Chapter 6

The ℓ-H Bessel function

6.1 Introduction

In this chapter, the classical Bessel function (1.9)

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \; z^{n+2k}}{2^{n+2k} \; k! \; \Gamma(1+n+k)}, \quad (n \text{ non negative integer})$$

is extended by means of ℓ-Hypergeometric function introduced in Chapter 4. In other words, besides the particular cases namely the ℓ-H exponential, trigonometric and hyperbolic functions, one more special case: the Bessel function $J_n(z)$ is now considered with similar objective of examining ℓ-extensions to its certain properties. As the main results, the differential equation, the generating function relation, and Contour integral representations including Schläfli's integral analogue are derived. With the aid of these, other results including some inequalities are also obtained. Thus the properties 1. to 18. as stated in Subsection 1.3.4 of Chapter 1, will be extended here.

It is worth mentioning that the proposed extension of $J_n(z)$ is suggested by the Laurent series expansion of the product of two ℓ-H exponential functions (4.29). The ℓ-H trigonometric functions (Subsection 4.3.2) then give rise to ℓ-analogues of certain properties of the Bessel function occurring in the literature hitherto (e.g. [59]).

In doing the ℓ-extension, the binomial coefficient also needs to be extended which consequently extends the binomial theorem.

Before all this done, the following definition and results of Chapter 4 ((6.1) to (6.7)) are required to be rementioned.

(i) The ℓ-H exponential function is denoted and defined by

$$e_H^\ell(z) = H\left[\begin{matrix} -; & & z \\ -; & (1:\ell); & \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{\ell n+1}} \tag{6.1}$$

for all $z \in \mathbb{C}$ and $\Re(\ell) \geq 0$. By replacing z by iz in (6.1), one finds the ℓ-H trigonometric functions (4.36) and (4.37) of Chapter 4. That is,

$$\text{(ii)}\quad e_H^\ell(iz) = \sum_{n=0}^{\infty} \frac{(-1)^n\ z^{2n}}{((2n)!)^{2\ell n+1}} + i\sum_{n=0}^{\infty} \frac{(-1)^n\ z^{2n+1}}{((2n+1)!)^{2\ell n+\ell+1}}, \tag{6.2}$$

$$\text{(iii)}\quad e_H^\ell(iz) = \cos_H^\ell(z) + i\sin_H^\ell(z). \tag{6.3}$$

Here

$$\text{(iv)}\quad \cos_H^\ell(-z) = \cos_H^\ell(z), \quad \text{and} \quad \sin_H^\ell(-z) = -\sin_H^\ell(z), \tag{6.4}$$

$$\text{(v)}\quad \cos_H^0(z) = \cos z, \quad \text{and} \quad \sin_H^0(z) = \sin z, \tag{6.5}$$

and

$$\text{(vi)}\quad \frac{1}{2}\left[e_H^\ell(iz) + e_H^\ell(-iz)\right] = \cos_H^\ell(z), \tag{6.6}$$

$$\text{(vii)}\quad \frac{1}{2i}\left[e_H^\ell(iz) - e_H^\ell(-iz)\right] = \sin_H^\ell(z). \tag{6.7}$$

The binomial coefficient and thereby binomial theorem may put into the following extended forms.

Definition 6.1. For $0 \leq k \leq n$, the ℓ-binomial coefficient may be denoted and defined by

$$\binom{n}{k}^{(\ell)} = \frac{(n!)^{\ell n+1}}{((n-k)!)^{\ell n-\ell k+1}\ (k!)^{\ell k+1}}.$$

For $z_1, z_2 \in \mathbb{C}$, denoting by $(z_1+z_2)^n_{(\ell)}$, the ℓ-analogue of $(z_1+z_2)^n$, the binomial theorem admits the extension in the form:

$$(z_1+z_2)^n_{(\ell)} = \sum_{k=0}^{n} \binom{n}{k}^{(\ell)} z_1^{n-k}\ z_2^k. \tag{6.8}$$

Remark 6.2. For $\ell = 0$, the ℓ-binomial coefficient and the theorem reduce to the usual binomial coefficient and theorem respectively.

In the light of the expansion (6.8), the ℓ-analogue of the identity:

$$e^{(z_1+z_2)} = e^{z_1}\ e^{z_2}$$

assumes the form which is given in

Lemma 6.3. *For the ℓ-H exponential function (6.1), there holds the identity:*

$$e_H^{\ell}(z_1 + z_2) \;=\; e_H^{\ell}(z_1)\; e_H^{\ell}(z_2). \tag{6.9}$$

Proof. From (6.1),

$$e_H^{\ell}(z_1 + z_2) \;=\; \sum_{n=0}^{\infty} \frac{(z_1 + z_2)_{(\ell)}^{n}}{(n!)^{\ell n+1}}. \tag{6.10}$$

In view of (6.8),

$$\begin{aligned}
e_H^{\ell}(z_1 + z_2) &= \sum_{n=0}^{\infty} \frac{1}{(n!)^{\ell n+1}} \sum_{k=0}^{n} \frac{(n!)^{\ell n+1}}{((n-k)!)^{\ell n-\ell k+1}\,(k!)^{\ell k+1}}\; z_1^{n-k}\; z_2^{k} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z_1^{n-k}\; z_2^{k}}{((n-k)!)^{\ell n-\ell k+1}\,(k!)^{\ell k+1}} \\
&= \sum_{n=0}^{\infty} \frac{z_1^{n}}{(n!)^{\ell n+1}} \sum_{k=0}^{\infty} \frac{z_2^{k}}{(k!)^{\ell k+1}} \\
&= e_H^{\ell}(z_1)\; e_H^{\ell}(z_2).
\end{aligned}$$

□

Remark 6.4. Since $e_H^{\ell}(0) = 1$, it follows from (6.9) with $z_2 = -z_1$ that

$$e_H^{\ell}(z_1)\; e_H^{\ell}(-z_1) \;=\; e_H^{\ell}(z_1 - z_1) = e_H^{\ell}(0) = 1. \tag{6.11}$$

With these prerequisites, the ℓ-extension of the Bessel function is defined as follows.

Definition 6.5. Let $\Re(\ell) \geq 0$ and $n \in \mathbb{N} \cup \{0\}$, then a new class of Bessel functions is denoted and defined by

$$J_{n,H}^{\ell}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k\;(z/2)^{n+2k}}{(k!)^{\ell k+1}\;((n+k)!)^{\ell n+\ell k+1}}. \tag{6.12}$$

This function will be referred to here as the ℓ-H Bessel function or briefly, the ℓ-HBF.

Remark 6.6. For $\ell = 0$, the ℓ-HBF (6.12) reduces to

$$J_{n,H}^{0}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k\;(z/2)^{n+2k}}{k!\;(n+k)!} = J_n(z). \tag{6.13}$$

Remark 6.7. For $\ell \in \mathbb{N} \cup \{0\}$, the ℓ-HBF (6.12) can be expressed as generalized ℓ-H function ((4.1))

$$J^{\ell}_{n,H}(z) = \frac{\left(\frac{z}{2}\right)^n}{(n!)^{\ell n+1}} \; {}_0H^2_{\ell n+1}\left[\begin{matrix} -; \\ n+1, n+1, \ldots, n+1; \, (n+1, 1:\ell); \end{matrix} \; \frac{-z^2}{4(n!)^{\ell}}\right]. \tag{6.14}$$

The series in (6.12) is indeed seen to be convergent yet the following theorem assures it.

