4.8}$$

Noticing from (2.10) that

$$\begin{aligned}(\theta + c_k - 1)^{\ell}\ z &= (1 + c_k - 1)^{\ell}\ z, \\ (\theta + c_k - 1)^{2\ell}\ z^2 &= (2 + c_k - 1)^{2\ell}\ z^2,\end{aligned}$$

in general, for $n \in \mathbb{N}$,

$$(\theta + c_k - 1)^{\ell n}\ z^n = (n + c_k - 1)^{\ell n}\ z^n, \tag{4.9}$$

one obtains from (4.8),

$$\begin{aligned}&{}_{(b)}\Lambda^{\theta}_{(c:\ell)} w \\ &= \left\{\prod_{k=1}^{p-1}\ {}_{\ell}\Delta^{\theta}_{c_k}\right\} \sum_{n=1}^{\infty} \left\{\prod_{j=1}^{s}(b_j + n - 1)\right\} A_n\ \frac{(c_p)^{\ell}_{n-1}}{(n-1)!}\ (c_p + n - 1)^{\ell n} z^n. \end{aligned} \tag{4.10}$$

Proceeding now by applying ${}_{\ell}\Delta^{\theta}_{c_k}$ for $k = 1, 2, \ldots, p-1$, yields

$$\begin{aligned}{}_{(b)}\Lambda^{\theta}_{(c:\ell)} w &= \sum_{n=1}^{\infty} \left\{\prod_{j=1}^{s}(b_j + n - 1)\right\} \left\{\prod_{k=1}^{p-1}(c_k)^{\ell}_{n-1}\ \ (c_k + n - 1)^{\ell n}\right\} A_n\ \frac{z^n}{(n-1)!} \\ &= \sum_{n=1}^{\infty} \frac{(a_1)_n\ (a_2)_n \cdots (a_r)_n}{(b_1)_{n-1}\ (b_2)_{n-1} \cdots (b_s)_{n-1}} \\ &\quad \times \frac{z^n}{(c_1)^{\ell n - \ell}_{n-1}\ (c_2)^{\ell n - \ell}_{n-1} \cdots (c_p)^{\ell n - \ell}_{n-1}\ (n-1)!} \qquad (4.11) \\ &= \sum_{n=0}^{\infty} \frac{(a_1)_{n+1}\ (a_2)_{n+1} \cdots (a_r)_{n+1}}{(b_1)_n\ (b_2)_n \cdots (b_s)_n\ (c_1)^{\ell n}_n\ (c_2)^{\ell n}_n \cdots (c_p)^{\ell n}_n}\ \frac{z^{n+1}}{n!} \qquad (4.12) \\ &= \sum_{n=0}^{\infty} f_n((a,r),(b,s),(c,p:\ell);z)\ \text{(say)}.\end{aligned}$$

To complete the proof, it remains to show that the series

$$\begin{aligned}&\sum_{n=0}^{\infty} \varphi_n\ f_n((a,r),(b,s),(c,p:\ell);z) \\ &\quad = \sum_{n=0}^{\infty} \frac{\left\{\prod_{i=1}^{r}(a_i)^2_n (a_i)_{n+1}\right\}}{\left\{\prod_{j=1}^{s}(b_j)^2_n\right\}\ \left\{\prod_{k=1}^{p}(c_k)^{2\ell n}_n\right\}}\ \frac{z^{n+1}}{(n!)^2}\end{aligned}$$

is convergent. For that take

$$\mu_n = \frac{\left\{\prod_{i=1}^{r}(a_i)_n^2\ (a_i+n)\right\}}{\left\{\prod_{j=1}^{s}(b_j)_n^2\right\}\ \left\{\prod_{k=1}^{p}(c_k)_n^{2\ell n}\right\}}\ \frac{1}{(n!)^2}$$

$$= \frac{\left\{\prod_{j=1}^{s}\Gamma^2(b_j)\right\}}{\left\{\prod_{i=1}^{r}\Gamma^2(a_i)\right\}}\ \frac{\left\{\prod_{i=1}^{r}\ (a_i+n)\ \Gamma^2(a_i+n)\right\}\left\{\prod_{k=1}^{p}\Gamma^{2\ell n}(c_k)\right\}}{\left\{\prod_{j=1}^{s}\Gamma^2(b_j+n)\right\}\left\{\prod_{k=1}^{p}\Gamma^{2\ell n}(c_k+n)\right\}\Gamma^2(n+1)}.$$

Then in view of the Stirling's asymptotic formula (4.3) for large $n$, one gets

$$|\mu_n|^{\frac{1}{n}} \sim \left|\frac{\prod_{j=1}^{s}\Gamma^2(b_j)}{\prod_{i=1}^{r}\Gamma^2(a_i)}\right|^{\frac{1}{n}} \frac{\left|\prod_{i=1}^{r}\left\{e^{-(a_i+n)}(a_i+n)^{a_i+n-\frac{1}{2}}\sqrt{2\pi}\right\}\right|^{\frac{2}{n}}}{\left|\prod_{j=1}^{s}\left\{e^{-(b_j+n)}(b_j+n)^{b_j+n-\frac{1}{2}}\sqrt{2\pi}\right\}\right|^{\frac{2}{n}}}$$

$$\times \frac{\left|\prod_{i=1}^{r}(a_i+n)^{\frac{1}{n}}\right|\ \left|\prod_{k=1}^{p}\Gamma^{2\ell}(c_k)\right|\ \left|e^{-(n+1)}(n+1)^{n+1-\frac{1}{2}}\sqrt{2\pi}\right|^{-\frac{2}{n}}}{\left|\prod_{k=1}^{p}\left\{e^{-(c_k+n)}(c_k+n)^{c_k+n-\frac{1}{2}}\sqrt{2\pi}\right\}\right|^{2\ell}}.$$

Using here the Cauchy-Hadamard formula, one finally obtains

$$\lim_{n\to\infty}\sup\ |\mu_n|^{\frac{1}{n}} \sim \lim_{n\to\infty}\sup\left\{\prod_{k=1}^{p}\left|\Gamma^{2\ell}(c_k)\right|\right\}\frac{\left|(2\pi)^{-\ell p}\ e^{2(s-r)}\right|\ \left|\frac{e}{n}\right|^{2np\ell}}{\left|n^{2(c_1+c_2+\cdots+c_p)\ell-\ell p+2(s-r+1)}\right|}$$
$$= 0,$$

provided that $\Re(\ell) \geq 0$ and $\Re\left(2(c_1+c_2+\cdots+c_p)\ell-\ell p+2(s-r+1)\right) > 0$. □

*Proof.* (of Theorem 4.10)
From (4.12),

$$_{(b)}\Lambda^{\theta}_{(c:\ell)}w = \sum_{n=0}^{\infty}\frac{(a_1)_{n+1}\ (a_2)_{n+1}\cdots(a_r)_{n+1}}{(b_1)_n\ (b_2)_n\cdots(b_s)_n\ (c_1)_n^{\ell n}\ (c_2)_n^{\ell n}\cdots(c_p)_n^{\ell n}}\ \frac{z^{n+1}}{n!}$$

$$= \sum_{n=0}^{\infty}\frac{(a_1)_n\ (a_2)_n\cdots(a_r)_n}{(b_1)_n\ (b_2)_n\cdots(b_s)_n\ (c_1)_n^{\ell n}\ (c_2)_n^{\ell n}\cdots(c_p)_n^{\ell n}}\ \left\{\prod_{i=1}^{r}(a_i+n)\right\}\ \frac{z^{n+1}}{n!}$$

$$= z\sum_{n=0}^{\infty}\frac{(a_1)_n\ (a_2)_n\cdots(a_r)_n}{(b_1)_n\ (b_2)_n\cdots(b_s)_n\ (c_1)_n^{\ell n}\ (c_2)_n^{\ell n}\cdots(c_p)_n^{\ell n}}\ \left\{\prod_{i=1}^{r}(\theta+a_i)\right\}\ \frac{z^n}{n!}$$

$$= \quad z\left\{\prod_{i=1}^{r}(\theta+a_i)\right\}\; w.$$

□

### 4.2.5 Contiguous function relations

In parallel to the theory of contiguous functions of ${}_pF_q[z]$ [46, p.82], here the functions that are contiguous to ${}_rH_s^p(\ell;z)$ are defined and obtain the relations amongst them as follows. Put

