Chapter 7

The q-ℓ-Ψ Bessel function

7.1 Introduction

The q-ℓ-Ψ exponential functions were introduced in Chapter 5. In this chapter, the product of such exponential functions will be considered and it will be shown that the product indeed generates the function which turns out to be a q-analogue of the ℓ-H Bessel function (ℓ-HBF) of Chapter 6.
In the following, the definitions of the q-ℓ-Ψ exponential function and the q-ℓ-Ψ trigonometric functions are restated.
(i) The q-ℓ-Ψ exponential function denoted and defined in Chapter 5 is given by (5.11)

$$e^{\ell}_{\Psi}(z;q) = {}_1\Psi^1_0\left[\begin{array}{lll} 0; & & q; \quad z \\ -; & (1:\ell); & \end{array}\right] = \sum_{n=0}^{\infty}\frac{z^n}{(q;q)_n^{\ell n+1}}. \tag{7.1}$$

with $|z| < \left|\sqrt{\frac{1+q}{1-q}}\,\Gamma_{q^2}\left(\frac{1}{2}\right)\right|^{\Re(\ell)}$ and $\Re(\ell) \geq 0$.
If $\ell = 0$ or $z = 0$,

$$e^{0}_{\Psi}(z;q) = e_q(z) \quad \text{and} \quad e^{\ell}_{\Psi}(0;\mathrm{q}) = 1. \tag{7.2}$$

By replacing z by iz in (7.1) provides in view of (5.45) and (5.46),

$$\text{(ii)}\quad e^{\ell}_{\Psi}(iz;q) = \sum_{n=0}^{\infty}\frac{(-1)^n\, z^{2n}}{(q;q)_{2n}^{2\ell n+1}} + i\sum_{n=0}^{\infty}\frac{(-1)^n\, z^{2n+1}}{(q;q)_{2n+1}^{2\ell n+\ell+1}} \tag{7.3}$$

$$= \cos^{\ell}_{\Psi}(z;q) + i\sin^{\ell}_{\Psi}(z;q). \tag{7.4}$$

$$\text{(iii)}\quad \frac{1}{2}\left[e_\Psi^\ell(iz;q) + e_\Psi^\ell(-iz;q)\right] \;=\; \cos_\Psi^\ell(z;q), \tag{7.5}$$

$$\text{(iv)}\quad \frac{1}{2i}\left[e_\Psi^\ell(iz;q) - e_\Psi^\ell(-iz;q)\right] \;=\; \sin_\Psi^\ell(z;q), \tag{7.6}$$

$$\text{(v)}\quad \cos_\Psi^\ell(-z;q) \;=\; \cos_\Psi^\ell(z;q), \quad \text{and} \quad \sin_\Psi^\ell(-z;q) = -\sin_\Psi^\ell(z;q), \tag{7.7}$$

and

$$\text{(vi)}\quad \cos_\Psi^0(z;q) \;=\; \cos_q(z), \quad \text{and} \quad \sin_\Psi^0(z;q) = \sin_q(z). \tag{7.8}$$

A q-analogue of ℓ-binomial coefficient may be defined as follows.

Definition 7.1. For $0 \le k \le n$, the q-ℓ-binomial coefficient may be denoted and defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q^{(\ell)} = \frac{(q;q)_n^{\ell n+1}}{(q;q)_{n-k}^{\ell n-\ell k+1}\ (q;q)_k^{\ell k+1}}.$$

Remark 7.2. For $\ell = 0$, the q-ℓ binomial coefficient reduces to the q-binomial coefficient .

For $z_1, z_2 \in \mathbb{C}$, if $(z_1+z_2)^n_{(\ell,q)}$ denotes a q-ℓ-analogue of $(z_1+z_2)^n$ then this suggests the q-ℓ binomial theorem in the form:

$$(z_1 + z_2)^n_{(\ell,q)} \;=\; \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q^{(\ell)} z_1^{n-k}\ z_2^k. \tag{7.9}$$

This theorem may be used to prove the following q-ℓ-analogue of the identity:

$$\exp(z_1 + z_2) = \exp(z_1)\ \exp(z_2)$$

which will be used later on.

Lemma 7.3. *For the q-ℓ-Ψ exponential function, there holds the identity:*

$$e_\Psi^\ell(z_1 + z_2;q) \;=\; e_\Psi^\ell(z_1;q)\ e_\Psi^\ell(z_2;q). \tag{7.10}$$

Proof. From (7.1),

$$e_\Psi^\ell(z_1 + z_2;q) \;=\; \sum_{n=0}^{\infty} \frac{(z_1+z_2)^n_{(\ell,q)}}{(q;q)_n^{\ell n+1}}. \tag{7.11}$$

In view of (7.9),

$$\begin{aligned} e^{\ell}_{\Psi}(z_1+z_2;q) &= \sum_{n=0}^{\infty} \frac{1}{(q;q)_n^{\ell n+1}} \sum_{k=0}^{n} \frac{(q;q)_n^{\ell n+1}}{(q;q)_{n-k}^{\ell n-\ell k+1}\ (q;q)_k^{\ell k+1}}\ z_1^{n-k}\ z_2^k \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{z_1^{n-k}\ z_2^k}{(q;q)_{n-k}^{\ell n-\ell k+1}\ (q;q)_k^{\ell k+1}} \\ &= \sum_{n=0}^{\infty} \frac{z_1^n}{(q;q)_n^{\ell n+1}} \sum_{k=0}^{\infty} \frac{z_2^k}{(q;q)_k^{\ell k+1}} \\ &= e^{\ell}_{\Psi}(z_1;q)\ e^{\ell}_{\Psi}(z_2;q). \end{aligned}$$

□

Remark 7.4. Since from (7.2), $e^{\ell}_{\Psi}(0;q) = 1$; it further follows from (7.10) with $z_2 = -z_1$ that

$$e^{\ell}_{\Psi}(z_1;q)\ e^{\ell}_{\Psi}(-z_1;q) = e^{\ell}_{\Psi}(z_1-z_1;q) = e^{\ell}_{\Psi}(0;q) = 1. \tag{7.12}$$

As in the case of ℓ-HBF and its GFR in Chapter 6, here also the Laurent series expansion of the product of two q-ℓ-Ψ exponential functions enables one to define an ℓ-extension of the q-Bessel function $J_n^{(1)}(z;q)$. The q-ℓ-Ψ trigonometric functions then help in deriving q-ℓ-analogues of certain properties of an extended q-Bessel function. This function is defined as follows.

Definition 7.5. Let $\Re(\ell) \geq 0, n \in \mathbb{N}\cup\{0\}$, then the q-ℓ-Ψ Bessel function or briefly, the q-ℓ-ΨBF is denoted and defined by

$$J^{\ell}_{n,\Psi}(z;q) = \sum_{k=0}^{\infty} \frac{(-1)^k\ (z/2)^{n+2k}}{(q;q)_k^{\ell k+1}\ (q;q)_{n+k}^{\ell n+\ell k+1}}. \tag{7.13}$$

Remark 7.6. For $\ell = 0$, the q-ℓ-ΨBF (7.13) reduces to q-analogue of Bessel function [23, Ex. 1.24, p.25]

$$J^{0}_{n,\Psi}(z;q) = \sum_{k=0}^{\infty} \frac{(-1)^k\ (z/2)^{n+2k}}{(q;q)_k\ (q;q)_{n+k}} = J_n^{(1)}(z;q). \tag{7.14}$$

The convergence of the series in (7.13) is examined in

Theorem 7.7. *If* $\Re(\ell) \geq 0$ *then* q-ℓ-ΨBF *is an analytic function of* z.