Theorem 6.8. *If $\Re(\ell) \geq 0$ and $\Re(2\ell n + \ell + 2) > 0$ then the ℓ-HBF is an entire function of z.*

Proof. Consider

$$J^{\ell}_{n,H}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \; (z/2)^{n+2k}}{(k!)^{\ell k+1} \; ((n+k)!)^{\ell n+\ell k+1}} = \sum_{k=0}^{\infty} \varphi_{n,k} \; z^{n+2k}$$

where

$$\varphi_{n,k} = \frac{(-1)^k}{(k!)^{\ell k+1} \; ((n+k)!)^{\ell n+\ell k+1} \; 4^k}.$$

Applying here the Stirling's formula [18]:

$$\Gamma(\alpha + k) \sim \sqrt{2\pi} \; e^{-(\alpha+k)} \; (\alpha + k)^{(\alpha+k-1/2)}$$

for large k and taking $\alpha = 1, n+1$ in turn, one gets

$$\begin{aligned} |\varphi_{n,k}|^{\frac{1}{k}} &= \left| \frac{1}{\Gamma^{\ell k+1}(k+1) \; \Gamma^{\ell n+\ell k+1}(n+k+1) \; 4^k} \right|^{\frac{1}{k}} \\ &\sim \frac{1}{4} \frac{\left| \sqrt{2\pi} \; e^{-(k+1)} \; (k+1)^{k+1-\frac{1}{2}} \right|^{-\frac{1}{k}-\Re(\ell)}}{\left| \sqrt{2\pi} \; e^{-(n+k+1)} \; (n+k+1)^{n+k+1-\frac{1}{2}} \right|^{\Re\left(\frac{\ell n}{k}+\ell+\frac{1}{k}\right)}}. \end{aligned}$$

Hence from the Cauchy-Hadamard formula,

$$\begin{aligned} \frac{1}{R} &= \lim_{k\to\infty} \sup \sqrt[k]{|\varphi_{n,k}|} \\ &= \frac{1}{4 \; (2\pi)^{\Re(\ell)}} \lim_{k\to\infty} \sup \frac{\left| e^{2\ell n+2\ell+2} \right|}{\left| e^{\ell n+2\ell} \; k^{2\ell n+\ell+2} \right|} \; \left| \frac{e}{k} \right|^{2k\Re(\ell)} \\ &= \frac{e^2}{4} \left| \frac{e^n}{2\pi} \right|^{\Re(\ell)} \lim_{k\to\infty} \sup \frac{1}{k^{2\ell n+\ell+2}} \; \left| \frac{e}{k} \right|^{2k\Re(\ell)} \\ &= 0, \end{aligned}$$

provided $\Re(\ell) \geq 0$ and $\Re(2\ell n + \ell + 2) > 0$. □

6.2 Main results

In order to derive further ℓ-analogues of certain properties of the Bessel function, it is required to obtain the ℓ-analogue of the relation: $J_{-n}(z) = (-1)^n J_n(z)$ when n is a negative integer [59, Ch. 2, p.15]. This is considered in

Lemma 6.9. *For a negative integer n and $\Re(\ell) \geq 0$,*

$$(-1)^n \; J^{\ell}_{n,H}(z) = J^{\ell}_{-n,H}(z). \tag{6.15}$$

Proof. Here

$$\begin{aligned}
(-1)^n \; J^{\ell}_{n,H}(z) &= (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s \; (z/2)^{n+2s}}{(s!)^{\ell s+1} \; ((n+s)!)^{\ell n+\ell s+1}} \\
&= \sum_{s=0}^{\infty} \frac{(-1)^{s+n} \; (z/2)^{2s+2n-n}}{(s!)^{\ell s+1} \; ((s+n)!)^{\ell s+\ell n+1}},
\end{aligned}$$

Taking $s + n = k$, this takes the form:

$$\begin{aligned}
(-1)^n \; J^{\ell}_{n,H}(z) &= \sum_{k=n}^{\infty} \frac{(-1)^k \; (z/2)^{2k-n}}{(k!)^{\ell k+1} \; ((k-n)!)^{\ell k-\ell n+1}} \\
&= J^{\ell}_{-n,H}(z).
\end{aligned}$$

□

For the ℓ-HBF, the generating function relation (GFR) will now be derived and then the differential equation and the integral representations.

6.2.1 Generating function relation

The derivation of the GFR of the ℓ-HBF uses the known finite summation formula [46, Lemma 12, p.112] which is outlined here as

Lemma 6.10. *For $n \geq 1$,*

$$\sum_{k=0}^{n} A(k,n) = \sum_{k=0}^{\left[\frac{n}{2}\right]} A(n-k,n) + \sum_{k=0}^{\left[\frac{n-1}{2}\right]} A(k,n). \tag{6.16}$$

Proof. First note that for $n \geq 1$,

$$n = 1 + \left[\frac{n}{2}\right] + \left[\frac{(n-1)}{2}\right],$$

in which $[*]$ is the usual greatest integer symbol then

$$\sum_{k=0}^{n} A(k,n) = \sum_{k=0}^{\left[\frac{n}{2}\right]} A(k,n) + \sum_{k=0}^{1+\left[\frac{n}{2}\right]+\left[\frac{(n-1)}{2}\right]} A(k,n). \tag{6.17}$$

Now replacing k by $n-k$ that is, k by $1+\left[\frac{n}{2}\right]+\left[\frac{(n-1)}{2}\right]-k$ in the last summation on the right hand side in (6.17), leads to (6.16). $\square$

By making use of this summation formula, an ℓ-extension of the GFR:

$$\exp\left(\frac{z}{2}\left(t-t^{-1}\right)\right) = \sum_{n=-\infty}^{\infty} J_n(z)\ t^n$$

is derived in

Theorem 6.11. *For $t \neq 0$ and for all finite $|z|$,*

$$\sum_{n=-\infty}^{\infty} J_{n,H}^{\ell}(z)\ t^n = e_H^{\ell}\left(\frac{z}{2}\left(t-t^{-1}\right)\right), \tag{6.18}$$

where $e_H^{\ell}(z)$ is as defined in (6.1).

Proof. The left hand side is

$$\begin{aligned}\sum_{n=-\infty}^{\infty} J_{n,H}^{\ell}(z)\ t^n &= \sum_{n=-\infty}^{-1} J_{n,H}^{\ell}(z)\ t^n + \sum_{n=0}^{\infty} J_{n,H}^{\ell}(z)\ t^n \\ &= \sum_{n=0}^{\infty} J_{-n-1,H}^{\ell}(z)\ t^{-n-1} + \sum_{n=0}^{\infty} J_{n,H}^{\ell}(z)\ t^n.\end{aligned}$$

In view of Lemma 6.9 and defining series (6.12), this further becomes

$$\begin{aligned}&\sum_{n=-\infty}^{\infty} J_{n,H}^{\ell}(z)\ t^n \\ &= \sum_{n=0}^{\infty}(-1)^{n+1}\ J_{n+1,H}^{\ell}(z)\ t^{-n-1} + \sum_{n=0}^{\infty} J_{n,H}^{\ell}(z)\ t^n\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n,k=0}^{\infty} \frac{(-1)^{n+k+1}\ (z/2)^{n+2k+1}\ t^{-n-1}}{(k!)^{\ell k+1}\ ((n+k+1)!)^{\ell n+\ell k+\ell+1}} + \sum_{n,k=0}^{\infty} \frac{(-1)^{k}\ (z/2)^{n+2k}\ t^{n}}{(k!)^{\ell k+1}\ ((n+k)!)^{\ell n+\ell k+1}} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^{n-2k+k+1}\ (z/2)^{n-2k+2k+1}\ t^{-n+2k-1}}{(k!)^{\ell k+1}\ ((n-2k+k+1)!)^{\ell n-2\ell k+\ell k+\ell+1}} \\
&\quad + \sum_{n=0}^{\infty} \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^{k}\ (z/2)^{n-2k+2k}\ t^{n-2k}}{(k!)^{\ell k+1}\ ((n-2k+k)!)^{\ell n-2\ell k+\ell k+1}} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \frac{(-1)^{n-k}\ (z/2)^{n}\ t^{-n+2k}}{(k!)^{\ell k+1}\ ((n-k)!)^{\ell n-\ell k+1}} + 1 \\
&\quad + \sum_{n=1}^{\infty} \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^{k}\ (z/2)^{n}\ t^{n-k-k}}{(k!)^{\ell k+1}\ ((n-k)!)^{\ell n-\ell k+1}} \\
&= 1 + \sum_{n=1}^{\infty} \left[\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \frac{(-1)^{n-k}\ (z/2)^{n}\ t^{-n+2k}}{(k!)^{\ell k+1}\ ((n-k)!)^{\ell n-\ell k+1}} + \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^{k}\ (z/2)^{n}\ t^{n-k-k}}{(k!)^{\ell k+1}\ ((n-k)!)^{\ell n-\ell k+1}} \right].
\end{aligned}
$$