$${}_rH_s^p \;=\; {}_rH_s^p\left[\begin{array}{llllllll} a_1, & a_2, & \ldots, & a_r; & & & & z \\ b_1, & b_2, & \ldots, & b_s; & (c_1, & c_2, & \ldots, & c_p:\ell); \end{array}\right];$$

and define the functions

$${}_rH_s^p(a_i+) = {}_rH_s^p\left[\begin{array}{lllllllll} a_1, & a_2, & \ldots, & a_{i-1}, & a_i+1, & a_{i+1}, & \ldots, & a_r; & z \\ b_1, & b_2, & \ldots, & b_s; & (c_1, & c_2, & \ldots, & c_p:\ell); & \end{array}\right],$$
$${}_rH_s^p(a_i-) = {}_rH_s^p\left[\begin{array}{lllllllll} a_1, & a_2, & \ldots, & a_{i-1}, & a_i-1, & a_{i+1}, & \ldots, & a_r; & z \\ b_1, & b_2, & \ldots, & b_s; & (c_1, & c_2, & \ldots, & c_p:\ell); & \end{array}\right],$$

and similarly, ${}_rH_s^p(b_j+)$, ${}_rH_s^p(b_j-)$, ${}_rH_s^p(c_k+)$, and ${}_rH_s^p(c_k-)$ as the functions contiguous to ${}_rH_s^p(\ell;z)$.
Now using the symbols

$$A=\prod_{i=1}^{r}a_i,\;\; B=\prod_{j=1}^{s}b_j,\;\;\; C=\prod_{k=1}^{p}(c_k)^{\ell},\;\; {}_rH_s^p \;=\; \sum_{n=0}^{\infty}\varphi_n\; z^n.$$

one finds at once that

$${}_rH_s^p(a_i+)=\sum_{n=0}^{\infty}\tfrac{a_i+n}{a_i}\;\varphi_n\; z^n, \qquad {}_rH_s^p(a_i-)=\sum_{n=0}^{\infty}\tfrac{a_i-1}{a_i+n-1}\;\varphi_n\; z^n,$$
$${}_rH_s^p(b_j+)=\sum_{n=0}^{\infty}\tfrac{b_j}{b_j+n}\;\varphi_n\; z^n, \qquad {}_rH_s^p(b_j-)=\sum_{n=0}^{\infty}\tfrac{b_j+n-1}{b_j-1}\;\varphi_n\; z^n,$$
$${}_rH_s^p(c_k+)=\sum_{n=0}^{\infty}\tfrac{(c_k)^{\ell n}}{(c_k+n)^{\ell n}}\;\varphi_n\; z^n, \quad {}_rH_s^p(c_k-)=\sum_{n=0}^{\infty}\tfrac{(c_k+n-1)^{\ell n}}{(c_k-1)^{\ell n}}\;\varphi_n\; z^n.$$

By making use of these, the contiguous functions relations are obtained in

**Theorem 4.12.** *For $j = 2, 3, \ldots, s$ and $\Re(\ell) \geq 0$ there hold the contiguous functions relations:*

$$(a_1 - b_j + 1)\ {}_rH_s^p = a_1\ {}_rH_s^p(a_1+) - (b_j - 1)\ {}_rH_s^p(b_j-). \tag{4.13}$$

*Whereas for $\ell \in \mathbb{N} \cup \{0\}$, there hold the following extended contiguous function relations.*

$$\begin{aligned} a_i\ {}_rH_s^p &= a_i\ {}_rH_s^p(a_i+) \\ &\quad - \frac{A}{B\ C}\ z\ {}_rH_{s+\ell}^p \left[ \begin{matrix} (a)+1; & \\ (b)+1, ((c)+1)^\ell; \quad ((c)+1:\ell); & \end{matrix} \frac{z}{C} \right], \end{aligned} \tag{4.14}$$

$$\begin{aligned} (b_j - 1)\ {}_rH_s^p &= (b_j - 1)\ {}_rH_s^p(b_j-) - \frac{A}{B\ C}\ z \\ &\quad \times {}_rH_{s+\ell}^p \left[ \begin{matrix} (a)+1; & \\ (b)+1, ((c)+1)^\ell; \quad ((c)+1:\ell); & \end{matrix} \frac{z}{C} \right], \end{aligned} \tag{4.15}$$

*and*

$$\begin{aligned} & {}_1H_{1+\ell}^1 \left[ \begin{matrix} a; & \\ b, c, c, \ldots, c; \quad (c:\ell); & \end{matrix} \frac{z}{(1-c)^\ell} \right] \\ &= (1 - ab^{-1})\ {}_1H_{1+\ell}^1 \left[ \begin{matrix} a; & \\ b+1, c, c, \ldots, c; \quad (c:\ell); & \end{matrix} \frac{z}{(1-c)^\ell} \right] \\ &\quad + ab^{-1}\ {}_1H_{1+\ell}^1 \left[ \begin{matrix} a+1; & \\ b+1, c, c, \ldots, c; \quad (c:\ell); & \end{matrix} \frac{z}{(1-c)^\ell} \right] \end{aligned} \tag{4.16}$$

*in which $(\alpha)+1$ stands for the array of the parameters: $\alpha_1+1, \alpha_2+1, \ldots, \alpha_m+1$.*

*Proof.* The function relations (4.13) may be obtained as follows. Choose the parameter $a_1$ from the set $\{a_i; i = 1, 2, \ldots, r\}$ of numerator parameters and consider

$$a_1\ {}_rH_s^p(a_1+) - a_i\ {}_rH_s^p(a_i+),$$

where $i \neq 1$. Then using the above definitions,

$$\begin{aligned} a_1\ {}_rH_s^p(a_1+) - a_i\ {}_rH_s^p(a_i+) &= a_1 \sum_{n=0}^{\infty} \frac{a_1+n}{a_1}\ \varphi_n\ z^n - a_i \sum_{n=0}^{\infty} \frac{a_i+n}{a_i}\ \varphi_n\ z^n \\ &= \sum_{n=0}^{\infty} (a_1 + n - a_i - n)\ \varphi_n\ z^n \end{aligned}$$

$$= \quad (a_1 - a_i)\ {}_rH_s^p. \tag{4.17}$$

Taking $z\frac{\mathrm{d}}{\mathrm{dz}} = \theta$, one gets

$$\theta\ {}_rH_s^p \quad = \quad \theta\ \sum_{n=0}^{\infty} \varphi_n\ z^n = \sum_{n=0}^{\infty} n\ \varphi_n\ z^n. \tag{4.18}$$

Hence

$$\begin{aligned}(\theta + a_i)\ {}_rH_s^p \quad &= \quad \theta\ \sum_{n=0}^{\infty} \varphi_n\ z^n + a_i\ \sum_{n=0}^{\infty} \varphi_n\ z^n \\ &= \quad a_i \sum_{n=0}^{\infty} \frac{a_i + n}{a_i}\ \varphi_n\ z^n \\ &= \quad a_i\ {}_rH_s^p(a_i+),\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}(\theta + b_j - 1)\ {}_rH_s^p \quad &= \quad \theta\ \sum_{n=0}^{\infty} \varphi_n\ z^n + (b_j - 1)\ \sum_{n=0}^{\infty} \varphi_n\ z^n \\ &= \quad (b_j - 1) \sum_{n=0}^{\infty} \frac{b_j + n - 1}{b_j - 1}\ \varphi_n\ z^n \\ &= \quad (b_j - 1)\ {}_rH_s^p(b_j-).\end{aligned} \tag{4.20}$$

From (4.17) and (4.19),

$$a_1\ {}_rH_s^p(a_1+) - (\theta + a_i)\ {}_rH_s^p \quad = \quad (a_1 - a_i)\ {}_rH_s^p,$$

that is

$$a_1\ {}_rH_s^p(a_1+) - \theta\ {}_rH_s^p \quad = \quad a_1\ {}_rH_s^p. \tag{4.21}$$

On adding (4.20) and (4.21), one arrives at the required contiguous functions relations:

$$(a_1 - b_j + 1)\ {}_rH_s^p = a_1\ {}_rH_s^p(a_1+) - (b_j - 1)\ {}_rH_s^p(b_j-).$$