Proof. Consider

$$J^{\ell}_{n,\Psi}(z;q) = \sum_{k=0}^{\infty} \frac{(-1)^k \ (z/2)^{n+2k}}{(q;q)_k^{\ell k+1} \ (q;q)_{n+k}^{\ell n+\ell k+1}} = \sum_{k=0}^{\infty} \xi_{n,k}.$$

Then in view of the formula

$$(a;q)_n = \frac{\Gamma_q(a+n)}{\Gamma_q(a)} \ (1-q)^n,$$

$$\begin{aligned}
|\xi_{n,k}|^{\frac{1}{k}} &= \left|\frac{z}{2}\right|^{\frac{n}{k}} \left| \frac{(-1)^k \ (z^2/4)^k}{(q;q)_k^{\ell k+1} \ (q;q)_{n+k}^{\ell n+\ell k+1}} \right|^{\frac{1}{k}} \\
&= \left|\frac{z}{2}\right|^{\frac{n}{k}} \left| \frac{|z^2/4|^k}{[\Gamma_q(1+k) \ (1-q)^k]^{\ell k+1} \ [\Gamma_q(1+n+k) \ (1-q)^{n+k}]^{\ell n+\ell k+1}} \right|^{\frac{1}{k}} \\
&= \left|\frac{z}{2}\right|^{\frac{n}{k}} \frac{|z^2/4|}{\left|\Gamma_q^{\ell+\frac{1}{k}}(1+k) \ \Gamma_q^{\frac{\ell n}{k}+\ell+\frac{1}{k}}(1+n+k) \ (1-q)^{\ell k+1+(n+k)\left(\frac{\ell n}{k}+\ell+\frac{1}{k}\right)}\right|}
\end{aligned}$$

Here using Stirling's formula of q-Gamma function [41, Eq.(2.25), p.482]:

$$\Gamma_q(z) \sim (1+q)^{\frac{1}{2}} \ \Gamma_{q^2}\left(\frac{1}{2}\right) (1-q)^{\frac{1}{2}-z} \ e^{\frac{\theta q^z}{1-q-q^z}}, \tag{7.15}$$

for large $|z|$ with $0<\theta<1$, one further gets

$$\begin{aligned}
|\xi_{n,k}|^{\frac{1}{k}} &\sim \frac{|z/2|^{\frac{n}{k}} \ |z^2/4| \ \left|(1-q)^{-2\ell n-2\ell k-2-\frac{\ell n^2}{k}-\frac{n}{k}}\right|}{\left|(1+q)^{\frac{1}{2}} \ \Gamma_{q^2}\left(\frac{1}{2}\right) (1-q)^{\frac{1}{2}-(1+k)} \ e^{\frac{\theta \ q^{1+k}}{1-q-q^{1+k}}}\right|^{\Re(\ell)+\frac{1}{k}}} \\
&\quad \times \frac{1}{\left|(1+q)^{\frac{1}{2}} \ \Gamma_{q^2}\left(\frac{1}{2}\right) (1-q)^{\frac{1}{2}-(1+n+k)} \ e^{\frac{\theta \ q^{1+n+k}}{1-q-q^{1+n+k}}}\right|^{\Re\left(\frac{\ell n}{k}+\ell+\frac{1}{k}\right)}} \\
&= \frac{|z/2|^{\frac{n}{k}} \ |z^2/4| \ (1-q)^{\ell+\frac{1}{k}}}{(1+q)^{\ell+\frac{(\ell n+1)}{2k}} \ \Gamma_{q^2}^{2\ell+\frac{(\ell n+2)}{k}} \ \left(\frac{1}{2}\right)} \ \frac{\left|e^{\frac{\theta \ q^{1+k}}{1-q-q^{1+k}}}\right|^{-\Re(\ell)-\frac{1}{k}}}{\left|e^{\frac{\theta \ q^{1+n+k}}{1-q-q^{1+n+k}}}\right|^{\Re\left(\frac{\ell n}{k}+\ell+\frac{1}{k}\right)}}
\end{aligned}$$

for large k. Now from the Cauchy's root test:

$$\frac{1}{R} = \limsup_{k\to\infty} \sqrt[k]{|\xi_{n,k}|}$$

$$= \left| \frac{|z^2/4| \; (1-q)^{\ell}}{(1+q)^{\ell} \; \Gamma_{q^2}^{2\ell}\left(\frac{1}{2}\right)} \right|,$$

hence it follows that the q-ℓ-ΨBF converges inside the open disk

$$|z| \;=\; 2 \left| \sqrt{\frac{1+q}{1-q}} \; \Gamma_{q^2}\left(\frac{1}{2}\right) \right|^{\Re(\ell)}. \tag{7.16}$$

□

Next, the relation: $J_{-n}(z;q) = (-1)^n J_n(z;q)$, for n negative integer, is also put in the q-ℓ-form which will be used later in obtaining certain properties.

Lemma 7.8. *For a negative integer n,*

$$(-1)^n \; J^{\ell}_{n,\Psi}(z;q) = J^{\ell}_{-n,\Psi}(z;q). \tag{7.17}$$

Proof. Consider the left hand member

$$\begin{aligned}(-1)^n \; J^{\ell}_{n,\Psi}(z;q) \;&=\; (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s \; (z/2)^{n+2s}}{(q;q)_s^{\ell s+1} \; (q;q)_{s+n}^{\ell n+\ell s+1}} \\ &=\; \sum_{s=0}^{\infty} \frac{(-1)^{s+n} \; (z/2)^{2s+2n-n}}{(q;q)_s^{\ell s+1} \; (q;q)_{s+n}^{\ell s+\ell n+1}}\end{aligned}$$

Taking $s+n=k$, this simplifies to

$$\begin{aligned}(-1)^n \; J^{\ell}_{n,\Psi}(z;q) \;&=\; \sum_{k=n}^{\infty} \frac{(-1)^k \; (z/2)^{2k-n}}{(q;q)_k^{\ell k+1} \; (q;q)_{k-n}^{\ell k-\ell n+1}} \\ &=\; J^{\ell}_{-n,\Psi}(z;q).\end{aligned}$$

□

7.2 Main results

For the q-ℓ-ΨBF, the generating function relation (GFR) will be derived first and then the difference equation and integral representation.

7.2.1 Generating function relation

The GFR is obtained in

Theorem 7.9. *For $t \neq 0$ and for all finite $|z|$,*

$$\sum_{n=-\infty}^{\infty} J^{\ell}_{n,\Psi}(z;q)\ t^n = e^{\ell}_{\Psi}\left(\frac{z}{2}\left(t - t^{-1}\right); q\right), \tag{7.18}$$

where $e^{\ell}_{\Psi}(z;q)$ is as defined in (7.1).