Finally, from the result stated in Lemma 6.10, one obtains

$$
\begin{aligned}
\sum_{n=-\infty}^{\infty} J_{n,H}^{\ell}(z)\ t^{n} &= 1 + \sum_{n=1}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{k}\ (z/2)^{n}\ t^{n-2k}}{(k!)^{\ell k+1}\ ((n-k)!)^{\ell n-\ell k+1}} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{k}\ (z/2)^{n+k}\ t^{n-k}}{(k!)^{\ell k+1}\ (n!)^{\ell n+1}} \\
&= \sum_{n=0}^{\infty} \frac{(z/2)^{n}\ t^{n}}{(n!)^{\ell n+1}} \sum_{k=0}^{\infty} \frac{(-1)^{k}\ (z/2)^{k}\ t^{-k}}{(k!)^{\ell k+1}} \\
&= e_{H}^{\ell}\left(\frac{zt}{2}\right) e_{H}^{\ell}\left(\frac{-z}{2t}\right).
\end{aligned}
$$

Thus, the GFR follows from Lemma 6.3. □

Remark 6.12. The special case $\ell = 0$ provides the GFR of $J_n(z)$.

6.2.2 Differential equation

The differential equation of the ℓ-HBF is obtained by means of the infinite order hyper-Bessel type differential operator which is already defined in Chapter 4. This is restated here.

Let $f(z) = \sum\limits_{n=1}^{\infty} a_n z^n$, $0 \neq z \in \mathbb{C}$, $p \in \mathbb{N} \cup \{0\}$ and $\alpha \in \mathbb{C}$. Define

$$ {}_p\Delta_{\alpha}^{\theta}(f(z)) = \begin{cases} \sum\limits_{n=1}^{\infty} a_n (\alpha)_{n-1}^{p} (\theta + \alpha - 1)^{pn} z^n, & \text{if} \quad p \in \mathbb{N} \\ f(z), & \text{if} \quad p = 0 \end{cases}, \tag{6.19} $$

where θ is the Euler differential operator $z\frac{d}{dz}$ and

$$ (\theta + \alpha)^r = \underbrace{(\theta + \alpha)(\theta + \alpha) \ldots (\theta + \alpha)}_{r \ times} $$

is a special case of the hyper-Bessel differential operators (2.8).
Also, the following operators will be needed.

Definition 6.13. For $f(z) = \sum\limits_{k=0}^{\infty} a_k \ z^{\alpha k}$, $\alpha \in \mathbb{R}$, define the lowering operator:

$$ \mathcal{O}_{\alpha-} f(z) = \sum_{k=0}^{\infty} a_k \ z^{(\alpha-1)k}, \tag{6.20} $$

the raising operator:

$$ \mathcal{O}_{\alpha+} f(z) = \sum_{k=0}^{\infty} a_k \ z^{(\alpha+1)k} \tag{6.21} $$

and as suggested by (6.19), the operator:

$$ {}_{\ell}\Lambda_M(f(z)) = \ {}_{\ell}\Delta_1^{\theta} \ (\theta(f(z))). \tag{6.22} $$

These operators combination will be symbolized as follows.

$$ {}_{\ell}\Omega_n^{(z)} \equiv \left(\mathcal{O}_{1+} \ {}_{\ell}\Lambda_M z^{-n} \ {}_{\ell}\Lambda_M \mathcal{O}_{2-} \right). \tag{6.23} $$

With these, the differential equation of the ℓ-HBF is derived in

Theorem 6.14. *For $\ell, n \in \mathbb{N} \cup \{0\}$ and $z \neq 0$, then the function $w = J_{n,H}^{\ell}(z)$ satisfies the differential equation*

$$ \left[{}_{\ell}\Omega_n^{(z)} + \frac{z^{-n+2}}{4} \right] w = 0, \tag{6.24} $$

where ${}_{\ell}\Omega_n^{(z)}$ is as defined in (6.23).

Proof. Here

$$
\begin{aligned}
&{}_{\ell}\Omega_n^{(z)}\ J_{n,H}^{\ell}(z)\\
&= \left(\mathcal{O}_{1+}\ {}_{\ell}\Lambda_M z^{-n}\ {}_{\ell}\Lambda_M \mathcal{O}_{2-}\right)\left(\sum_{k=0}^{\infty}\frac{(-1)^k\ (z/2)^{n+2k}}{(k!)^{\ell k+1}\ ((n+k)!)^{\ell n+\ell k+1}}\right)\\
&= \mathcal{O}_{1+}\ {}_{\ell}\Lambda_M z^{-n}\ {}_{\ell}\Lambda_M\left(\sum_{k=0}^{\infty}\frac{(-1)^k\ z^{n+k}}{2^{n+2k}\ (k!)^{\ell k+1}\ ((n+k)!)^{\ell n+\ell k+1}}\right)\\
&= \mathcal{O}_{1+}\ {}_{\ell}\Lambda_M z^{-n}\ {}_{\ell}\Delta_M\left(\sum_{k=0}^{\infty}\frac{(-1)^k\ z^{n+k}}{2^{n+2k}\ (k!)^{\ell k+1}\ ((n+k)!)^{\ell n+\ell k}\ (n+k-1)!}\right)\\
&= \mathcal{O}_{1+}\ {}_{\ell}\Lambda_M z^{-n}\ \left(\sum_{k=0}^{\infty}\frac{(-1)^k\ (1)_{n+k-1}^{\ell}\ \theta^{\ell n+\ell k}\ z^{n+k}}{2^{n+2k}\ (k!)^{\ell k+1}\ ((n+k)!)^{\ell n+\ell k}\ (n+k-1)!}\right).
\end{aligned}
$$

Since $\theta^{\ell n+\ell k}\ z^{n+k}=(n+k)^{\ell n+\ell k}\ z^{n+k}$ (cf. (4.34)),

$$
\begin{aligned}
&{}_{\ell}\Omega_n^{(z)}\ J_{n,H}^{\ell}(z)\\
&= \mathcal{O}_{1+}\ {}_{\ell}\Lambda_M z^{-n}\ \left(\sum_{k=0}^{\infty}\frac{(-1)^k\ z^{n+k}}{2^{n+2k}\ (k!)^{\ell k+1}\ ((n+k-1)!)^{\ell n+\ell k-\ell+1}}\right)\\
&= \mathcal{O}_{1+}\ {}_{\ell}\Lambda_M\ \left(\sum_{k=0}^{\infty}\frac{(-1)^k\ z^{k}}{2^{n+2k}\ (k!)^{\ell k+1}\ ((n+k-1)!)^{\ell n+\ell k-\ell+1}}\right)\\
&= \mathcal{O}_{1+}\ {}_{\ell}\Delta_M\ \left(\sum_{k=1}^{\infty}\frac{(-1)^k\ z^{k}}{2^{n+2k}\ (k!)^{\ell k}\ (k-1)!\ ((n+k-1)!)^{\ell n+\ell k-\ell+1}}\right)\\
&= \mathcal{O}_{1+}\left(\sum_{k=1}^{\infty}\frac{(-1)^k\ (1)_{k-1}^{\ell}\ \theta^{\ell k}\ z^{k}}{2^{n+2k}\ (k!)^{\ell k}\ (k-1)!\ ((n+k-1)!)^{\ell n+\ell k-\ell+1}}\right)\\
&= \mathcal{O}_{1+}\left(\sum_{k=1}^{\infty}\frac{(-1)^k\ z^{k}}{2^{n+2k}\ ((k-1)!)^{\ell k-\ell+1}\ ((n+k-1)!)^{\ell n+\ell k-\ell+1}}\right)\\
&= \sum_{k=1}^{\infty}\frac{(-1)^k\ z^{2k}}{2^{n+2k}\ ((k-1)!)^{\ell k-\ell+1}\ ((n+k-1)!)^{\ell n+\ell k-\ell+1}}\\
&= \sum_{k=0}^{\infty}\frac{(-1)^{k+1}\ z^{2k+2}}{2^{n+2k+2}\ (k!)^{\ell k+1}\ ((n+k)!)^{\ell n+\ell k+1}}\\
&= -\frac{z^{-n+2}}{4}\ J_{n,H}^{\ell}(z).
\end{aligned}
$$

$\square$

Remark 6.15. The zero order ℓ-HBF, that is $w=J_{0,H}^{\ell}(z)$ satisfies the differential equation:

$$\left[{}_{\ell}\Omega_0^{(z)}+\frac{z}{4}\right]w=0,$$

where ${}_{\ell}\Omega_0^{(z)} = \mathcal{O}_{1+}\ {}_{\ell}\Lambda_M^2\ \mathcal{O}_{2-}$.