If the parameter $a_1$ is replaced by $a_m$ with $m \neq i$ in (4.17) then (4.13) gives rise to a set of contiguous functions relations:

$$(a_m - b_j + 1)\ {}_rH_s^p = a_m\ {}_rH_s^p(a_m+) - (b_j - 1)\ {}_rH_s^p(b_j-).$$

Now if $\ell \in \mathbb{N} \cup \{0\}$ then

$$\begin{aligned}
\theta\ {}_rH_s^p &= \sum_{n=0}^{\infty} \varphi_n\ \theta z^n \\
&= \sum_{n=0}^{\infty} \frac{\left\{\prod_{i=1}^{r}(a_i)_{n+1}\right\}}{\left\{\prod_{j=1}^{s}(b_j)_{n+1}\right\}\ \left\{\prod_{k=1}^{p}(c_k)_{n+1}\right\}^{\ell n+\ell}} \frac{z^{n+1}}{n!} \\
&= z\sum_{n=0}^{\infty} \frac{\left\{\prod_{i=1}^{r}(a_i)_{n+1}\right\}}{\left\{\prod_{j=1}^{s}(b_j)_{n+1}\right\}\ \left\{\prod_{k=1}^{p}(c_k)_n\ (c_k+n)\right\}^{\ell n+\ell}} \frac{z^{n}}{n!} \\
&= z\sum_{n=0}^{\infty} \frac{\left\{\prod_{i=1}^{r}(a_i)_{n+1}\right\}\left\{\prod_{k=1}^{p}\left(\frac{c_k}{c_k+n}\right)\right\}^{\ell n}}{\left\{\prod_{j=1}^{s}(b_j)_{n+1}\right\}\left\{\prod_{k=1}^{p}(c_k)_{n+1}^{\ell}\ (c_k)_n^{\ell n}\right\}\left\{\prod_{k=1}^{p}(c_k)^{\ell}\right\}^{n}} \frac{z^{n}}{n!} \\
&= \frac{A}{B\ C}\ z\sum_{n=0}^{\infty} \frac{\left\{\prod_{i=1}^{r}\frac{(a_i+n)}{a_i}\right\}}{\left\{\prod_{j=1}^{s}\frac{(b_j+n)}{b_j}\right\}} \frac{\left\{\prod_{k=1}^{p}\frac{c_k}{(c_k+n)}\right\}^{\ell n}}{\left\{\prod_{k=1}^{p}\frac{(c_k+n)}{c_k}\right\}^{\ell}} \frac{\varphi_n\ z^n}{\left\{\prod_{k=1}^{p}(c_k)^{\ell n}\right\}} \\
&= \frac{A}{B\ C}\ z\ {}_rH_{s+\ell}^p\left[\begin{array}{cc} (a)+1; & \frac{z}{C} \\ (b)+1,((c)+1)^{\ell};\quad ((c)+1:\ell); & \end{array}\right]. \qquad (4.22)
\end{aligned}$$

Here by eliminating $\theta$ from (4.19) and (4.22) gives the desired relation:

$$\begin{aligned}
a_i\ {}_rH_s^p &= a_i\ {}_rH_s^p(a_i+) \\
&\quad -\frac{A}{B\ C}\ z\ {}_rH_{s+\ell}^p\left[\begin{array}{cc} (a)+1; & \frac{z}{C} \\ (b)+1,((c)+1)^{\ell};\quad ((c)+1:\ell); & \end{array}\right].
\end{aligned}$$

On the other hand, the relation:

$$\begin{aligned}
(b_j-1)\ {}_rH_s^p &= (b_j-1)\ {}_rH_s^p(b_j-) - \frac{A}{B\ C}\ z \\
&\quad \times {}_rH_{s+\ell}^p\left[\begin{array}{cc} (a)+1; & \frac{z}{C} \\ (b)+1,((c)+1)^{\ell};\quad ((c)+1:\ell); & \end{array}\right]
\end{aligned}$$

is obtained by the eliminating $\theta$ from (4.20) and (4.22).

To prove (4.16), let $\ell \in \mathbb{N} \cup \{0\}$ in

$$ {}_1H_1^1 = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n\ (c)_n^{\ell n}} \frac{z^n}{n!}. $$

Then

$$
\begin{aligned}
\theta\ {}_1H_1^1(c-) &= \sum_{n=1}^{\infty} \frac{(a)_n}{(b)_n\ (c-1)_n^{\ell n}} \frac{z^n}{(n-1)!} \\
&= \sum_{n=1}^{\infty} \frac{(a)_n}{(b)_n\ (c-1)^{\ell n}\ (c)_{n-1}^{\ell n}} \frac{z^n}{(n-1)!} \\
&= z \sum_{n=0}^{\infty} \frac{a+n}{b+n} \frac{(a)_n}{(b)_n\ (c-1)^{\ell n+\ell}\ (c)_n^{\ell n+\ell}} \frac{z^n}{n!} \\
&= \frac{z}{(1-c)^{\ell}} \sum_{n=0}^{\infty} \frac{a+n}{b+n} \frac{(a)_n}{(b)_n\ (c)_n^{\ell}\ (c)_n^{\ell n}} \frac{z^n/(1-c)^{\ell n}}{n!}.
\end{aligned}
$$

Now, writing

$$ \frac{a+n}{b+n} = 1 + \frac{a-b}{b+n}, $$

this further gives

$$
\begin{aligned}
\theta\ {}_1H_1^1(c-) &= \frac{z}{(1-c)^{\ell}}\ {}_1H_{1+\ell}^1 \left[ \begin{matrix} a; & & \frac{z}{(1-c)^{\ell}} \\ b, c, c, \ldots, c; & (c:\ell); & \end{matrix} \right] \\
&\quad + \frac{z(a-b)}{b\,(1-c)^{\ell}}\ {}_1H_{1+\ell}^1 \left[ \begin{matrix} a; & & \frac{z}{(1-c)^{\ell}} \\ b+1, c, c, \ldots, c; & (c:\ell); & \end{matrix} \right]
\end{aligned} \tag{4.23}
$$

Also

$$
\begin{aligned}
\theta\ {}_1H_1^1 &= \sum_{n=1}^{\infty} \frac{(a)_n}{(b)_n\ (c)_n^{\ell n}} \frac{z^n}{(n-1)!} \\
&= \frac{az}{b} \sum_{n=1}^{\infty} \frac{(a+1)_{n-1}}{(b+1)_{n-1}\ (c+1)_{n-1}^{\ell n}} \frac{z^{n-1}}{c^{\ell n}(n-1)!} \\
&= ab^{-1}z \sum_{n=0}^{\infty} \frac{(a+1)_n}{(b+1)_n\ (c+1)_n^{\ell n+\ell}} \frac{z^n}{c^{\ell n+\ell} n!} \\
&= \frac{az}{b\ c^{\ell}} \sum_{n=0}^{\infty} \frac{(a+1)_n}{(b+1)_n\ (c+1)_n^{\ell}\ (c+1)_n^{\ell n}} \frac{z^n/c^{\ell}}{n!} \\
&= \frac{az}{b\ c^{\ell}}\ {}_1H_{1+\ell}^1 \left[ \begin{matrix} a+1; & & \frac{z}{c^{\ell}} \\ b+1, c+1, c+1, \ldots, c+1; & (c+1:\ell); & \end{matrix} \right].
\end{aligned}
$$

Hence

$$\theta\ {}_1H_1^1(c-) = \frac{az}{b\,(c-1)^{\ell}}\ {}_1H_{1+\ell}^1\left[\begin{array}{ccc} a+1; & & \frac{z}{(c-1)^{\ell}} \\ b+1,c,c,\ldots,c; & (c:\ell); & \end{array}\right]. \quad (4.24)$$

Elimination of $\theta\ {}_1H_1^1(c-)$ from (4.23) and (4.24) yields (4.16). □

*Remark* 4.13. By taking $\ell = 0$ in (4.13), (4.14), (4.15) and (4.16), they reduce to the contiguous function relations of hypergeometric function respectively which are given below. For $j = 2, 3, \ldots, s$,

$$\begin{aligned}(a_1 - b_j + 1)\ {}_pF_q &= a_1\ {}_pF_q(a_1+) - (b_j - 1)\ {}_pF_q(b_j-),\\ a_i\ {}_pF_q &= a_i\ {}_pF_q(a_i+) - \frac{A}{B}\ z\ {}_pF_q\left[\begin{array}{cc}(a)+1; & z\\ (b)+1; & \end{array}\right],\\ (b_j - 1)\ {}_pF_q &= (b_j - 1)\ {}_pF_q(b_j-) - \frac{A}{B}\ {}_pF_q\left[\begin{array}{cc}(a)+1; & z\\ (b)+1; & \end{array}\right],\end{aligned}$$

and

$${}_1F_1\left[\begin{array}{cc} a; & z\\ b; & \end{array}\right] = (1 - ab^{-1})\ {}_1F_1\left[\begin{array}{cc} a; & z\\ b+1; & \end{array}\right] + ab^{-1}\ {}_1F_1\left[\begin{array}{cc} a+1; & z\\ b+1; & \end{array}\right].$$

These are found to be true.