Proof. Here the left hand member

$$\begin{aligned}\sum_{n=-\infty}^{\infty} J^{\ell}_{n,\Psi}(z;q)\ t^n &= \sum_{n=-\infty}^{-1} J^{\ell}_{n,\Psi}(z;q)\ t^n + \sum_{n=0}^{\infty} J^{\ell}_{n,\Psi}(z;q)\ t^n \\ &= \sum_{n=0}^{\infty} J^{\ell}_{-n-1,\Psi}(z;q)\ t^{-n-1} + \sum_{n=0}^{\infty} J^{\ell}_{n,\Psi}(z;q)\ t^n.\end{aligned}$$

In view of Lemma 7.8 and defining series (7.13), this may further be written as

$$\begin{aligned}&\sum_{n=-\infty}^{\infty} J^{\ell}_{n,\Psi}(z;q)\ t^n \\ &= \sum_{n=0}^{\infty}(-1)^{n+1}\ J^{\ell}_{n+1,\Psi}(z;q)\ t^{-n-1} + \sum_{n=0}^{\infty} J^{\ell}_{n,\Psi}(z;q)\ t^n \\ &= \sum_{n,k=0}^{\infty} \frac{(-1)^{n+k+1}\ (z/2)^{n+2k+1}\ t^{-n-1}}{(q;q)_k^{\ell k+1}\ (q;q)_{n+k+1}^{\ell n+\ell k+\ell+1}} + \sum_{n,k=0}^{\infty} \frac{(-1)^k\ (z/2)^{n+2k}\ t^n}{(q;q)_k^{\ell k+1}\ (q;q)_{n+k}^{\ell n+\ell k+1}} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^{n-2k+k+1}\ (z/2)^{n-2k+2k+1}\ t^{-n+2k-1}}{(q;q)_k^{\ell k+1}\ (q;q)_{n-2k+k+1}^{\ell n-2\ell k+\ell k+\ell+1}} \\ &\quad + \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k\ (z/2)^{n-2k+2k}\ t^{n-2k}}{(q;q)_k^{\ell k+1}\ (q;q)_{n-2k+k}^{\ell n-2\ell k+\ell k+1}} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \frac{(-1)^{n-k}\ (z/2)^n\ t^{-n+2k}}{(q;q)_k^{\ell k+1}\ (q;q)_{n-k}^{\ell n-\ell k+1}} + 1 + \sum_{n=1}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k\ (z/2)^n\ t^{n-k-k}}{(q;q)_k^{\ell k+1}\ (q;q)_{n-k}^{\ell n-\ell k+1}} \\ &= 1 + \sum_{n=1}^{\infty}\left[\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \frac{(-1)^{n-k}\ (z/2)^n\ t^{-n+2k}}{(q;q)_k^{\ell k+1}\ (q;q)_{n-k}^{\ell n-\ell k+1}} + \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k\ (z/2)^n\ t^{n-k-k}}{(q;q)_k^{\ell k+1}\ (q;q)_{n-k}^{\ell n-\ell k+1}}\right].\end{aligned}$$

Next, using Lemma 6.10, this simplifies to

$$\begin{aligned}\sum_{n=-\infty}^{\infty} J^{\ell}_{n,\Psi}(z;q)\ t^n &= 1+\sum_{n=1}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k\ (z/2)^n\ t^{n-2k}}{(q;q)_k^{\ell k+1}\ (q;q)_{n-k}^{\ell n-\ell k+1}}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k\ (z/2)^{n+k}\ t^{n-k}}{(q;q)_k^{\ell k+1}\ (q;q)_n^{\ell n+1}}\\ &= \sum_{n=0}^{\infty}\frac{(z/2)^n\ t^n}{(q;q)_n^{\ell n+1}}\sum_{k=0}^{\infty}\frac{(-1)^k\ (z/2)^k\ t^{-k}}{(q;q)_k^{\ell k+1}}\\ &= e^{\ell}_{\Psi}\left(\frac{zt}{2};q\right) e^{\ell}_{\Psi}\left(\frac{-z}{2t};q\right).\end{aligned}$$

Finally, the GFR follows from Lemma 7.3. □

7.2.2 Difference equation

The difference equation of the q-ℓ-ΨBF is obtained by means of the infinite order q-hyper-Bessel type difference operator which was defined in (3.5). This is restated here.

Let $f(z) = \sum_{n=1}^{\infty} a_{n,q}\ z^n$, $0 \neq z \in \mathbb{C},\ p \in \mathbb{N}\cup\{0\},\ \alpha \in \mathbb{C}$ and the difference operator θ_q be defined by

$$\theta_q f(z) = f(z) - f(zq). \tag{7.19}$$

Define

$$ {}_p\Delta^{\theta_q}_{\alpha} f(z) = \begin{cases} \sum_{n=1}^{\infty} a_{n,q}\ (\alpha;q)^p_{n-1}\ (q^{\alpha-1}\ \theta_q - q^{\alpha-1}+1)^{pn}\ z^n, & \text{if } p\in\mathbb{N}\\ f(z), & \text{if } p=0. \end{cases} \tag{7.20}$$

Also, the lowering and raising operators defined in (6.20) and (6.21) are:

$$\mathcal{O}_{\alpha-} f(z) = \sum_{k=0}^{\infty} a_{k,q}\ z^{(\alpha-1)k},$$

and

$$\mathcal{O}_{\alpha+} f(z) = \sum_{k=0}^{\infty} a_{k,q}\ z^{(\alpha+1)k},$$

where $f(z) = \sum_{k=0}^{\infty} a_{k,q}\ z^{\alpha k},\quad \alpha\in\mathbb{R}$, and as suggested by (7.20), the operator:

$${}_{\ell}\Lambda^{q}_{M}(f(z)) = {}_{\ell}\Delta^{\theta_q}_{1}\ (\theta_q(f(z))). \tag{7.21}$$

The following operator symbol will be used.

$$ {}_{\ell}\mho_n^{(z;q)} \equiv \left(\mathcal{O}_{1+}\ {}_{\ell}\Lambda_M^q z^{-n}\ {}_{\ell}\Lambda_M^q \mathcal{O}_{2-}\right). \tag{7.22} $$

With these operators, the difference equation of the q-ℓ-ΨBF is derived in

Theorem 7.10. *For $\ell, n \in \mathbb{N}\cup\{0\}$ and $z \neq 0$, $w = J^{\ell}_{n,\Psi}(z;q)$ satisfies the difference equation*

$$ \left[\ {}_{\ell}\mho_n^{(z;q)} + \frac{z^{-n+2}}{4}\right] w = 0, \tag{7.23} $$

where ${}_{\ell}\mho_n^{(z;q)}$ is as defined in (7.22).

Proof. Beginning with the left hand side operator:

$$
\begin{aligned}
&{}_{\ell}\mho_n^{(z;q)}\ J^{\ell}_{n,\Psi}(z;q)\\
&= \left(\mathcal{O}_{1+}\ {}_{\ell}\Lambda_M^q z^{-n}\ {}_{\ell}\Lambda_M^q \mathcal{O}_{2-}\right)\left(\sum_{k=0}^{\infty}\frac{(-1)^k\ (z/2)^{n+2k}}{(q;q)_k^{\ell k+1}\ (q;q)_{n+k}^{\ell n+\ell k+1}}\right)\\
&= \mathcal{O}_{1+}\ {}_{\ell}\Lambda_M^q z^{-n}\ {}_{\ell}\Lambda_M^q \left(\sum_{k=0}^{\infty}\frac{(-1)^k\ z^{n+k}}{2^{n+2k}\ (q;q)_k^{\ell k+1}\ (q;q)_{n+k}^{\ell n+\ell k+1}}\right)\\
&= \mathcal{O}_{1+}\ {}_{\ell}\Lambda_M^q z^{-n}\ {}_{\ell}\Delta_M^q \left(\sum_{k=0}^{\infty}\frac{(-1)^k\ \theta_q(z^{n+k})}{2^{n+2k}\ (q;q)_k^{\ell k+1}\ (q;q)_{n+k}^{\ell n+\ell k+1}}\right)\\
&= \mathcal{O}_{1+}\ {}_{\ell}\Lambda_M^q z^{-n}\ {}_{\ell}\Delta_M^q \left(\sum_{k=0}^{\infty}\frac{(-1)^k\ z^{n+k}}{2^{n+2k}\ (q;q)_k^{\ell k+1}\ (q;q)_{n+k}^{\ell n+\ell k}\ (q;q)_{n+k-1}}\right)\\
&= \mathcal{O}_{1+}\ {}_{\ell}\Lambda_M^q z^{-n} \left(\sum_{k=0}^{\infty}\frac{(-1)^k\ (q;q)_{n+k-1}^{\ell}\ \theta_q^{\ell n+\ell k}\ z^{n+k}}{2^{n+2k}\ (q;q)_k^{\ell k+1}\ (q;q)_{n+k}^{\ell n+\ell k}\ (q;q)_{n+k-1}}\right).
\end{aligned}
$$

