A STUDY OF FOURIER SERIES AND FUNCTIONS OF GENERALIZED BOUNDED VARIATIONS ON GROUPS

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BY

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DECLARATION

I hereby declare that the matter embodied in this Ph. D. Thesis is the result of the researches carried out by me in the Department of Mathematics, Faculty of Science, The Maharaja Sayajirao University of Baroda, Vadodara – 390002, under the guidance of Dr. J. R. Patadia, Professor of Mathematics (Retd.), Department of Mathematics, Faculty of Science, The Maharaja Sayajirao University of Baroda, Vadodara – 390002, and it has not been submitted for the award of any degree of any other university or institution.

Bhikha Lila Ghodadra

Place: Vadodara
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CERTIFICATE

This is to certify that Mr. Bhikha Lila Ghodadra has worked under my guidance and supervision for his Ph. D. Thesis in the subject of Mathematics on the topic entitled

"*A STUDY OF FOURIER SERIES AND FUNCTIONS OF GENERALIZED BOUNDED VARIATIONS ON GROUPS*"

and has completed it to my satisfaction. This thesis is being submitted herewith for the award of the degree of

Doctor of Philosophy
in
Mathematics

and, to the best of my knowledge, no part of this thesis has been submitted by him to any other university or institution for the same or any other degree.

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Finally I express my immense sense of gratitude to my mother, my younger brother and my better half for their co-operation and sustained support during this time. And, of course, I don't have proper words to express my feelings for my little son and daughter who has spared, often reluctantly, many of their rightful hours for letting me spend them for my these investigations during this period.

Bhikha Lila Ghodadra

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DEDICATED
TO
MY PARENTS

Smt. Liriben Lilabhai Ghodadra

and

Late Shri Lilabhai Vasarambhai Ghodadra

who imbibed in me

Courage, Conviction
and
Academic Values

P R E F A C E

The material of the present thesis is based on the researches that I have been carrying out since November, 2005 at The Maharaja Sayajirao University of Baroda, Vadodara, on *'A study of Fourier series and functions of generalized bounded variations on groups'* under the able guidance and supervision of Professor Jamanadas R. Patadia, M. Sc., Ph. D., (Retired) Professor of Mathematics, Department of Mathematics, Faculty of Science, The Maharaja Sayajirao University of Baroda.

The present thesis contains five chapters. The two dimensional analogue of the first half of results of Chapter 2 is published in the form of a paper in *Acta Mathematica Hungarica*, **128 (4)** (2010), 328 – 343 (DOI: 10.1007/s10474-010-9202-y). These results were first presented at the National Seminar on Analysis, Differential Equations and Applications held at the Department of Mathematics, Faculty of Science, The Maharaja Sayajirao University of Baroda during January 30 – 31, 2009. The results of second half of this chapter are accepted for publication in *Acta Scientiarum Mathematicarum*, János Bolyai Mathematical Institute, University of Szeged, Szeged, Hungary. The first one fourth of results of Chapter 3 is published in the form of a paper in the *Journal of Inequalities in Pure and Applied Mathematics*, **9 (2)** (2008), Article 44, 7 pp.. The second one fourth is published in the form of a paper in *Mathematics Today*, **24** (2008), 47 – 54. These results were also presented at the Mathematics Meet – 2008, held at Department of Mathematics, Gujarat University, Ahmedabad, during March 28 – 29, 2008. The first half of results of Chapter 4 is published in the form of a paper in the *Journal of Indian Mathematical Society*, **75 (1–4)** (2008), 93 – 104. These were first presented at the 73rd Annual Conference of the Indian Mathematical Society held at Department of Mathematics, University of Pune, College of Engineering, Pune and Fergussion College, Pune during December 27 – 30, 2007 for competition for paper presentation in the section of *Analysis* and won the "V. M. Shah Prize for the Year 2007" for presenting the best research paper in Analysis. The second half of results of Chapter 4 are accepted for publication in *Kyoto Journal of Mathematics*, Kyoto University, Kyoto, Japan. These results were presented at 23rd Annual Meeting of Ramanujan Mathematical Society organized by Department of Mathematics, Indian Institute of Technology Kanpur, Kanpur during May 19 – 21, 2008. The rest of the results of the present thesis are under communication for publication. Prof. Patadia has also collaborated with me in some of the investigations [13].