### 4.2.6 Eigen function property

The eigen function property of ℓ-H function ${}_2H_1^2(\ell : z)$ obtained in Theorem 2.20 when extended in straight forward manner yields the eigen function property of the generalized ℓ-H function. For that the operator (2.17) must be put in generalized form. This is given below.

**Definition 4.14.** Let $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$, $0 \neq |z| < R, R > 0$, $l, m, n \in \mathbb{N}$, and $\alpha_i, \beta_j, \gamma_k \in \mathbb{C}$ with $\Re(\alpha_i) \geq 0$, and as before $\theta = z\ \frac{\mathrm{d}}{\mathrm{d}z}$. Define the operator

$$\begin{aligned}&{}_{(\alpha,l)}\mathcal{H}_{(\beta,m)}^{(\gamma,n)} f(z)\\ &= \left[\left\{\prod_{i=1}^{l} I_{\alpha_i}\right\} z^{-1} \left\{\prod_{k=1}^{n}\ {}_{\ell}\Delta_{\gamma_k}^{\theta}\right\} \left\{\prod_{j=1}^{m}(\theta + \beta_j - 1)\right\} \theta\right] f(z), \quad (4.25)\end{aligned}$$

where

$$I_\alpha(f(z)) = z^{-\alpha} \int\limits_0^z t^{\alpha-1} f(t) \ \mathrm{dt} \tag{4.26}$$

and ${}_\ell\Delta^\theta_{\gamma_k}$ is as defined in (4.6).

This operator enables one to derive the eigen function property which is proved in

**Theorem 4.15.** *If $z \neq 0$, and $\Re(\alpha_i) \geq 0, \forall i = 1, 2, \ldots, r$, then the function*

$$w = {}_rH_s^p \left[ \begin{array}{llllllll} a_1, & a_2, & \ldots, & a_r; & & & & z \\ b_1, & b_2, & \ldots, & b_s; & (c_1, & c_2, & \ldots, & c_p : \ell); \end{array} \right]$$

*is an eigen function with respect to the operator ${}_{(a,r)}\mathcal{H}^{(c,p)}_{(b,s)}$ defined in (4.25).*

$$\textit{That is,} \quad {}_{(a,r)}\mathcal{H}^{(c,p)}_{(b,s)} \left({}_rH_s^p(\ell; \lambda z)\right) = \lambda \ {}_rH_s^p(\ell; \lambda z), \ \ \lambda \in \mathbb{C}. \tag{4.27}$$

**Note 4.16.** *Once again in view of the Lemma 4.11, the applicability of the operator ${}_{(a,r)}\mathcal{H}^{(c,p)}_{(b,s)}$ to the generalized ℓ-H function follows.*

*Proof.* In (4.11) using the notation

$$B_n = \frac{(a_1)_n \ (a_2)_n \cdots (a_r)_n \ \lambda^n}{(b_1)_{n-1} \ (b_2)_{n-1} \cdots (b_s)_{n-1} \ (c_1)^{\ell n-\ell}_{n-1} \ (c_2)^{\ell n-\ell}_{n-1} \cdots (c_p)^{\ell n-\ell}_{n-1} \ (n-1)!},$$

one gets

$$\begin{aligned}
&{}_{(a,r)}\mathcal{H}^{(c,p)}_{(b,s)} \left({}_rH_s^p(\ell; \lambda z)\right) \\
&= \left[ \left\{ \prod_{i=1}^r I_{a_i} \right\} z^{-1} \left\{ \prod_{k=1}^p {}_\ell\Delta^\theta_{c_k} \right\} \left\{ \prod_{j=1}^s (\theta + b_j - 1) \right\} \theta \right] \sum_{n=0}^\infty \varphi_n \ \lambda^n \ z^n \\
&= \left\{ \prod_{i=1}^r I_{a_i} \right\} z^{-1} \sum_{n=1}^\infty B_n \ z^n \\
&= \left\{ \prod_{i=1}^{r-1} I_{a_i} \right\} I_{a_r} \sum_{n=1}^\infty B_n \ z^{n-1} \\
&= \left\{ \prod_{i=1}^{r-1} I_{a_i} \right\} \ z^{-a_r} \int\limits_0^z t^{a_r-1} \ \sum_{n=1}^\infty B_n \ t^{n-1} \\
&= \left\{ \prod_{i=1}^{r-1} I_{a_i} \right\} \ z^{-a_r} \sum_{n=1}^\infty \frac{B_n}{a_r + n - 1} \ z^{a_r+n-1}
\end{aligned}$$

$$= \left\{\prod_{i=1}^{r-1} I_{a_i}\right\} \sum_{n=1}^{\infty} \frac{B_n}{a_r+n-1}\, z^{n-1}.$$

Applying in this manner the operator $I_{\alpha_i}$ for $i = 1, 2, \ldots, r-1$, one finally obtains

$$\begin{aligned} {}_{(a,r)}\mathcal{H}_{(b,s)}^{(c,p)}\left({}_rH_s^p(\ell; \lambda z)\right) &= \sum_{n=1}^{\infty} \left\{\prod_{i=1}^{r}(a_i+n-1)\right\}^{-1} B_n\, z^{n-1} \\ &= \sum_{n=0}^{\infty} \varphi_n\, \lambda^{n+1}\, z^n \\ &= \lambda\, {}_rH_s^p(\ell; \lambda z). \end{aligned}$$

□

## 4.3 Special cases

When the parameters $a_i$'s; $i = 2, 3, \ldots r$ and $b_j$'s are all absent and $a_1 = c_1 = \ell = 1$ in (4.1), then

$${}_1H_0^1\left[\begin{matrix} 1; & & z \\ -; & (1:1); & \end{matrix}\right] = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!^n}.$$

Thus,

$${}_1H_0^1\left[\begin{matrix} 1; & & z \\ -; & (1:1); & \end{matrix}\right] - 1$$

gives the Sikemma's function (1.8).

### 4.3.1 The ℓ-H exponential function

In (4.1), if $r = s, p = 1$ with all $a_i = b_j$ and $c_1 = c$, then

$${}_0H_0^1\left[\begin{matrix} -; & & z \\ -; & (c:\ell); & \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{z^n}{(c)_n^{\ell n}\; n!}. \tag{4.28}$$

This defines the ℓ-H exponential function as follows.

**Definition 4.17.** The ℓ-H exponential function is denoted and defined by

$$e_H^{\ell}(z) = {}_0H_0^1\left[\begin{matrix} -; & & z \\ -; & (1:\ell); & \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{\ell n+1}}, \tag{4.29}$$

for all $z \in \mathbb{C}$ and $\Re(\ell) \geq 0$.

*Remark* 4.18. Obviously, $e_H^0(z) = e^z$ *and* $e_H^\ell(0) = 1$.

*Remark* 4.19. The differential equation (4.7) when $\ell = 0$, gets reduced to the differential equation

$$(\theta - z)\; w = 0,$$

where $w = e^z$.

In order to derive the eigen function property for the $\ell$-H exponential function, the operator defined in Definition 4.14 is particularized by removing all the parameters $a_i$'s and $b_j$'s, and $c_k$'s for $k = 2, 3, \ldots, p$ and taking $c_1 = 1$.

**Definition 4.20.** Let $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$, $|z| < R, R > 0$. Define the operator

$$ {}_p\mathcal{D}_M^{(z)}\,(f(z)) = z^{-1}\;{}_p\Delta_1^\theta\;(\theta(f(z))), \tag{4.30}$$

where $z \neq 0, p \in \mathbb{N} \cup \{0\}$ and the operator ${}_p\Delta_1^\theta$ is as defined in (2.7).