Now in (3.10), $c = 1$ gives

$$ \theta_q^{\ell n+\ell k}\ z^{n+k} = \left(1 - q^{n+k}\right)^{\ell n+\ell k}\ z^{n+k}, $$

Hence

$$
\begin{aligned}
&{}_{\ell}\mho_n^{(z;q)}\ J^{\ell}_{n,\Psi}(z;q)\\
&= \mathcal{O}_{1+}\ {}_{\ell}\Lambda_M^q\ z^{-n}\left(\sum_{k=0}^{\infty}\frac{(-1)^k\ z^{n+k}}{2^{n+2k}\ (q;q)_k^{\ell k+1}\ (q;q)_{n+k-1}^{\ell n+\ell k-\ell+1}}\right)\\
&= \mathcal{O}_{1+}\ {}_{\ell}\Lambda_M^q\left(\sum_{k=0}^{\infty}\frac{(-1)^k\ z^{k}}{2^{n+2k}\ (q;q)_k^{\ell k+1}\ (q;q)_{n+k-1}^{\ell n+\ell k-\ell+1}}\right)
\end{aligned}
$$

$$\begin{aligned}
&= \mathcal{O}_{1+}\ {}_{\ell}\Delta_M^q \left(\sum_{k=1}^{\infty} \frac{(-1)^k\ z^k}{2^{n+2k}\ (q;q)_k^{\ell k}\ (q;q)_{k-1}\ (q;q)_{n+k-1}^{\ell n+\ell k-\ell+1}}\right)\\
&= \mathcal{O}_{1+} \left(\sum_{k=1}^{\infty} \frac{(-1)^k\ (q;q)_{k-1}^{\ell}\ \theta_q^{\ell k}\ z^k}{2^{n+2k}\ (q;q)_k^{\ell k}\ (q;q)_{k-1}\ (q;q)_{n+k-1}^{\ell n+\ell k-\ell+1}}\right)\\
&= \mathcal{O}_{1+} \left(\sum_{k=1}^{\infty} \frac{(-1)^k\ (q;q)_{k-1}^{\ell}\ (1-q^k)^{\ell k}\ z^k}{2^{n+2k}\ (q;q)_k^{\ell k}\ (q;q)_{k-1}\ (q;q)_{n+k-1}^{\ell n+\ell k-\ell+1}}\right)\\
&= \mathcal{O}_{1+} \left(\sum_{k=1}^{\infty} \frac{(-1)^k\ z^k}{2^{n+2k}\ (q;q)_{k-1}^{\ell k-\ell+1}\ (q;q)_{n+k-1}^{\ell n+\ell k-\ell+1}}\right)\\
&= \sum_{k=1}^{\infty} \frac{(-1)^k\ z^{2k}}{2^{n+2k}\ (q;q)_{k-1}^{\ell k-\ell+1}\ (q;q)_{n+k-1}^{\ell n+\ell k-\ell+1}}\\
&= \sum_{k=0}^{\infty} \frac{(-1)^{k+1}\ z^{2k+2}}{2^{n+2k+2}\ (q;q)_k^{\ell k+1}\ (q;q)_{n+k}^{\ell n+\ell k+1}}\\
&= -\frac{z^{-n+2}}{4}\ J_{n,\Psi}^{\ell}(z;q).
\end{aligned}$$

□

Remark 7.11. The zero order q-ℓ-ΨBF, that is $w = J_{0,\Psi}^{\ell}(z;q)$ satisfies the difference equation:

$$\left[{}_{\ell}\mho_0^{(z;q)} + \frac{z}{4}\right] w = 0,$$

where ${}_{\ell}\mho_n^{(z;q)} \equiv \left(\mathcal{O}_{1+}\ ({}_{\ell}\Lambda_M^q)^2\ \mathcal{O}_{2-}\right).$

7.2.3 The q-ℓ-ΨBF integral

The q-Laurent's theorem as considered by Ahmed Salem [47, Thm. 4.11, p.148-150] reads in the following form.

Theorem 7.12. *If $f(z)$ is analytic function on the annulus surrounded by two concentric circles C and C' of center at the origin point with the point z_0 lies inside C' where C' is completely contained in C, then at any point of the annulus, $f(z)$ can be expanded in the form*

$$f(z) = \sum_{k=0}^{\infty} a_k \left(\frac{z_0}{z};q\right)_k\ z^k + \sum_{k=1}^{\infty} \frac{b_k\ z^{-k}}{\left(\frac{z_0}{z};q\right)_k}, \tag{7.24}$$

where

$$a_k = \frac{1}{2\pi i}\int_C \frac{f(w)}{w^{k+1}\ \left(\frac{z_0}{w};q\right)_{k+1}}\ \mathrm{d}w$$

and

$$b_k = \frac{1}{2\pi i} \int\limits_{C'} w^{k-1} \ \left(\frac{z_0}{w}; q\right)_{k-1} \ f(w) \ \mathrm{d}w.$$

He remarks in [47, Rem. 4.3, p.150] that the q-Laurent series (7.24) is equivalent to the classical Laurent series when $z_0 = 0$. In view of this and using the q-ℓ-Ψ trigonometric functions (7.4), the q-ℓ-analogue of the Bessel's integral is obtained in

Theorem 7.13. *For $n \in \mathbb{Z}$ and $|z| < \left|\sqrt{\frac{1+q}{1-q}}\ \Gamma_{q^2}\left(\frac{1}{2}\right)\right|^{\Re(\ell)}$,*

$$J^{\ell}_{n,\Psi}(z;q) = \frac{1}{\pi} \int\limits_{0}^{\pi} \left[\cos n\theta \ \ \cos^{\ell}_{\Psi}(z \sin\theta; q) + \sin n\theta \ \ \sin^{\ell}_{\Psi}(z\sin\theta;q)\right] \mathrm{d}\theta. \qquad (7.25)$$