6.2.3 The ℓ-HBF integral

By using the ℓ-trigonometric functions (6.3), the ℓ-analogue of the Bessel's integral [46, Theorem 40, p.114] is obtained in

Theorem 6.16. *For $n \in \mathbb{Z}$,*

$$J_{n,H}^{\ell}(z) = \frac{1}{\pi}\int_0^{\pi} \left[\cos n\theta\ \ \cos_H^{\ell}(z\sin\theta) + \sin n\theta\ \ \sin_H^{\ell}(z\sin\theta)\right] \mathrm{d}\theta. \tag{6.25}$$

Proof. The generating function relation of the ℓ-HBF (6.18) if regarded as the Laurent series expansion of the function $e_H^{\ell}\left(\frac{z}{2}(t-t^{-1})\right)$; valid near the essential singularity $t=0$ then the coefficient

$$J_{n,H}^{\ell}(z) \;=\; \frac{1}{2\pi i}\int^{(0+)} u^{-n-1}\ e_H^{\ell}\left(\frac{z}{2}(u-u^{-1})\right) \mathrm{du} \tag{6.26}$$

in which the contour $(0+)$ is a simple closed path encircling the origin $u=0$ once in the positive direction.
In (6.26), choosing the particular path

$$u = e^{i\theta} = \cos\theta + i\sin\theta,$$

where θ runs from $-\pi$ to π, then $u^{-1} = \cos\theta - i\sin\theta$ and (6.26) yields

$$\begin{aligned}
J_{n,H}^{\ell}(z) &= \frac{1}{2\pi i}\int_{-\pi}^{\pi} e^{(-n-1)i\theta}\ e_H^{\ell}\left(\frac{z}{2}(\cos\theta + i\sin\theta - \cos\theta + i\sin\theta)\right) ie^{i\theta}\mathrm{d}\theta \\
&= \frac{1}{2\pi i}\int_{-\pi}^{\pi} e^{(-n-1)i\theta}\ e_H^{\ell}\left(\frac{z}{2}(2i\sin\theta)\right) ie^{i\theta}\mathrm{d}\theta \qquad (6.27)\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} (\cos n\theta - i\sin n\theta)\left[\cos_H^{\ell}(z\sin\theta) + i\sin_H^{\ell}(z\sin\theta)\right] \mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos n\theta\ \ \cos_H^{\ell}(z\sin\theta)\ \mathrm{d}\theta + \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin \mathrm{n}\theta\ \ \sin_{\mathrm{H}}^{\ell}(\mathrm{z}\sin\theta)\ \mathrm{d}\theta \\
&\quad - \frac{i}{2\pi}\left[\int_{-\pi}^{\pi} \sin n\theta\ \ \cos_H^{\ell}(z\sin\theta)\ \mathrm{d}\theta - \int_{-\pi}^{\pi} \cos \mathrm{n}\theta\ \ \sin_{\mathrm{H}}^{\ell}(\mathrm{z}\sin\theta)\ \mathrm{d}\theta\right]
\end{aligned}$$

$$= \quad I_1 + I_2 + I_3 + I_4 \quad \text{(say)}.$$

Here the integrands in I_1 and I_2 are even functions of θ. But the integrands in I_3 and I_4 are odd functions of θ, by (6.4), hence $I_3 = I_4 = 0$. □

Remark 6.17. When ℓ=0, then in view of (6.5) and (6.14), (6.25) reduces to the Bessel's integral [46, Theorem 40, p.114]:

$$J_n(z) \quad = \quad \frac{1}{\pi}\int_0^{\pi} \cos\left(n\theta - z\sin\theta\right) \mathrm{d}\theta,$$

for integral n.

6.2.4 Schlăfli's Integral analogue

Here the integral of Theorem 6.16 is modified so as to include the non integral values of n. For non integral n, the ℓ-HBF takes the form:

$$J^{\ell}_{n,H}(z) \quad = \quad \sum_{k=0}^{\infty} \frac{(-1)^k \; (z/2)^{n+2k}}{(k!)^{\ell k+1} \; \Gamma^{\ell n+\ell k+1}(n+k+1)}. \tag{6.28}$$

Theorem 6.18. *If* $\Re(z) > 0$*, then for any* n*,*

$$\begin{aligned} J^{\ell}_{n,H}(z) \quad = \quad & \frac{1}{\pi}\left\{ \int_0^{\pi} \cos n\theta \;\; \cos^{\ell}_{H}(z\sin\theta) + \sin n\theta \;\; \sin^{\ell}_{H}(z\sin\theta) \mathrm{d}\theta \right. \\ & \left. - \frac{\sin n\pi}{\pi} \int_0^{\infty} e^{-n\theta} \;\; e^{\ell}_{H}\left(-z\sinh\theta\right) \mathrm{d}\theta \right\}. \end{aligned} \tag{6.29}$$

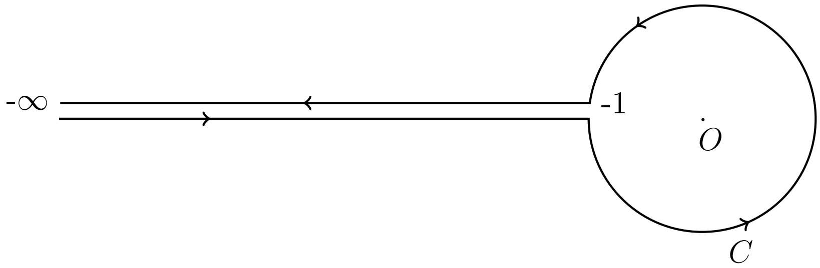
Proof. The integral in (6.26) is

$$J^{\ell}_{n,H}(z) = \frac{1}{2\pi i}\int^{(0+)} u^{-n-1} \; e^{\ell}_{H}\left(\frac{z}{2}(u-u^{-1})\right) \mathrm{du}.$$

Now the branch

$$u^{-n-1} \; e^{\ell}_{H}\left(\frac{z}{2}(u-u^{-1})\right) \qquad (|u| > 0, \; -\pi < \arg u < \pi)$$

when integrated with branch cut $\arg u = \pi$ around the contour C which is traced out by a point moving (i) along the lower edge of the cut from $-\infty$ to -1, then (ii) around the circle $|u| = 1$ in positive direction and finally (iii) along the upper edge of the cut from -1 to $-\infty$ (as shown in the figure),



then for $\Re(z) > 0$, one gets

$$\begin{aligned} J^{\ell}_{n,H}(z) &= \frac{1}{2\pi i}\int\limits_{C} u^{-n-1}\ e^{\ell}_{H}\left(\frac{z}{2}(u-u^{-1})\right)\mathrm{du} \\ &= \frac{1}{2\pi i}\left\{\int\limits_{-\infty}^{-1} + \int\limits_{|u|=1} + \int\limits_{-1}^{-\infty}\right\} u^{-n-1}\ e^{\ell}_{H}\left(\frac{z}{2}(u-u^{-1})\right)\mathrm{du}. \end{aligned}$$