*Remark* 4.21. When the parameters $a_i$'s; $b_j$'s and $c_k$'s; $k = 2, 3, \ldots, p$ are absent and $c_1 = 1$ then

$$ {}_p\mathcal{D}_M^{(z)} = \mathcal{H}^{(1:1)}$$

where the operator $\mathcal{H}^{(1:1)}$ is defined in Definition 4.14 as $z^{-1}\;{}_p\Delta_1^\theta\;\theta$.

It can be noticed that if $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum\limits_{n=0}^{\infty} b_n z^n$, $|z| < R$ then for $\alpha, \beta \in \mathbb{R}$

$$ {}_p\mathcal{D}_M^{(z)}\,(\alpha f(z) + \beta g(z)) = \alpha\;\;{}_p\mathcal{D}_M^{(z)}(f(z)) + \beta\;{}_p\mathcal{D}_M^{(z)}(g(z)). \tag{4.31}$$

In view of Lemma 4.11, the operator ${}_\ell\mathcal{D}_M^{(z)}$ is applicable to the $\ell$-H exponential function, consequently leading to

**Theorem 4.22.** *With $\ell \in \mathbb{N} \cup \{0\}$, the $\ell$-H exponential function is the eigen function of the operator ${}_\ell\mathcal{D}_M^{(z)}$ as defined in (4.30), that is,*

$$ {}_\ell\mathcal{D}_M^{(z)}\,\left(e_H^\ell(\lambda z)\right) = \lambda\; e_H^\ell(\lambda z),\;\; \lambda \in \mathbb{C}. \tag{4.32}$$

*Proof.* From (4.30),

$$ {}_\ell\mathcal{D}_M^{(z)}\,\left(e_H^\ell(\lambda z)\right) \;\;=\;\; z^{-1}\;{}_p\Delta_1^\theta\left(\sum_{n=0}^{\infty} \frac{\lambda^n}{(n!)^{\ell n+1}}(\theta z^n)\right)$$

$$
\begin{aligned}
&= z^{-1} \sum_{n=1}^{\infty} \frac{\lambda^n}{(n!)^{\ell n}(n-1)!} \; {}_{\ell}\Delta_1^{\theta}(z^n) \\
&= z^{-1} \sum_{n=1}^{\infty} \frac{\lambda^n}{(n!)^{\ell n}(n-1)!} \; ((n-1)!)^{\ell} \; \theta^{\ell n}(z^n). \qquad (4.33)
\end{aligned}
$$

Now

$$
\begin{aligned}
(\theta)^{\ell} z &= z (= (1^1)^{\ell} z), \\
(\theta)^{2\ell} z^2 &= \underbrace{\left( z\frac{d}{dz} z\frac{d}{dz} \dots \frac{d}{dz} z\frac{d}{dz} \right)}_{2\ell \ derivatives} z^2 \\
&= (2^2)^{\ell} z^2,
\end{aligned}
$$

and in general,

$$
(\theta)^{\ell n} z^n = n^{\ell n} z^n, \; n = 1, 2, \dots, \qquad (4.34)
$$

hence using (4.34) in (4.33), one gets

$$
\begin{aligned}
{}_{\ell}\mathcal{D}_M^{(z)} \left( e_H^{\ell}(\lambda z) \right) &= z^{-1} \sum_{n=1}^{\infty} \frac{\lambda^n}{(n!)^{\ell n}((n-1)!)^{1-\ell}} \; n^{\ell n} z^n \\
&= \sum_{n=1}^{\infty} \frac{\lambda^n}{((n-1)!)^{\ell n+1-\ell}} \; z^{n-1} \\
&= \sum_{n=0}^{\infty} \frac{\lambda^{n+1} \; z^n}{(n!)^{\ell n+1}} \\
&= \lambda \; e_H^{\ell}(\lambda z).
\end{aligned}
$$

□

It is interesting to see that from (4.29), one further finds

$$
\begin{aligned}
e_H^{\ell}(iz) &= \sum_{n=0}^{\infty} \frac{(iz)^n}{(n!)^{\ell n+1}} \\
&= \sum_{n=0}^{\infty} \frac{(i)^{2n} \; z^{2n}}{((2n)!)^{2\ell n+1}} + \sum_{n=0}^{\infty} \frac{(i)^{2n+1} \; z^{2n+1}}{((2n+1)!)^{2\ell n+\ell+1}} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n \; z^{2n}}{((2n)!)^{2\ell n+1}} + i \sum_{n=0}^{\infty} \frac{(-1)^n \; z^{2n+1}}{((2n+1)!)^{2\ell n+\ell+1}}. \qquad (4.35)
\end{aligned}
$$

These infinite series are resembling with those of cosine and sine series. They are further taken up in the next Subsection.

### 4.3.2 The $\ell$-H trigonometric functions

The first and second series on the right hand side of (4.35), give rise to the extended cosine and sine functions respectively which are denoted here by $\cos_H^\ell(z)$ and $\sin_H^\ell(z)$. In fact, for any $z \in \mathbb{C}$,

$$\Re(e_H^\ell(iz)) = \Re\left({}_0H_0^1\left[\begin{matrix} -; & & iz \\ -; & (1:\ell); & \end{matrix}\right]\right) := \cos_H^\ell(z), \tag{4.36}$$

and

$$\Im(e_H^\ell(iz)) = \Im\left({}_0H_0^1\left[\begin{matrix} -; & & iz \\ -; & (1:\ell); & \end{matrix}\right]\right) := \sin_H^\ell(z), \tag{4.37}$$

whence one obtains from (4.35),

$$e_H^\ell(iz) = \cos_H^\ell(z) + i\sin_H^\ell(z). \tag{4.38}$$

*Remark* 4.23. It is noteworthy that $\cos_H^0(z) = \cos z$, and $\sin_H^0(z) = \sin z$.

Further,

$$\begin{aligned}
\frac{1}{2}\left[e_H^\ell(iz) + e_H^\ell(-iz)\right] &= \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{(iz)^n}{(n!)^{\ell n+1}} + \sum_{n=0}^{\infty}\frac{(-iz)^n}{(n!)^{\ell n+1}}\right] \\
&= \frac{1}{2}\Bigg[1 + \frac{iz}{(1!)^{\ell+1}} + \frac{(iz)^2}{(2!)^{2\ell+1}} + \dots \\
&\qquad +1 + \frac{-iz}{(1!)^{\ell+1}} + \frac{(iz)^2}{(2!)^{2\ell+1}} + \dots\Bigg] \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n\ (z)^{2n}}{((2n)!)^{2\ell n+1}} \\
&= \cos_H^\ell(z),
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2i}\left[e_H^\ell(iz) - e_H^\ell(-iz)\right] &= \frac{1}{2i}\left[\sum_{n=0}^{\infty}\frac{(iz)^n}{(n!)^{\ell n+1}} - \sum_{n=0}^{\infty}\frac{(-iz)^n}{(n!)^{\ell n+1}}\right] \\
&= \frac{1}{2i}\Bigg[1 + \frac{iz}{(1!)^{\ell+1}} + \frac{(iz)^2}{(2!)^{2\ell+1}} + \dots \\
&\qquad -1 - \frac{-iz}{(1!)^{\ell+1}} - \frac{(iz)^2}{(2!)^{2\ell+1}} - \dots\Bigg]
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \; z^{2n+1}}{((2n+1)!)^{2\ell n+\ell+1}}$$
$$= \sin_H^\ell(z).$$

Also,

$$\cos_H^\ell(0) = \frac{1}{2}\left[e_H^\ell(0) + e_H^\ell(0)\right] = 1,$$
$$\sin_H^\ell(0) = \frac{1}{2i}\left[e_H^\ell(0) - e_H^\ell(0)\right] = 0.$$

*Remark* 4.24. The operator (4.30) yields the identities:

1. ${}_\ell\mathcal{D}_M^{(z)}(\cos_H^\ell(z)) = -\sin_H^\ell(z)$,
2. ${}_\ell\mathcal{D}_M^{(z)}(\sin_H^\ell(z)) = \cos_H^\ell(z)$.