Proof. The GFR of the q-ℓ-ΨBF obtained in Theorem 7.9 may be regarded as the Laurent series expansion of the function $e^{\ell}_{\Psi}\left(\frac{z}{2}(t - t^{-1}); q\right)$; valid near the essential singularity $t = 0$. Then the coefficient in this series is given by

$$J^{\ell}_{n,\Psi}(z;q) \ = \ \frac{1}{2\pi i} \int^{(0+)} u^{-n-1} \ e^{\ell}_{\Psi}\left(\frac{z}{2}(u - u^{-1}); q\right) \mathrm{du}, \qquad (7.26)$$

where the contour $(0+)$ is a simple closed path encircling the origin $u = 0$ once in the positive direction.
In (7.26), by choosing the particular path

$$u = e^{i\theta} = \cos\theta + i\sin\theta,$$

where θ runs from $-\pi$ to π, one finds

$$\begin{aligned} J^{\ell}_{n,\Psi}(z;q) \ &= \ \frac{1}{2\pi i} \int\limits_{-\pi}^{\pi} e^{(-n-1)i\theta} \ e^{\ell}_{\Psi}\left(\frac{z}{2}(2i\sin\theta); q\right) ie^{i\theta} \mathrm{d}\theta \qquad (7.27) \\ &= \ \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} (\cos n\theta - i\sin n\theta) \left[\cos^{\ell}_{\Psi}(z\sin\theta;q) + i\sin^{\ell}_{\Psi}(z\sin\theta;q)\right] \mathrm{d}\theta \\ &= \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \cos n\theta \ \ \cos^{\ell}_{\Psi}(z\sin\theta;q) \ \mathrm{d}\theta + \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \sin \mathrm{n}\theta \ \ \sin^{\ell}_{\Psi}(\mathrm{z}\sin\theta;\mathrm{q}) \ \mathrm{d}\theta \\ & \quad -\frac{i}{2\pi}\left[\int\limits_{-\pi}^{\pi} \sin n\theta \ \ \cos^{\ell}_{\Psi}(z\sin\theta;q) \ \mathrm{d}\theta + \int\limits_{-\pi}^{\pi} \cos \mathrm{n}\theta \ \ \sin^{\ell}_{\Psi}(\mathrm{z}\sin\theta;\mathrm{q}) \ \mathrm{d}\theta\right] \\ &= \ I_1 + I_2 + I_3 + I_4 \ \ (\text{say}). \end{aligned}$$

In view of (7.7), it may be seen that the integrands in I_1 and I_2 are even functions of θ whereas the integrands in I_3 and I_4 are odd functions of θ, giving $I_3 = I_4 = 0$. □

Remark 7.14. For $\ell = 0$, $|z| < \left|\sqrt{\frac{1+q}{1-q}}\,\Gamma_{q^2}\left(\frac{1}{2}\right)\right|^{\Re(\ell)}$ implies $|z| < 1$; and (7.25) reduces to the q-Bessel integral (cf. [53, Thm. 3.13, p.38])

$$J_\nu^{(1)}(z;q) \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}\,\frac{1}{\left(\frac{ze^{i\theta}}{2},\frac{-ze^{-i\theta}}{2};q\right)_\infty}\,\mathrm{d}\theta.$$

Further, if $q \to 1$, then in view of (7.8) and (7.14), (7.25) reduces to the Bessel's integral [46, Theorem 40, p.114]:

$$J_n(z) \;=\; \frac{1}{\pi}\int_0^{\pi}\cos\left(n\theta - z\sin\theta\right)\mathrm{d}\theta,$$

for integral n.

7.3 Other properties

Here a differential recurrence relation is derived and then with the help of the GFR and the q-ℓ-ΨBF integral representation, some other properties will be deduced.

7.3.1 Differential recurrence relation

In Chapter 5, it was proved that with respect to the operator defined in (5.37), the q-ℓ-Ψ exponential function is an eigen function. It is restated below.
In Chapter 5, the following operators was introduced.
For $f(z) = \sum\limits_{n=0}^{\infty} a_{n,q}z^n$, $|z| < R, R > 0,\ p \in \mathbb{N}\cup\{0\}$,

$${}_p\Omega_\alpha^{\mathbb{D}_q^z} f(z) = \begin{cases} \sum\limits_{n=1}^{\infty} a_{n,q}\ (\alpha;q)_{n-1}^p\ \left((1-q)\ \mathbb{D}_q^z\right)^{pn}\ z^n, & \text{if}\quad p\in\mathbb{N} \\ f(z;q), & \text{if}\quad p=0 \end{cases}, \tag{7.28}$$

where $\mathbb{D}_q^z$ is the q-analogue of hyper-Bessel differential operators (2.8) given by

$$(\mathbb{D}_q^z)^n = \underbrace{D_qzD_q\ldots D_qzD_q}_{n\text{ derivatives}},$$

and

$$ {}_{\ell}\mathcal{D}_M^{(z;q)} = \left\{ {}_{\ell}\Omega_1^{\mathbb{D}_q^z} \right\} \theta_q. \tag{7.29}$$

This operator is also useful in deriving the differential recurrence relation.

Theorem 7.15. *For $\ell \in \mathbb{N} \cup \{0\}$,*

$$2\ {}_{\ell}\mathcal{D}_M^{(z;q)} J_{n,\Psi}^{\ell}(z;q) = J_{n-1,\Psi}^{\ell}(z;q) - J_{n+1,\Psi}^{\ell}(z;q). \tag{7.30}$$

The theorem is proved by using the following lemma which describes the eigen function property of the q-ℓ-Ψ exponential function.

Lemma 7.16. *The q-ℓ-Ψ exponential function $e_{\Psi}^{\ell}\left(\frac{z}{2}\left(t-t^{-1}\right);q\right)$ with t fixed, is an eigen function with respect to the operator ${}_{\ell}\mathcal{D}_M^{(z;q)}$ as defined in (7.29). That is,*

$${}_{\ell}\mathcal{D}_M^{(z;q)}\left[e_{\Psi}^{\ell}\left(\frac{z}{2}\left(t-t^{-1}\right);q\right)\right] = \frac{1}{2}\left(t-t^{-1}\right)\ e_{\Psi}^{\ell}\left(\frac{z}{2}\left(t-t^{-1}\right);q\right) \tag{7.31}$$

for fixed t.

Proof. Here

$$\begin{aligned}
{}_{\ell}\mathcal{D}_M^{(z;q)}\left[e_{\Psi}^{\ell}\left(\frac{z}{2}\left(t-t^{-1};q\right)\right)\right] &= {}_{\ell}\Omega_1^{\mathbb{D}_q^z}\left(\theta_q \sum_{n=0}^{\infty} \frac{\left(t-t^{-1}\right)^n\ z^n}{2^n\ (q;q)_n^{\ell n+1}}\right) \\
&= {}_{\ell}\Omega_1^{\mathbb{D}_q^z}\left(\sum_{n=1}^{\infty} \frac{\left(t-t^{-1}\right)^n\ \left(z^n-(zq)^n\right)}{2^n\ (q;q)_n^{\ell n+1}}\right) \\
&= \sum_{n=1}^{\infty} \frac{\left(t-t^{-1}\right)^n\ (q;q)_{n-1}^{\ell}\ \left((1-q)\ \mathbb{D}_q^z\right)^{\ell n}\ z^n}{2^n\ (q;q)_n^{\ell n}\ (q;q)_{n-1}}.
\end{aligned}$$

Now applying the same procedure of deriving (5.42), one gets for $n \in \mathbb{N}$,

$$(1-q)^{\ell n}\left(\mathbb{D}_q^z\right)^{\ell n}\ z^n = (1-q^n)^{\ell n}\ z^{n-1}.$$

Hence

$$\begin{aligned}
{}_{\ell}\mathcal{D}_M^{(z;q)}\left[e_{\Psi}^{\ell}\left(\frac{z}{2}\left(t-t^{-1};q\right)\right)\right] &= \sum_{n=1}^{\infty} \frac{\left(t-t^{-1}\right)^n\ (q;q)_{n-1}^{\ell}\ (1-q^n)^{\ell n}\ z^{n-1}}{2^n\ (q;q)_n^{\ell n}\ (q;q)_{n-1}} \\
&= \sum_{n=1}^{\infty} \frac{\left(t-t^{-1}\right)^n\ z^{n-1}}{2^n\ (q;q)_{n-1}^{\ell n-\ell+1}} \\
&= \sum_{n=0}^{\infty} \frac{\left(t-t^{-1}\right)^{n+1}\ z^n}{2^{n+1}\ (q;q)_n^{\ell n+1}}
\end{aligned}$$

$$= \frac{1}{2}\left(t - t^{-1}\right)\ e^{\ell}_{\Psi}\left(\frac{z}{2}\left(t - t^{-1}; q\right)\right).$$