Writing $u = e^{\mp i\pi}\ t$ in the first and third integrals respectively and $u = e^{i\theta}, -\pi < \theta < \pi$ in the second, one further gets

$$\begin{aligned} J^{\ell}_{n,H}(z) &= \frac{1}{2\pi i}\int\limits_{-\pi}^{\pi} e^{(-n-1)i\theta}\ e^{\ell}_{H}\left(\frac{z}{2}(\cos\theta + i\sin\theta - \cos\theta + i\sin\theta)\right) ie^{i\theta}\mathrm{d}\theta \\ &\quad -\frac{1}{2\pi i}\int\limits_{\infty}^{1} e^{(-n-1)(-i\pi)}t^{-n-1}\ e^{\ell}_{H}\left(\frac{z}{2}\left(e^{-i\pi}t - \frac{e^{i\pi}}{t}\right)\right)\mathrm{dt} \\ &\quad -\frac{1}{2\pi i}\int\limits_{1}^{\infty} e^{(-n-1)(i\pi)}t^{-n-1}\ e^{\ell}_{H}\left(\frac{z}{2}\left(e^{i\pi}t - \frac{e^{-i\pi}}{t}\right)\right)\mathrm{dt} \\ &= \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} e^{-in\theta}e^{\ell}_{H}(iz\sin\theta)\mathrm{d}\theta \\ &\quad +\int\limits_{1}^{\infty} t^{-n-1}\ e^{\ell}_{H}\left(\frac{z}{2}\left(-t+t^{-1}\right)\right)\left[\frac{e^{(n+1)\pi i} - e^{-(n+1)\pi i}}{2\pi i}\right]\mathrm{dt} \\ &= \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} e^{-in\theta}e^{\ell}_{H}(iz\sin\theta)\mathrm{d}\theta \\ &\quad +\frac{\sin(n+1)\pi}{\pi}\int\limits_{1}^{\infty} t^{-n-1}\ e^{\ell}_{H}\left(\frac{z}{2}\left(t^{-1}-t\right)\right)\mathrm{dt}. \end{aligned}$$

Now evaluating the first integral by following the procedure of obtaining (6.25) from (6.27), and evaluating the second integral by putting $t = e^{\theta}$, one finally finds

$$\begin{aligned} J^{\ell}_{n,H}(z) &= \frac{1}{\pi}\int_0^{\pi}\left[\cos n\theta \ \ \cos^{\ell}_{H}(z\sin\theta) + \sin n\theta \ \ \sin^{\ell}_{H}(z\sin\theta)\right] \mathrm{d}\theta \\ &\quad + \frac{\sin(n+1)\pi}{\pi}\int_0^{\infty} e^{-n\theta} \ \ e^{\ell}_{H}\left(\frac{z}{2}\left(e^{-\theta}-e^{\theta}\right)\right) \mathrm{d}\theta \\ &= \frac{1}{\pi}\int_0^{\pi}\left[\cos n\theta \ \ \cos^{\ell}_{H}(z\sin\theta) + \sin n\theta \ \ \sin^{\ell}_{H}(z\sin\theta)\right] \mathrm{d}\theta \\ &\quad - \frac{\sin n\pi}{\pi}\int_0^{\infty} e^{-n\theta} \ \ e^{\ell}_{H}\left(-z\sinh\theta\right) \mathrm{d}\theta. \end{aligned}$$

$\square$

Remark 6.19. (1) The integral (6.29) may be termed as the *ℓ-Schläfli's integral*. If $\ell = 0$ then (6.29) yields the Schläfli's integral for Bessel function $J_n(z)$ given by [60, Sec.17.231, p.362]

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi}\cos n(\theta - z\sin\theta)\ \mathrm{d}\theta - \frac{\sin n\pi}{\pi}\int_0^{\infty} e^{-n\theta - z\sinh\theta}\mathrm{d}\theta.$$

(2) If n is an integer, then the integral (6.29) reduces to the ℓ-HBF integral (6.25).

6.3 Other properties

In this section, a differential recurrence relation is derived and then with the help of the GFR and the ℓ-HBF integral representation, some other properties will be deduced.

6.3.1 Differential recurrence relation

Let ${}_{\ell}\Delta^{\theta}_{1}$ be as defined in (6.19). Then by using the operator defined in (4.30):

$${}_{\ell}\mathcal{D}^{(z)}_{M}\left(f(z)\right) = \left(z^{-1} \ {}_{\ell}\Delta^{\theta}_{1} \ \theta\right)\left(f(z)\right) \quad \text{or} \quad \left(\equiv \ {}_{\ell}\Delta^{\mathbb{D}_z}_{1} \ \theta(f(z))\right), \tag{6.30}$$

the differential recurrence relation is derived in

Theorem 6.20. *For $\ell \in \mathbb{N} \cup \{0\}$,*

$$2\ {}_{\ell}\mathcal{D}_M^{(z)} J_{n,H}^{\ell}(z) = J_{n-1,H}^{\ell}(z) - J_{n+1,H}^{\ell}(z). \tag{6.31}$$

The theorem is proved by using the following lemma which describes the eigen function property of ℓ-exponential function.

Lemma 6.21. *The ℓ-exponential function $e_H^{\ell}\left(\frac{z}{2}\left(t-t^{-1}\right)\right)$ with t fixed, is an eigen function with respect to the operator ${}_{\ell}\mathcal{D}_M^{(z)}$ as defined in (6.30). That is,*

$${}_{\ell}\mathcal{D}_M^{(z)}\left[e_H^{\ell}\left(\frac{z}{2}\left(t-t^{-1}\right)\right)\right] = \frac{1}{2}\left(t-t^{-1}\right)\ e_H^{\ell}\left(\frac{z}{2}\left(t-t^{-1}\right)\right) \tag{6.32}$$

for fixed t.

Proof. The left hand side is

$$\begin{aligned}
{}_{\ell}\mathcal{D}_M^{(z)}\left[e_H^{\ell}\left(\frac{z}{2}\left(t-t^{-1}\right)\right)\right] &= {}_{\ell}\Delta_1^{\mathbb{D}_z}\left(\theta \sum_{n=0}^{\infty} \frac{\left(t-t^{-1}\right)^n\ z^n}{2^n\ (n!)^{\ell n+1}}\right) \\
&= {}_{\ell}\Delta_1^{\mathbb{D}_z}\left(\sum_{n=1}^{\infty} \frac{\left(t-t^{-1}\right)^n\ z^n}{2^n\ (n!)^{\ell n}\ (n-1)!}\right) \\
&= \sum_{n=1}^{\infty} \frac{\left(t-t^{-1}\right)^n\ (1)_{n-1}^{\ell}\ (\mathbb{D}^z)^n\ z^n}{2^n\ (n!)^{\ell n}\ (n-1)!}.
\end{aligned}$$

Now from (4.34)

$$(\mathbb{D}_z)^{\ell n}\ z^n = \underbrace{\frac{d}{dz} z \frac{d}{dz} \cdots \frac{d}{dz} z \frac{d}{dz}}_{\ell n \text{ derivatives}}\ z^n = n^{\ell n}\ z^{n-1},$$

one gets

$$\begin{aligned}
{}_{\ell}\mathcal{D}_M^{(z)}\left[e_H^{\ell}\left(\frac{z}{2}\left(t-t^{-1}\right)\right)\right] &= \sum_{n=1}^{\infty} \frac{\left(t-t^{-1}\right)^n\ (1)_{n-1}^{\ell}\ n^{\ell n}\ z^{n-1}}{2^n\ (n!)^{\ell n}\ (n-1)!} \\
&= \sum_{n=1}^{\infty} \frac{\left(t-t^{-1}\right)^n\ z^{n-1}}{2^n\ ((n-1)!)^{\ell n-\ell+1}} \\
&= \frac{1}{2}\left(t-t^{-1}\right)\ e_H^{\ell}\left(\frac{z}{2}\left(t-t^{-1}\right)\right).
\end{aligned}$$

□

Proof. (of Theorem 6.20)
From Theorem 6.11,

$$e_H^{\ell}\left(\frac{z}{2}\left(t-t^{-1}\right)\right) = \sum_{n=-\infty}^{\infty} J_{n,H}^{\ell}(z)\ t^n.$$