Just as the functions $\sin z$ and $\cos z$ are solutions of the equation $\frac{d^2y}{dz^2}+y=0$, the $\ell$-H sine and the $\ell$-H cosine functions are also solution of a differential equation. This is shown in

**Theorem 4.25.** *The $\ell$-H cosine and the $\ell$-H sine functions are solutions of the differential equation*

$$\left({}_\ell\mathcal{D}_M^{(z)}\right)^2 \nu + \nu = 0.$$

*Proof.* It may be noted from Theorem 4.22, that

$${}_\ell\mathcal{D}_M^{(z)}(e_H^\ell(iz)) = i(e_H^\ell(iz))).$$

Hence,

$$\left({}_\ell\mathcal{D}_M^{(z)}\right)^2 (e_H^\ell(iz)) = {}_\ell\mathcal{D}_M^{(z)}(\;{}_\ell\mathcal{D}_M^{(z)}(e_H^\ell(iz))) = {}_\ell\mathcal{D}_M^{(z)}(i(e_H^\ell(iz))) = -e_H^\ell(iz).$$

Now in view of (4.38), this may be written as

$$\left({}_\ell\mathcal{D}_M^{(z)}\right)^2 (\cos_H^\ell(z) + i\sin_H^\ell(z)) = -(\cos_H^\ell(z) + i\sin_H^\ell(z)).$$

By making an appeal to the property (4.31) and comparing the real and imaginary parts, one finds

$$\left({}_\ell\mathcal{D}_M^{(z)}\right)^2 (\cos_H^\ell(z)) + \cos_H^\ell(z) = 0 \quad \textit{and} \quad \left({}_\ell\mathcal{D}_M^{(z)}\right)^2 (\sin_H^\ell(z)) + \sin_H^\ell(z) = 0.$$

□

### 4.3.3 The ℓ-H hyperbolic functions

Again splitting the series of the ℓ-H exponential function (4.29) into even-odd powers of $z$, it takes the form:

$$\begin{aligned} e_H^\ell(z) &= \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{\ell n+1}} \\ &= \sum_{n=0}^{\infty} \frac{z^{2n}}{((2n)!)^{2\ell n+1}} + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{((2n+1)!)^{2\ell n+\ell+1}}. \end{aligned} \tag{4.39}$$

If the first series (with even powers of $z$) on the right hand side is denoted by (cf. [14])

$$\mathcal{E}(e_H^\ell(z)) = \mathcal{E}\left({}_0H_0^1\left[\begin{matrix} -; & & z \\ -; & (1:\ell); & \end{matrix}\right]\right) = \cosh_H^\ell(z) \tag{4.40}$$

which may be called the hyperbolic ℓ-H cosine function and the second series (with odd powers of $z$) on right hand side by (cf. [14])

$$\mathcal{O}(e_H^\ell(z)) = \mathcal{O}\left({}_0H_0^1\left[\begin{matrix} -; & & z \\ -; & (1:\ell); & \end{matrix}\right]\right) = \sinh_H^\ell(z) \tag{4.41}$$

which may be termed as the hyperbolic ℓ-H sine function, then from (4.39),

$$e_H^\ell(z) = \cosh_H^\ell(z) + \sinh_H^\ell(z). \tag{4.42}$$

*Remark* 4.26. $\cosh_H^0(z) = \cosh(z)$, and $\sinh_H^0(z) = \sinh(z)$.

Also

$$\begin{aligned} \frac{1}{2}\left[e_H^\ell(z) + e_H^\ell(-z)\right] &= \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{z^n}{(n!)^{\ell n+1}} + \sum_{n=0}^{\infty} \frac{(-z)^n}{(n!)^{\ell n+1}}\right] \\ &= \frac{1}{2}\left[1 + \frac{z}{(1!)^{\ell+1}} + \frac{z^2}{(2!)^{2\ell+1}} + \dots \right. \\ &\quad \left. +1 + \frac{-z}{(1!)^{\ell+1}} + \frac{z^2}{(2!)^{2\ell+1}} + \dots\right] \\ &= \sum_{n=0}^{\infty} \frac{z^{2n}}{((2n)!)^{2\ell n+1}} \\ &= \cosh_H^\ell(z), \end{aligned} \tag{4.43}$$

and

$$\begin{aligned}
\frac{1}{2}\left[e_H^\ell(z) - e_H^\ell(-z)\right] &= \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{z^n}{(n!)^{\ell n+1}} - \sum_{n=0}^{\infty}\frac{(-z)^n}{(n!)^{\ell n+1}}\right]\\
&= \frac{1}{2}\Bigg[1 + \frac{z}{(1!)^{\ell+1}} + \frac{z^2}{(2!)^{2\ell+1}} + \dots\\
&\qquad -1 - \frac{-z}{(1!)^{\ell+1}} - \frac{z^2}{(2!)^{2\ell+1}} - \dots\Bigg]\\
&= \sum_{n=0}^{\infty}\frac{z^{2n+1}}{((2n+1)!)^{2\ell n+\ell+1}}\\
&= \sinh_H^\ell(z).
\end{aligned}$$

In particular,

$$\cosh_H^\ell(0) = \frac{1}{2}\left[e_H^\ell(0) + e_H^\ell(0)\right] = 1,\ \sinh_H^\ell(0) = \frac{1}{2}\left[e_H^\ell(0) - e_H^\ell(0)\right] = 0.$$

In parallel to Theorem 4.25, the following is

**Theorem 4.27.** *The hyperbolic ℓ-H cosine and the hyperbolic ℓ-H sine functions are solutions of the differential equation*

$$\left({}_\ell\mathcal{D}_M^{(z)}\right)^2 \nu - \nu = 0.$$

*Proof.* One can see that from (4.43), (4.31) and (4.32),

$$\begin{aligned}
&\left({}_\ell\mathcal{D}_M^{(z)}\right)^2 (\cosh_H^\ell(z)) - \cosh_H^\ell(z)\\
&= \left({}_\ell\mathcal{D}_M^{(z)}\right)^2 \left(\frac{e_H^\ell(z) + e_H^\ell(-z)}{2}\right) - \left(\frac{e_H^\ell(z) + e_H^\ell(-z)}{2}\right)\\
&= \frac{1}{2}\left[e_H^\ell(z) + e_H^\ell(-z) - e_H^\ell(z) - e_H^\ell(-z)\right]\\
&= 0.
\end{aligned}$$

Likewise,

$$\begin{aligned}
&\left({}_\ell\mathcal{D}_M^{(z)}\right)^2 (\sinh_H^\ell(z)) - \sinh_H^\ell(z)\\
&= \left({}_\ell\mathcal{D}_M^{(z)}\right)^2 \left(\frac{e_H^\ell(z) - e_H^\ell(-z)}{2}\right) - \left(\frac{e_H^\ell(z) - e_H^\ell(-z)}{2}\right)\\
&= \frac{1}{2}\left[e_H^\ell(z) - e_H^\ell(-z) - e_H^\ell(z) + e_H^\ell(-z)\right]\\
&= 0.
\end{aligned}$$

□

*Remark* 4.28. The new functions (4.1) can evidently be considered as extensions of the generalized hypergeometric function ${}_rF_{s+p}$ (see [18, Ch.4]), reduced to the so-called hyper-Bessel functions ${}_0F_p$ if $r = s; a_i = b_j$, and being eigen functions of the hyper-Bessel operators ([33, 36]) where $p$ in the second index goes to infinity together with the summation index $n$ in the power series.

### 4.3.4 Graphs

Following are the graphs of particular ℓ-H functions.