□

Proof. (of Theorem 7.15)
From Theorem 7.9,

$$e^{\ell}_{\Psi}\left(\frac{z}{2}\left(t - t^{-1}\right); q\right) = \sum_{n=-\infty}^{\infty} J^{\ell}_{n,\Psi}(z;q)\ t^{n}.$$

By applying the operator ${}_{\ell}\mathcal{D}^{(z;q)}_{M}$ both the sides, it gives

$${}_{\ell}\mathcal{D}^{(z;q)}_{M}\left[e^{\ell}_{\Psi}\left(\frac{z}{2}\left(t - t^{-1}; q\right)\right)\right] = \sum_{n=-\infty}^{\infty} {}_{\ell}\mathcal{D}^{(z;q)}_{M}\left(J^{\ell}_{n,\Psi}(z;q)\right)\ t^{n}.$$

From Lemma 7.16, this may be rewritten as

$$\frac{1}{2}\left(t - t^{-1}\right)\ e^{\ell}_{\Psi}\left(\frac{z}{2}\left(t - t^{-1}; q\right)\right) = \sum_{n=-\infty}^{\infty} {}_{\ell}\mathcal{D}^{(z;q)}_{M}\left(J^{\ell}_{n,\Psi}(z;q)\right)\ t^{n}.$$

Once again using Theorem 7.9, one finds

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} {}_{\ell}\mathcal{D}^{(z;q)}_{M}\left(J^{\ell}_{n,\Psi}(z;q)\right)\ t^{n} \\
&= \frac{t}{2}\sum_{n=-\infty}^{\infty} J^{\ell}_{n,\Psi}(z;q)\ t^{n} + \left(\frac{-1}{2t}\right)\sum_{n=-\infty}^{\infty} J^{\ell}_{n,\Psi}(z;q)\ t^{n} \\
&= \frac{1}{2}\sum_{n=-\infty}^{\infty} J^{\ell}_{n,\Psi}(z;q)\ t^{n+1} - \frac{1}{2}\sum_{n=-\infty}^{\infty} J^{\ell}_{n,\Psi}(z;q)\ t^{n-1} \\
&= \frac{1}{2}\sum_{n=-\infty}^{\infty} J^{\ell}_{n-1,\Psi}(z;q)\ t^{n} - \frac{1}{2}\sum_{n=-\infty}^{\infty} J^{\ell}_{n+1,\Psi}(z;q)\ t^{n}.
\end{aligned}$$

On comparing the coefficients of t^n both sides, the result follows. □

The iteration of the relation (7.30) yields the following general formula.

Theorem 7.17. *If $n \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$ then (cf. [46, Ex. 7, p.121])*

$$2^{k}\left({}_{\ell}\mathcal{D}^{(z;q)}_{M}\right)^{k}\ J^{\ell}_{n,\Psi}(z;q) = \sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m} J^{\ell}_{n+k-2m,\Psi}(z;q). \tag{7.32}$$

Proof. For $k = 1$, Theorem holds true from Theorem 7.15. That is,

$$2\ {}_\ell\mathcal{D}_M^{(z;q)} J_{n,\Psi}^\ell(z;q) = J_{n-1,\Psi}^\ell(z;q) - J_{n+1,\Psi}^\ell(z;q).$$

Here, applying the operator $2\ {}_\ell\mathcal{D}_M^{(z;q)}$ both the sides, one finds that

$$\begin{aligned} 2^2 \left({}_\ell\mathcal{D}_M^{(z;q)} \right)^2 J_{n,\Psi}^\ell(z;q) &= 2\ {}_\ell\mathcal{D}_M^{(z;q)} J_{n-1,\Psi}^\ell(z;q) - 2\ {}_\ell\mathcal{D}_M^{(z;q)} J_{n+1,\Psi}^\ell(z;q) \\ &= J_{n-2,\Psi}^\ell(z;q) - 2J_{n,\Psi}^\ell(z;q) + J_{n+2,\Psi}^\ell(z;q) \\ &= \sum_{m=0}^{2} (-1)^{2-m} \binom{2}{m} J_{n+2-2m,\Psi}^\ell(z;q). \end{aligned} \tag{7.33}$$

Similarly,

$$\begin{aligned} & 2^3 \left({}_\ell\mathcal{D}_M^{(z;q)} \right)^3 J_{n,\Psi}^\ell(z;q) \\ &= 2\ {}_\ell\mathcal{D}_M^{(z;q)} J_{n-2,\Psi}^\ell(z;q) - 2\ {}_\ell\mathcal{D}_M^{(z;q)} \left[2J_{n,\Psi}^\ell(z;q) \right] + 2\ {}_\ell\mathcal{D}_M^{(z;q)} J_{n+2,\Psi}^\ell(z;q) \\ &= J_{n-3,\Psi}^\ell(z;q) - 3J_{n-1,\Psi}^\ell(z;q) + 3J_{n+1,\Psi}^\ell(z;q) - J_{n+3,\Psi}^\ell(z;q) \\ &= \sum_{m=0}^{3} (-1)^{3-m} \binom{3}{m} J_{n+3-2m,\Psi}^\ell(z;q). \end{aligned}$$

The recursive procedure k-times, leads to the theorem. □

Alternatively, this theorem can also be proved by using the principle of mathematical induction on k.

Proof. (Alternate proof Theorem 7.17)
For $k = 1$, it is true from Theorem 7.15. Now suppose that the theorem holds for $k =$some positive integer r. That is,

$$2^r \left({}_\ell\mathcal{D}_M^{(z;q)} \right)^r J_{n,\Psi}^\ell(z;q) = \sum_{m=0}^{r} (-1)^{r-m} \binom{r}{m} J_{n+r-2m,\Psi}^\ell(z;q). \tag{7.34}$$

Then to prove that for $k = r + 1$ the theorem holds true. In fact,

$$\begin{aligned} & 2^{r+1} \left({}_\ell\mathcal{D}_M^{(z;q)} \right)^{r+1} J_{n,\Psi}^\ell(z;q) \\ &= 2\ {}_\ell\mathcal{D}_M^{(z;q)} \left(2^r \left({}_\ell\mathcal{D}_M^{(z;q)} \right)^r J_{n,\Psi}^\ell(z;q) \right) \\ &= 2\ {}_\ell\mathcal{D}_M^{(z;q)} \left(\sum_{m=0}^{r} (-1)^{r-m} \binom{r}{m} J_{n+r-2m,\Psi}^\ell(z;q) \right) \\ &= 2\ {}_\ell\mathcal{D}_M^{(z;q)} \left[(-1)^r J_{n+r,\Psi}^\ell(z;q) + r(-1)^{r-1} J_{n+r-2,\Psi}^\ell(z;q) + \ldots + J_{n-r,\Psi}^\ell(z;q) \right]. \end{aligned}$$

Once again using Theorem 7.15, one gets

$$\begin{aligned}
&2^{r+1}\left({}_{\ell}\mathcal{D}_{M}^{(z;q)}\right)^{r+1} \quad J^{\ell}_{n,\Psi}(z;q) \\
&= (-1)^r\left[J^{\ell}_{n-1+r,\Psi}(z;q)-J^{\ell}_{n+1+r,\Psi}(z;q)\right]+r(-1)^{r-1}\left[J^{\ell}_{n-3+r,\Psi}(z;q)-J^{\ell}_{n-1+r,\Psi}(z;q)\right] \\
&\quad +\ldots+\left[J^{\ell}_{n-1-r,\Psi}(z;q)-J^{\ell}_{n+1-r,\Psi}(z;q)\right] \\
&= (-1)^{r+1}J^{\ell}_{n+1+r,\Psi}(z;q)+(-1)^r(r+1)J^{\ell}_{n-1+r,\Psi}(z;q)+\ldots+J^{\ell}_{n-1-r,\Psi}(z;q) \\
&= \sum_{m=0}^{r+1}(-1)^{r+1-m}\binom{r+1}{m}\; J^{\ell}_{n+r+1-2m,\Psi}(z;q)
\end{aligned}$$

as desired. □

In the following theorems, some other properties are extended.