Applying the operator ${}_{\ell}\mathcal{D}_M^{(z)}$ both the sides, one obtains

$${}_{\ell}\mathcal{D}_M^{(z)}\left[e_H^{\ell}\left(\frac{z}{2}\left(t-t^{-1}\right)\right)\right] = \sum_{n=-\infty}^{\infty} {}_{\ell}\mathcal{D}_M^{(z)}\left(J_{n,H}^{\ell}(z)\right)\ t^n.$$

In view of Lemma 6.21, this changes to

$$\frac{1}{2}\left(t-t^{-1}\right)\ e_H^{\ell}\left(\frac{z}{2}\left(t-t^{-1}\right)\right)\ = \sum_{n=-\infty}^{\infty} {}_{\ell}\mathcal{D}_M^{(z)}\left(J_{n,H}^{\ell}(z)\right)\ t^n.$$

Once again using Theorem 6.11,

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} {}_{\ell}\mathcal{D}_M^{(z)}\left(J_{n,H}^{\ell}(z)\right)\ t^n \\
&= \frac{t}{2}\sum_{n=-\infty}^{\infty} J_{n,H}^{\ell}(z)\ t^n + \left(\frac{-1}{2t}\right)\sum_{n=-\infty}^{\infty} J_{n,H}^{\ell}(z)\ t^n \\
&= \frac{1}{2}\sum_{n=-\infty}^{\infty} J_{n,H}^{\ell}(z)\ t^{n+1} - \frac{1}{2}\sum_{n=-\infty}^{\infty} J_{n,H}^{\ell}(z)\ t^{n-1} \\
&= \frac{1}{2}\sum_{n=-\infty}^{\infty} J_{n-1,H}^{\ell}(z)\ t^{n} - \frac{1}{2}\sum_{n=-\infty}^{\infty} J_{n+1,H}^{\ell}(z)\ t^{n}.
\end{aligned}$$

On comparing the coefficients of t^n both sides, (6.31) follows. □

The iteration of the relation (6.31) yields the following general formula.

Theorem 6.22. *If* $n \in \mathbb{N} \cup \{0\}$ *and* $k \in \mathbb{N}$ *then (cf. [46, Ex. 7, p.121] with* $\ell = 0$*)*

$$2^k\left({}_{\ell}\mathcal{D}_M^{(z)}\right)^k\ J_{n,H}^{\ell}(z) = \sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}J_{n+k-2m,H}^{\ell}(z). \tag{6.33}$$

Proof. For $k=1$, Theorem holds true from Theorem 6.20. That is,

$$2\ {}_{\ell}\mathcal{D}_M^{(z)} J_{n,H}^{\ell}(z) = J_{n-1,H}^{\ell}(z) - J_{n+1,H}^{\ell}(z).$$

Here, applying the operator $2\ {}_\ell\mathcal{D}_M^{(z)}$ both the sides and then making use of Theorem 6.20, one obtains

$$\begin{aligned} 2^2\left({}_\ell\mathcal{D}_M^{(z)}\right)^2 J_{n,H}^\ell(z) &= 2\ {}_\ell\mathcal{D}_M^{(z)} J_{n-1,H}^\ell(z) - 2\ {}_\ell\mathcal{D}_M^{(z)} J_{n+1,H}^\ell(z) \\ &= J_{n-2,H}^\ell(z) - 2J_{n,H}^\ell(z) + J_{n+2,H}^\ell(z) \\ &= \sum_{m=0}^{2}(-1)^{2-m}\binom{2}{m} J_{n+2-2m,H}^\ell(z). \end{aligned} \tag{6.34}$$

Similarly,

$$\begin{aligned} &2^3\left({}_\ell\mathcal{D}_M^{(z)}\right)^3 J_{n,H}^\ell(z) \\ &= 2\ {}_\ell\mathcal{D}_M^{(z)} J_{n-2,H}^\ell(z) - 2\ {}_\ell\mathcal{D}_M^{(z)}\left[2J_{n,H}^\ell(z)\right] + 2\ {}_\ell\mathcal{D}_M^{(z)} J_{n+2,H}^\ell(z) \\ &= J_{n-3,H}^\ell(z) - 3J_{n-1,H}^\ell(z) + 3J_{n+1,H}^\ell(z) - J_{n+3,H}^\ell(z) \\ &= \sum_{m=0}^{3}(-1)^{3-m}\binom{3}{m} J_{n+3-2m,H}^\ell(z). \end{aligned}$$

The recursive procedure k-times, leads to the theorem. □

Alternatively, this theorem can also be proved by using the principle of mathematical induction on k as follows.

Proof. (of Theorem 6.22 by the principle of Mathematical induction)
For $k = 1$, it is true from Theorem 6.20. Now suppose that the theorem holds for $k =$some positive integer r. That is,

$$2^r\left({}_\ell\mathcal{D}_M^{(z)}\right)^r J_{n,H}^\ell(z) = \sum_{m=0}^{r}(-1)^{r-m}\binom{r}{m} J_{n+r-2m,H}^\ell(z). \tag{6.35}$$

Then to prove that for $k = r + 1$, the theorem holds true. In fact, from Theorem 6.20,

$$\begin{aligned} &2^{r+1}\left({}_\ell\mathcal{D}_M^{(z)}\right)^{r+1} J_{n,H}^\ell(z) \\ &= 2\ {}_\ell\mathcal{D}_M^{(z)}\left(2^r\left({}_\ell\mathcal{D}_M^{(z)}\right)^r J_{n,H}^\ell(z)\right) \\ &= 2\ {}_\ell\mathcal{D}_M^{(z)}\left(\sum_{m=0}^{r}(-1)^{r-m}\binom{r}{m} J_{n+r-2m,H}^\ell(z)\right) \\ &= 2\ {}_\ell\mathcal{D}_M^{(z)}\left[(-1)^r J_{n+r,H}^\ell(z) + r(-1)^{r-1} J_{n+r-2,H}^\ell(z) + \ldots + J_{n-r,H}^\ell(z)\right] \\ &= (-1)^r\left[J_{n-1+r,H}^\ell(z) - J_{n+1+r,H}^\ell(z)\right] + r(-1)^{r-1}\left[J_{n-3+r,H}^\ell(z) - J_{n-1+r,H}^\ell(z)\right] \end{aligned}$$

$$
\begin{aligned}
&\quad +\ldots+\left[J^{\ell}_{n-1-r,H}(z)-J^{\ell}_{n+1-r,H}(z)\right]\\
&= (-1)^{r+1}J^{\ell}_{n+1+r,H}(z)+(-1)^{r}(r+1)J^{\ell}_{n-1+r,H}(z)+\ldots+J^{\ell}_{n-1-r,H}(z)\\
&= \sum_{m=0}^{r+1}(-1)^{r+1-m}\binom{r+1}{m}\,J^{\ell}_{n+r+1-2m,H}(z).
\end{aligned}
$$

□

In the following two theorems, the properties involving series and definite integrals are extended.