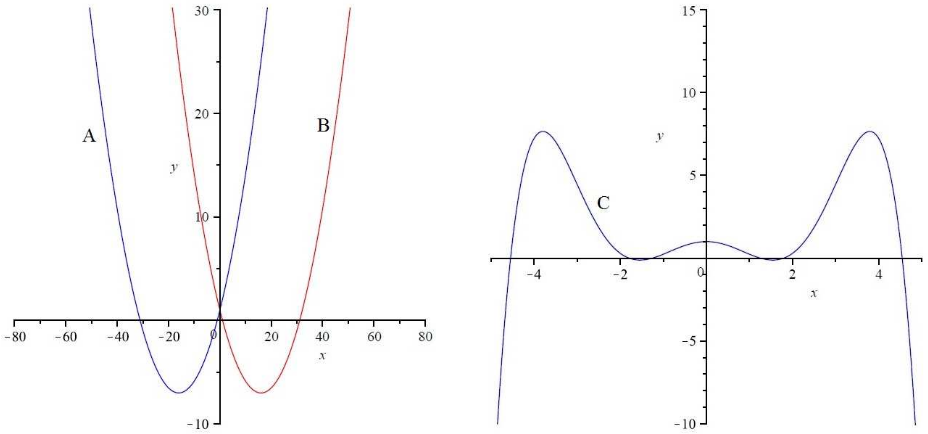


FIGURE 4.1: A: Graph of $e_H^2(x)$; B: Graph of $e_H^2(-x)$ and C: Graph of $e_H^{\frac{1}{2}}(-x^2)$

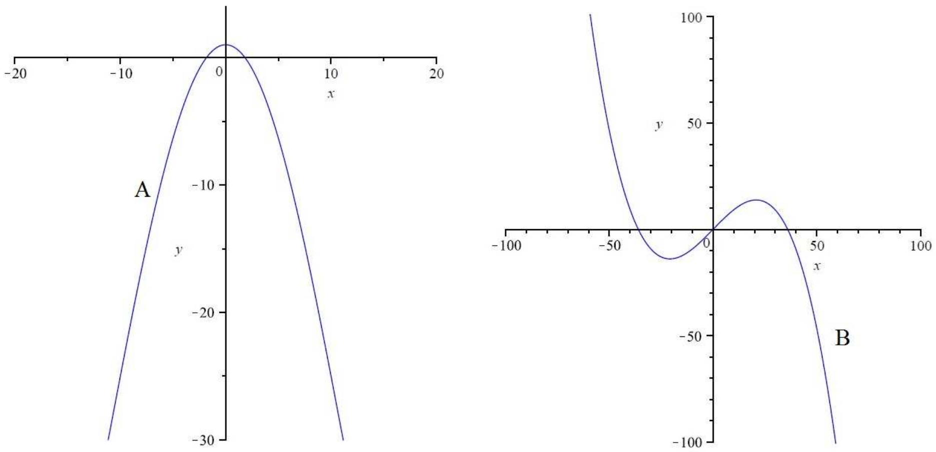


FIGURE 4.2: A: Graph of $\cos_H^{\frac{5}{14}}(x)$ and B: Graph of $\sin_H^1(x)$

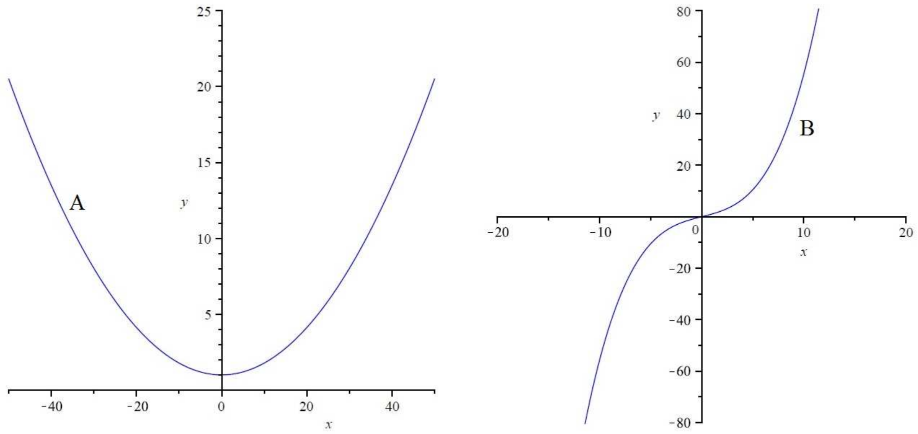


FIGURE 4.3: A: Graph of $\cosh_H^3(x)$ and B: Graph of $\sinh_H^{\frac{1}{4}}(x)$

### 4.3.5 $\ell$-Analogues of Ramanujan's theorem and Kummer's first formula

It is noteworthy that the generalized $\ell$-H-function provides $\ell$-extensions to (i) the Ramanujan's theorem [46, Ex.5, p.106]:

$$ {}_1F_1\begin{bmatrix} a; & x \\ b; & \end{bmatrix} {}_1F_1\begin{bmatrix} a; & -x \\ b; & \end{bmatrix} = {}_2F_3\begin{bmatrix} & a, & b-a; & \frac{x^2}{4} \\ b, & \frac{b}{2}, & \frac{b}{2}+\frac{1}{2}; & \end{bmatrix}, \tag{4.44}$$

and (ii) the Kummer's first formula [46, p.125]:

$$ e^{-z}\ {}_1F_1\begin{bmatrix} a; & z \\ b; & \end{bmatrix} = {}_1F_1\begin{bmatrix} b-a; & -z \\ b; & \end{bmatrix}. \tag{4.45}$$

This is shown below.

**Theorem 4.29.** *($\ell$-Analogue of Ramanujan's Theorem)*
*If $\ell \in \mathbb{N}\cup\{0\}$ then*

$$ {}_1H_1^1\begin{bmatrix} a; & & x \\ b; & (c:\ell); & \end{bmatrix} {}_1H_1^1\begin{bmatrix} a; & & -x \\ b; & (c:\ell); & \end{bmatrix} $$
$$ = \sum_{n=0}^{\infty} \frac{(a)_n\ x^n}{(b)_n\ (c)_n^{\ell n}\ n!} $$
$$ \times\ {}_{3+\ell n}H_2^2\begin{bmatrix} -n, a, 1-b-n, (1-c-n)^{\ell n}; & & (-1)^{\ell(n-1)}(c)_n^{\ell} \\ b, 1-a-n; & (c, 1-c-n:\ell); & \end{bmatrix}. $$

*Proof.* From the definition of the generalized $\ell$-function (4.1) and the formula:

$$(\alpha)_{n-k} = \frac{(-1)^k\,(\alpha)_n}{(1-\alpha-n)_k},$$

$$\begin{aligned}
&{}_1H_1^1\left[\begin{matrix} a; & & x \\ b; & (c:\ell); & \end{matrix}\right]\;{}_1H_1^1\left[\begin{matrix} a; & & -x \\ b; & (c:\ell); & \end{matrix}\right] \\
&= \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n\,(c)_n^{\ell n}} \frac{x^n}{n!} \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k\,(c)_k^{\ell k}} \frac{(-x)^k}{k!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(a)_{n-k}}{(b)_{n-k}\,(c)_{n-k}^{\ell n-\ell k}} \frac{x^n}{(n-k)!}\; \frac{(a)_k}{(b)_k\,(c)_k^{\ell k}} \frac{(-1)^k}{k!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k (a)_n}{(1-a-n)_k} \frac{(1-b-n)_k}{(-1)^k (b)_n} \left[\frac{(1-c-n)_k}{(-1)^k (c)_n}\right]^{\ell n-\ell k} \\
&\quad \times \frac{(-n)_k}{(-1)^k n!}\; \frac{(a)_k}{(b)_k\,(c)_k^{\ell k}} \frac{(-1)^k}{k!}\; x^n \\
&= \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n\,(c)_n^{\ell n}} \frac{x^n}{n!} \\
&\quad \times \sum_{k=0}^{n} (-1)^{\ell(n-1)k}\; \frac{(-n)_k\,(a)_k\,(1-b-n)_k\,(1-c-n)_k^{\ell n}\,(c)_n^{\ell k}}{(b)_k\,(1-a-n)_k\,(c)_k^{\ell k}\,(1-c-n)_k^{\ell k}\;k!} \\
&= \sum_{n=0}^{\infty} \frac{(a)_n\,x^n}{(b)_n\,(c)_n^{\ell n}\;n!} \\
&\quad \times {}_{3+\ell n}H_2^2\left[\begin{matrix} -n, a, 1-b-n, (1-c-n)^{\ell n}; & & (-1)^{\ell(n-1)}(c)_n^{\ell} \\ b, 1-a-n; & (c, 1-c-n:\ell); & \end{matrix}\right].
\end{aligned}$$

□

*Remark* 4.30. It may be seen that for $\ell = 0$, Theorem 4.29 gets reduced to the Ramanujan's theorem (4.44). This may be verified as follows.