Theorem 7.18. *There hold the identities (cf. [46, Ex.2, p.120]):*

$$\cos^{\ell}_{\Psi}(z;q) = J^{\ell}_{0,\Psi}(z;q)+2\sum_{n=1}^{\infty}(-1)^n\; J^{\ell}_{2n,\Psi}(z;q), \tag{7.35}$$

$$\sin^{\ell}_{\Psi}(z;q) = 2\sum_{n=0}^{\infty}(-1)^n\; J^{\ell}_{2n+1,\Psi}(z;q). \tag{7.36}$$

Proof. From the GFR of the q-ℓ-ΨBF (7.18) and the identity (7.17),

$$\begin{aligned}
e^{\ell}_{\Psi}\left(\frac{z}{2}\left(t-t^{-1};q\right)\right) &= \sum_{n=-\infty}^{\infty} J^{\ell}_{n,\Psi}(z;q)\; t^n \\
&= J^{\ell}_{0,\Psi}(z;q)+\sum_{n=1}^{\infty}J^{\ell}_{n,\Psi}(z;q)\; t^n+\sum_{n=1}^{\infty}J^{\ell}_{-n,\Psi}(z;q)\; t^{-n} \\
&= J^{\ell}_{0,\Psi}(z;q)+\sum_{n=1}^{\infty}J^{\ell}_{n,\Psi}(z;q)\;[t^n+(-1)^n\; t^{-n}]. \qquad (7.37)
\end{aligned}$$

Taking $t=i$, and thereby $t^{-1}=-i$, one gets

$$\begin{aligned}
e^{\ell}_{\Psi}(iz;q) &= J^{\ell}_{0,\Psi}(z;q)+\sum_{n=1}^{\infty}J^{\ell}_{n,\Psi}(z;q)\;[i^n+(-1)^n\; i^{-n}] \\
&= J^{\ell}_{0,\Psi}(z;q)+\sum_{n=1}^{\infty}J^{\ell}_{2n,\Psi}(z;q)\;[i^{2n}+(-1)^{2n}\; i^{-2n}] \\
&\quad +\sum_{n=0}^{\infty}J^{\ell}_{2n+1,\Psi}(z;q)\;[i^{2n+1}+(-1)^{2n+1}\; i^{-(2n+1)}].
\end{aligned}$$

Finally, in view of (7.4), this gives

$$\begin{aligned}\cos_\Psi^\ell(z;q)+i\sin_\Psi^\ell(z;q) &= J_{0,\Psi}^\ell(z;q)+2\sum_{n=1}^\infty(-1)^n\ J_{2n,\Psi}^\ell(z;q)\\ &\quad +2i\ \sum_{n=0}^\infty(-1)^n\ J_{2n+1,\Psi}^\ell(z;q).\end{aligned}$$

On comparing the real and imaginary parts, the theorem follows. □

Theorem 7.19. *There hold the identities (cf. [46, Ex.3, p.120]):*

$$\cos_\Psi^\ell(z\sin_\Psi^\ell(\theta;q);q) = J_{0,\Psi}^\ell(z;q)+2\sum_{n=1}^\infty J_{2n,\Psi}^\ell(z;q)\ \cos_\Psi^\ell(2n\theta;q), \quad (7.38)$$

$$\sin_\Psi^\ell(z\sin_\Psi^\ell(\theta;q);q) = 2\sum_{n=0}^\infty J_{2n+1,\Psi}^\ell(z;q)\ \sin_\Psi^\ell((2n+1)\theta;q). \quad (7.39)$$

Proof. In (7.37), substituting $t=e_\Psi^\ell(i\theta)$, the q-ℓ-Ψ exponential function, one finds

$$\begin{aligned}&e_\Psi^\ell\left(\frac{z}{2}\left(e_\Psi^\ell(i\theta;q)-e_\Psi^\ell(-i\theta;q)\right);q\right)\\ &= J_{0,\Psi}^\ell(z;q)+\sum_{n=1}^\infty J_{n,\Psi}^\ell(z;q)\ [e_\Psi^\ell(in\theta;q)+(-1)^n\ e_\Psi^\ell(-in\theta;q)]\\ &= J_{0,\Psi}^\ell(z;q)+\sum_{n=1}^\infty J_{2n,\Psi}^\ell(z;q)\ [e_\Psi^\ell(2in\theta;q)+(-1)^{2n}\ e_\Psi^\ell(-2in\theta;q)]\\ &\quad +\sum_{n=0}^\infty J_{2n+1,\Psi}^\ell(z;q)\ [e_\Psi^\ell((2n+1)i\theta;q)+(-1)^{2n+1}\ e_\Psi^\ell(-(2n+1)i\theta;q)]\\ &= J_{0,\Psi}^\ell(z;q)+2\sum_{n=1}^\infty J_{2n,\Psi}^\ell(z;q)\cos_\Psi^\ell(2n\theta;q)\\ &\quad +2i\ \sum_{n=0}^\infty J_{2n+1,\Psi}^\ell(z;q)\ \sin_\Psi^\ell((2n+1)\theta;q).\end{aligned}$$

From (7.6),

$$\begin{aligned}&e_\Psi^\ell\left(\frac{z}{2}\left(e_\Psi^\ell(i\theta;q)-e_\Psi^\ell(-i\theta;q)\right);q\right)\\ &= e_\Psi^\ell(iz\sin_\Psi^\ell(\theta;q);q)\\ &= \cos_\Psi^\ell(z\sin_\Psi^\ell(\theta;q);q)+i\sin_\Psi^\ell(z\sin_\Psi^\ell(\theta;q);q),\end{aligned}$$

hence the result follows by comparison of the real and imaginary parts. □

Theorem 7.20. *For $n \in \mathbb{Z}$, there hold (cf. [46, Ex.3, p.120]):*

$$[1+(-1)^n]\ J^{\ell}_{n,\Psi}(z;q) \ = \ \frac{2}{\pi}\int_0^{\pi} \cos n\theta\ \ \cos^{\ell}_{\Psi}(z\sin\theta;q)\ \mathrm{d}\theta, \qquad (7.40)$$

$$[1-(-1)^n]\ J^{\ell}_{n,\Psi}(z;q) \ = \ \frac{2}{\pi}\int_0^{\pi} \sin n\theta\ \ \sin^{\ell}_{\Psi}(z\sin\theta;q)\ \mathrm{d}\theta. \qquad (7.41)$$