Contents

Chapter 1

Introduction and Preliminaries

1.1 Order of Magnitude of Fourier Coefficients

Let f be a Lebesgue integrable function on the circle group $\mathbb{T}$ (identified with an interval of length 2π, say $[0, 2\pi)$) and let the series

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)\mathrm{e}^{inx}$$

denote its Fourier series where, for each integer n,

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x)\mathrm{e}^{-inx}\, dx$$

denotes the n^{th} Fourier coefficient of f. The fundamental property of Fourier coefficients of such a function f is the well known Riemann-Lebesgue lemma which states that $\hat{f}(n) = o(1)$ as $|n| \to \infty$ (see, for example, [9, 2.3.8]). However, in general, there is no definite rate at which the Fourier coefficients tend to zero — even for continuous functions on $\mathbb{T}$ (see, for example, [9, 2.3.9] or [3, Vol. I, p. 229]). In fact, it is known that the Fourier coefficients can tend to zero as slow as possible — in the sense that given a sequence $\{a_n\}$ of positive numbers tending to zero, one can find $f \in \mathrm{L}^1(\mathbb{T})$ such that $|\hat{f}(n)| \geq a_n$, for all n (see, for example, [9, 7.4] or [3, Vol. I, p. 229]). Therefore it is interesting to know for functions of which subspaces of the space of integrable functions, there is some definite rate at which the Fourier coefficients tend to zero. Let us discuss the development in this regard in the setting of certain groups. Since the study in the case of circle group is classical and has motivated the study in the setting of other groups, we begin with the case of circle group. We have carried out the study in the setting of other than the circle group and we shall mention our work at the end of each case.

(A). In the case of the **circle group** $\mathbb{T}$, the study of definite rate at which Fourier coefficients tend to zero has been carried out intensively for functions of various subspaces of $\mathrm{L}^1(\mathbb{T})$. In fact, following results are well known in which $I = [a, b]$ is an interval and when $I = [0, 2\pi]$, we drop writing I or write $\mathbb{T}$.

Theorem 1.1.1 ([3, Vol. I, p. 38, (25.6) and p. 71, (21.5)]). *$\hat{f}(n) = O(1/|n|^\alpha)$ as $|n| \to \infty$ for $f \in \mathrm{Lip}\,\alpha(\mathbb{T})$, $0 < \alpha \leq 1$, where $\mathrm{Lip}\,\alpha(\mathbb{T})$ is the subspace of $\mathrm{C}(\mathbb{T}) \subset \mathrm{L}^1(\mathbb{T})$ of functions satisfying Lipschitz condition of order α over $[0, 2\pi]$. (Recall that a real-valued function f is in $\mathrm{Lip}\,\alpha(\mathbb{T})$ if there exists a constant C such that $|f(x) - f(y)| \leq C|x - y|^\alpha$ for all $x, y \in \mathbb{T}$.)*

Theorem 1.1.2 ([9, p. 33, Remark (4)]). *$\hat{f}(n) = o(1/|n|)$ as $|n| \to \infty$ for $f \in \mathrm{AC}(\mathbb{T})$, the subspace of $\mathrm{C}(\mathbb{T}) \subset \mathrm{L}^1(\mathbb{T})$ of functions which are absolutely continuous over $[0, 2\pi]$. (Recall that a real-valued function f is in $\mathrm{AC}(\mathbb{T})$ if for every $\epsilon > 0$, there is a $\delta > 0$ such that $\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon$ for every finite collection $\{(a_k, b_k)\}_{k=1}^{n}$ of non-overlapping subintervals of $\mathbb{T}$ with $\sum_{k=1}^{n} |b_k - a_k| < \delta$.)*

Theorem 1.1.3 ([9, p. 33, 2.3.6]). *$\hat{f}(n) = O(1/|n|)$ as $|n| \to \infty$ for $f \in \mathrm{BV}(\mathbb{T})$, the subspace of $\mathrm{L}^1(\mathbb{T})$ of functions of bounded variation over $[0, 2\pi]$, that is,*

$$\mathrm{BV}(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{R} : V(f, \mathbb{T}) < \infty\}, \tag{1.1}$$

where

$$V(f; \mathbb{T}) = \sup \left\{ \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \right\},$$

in which the supremum is taken over all the partitions $0 = x_0 < x_1 < ... < x_n = 2\pi$ of $[0, 2\pi]$.

Theorem 1.1.4 ([59]). *$\hat{f}(n) = O\left(1/|n|^{1/p}\right)$ as $|n| \to \infty$ for $f \in \mathrm{BV}^{(p)}(\mathbb{T})$, $p \geq 1$, the subspace of $\mathrm{L}^1(\mathbb{T})$ of functions of bounded variation of order $p \geq 1$ over $[0, 2\pi]$, that is,*

$$\mathrm{BV}^{(p)}(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{R} : V(f, p, \mathbb{T}) < \infty\}, \tag{1.2}$$

where

$$V_p(f;\mathbb{T}) = \sup\left\{\left(\sum_{k=1}^{n}|f(x_k)-f(x_{k-1})|^p\right)^{1/p}\right\},$$

in which the supremum is taken over all the partitions $0 = x_0 < x_1 < ... < x_n = 2\pi$ *of* $[0, 2\pi]$.