For $\ell = 0$, from Remark 4.3, Theorem 4.29 takes the form

$${}_1F_1\left[\begin{matrix} a; & x \\ b; & \end{matrix}\right]\;{}_1F_1\left[\begin{matrix} a; & -x \\ b; & \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a)_n\,x^n}{(b)_n\;n!}\;{}_3F_2\left[\begin{matrix} -n, & a, & 1-b-n; & 1 \\ & 1-a-n, & b; & \end{matrix}\right].$$

Replacing $n$ by $2n$, this becomes

$${}_1F_1\left[\begin{matrix} a; & x \\ b; & \end{matrix}\right]\;{}_1F_1\left[\begin{matrix} a; & -x \\ b; & \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a)_{2n}\,x^{2n}}{(b)_{2n}\;(2n)!}\;{}_3F_2\left[\begin{matrix} -2n, & a, & 1-b-2n; & 1 \\ & 1-a-2n, & b; & \end{matrix}\right].$$

Then Theorem 1.9 together with the formula:

$$(\alpha)_{2n} = 2^{2n} \left(\frac{\alpha}{2}\right)_n \left(\frac{\alpha+1}{2}\right)_n, \; n \in \mathbb{N}, \; \alpha \in \mathbb{C}$$

leads to

$$\begin{aligned} {}_1F_1\left[\begin{matrix} a; & x \\ b; & \end{matrix}\right] {}_1F_1\left[\begin{matrix} a; & -x \\ b; & \end{matrix}\right] &= \sum_{n=0}^{\infty} \frac{(a)_{2n}\; x^{2n}}{(b)_{2n}\; (2n)!} \frac{(2n)!\; (a)_n\; (b-a)_n}{(a)_{2n}\; (b)_n\; n!} \\ &= \sum_{n=0}^{\infty} \frac{(a)_n\; (b-a)_n\; x^{2n}}{2^{2n}\; \left(\frac{b}{2}\right)_n\; \left(\frac{b}{2}+\frac{1}{2}\right)_n\; (b)_n\; n!} \\ &= {}_2F_3\left[\begin{matrix} & a, & b-a; & \frac{x^2}{4} \\ b, & \frac{b}{2}, & \frac{b}{2}+\frac{1}{2}; & \end{matrix}\right]. \end{aligned}$$

**Theorem 4.31.** *(ℓ-Analogue of Kummer's first formula)*
*If $\ell \in \mathbb{N} \cup \{0\}$ and $z \in \mathbb{C}$ then*

$$\begin{aligned} & e_H^{\ell}(-z)\; {}_1H_1^1\left[\begin{matrix} a; & & z \\ b; & (c:\ell); & \end{matrix}\right] \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n\; z^n}{(n!)^{\ell n+1}} \\ &\quad \times {}_{2+\ell n}H_1^2\left[\begin{matrix} -n, -n, \ldots, -n, & a; & (1)_n^{\ell}(-1)^{\ell(n-1)} \\ b; & (c, -n:\ell); & \end{matrix}\right], \end{aligned} \tag{4.46}$$

*where $e_H^{\ell}(z)$ is the ℓ-H exponential function as defined in (4.29).*

*Proof.* The left hand side

$$\begin{aligned} & e_H^{\ell}(-z)\; {}_1H_1^1\left[\begin{matrix} a; & & z \\ b; & (c:\ell); & \end{matrix}\right] \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n\; z^n}{(n!)^{\ell n+1}} \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k\; (c)_k^{\ell k}} \frac{z^k}{k!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{n-k} \frac{(a)_k}{[(n-k)!]^{\ell n-\ell k+1}\; (b)_k\; (c)_k^{\ell k}} \frac{z^n}{k!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^{n-k} \left[\frac{(-1)^k\; (-n)_k}{n!}\right]^{\ell n-\ell k+1} \frac{(a)_k\; z^n}{(b)_k\; (c)_k^{\ell k}\; k!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n\; z^n}{(n!)^{\ell n+1}} \sum_{k=0}^{n} \frac{(-n)_k^{\ell n+1}\; n!^{\ell k}\; (a)_k}{(-n)_k^{\ell k}\; (b)_k\; (c)_k^{\ell k}\; k!} (-1)^{\ell n k-\ell k} \end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\ z^n}{(n!)^{\ell n+1}} \sum_{k=0}^{n} \frac{(-n)_k^{\ell n+1}\ (a)_k\ \left[(n!)^{\ell}(-1)^{\ell(n-1)}\right]^k}{(-n)_k^{\ell k}\ (b)_k\ (c)_k^{\ell k}\ k!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\ z^n}{(n!)^{\ell n+1}}\ {}_{2+\ell n}H_1^2 \left[\begin{matrix} -n, -n, \ldots, -n, a; & & (n!)^{\ell}(-1)^{\ell(n-1)} \\ b; & (c, -n : \ell); & \end{matrix}\right].$$

□

*Remark* 4.32. When $\ell = 0$, Theorem 4.31 reduces to the Kummer's first formula (4.45). This can also be verified as follows.

From Remarks 4.18 and 4.3, (4.46) reduced to

$$e^{-z}\ {}_1F_1\left[\begin{matrix} a; & z \\ b; & \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(-1)^n\ z^n}{n!}\ {}_2F_1\left[\begin{matrix} -n, & a; & 1 \\ & b; & \end{matrix}\right].$$

Now the application of Gauss summation formula (stated in Chapter 1 as Theorem 1.8) yields

$$\begin{aligned} e^{-z}\ {}_1F_1\left[\begin{matrix} a; & z \\ b; & \end{matrix}\right] &= \sum_{n=0}^{\infty} \frac{(b-a)_n}{(b)_n} \frac{(-z)^n}{n!} \\ &= {}_1F_1\left[\begin{matrix} b-a; & -z \\ b; & \end{matrix}\right]. \end{aligned}$$

## 4.4 Generalized Maclaurin's Theorem

Let $f$ be an $n$-times differentiable function at $z_0$, then its Taylor series expansion is given by [5] :

$$f(z) = \sum_{n=0}^{\infty} \frac{D^n f(z_0)}{n!}\ (z - z_0)^n, \tag{4.47}$$

where $D^n f$ is the usual $n^{\text{th}}$ order derivative of $f$. At $z_0 = 0$, this expansion is well known as Maclaurin's expansion of $f$. Here a generalized $\ell$-H function analogue of the Maclaurin's expansion of $f(z) = {}_rH_s^p(\ell : z)$ is proved as

**Theorem 4.33.** *For the generalized ℓ-H function,*

$${}_rH_s^p(\ell : z) = \sum_{n=0}^{\infty} \left[D^n\ {}_rH_s^p(\ell : z)\right]_{z=0} \frac{z^n}{n!}, \tag{4.48}$$

*where* $D = \dfrac{d}{dz}$.

*Proof.* From the definition of the generalized ℓ-H function (4.1),

$$\begin{aligned} D\left( {}_rH_s^p(\ell : z)\right) &= \sum_{n=1}^{\infty} \frac{(a_1, a_2, \ldots, a_r)_n}{(b_1, b_2, \ldots, b_s)_n \ (c_1, c_2, \ldots, c_p; q)_n^{\ell n}} \frac{z^{n-1}}{(n-1)!}, \\ D^m\left( {}_rH_s^p(\ell : z)\right) &= \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r)_{n+m}}{(b_1, b_2, \ldots, b_s)_{n+m} \ (c_1, c_2, \ldots, c_p; q)_{n+m}^{\ell n+\ell m}} \ \frac{z^n}{n!}, \end{aligned}$$

and

$$\left[D^m\left( {}_rH_s^p(\ell : z)\right)\right]_{z=0} = \frac{(a_1, a_2, \ldots, a_r)_m}{(b_1, b_2, \ldots, b_s)_m \ (c_1, c_2, \ldots, c_p; q)_m^{\ell m}}.$$

Hence,

$$\begin{aligned} &\sum_{n=0}^{\infty} \left[D^n \ {}_rH_s^p(\ell : z)\right]_{z=0} \ \frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r)_n}{(b_1, b_2, \ldots, b_s)_n \ (c_1, c_2, \ldots, c_p; q)_n^{\ell n}} \ \frac{z^n}{n!} \\ &= {}_rH_s^p(\ell : z). \end{aligned}$$

□

*Remark* 4.34. When $\ell = 0$, (4.48) reduces to the form

$${}_rF_s[z] = \sum_{n=0}^{\infty} \left(D^n \ {}_rF_s[z]\right)_{z=0} \ \frac{z^n}{n!}.$$