Theorem 6.23. *For an integer n, (cf. [46, Ex.2, p.120])*

$$
\cos^{\ell}_{H}(z) = J^{\ell}_{0,H}(z)+2\sum_{n=1}^{\infty}(-1)^{n}\;J^{\ell}_{2n,H}(z) \tag{6.36}
$$

$$
\sin^{\ell}_{H}(z) = 2\sum_{n=0}^{\infty}(-1)^{n}\;J^{\ell}_{2n+1,H}(z). \tag{6.37}
$$

Proof. From the GFR of the ℓ-HBF (6.18) and the identity (6.15),

$$
\begin{aligned}
e^{\ell}_{H}\left(\frac{z}{2}\left(t-t^{-1}\right)\right) &= \sum_{n=-\infty}^{\infty}J^{\ell}_{n,H}(z)\;t^{n}\\
&= J^{\ell}_{0,H}(z)+\sum_{n=1}^{\infty}J^{\ell}_{n,H}(z)\;t^{n}+\sum_{n=1}^{\infty}J^{\ell}_{-n,H}(z)\;t^{-n}\\
&= J^{\ell}_{0,H}(z)+\sum_{n=1}^{\infty}J^{\ell}_{n,H}(z)\;[t^{n}+(-1)^{n}\;t^{-n}].
\end{aligned} \tag{6.38}
$$

Taking $t=i$, one gets

$$
\begin{aligned}
e^{\ell}_{H}(iz) &= J^{\ell}_{0,H}(z)+\sum_{n=1}^{\infty}J^{\ell}_{n,H}(z)\;[i^{n}+(-1)^{n}\;i^{-n}]\\
&= J^{\ell}_{0,H}(z)+\sum_{n=1}^{\infty}J^{\ell}_{2n,H}(z)\;[i^{2n}+(-1)^{2n}\;i^{-2n}]\\
&\quad +\sum_{n=0}^{\infty}J^{\ell}_{2n+1,H}(z)\;[i^{2n+1}+(-1)^{2n+1}\;i^{-(2n+1)}].
\end{aligned}
$$

In view of (6.3), this gives

$$
\cos^{\ell}_{H}(z)+i\sin^{\ell}_{H}(z)=J^{\ell}_{0,H}(z)+2\sum_{n=1}^{\infty}(-1)^{n}\;J^{\ell}_{2n,H}(z)+2i\sum_{n=0}^{\infty}(-1)^{n}\;J^{\ell}_{2n+1,H}(z).
$$

On comparing the real and imaginary parts, the required result is obtained. □

Theorem 6.24. *For $n \in \mathbb{Z}$, there hold (cf. [46, Ex.3, p.120]):*

$$[1+(-1)^n]\ J^{\ell}_{n,H}(z) = \frac{2}{\pi}\int_0^{\pi} \cos n\theta\ \cos^{\ell}_{H}(z\sin\theta)\mathrm{d}\theta, \tag{6.39}$$

$$[1-(-1)^n]\ J^{\ell}_{n,H}(z) = \frac{2}{\pi}\int_0^{\pi} \sin n\theta\ \sin^{\ell}_{H}(z\sin\theta)\mathrm{d}\theta. \tag{6.40}$$

Further,

$$J^{\ell}_{2n,H}(z) = \frac{1}{\pi}\int_0^{\pi} \cos 2n\theta\ \cos^{\ell}_{H}(z\sin\theta)\mathrm{d}\theta, \tag{6.41}$$

$$J^{\ell}_{2n+1,H}(z) = \frac{1}{\pi}\int_0^{\pi} \sin(2n+1)\theta\ \sin^{\ell}_{H}(z\sin\theta)\mathrm{d}\theta \tag{6.42}$$

and

$$\int_0^{\pi} \cos(2n+1)\theta\ \cos^{\ell}_{H}(z\sin\theta)\mathrm{d}\theta = 0, \tag{6.43}$$

$$\int_0^{\pi} \sin 2n\theta\ \sin^{\ell}_{H}(z\sin\theta)\mathrm{d}\theta = 0. \tag{6.44}$$

Proof. From the ℓ-HBF integral considered in Theorem 6.16, it follows that

$$J^{\ell}_{n,H}(z) = \frac{1}{\pi}\int_0^{\pi} \left[\cos n\theta\ \cos^{\ell}_{H}(z\sin\theta) + \sin n\theta\ \sin^{\ell}_{H}(z\sin\theta)\right] \mathrm{d}\theta \tag{6.45}$$

hence,

$$J^{\ell}_{-n,H}(z) = \frac{1}{\pi}\int_0^{\pi} \left[\cos(-n\theta)\ \cos^{\ell}_{H}(z\sin\theta) + \sin(-n\theta)\ \sin^{\ell}_{H}(z\sin\theta)\right] \mathrm{d}\theta,$$

that is,

$$(-1)^n\ J^{\ell}_{n,H}(z) = \frac{1}{\pi}\int_0^{\pi} \left[\cos n\theta\ \cos^{\ell}_{H}(z\sin\theta) - \sin n\theta\ \sin^{\ell}_{H}(z\sin\theta)\right] \mathrm{d}\theta. \tag{6.46}$$

Here adding (or subtracting) (6.45) and (6.46) one obtains (6.39) (or (6.40)). By considering the even (or odd) order ℓ-HBF in (6.39) (or (6.40)) yields (6.41) (or (6.42)), whereas considering odd (or even) order ℓ-HBF in (6.39) (or (6.40)), one obtains the formula (6.43) (or (6.44)).

□

6.3.2 ℓ-Analogue of Jacobi's expansion

In the following theorem, the Jacobi's expansion in series of Bessel coefficients for $n \in \mathbb{Z}$ ([46, Ex.3, p.120], [59, Sec. 2.22, p.22]), given by

$$\exp(z\sin\theta) = J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z)\cos 2n\theta + 2i\sum_{n=0}^{\infty} J_{2n+1}(z)\ \sin(2n+1)\theta$$

is extended.

Theorem 6.25. *For $n \in \mathbb{Z}$,*

$$e_H^\ell(z\sin_H^\ell\theta) = J_{0,H}^\ell(z) + 2\sum_{n=1}^{\infty} J_{2n,H}^\ell(z)\cos_H^\ell 2n\theta + 2i\sum_{n=0}^{\infty} J_{2n+1,H}^\ell(z)\ \sin_H^\ell(2n+1)\theta.$$

That is,

$$\cos_H^\ell(z\sin_H^\ell\theta) = J_{0,H}^\ell(z) + 2\sum_{n=1}^{\infty} J_{2n,H}^\ell(z)\ \cos_H^\ell 2n\theta, \tag{6.47}$$

$$\sin_H^\ell(z\sin_H^\ell\theta) = 2\sum_{n=0}^{\infty} J_{2n+1,H}^\ell(z)\ \sin_H^\ell(2n+1)\theta. \tag{6.48}$$

Proof. In (6.38), substituting $t = e_H^\ell(i\theta)$ the ℓ-H exponential function, it gives

$$\begin{aligned}
&e_H^\ell\left(\frac{z}{2}\left(e_H^\ell(i\theta) - e_H^\ell(-i\theta)\right)\right)\\
&= J_{0,H}^\ell(z) + \sum_{n=1}^{\infty} J_{n,H}^\ell(z)\ [e_H^\ell(in\theta) + (-1)^n\ e_H^\ell(-in\theta)]\\
&= J_{0,H}^\ell(z) + \sum_{n=1}^{\infty} J_{2n,H}^\ell(z)\ [e_H^\ell(2in\theta) + (-1)^{2n}\ e_H^\ell(-2in\theta)]\\
&\quad + \sum_{n=0}^{\infty} J_{2n+1,H}^\ell(z)\ [e_H^\ell((2n+1)i\theta) + (-1)^{2n+1}\ e_H^\ell(-(2n+1)i\theta)]\\
&= J_{0,H}^\ell(z) + 2\sum_{n=1}^{\infty} J_{2n,H}^\ell(z)\cos_H^\ell 2n\theta + 2i\sum_{n=0}^{\infty} J_{2n+1,H}^\ell(z)\ \sin_H^\ell(2n+1)\theta.
\end{aligned}$$

Finally from (6.7),

$$e_H^\ell\left(\frac{z}{2}\left(e_H^\ell(i\theta)-e_H^\ell(-i\theta)\right)\right)=e_H^\ell(iz\sin_H^\ell\theta)=\cos_H^\ell(z\sin_H^\ell\theta)+i\sin_H^\ell(z\sin_H^\ell\theta),$$

hence the result follows by comparison of the real and imaginary parts. □

6.3.3 ℓ-Analogue of Bessel's inequality due to Cauchy

The inequality [59, p.16]

$$|J_n(z)|\le\frac{\left|\frac{z}{2}\right|^n}{n!}\ \exp\left(\left|\frac{z^2}{4}\right|\right)$$

due to Cauchy is known as Bessel's inequality. In order to obtain its ℓ-analogue, the ℓ-analogue of the inequality

$$\frac{(n+r)!}{n!}\ge(n+1)^r$$

will be obtained first.