Further,

$$J^{\ell}_{2n,\Psi}(z;q) \ = \ \frac{1}{\pi}\int_0^{\pi} \cos 2n\theta\ \ \cos^{\ell}_{\Psi}(z\sin\theta;q)\ \mathrm{d}\theta, \qquad (7.42)$$

$$J^{\ell}_{2n+1,\Psi}(z;q) \ = \ \frac{1}{\pi}\int_0^{\pi} \sin(2n+1)\theta\ \ \sin^{\ell}_{\Psi}(z\sin\theta;q)\ \mathrm{d}\theta \qquad (7.43)$$

and

$$\int_0^{\pi} \cos(2n+1)\theta\ \ \cos^{\ell}_{\Psi}(z\sin\theta;q)\ \mathrm{d}\theta \ = \ 0, \qquad (7.44)$$

$$\int_0^{\pi} \sin 2n\theta\ \ \sin^{\ell}_{\Psi}(z\sin\theta;q)\ \mathrm{d}\theta \ = \ 0. \qquad (7.45)$$

Proof. The q-ℓ-ΨBF integral (7.25),

$$J^{\ell}_{n,\Psi}(z;q) = \frac{1}{\pi}\int_0^{\pi}\left[\cos n\theta\ \ \cos^{\ell}_{\Psi}(z\sin\theta;q) + \sin n\theta\ \ \sin^{\ell}_{\Psi}(z\sin\theta;q)\right]\mathrm{d}\theta. \quad (7.46)$$

Hence

$$J^{\ell}_{-n,\Psi}(z;q) = \frac{1}{\pi}\int_0^{\pi}\left[\cos(-n\theta)\ \ \cos^{\ell}_{\Psi}(z\sin\theta;q) + \sin(-n\theta)\ \ \sin^{\ell}_{\Psi}(z\sin\theta;q)\right]\mathrm{d}\theta,$$

that is,

$$(-1)^n J^{\ell}_{n,\Psi}(z;q) = \frac{1}{\pi}\int_0^{\pi}\left[\cos n\theta \cos^{\ell}_{\Psi}(z\sin\theta;q) - \sin n\theta \sin^{\ell}_{\Psi}(z\sin\theta;q)\right]\mathrm{d}\theta. (7.47)$$

Here adding (subtracting) (7.46) and (7.47) one obtains (7.40) ((7.41)).
By considering even ordered q-ℓ-ΨBF in (7.40) and odd ordered q-ℓ-ΨBF in (7.41)

yield (7.42) and (7.43) respectively.
Similarly, if the order n is an odd (even) integer, then (7.40) ((7.41)) furnishes (7.44) ((7.45)). □

7.3.2 Addition formula

Theorem 7.21. *For the q-ℓ-ΨBF, there holds (cf. [59, Sec. 2.4, 2.5, p.30])*

$$J^{\ell}_{n,\Psi}(z_1+z_2;q) \;=\; \sum_{m=-\infty}^{\infty} J^{\ell}_{m,\Psi}(z_1;q)\; J^{\ell}_{n-m,\Psi}(z_2;q). \tag{7.48}$$

Proof. On substituting $z = z_1 + z_2$ in the contour integral (7.26) and then using the property (7.10), one obtains

$$\begin{aligned}
J^{\ell}_{n,\Psi}(z_1+z_2;q) &= \frac{1}{2\pi i}\int^{(0+)} u^{-n-1}\; e^{\ell}_{\Psi}\left(\frac{(z_1+z_2)\,(u-u^{-1})}{2};q\right) \mathrm{du} \\
&= \frac{1}{2\pi i}\int^{(0+)} u^{-n-1}\; e^{\ell}_{\Psi}\left(\frac{z_1}{2}(u-u^{-1});q\right)\; e^{\ell}_{\Psi}\left(\frac{z_2}{2}(u-u^{-1});q\right) \mathrm{du} \\
&= \frac{1}{2\pi i}\int^{(0+)} u^{-n-1} \sum_{m=-\infty}^{\infty} J^{\ell}_{m,\Psi}(z_1;q)\; u^m\; e^{\ell}_{\Psi}\left(\frac{z_2}{2}(u-u^{-1});q\right) \mathrm{du} \\
&= \sum_{m=-\infty}^{\infty} J^{\ell}_{m,\Psi}(z_1;q)\; \frac{1}{2\pi i}\int^{(0+)} u^{m-n-1}\; e^{\ell}_{\Psi}\left(\frac{z_2}{2}(u-u^{-1});q\right) \mathrm{du}.
\end{aligned}$$

Once again making an appeal to (7.26), (7.48) occurs. □

Corollary 7.22. *The series relation:*

$$J^{\ell}_{n,\Psi}(2z;q) = \sum_{m=0}^{n} J^{\ell}_{m,\Psi}(z;q)\; J^{\ell}_{n-m,\Psi}(z;q) + 2\sum_{m=1}^{\infty}(-1)^m\; J^{\ell}_{m,\Psi}(z;q)\; J^{\ell}_{n+m,\Psi}(z;q) \tag{7.49}$$

holds.

Proof. If $z_1 = z_2 = z$, then (7.48) reduces to

$$\begin{aligned}
J^{\ell}_{n,\Psi}(2z;q) &= \sum_{m=-\infty}^{\infty} J^{\ell}_{m,\Psi}(z;q)\; J^{\ell}_{n-m,\Psi}(z;q) \\
&= \sum_{m=-\infty}^{-1} J^{\ell}_{m,\Psi}(z;q)\; J^{\ell}_{n-m,\Psi}(z;q) + \sum_{m=0}^{\infty} J^{\ell}_{m,\Psi}(z;q)\; J^{\ell}_{n-m,\Psi}(z;q)
\end{aligned}$$

$$= \sum_{m=-\infty}^{-1} J^{\ell}_{m,\Psi}(z;q)\ J^{\ell}_{n-m,\Psi}(z;q) + \sum_{m=0}^{n} J^{\ell}_{m,\Psi}(z;q)\ J^{\ell}_{n-m,\Psi}(z;q)$$
$$+ \sum_{m=n+1}^{\infty} J^{\ell}_{m,\Psi}(z;q)\ J^{\ell}_{n-m,\Psi}(z;q).$$

This in view of (7.17), gives

$$\begin{aligned} J^{\ell}_{n,\Psi}(2z;q) &= \sum_{m=0}^{n} J^{\ell}_{m,\Psi}(z;q)\ J^{\ell}_{n-m,\Psi}(z;q) + \sum_{m=1}^{\infty}(-1)^m\ J^{\ell}_{m,\Psi}(z;q)\ J^{\ell}_{n+m,\Psi}(z;q) \\ &\quad + \sum_{m=1}^{\infty}(-1)^m\ J^{\ell}_{m,\Psi}(z;q)\ J^{\ell}_{n+m,\Psi}(z;q) \\ &= \sum_{m=0}^{n} J^{\ell}_{m,\Psi}(z;q)\ J^{\ell}_{n-m,\Psi}(z;q) + 2\sum_{m=1}^{\infty}(-1)^m\ J^{\ell}_{m,\Psi}(z;q)\ J^{\ell}_{n+m,\Psi}(z;q). \end{aligned}$$

Remark 7.23. For $\ell = 0$ and $q \to 1$, this result reduces to

$$J_n(2z) = \sum_{m=0}^{n} J_m(z)\ J_{n-m}(z) + 2\sum_{m=1}^{\infty}(-1)^m\ J_m(z)\ J_{n+m}(z).$$

□