Lemma 6.26. *If $\Re(\ell)\ge 0$ and $r\in\mathbb{N}\cup\{0\}$ then*

$$\frac{((n+r)!)^{\ell n+1}}{(n!)^{\ell n+1}}\ \ge\ (n+1)^{r(\ell n+1)}. \tag{6.49}$$

Proof. For $r=0$, (6.49) is evident. For the remaining values of r, the principle of mathematical induction is used.
For $r=1$,

$$\text{L.H.S.}\ =\frac{((n+1)!)^{\ell n+1}}{(n!)^{\ell n+1}}=(n+1)^{\ell n+1}=\ \text{R.H.S.}$$

Suppose that (6.49) is true for $r=$ some positive integer k that is,

$$\frac{((n+k)!)^{\ell n+1}}{(n!)^{\ell n+1}}\ \ge\ (n+1)^{k(\ell n+1)} \tag{6.50}$$

holds true. Then for $r=k+1$, it suffice to prove

$$\frac{((n+k+1)!)^{\ell n+1}}{(n!)^{\ell n+1}}\ \ge\ (n+1)^{(k+1)(\ell n+1)}.$$

The left hand member

$$\begin{aligned}
\frac{((n+k+1)!)^{\ell n+1}}{(n!)^{\ell n+1}} &= (n+k+1)^{\ell n+1}\,\frac{((n+k)!)^{\ell n+1}}{(n!)^{\ell n+1}} \\
&\geq (n+k+1)^{\ell n+1}\,(n+1)^{k(\ell n+1)} \\
&\geq (n+1)^{\ell n+1}\,(n+1)^{k(\ell n+1)} \\
&= (n+1)^{(k+1)(\ell n+1)}
\end{aligned}$$

whenever $\Re(\ell) \geq 0$. □

Using this, the ℓ-Bessel's inequality due to Cauchy is established in

Theorem 6.27. *For $\ell, n \in \mathbb{N} \cup \{0\}$,*

$$\left|J^{\ell}_{n,H}(z)\right| \leq \frac{\left|\frac{z}{2}\right|^n}{(n!)^{\ell n+1}}\; e^{\ell}_{H}\left(\left|\frac{z^2}{4}\right|\right). \tag{6.51}$$

Proof. From the Definition 6.5 (of the ℓ-HBF),

$$\left|J^{\ell}_{n,H}(z)\right| \;\leq\; \left|\frac{z}{2}\right|^n \sum_{k=0}^{\infty} \frac{\left|\frac{z}{2}\right|^{2k}}{((n+k)!)^{\ell n+\ell k+1}\;(k!)^{\ell k+1}}. \tag{6.52}$$

Since

$$\frac{1}{((n+k)!)^{\ell n+\ell k+1}} \;\leq\; \frac{1}{((n+k)!)^{\ell k}\;(n!)^{\ell n+1}\;(n+1)^{k(\ell n+1)}}$$

in view of Lemma 6.26, (6.52) yields

$$\begin{aligned}
\left|J^{\ell}_{n,H}(z)\right| &\leq \frac{\left|\frac{z}{2}\right|^n}{(n!)^{\ell n+1}} \sum_{k=0}^{\infty} \frac{\left|\frac{z}{2}\right|^{2k}}{((n+k)!)^{\ell k}\;(n+1)^{k(\ell n+1)}\;(k!)^{\ell k+1}} \\
&\leq \frac{\left|\frac{z}{2}\right|^n}{(n!)^{\ell n+1}} \sum_{k=0}^{\infty} \frac{\left|\frac{z}{2}\right|^{2k}}{(k!)^{\ell k+1}} \\
&= \frac{\left|\frac{z}{2}\right|^n}{(n!)^{\ell n+1}}\; e^{\ell}_{H}\left(\left|\frac{z^2}{4}\right|\right),
\end{aligned}$$

when $\ell, n \in \mathbb{N} \cup \{0\}$. □

6.3.4 Addition formula

Theorem 6.28. *For the ℓ-HBF (cf. [59, Sec. 2.4, 2.5, p.30]),*

$$J^{\ell}_{n,H}(z_1+z_2) \;=\; \sum_{m=-\infty}^{\infty} J^{\ell}_{m,H}(z_1)\; J^{\ell}_{n-m,H}(z_2). \tag{6.53}$$

Proof. On substituting $z = z_1 + z_2$ in the contour integral (6.26) and then using the property (6.9), one obtains

$$\begin{aligned}
J^{\ell}_{n,H}(z_1+z_2) &= \frac{1}{2\pi i}\int^{(0+)} u^{-n-1}\; e^{\ell}_{H}\left(\frac{(z_1+z_2)\;(u-u^{-1})}{2}\right) \mathrm{du} \\
&= \frac{1}{2\pi i}\int^{(0+)} u^{-n-1}\; e^{\ell}_{H}\left(\frac{z_1}{2}(u-u^{-1})\right)\; e^{\ell}_{H}\left(\frac{z_2}{2}(u-u^{-1})\right) \mathrm{du} \\
&= \frac{1}{2\pi i}\int^{(0+)} u^{-n-1} \sum_{m=-\infty}^{\infty} J^{\ell}_{m,H}(z_1)\; u^{m}\; e^{\ell}_{H}\left(\frac{z_2}{2}(u-u^{-1})\right) \mathrm{du} \\
&= \sum_{m=-\infty}^{\infty} J^{\ell}_{m,H}(z_1)\; \frac{1}{2\pi i}\int^{(0+)} u^{m-n-1}\; e^{\ell}_{H}\left(\frac{z_2}{2}(u-u^{-1})\right) \mathrm{du}.
\end{aligned}$$

Once again making an appeal to (6.26), one is led to (6.53). $\square$

Corollary 6.29. *There holds the series relation:*

$$J^{\ell}_{n,H}(2z) = \sum_{m=0}^{n} J^{\ell}_{m,H}(z)\; J^{\ell}_{n-m,H}(z) + 2\sum_{m=1}^{\infty}(-1)^m\; J^{\ell}_{m,H}(z)\; J^{\ell}_{n+m,H}(z). \tag{6.54}$$

Proof. If $z_1 = z_2 = z$ in (6.53) then

$$\begin{aligned}
J^{\ell}_{n,H}(2z) &= \sum_{m=-\infty}^{\infty} J^{\ell}_{m,H}(z)\; J^{\ell}_{n-m,H}(z) \\
&= \sum_{m=-\infty}^{-1} J^{\ell}_{m,H}(z)\; J^{\ell}_{n-m,H}(z) + \sum_{m=0}^{\infty} J^{\ell}_{m,H}(z)\; J^{\ell}_{n-m,H}(z) \\
&= \sum_{m=-\infty}^{-1} J^{\ell}_{m,H}(z)\; J^{\ell}_{n-m,H}(z) + \sum_{m=0}^{n} J^{\ell}_{m,H}(z)\; J^{\ell}_{n-m,H}(z) \\
&\quad + \sum_{m=n+1}^{\infty} J^{\ell}_{m,H}(z)\; J^{\ell}_{n-m,H}(z).
\end{aligned}$$

This, in view of (6.15) gives

$$\begin{aligned} J^{\ell}_{n,H}(2z) &= \sum_{m=0}^{n} J^{\ell}_{m,H}(z)\ J^{\ell}_{n-m,H}(z) + \sum_{m=1}^{\infty}(-1)^m\ J^{\ell}_{m,H}(z)\ J^{\ell}_{n+m,H}(z) \\ &\quad + \sum_{m=1}^{\infty}(-1)^m\ J^{\ell}_{m,H}(z)\ J^{\ell}_{n+m,H}(z) \\ &= \sum_{m=0}^{n} J^{\ell}_{m,H}(z)\ J^{\ell}_{n-m,H}(z) + 2\sum_{m=1}^{\infty}(-1)^m\ J^{\ell}_{m,H}(z)\ J^{\ell}_{n+m,H}(z). \end{aligned}$$

□