Note that $\mathrm{BV}^{(1)} = \mathrm{BV}$. Note also that the class $\mathrm{BV}^{(p)}(I)$ can be defined as in (1.2) by replacing $\mathbb{T}$ by I. The class $\mathrm{BV}^{(p)}$ was introduced by Wiener in 1924 [77]. This concept of bounded variation of order $p \geq 1$ was subsequently generalized by L. C. Young [78] in 1937 who introduced the following class $\phi\mathrm{BV}$ of functions of ϕ-bounded variation.

Definition 1.1.5. Given a continuous function ϕ, defined on $[0, \infty)$ and strictly increasing from 0 to ∞, we say that $f \in \phi\mathrm{BV}(I)$ (that is, f is of ϕ-*bounded variation* over I) if

$$V_\phi(f;I) = \sup\left\{\sum_{k=1}^{n}\phi(|f(x_k)-f(x_{k-1})|)\right\} < \infty,$$

where the supremum is taken over all the partitions $0 = x_0 < x_1 < ... < x_n = 2\pi$ of $[0, 2\pi]$.

Clearly, $\phi(u) = u$ gives the Jordan's class $\mathrm{BV}(I)$ and $\phi(u) = u^p$ gives the Wiener's class $\mathrm{BV}^{(p)}(I)$. It is customary to consider ϕ a convex function such that $\phi(0) = 0$, $\frac{\phi(x)}{x} \to 0$ $(x \to 0_+)$, $\frac{\phi(x)}{x} \to \infty$ $(x \to \infty)$; such a function is called an N-function and is necessarily continuous and strictly increasing on $[0, \infty)$. For functions of $\phi\mathrm{BV}(\mathbb{T})$, the following result is well known (which follows from [57, Corollary to Theorem 1]).

Theorem 1.1.6. $\hat{f}(n) = O\left(\phi^{-1}\left(\frac{1}{|n|}\right)\right)$ *as* $|n| \to \infty$ *for* $f \in \phi\mathrm{BV}(\mathbb{T})$.

Another class, directly influenced by the study of the convergence problems in the theory of Fourier series, namely the following class $\Lambda\mathrm{BV}$ of functions of Λ-bounded variation appeared in Waterman's paper [76] in 1972.

Definition 1.1.7. Given a sequence $\Lambda = \{\lambda_k\}_{k\in\mathbb{N}}$ of non-decreasing positive numbers λ_k such that $\sum_{k=1}^{\infty}\frac{1}{\lambda_k}$ diverges, we say that $f \in \Lambda\mathrm{BV}(I)$ (that is, f is of Λ-*bounded variation* over I) if

$$V_\Lambda(f,I) = \sup_{\{I_k\}}\left\{\sum_k \frac{|f(b_k)-f(a_k)|}{\lambda_k}\right\} < \infty,$$

where $\{I_k = [a_k, b_k]\}$ is a sequence of non-overlapping subintervals of I.

Note that if $\lambda_k \equiv 1$, one gets from this the class $\mathrm{BV}(I)$ and if $\lambda_k = k$ $\forall k$ then one gets the class $\mathrm{HBV}(I)$ of functions of harmonic bounded variation over I. For functions of $\Lambda\mathrm{BV}(\mathbb{T})$, Schramm and Waterman [57] proved the following result.

Theorem 1.1.8. *$\hat{f}(n) = O\left(1/\sum_{j=1}^{|n|} \frac{1}{\lambda_j}\right)$ as $|n| \to \infty$ for $f \in \Lambda\mathrm{BV}(\mathbb{T})$.*

Later, in 1980, Shiba [58] introduced the following class $\Lambda\mathrm{BV}^{(p)}$ of functions of p-Λ-bounded variation.

Definition 1.1.9. Given a sequence Λ as in Definition 1.1.7 and a real number p, $1 \leq p < \infty$, we say that $f \in \Lambda\mathrm{BV}^{(p)}(I)$ (that is, f is of p-Λ-*bounded variation* over I) if

$$V_{p\Lambda}(f; I) = \sup_{\{I_k\}} \left\{ \left(\sum_k \frac{|f(b_k) - f(a_k)|^p}{\lambda_k} \right)^{1/p} \right\} < \infty,$$

where $\{I_k\}$ is as in Definition 1.1.7.

Note that if $p = 1$ one gets from this the class $\Lambda\mathrm{BV}(I)$; if $\lambda_k \equiv 1$, one gets the class $\mathrm{BV}^{(p)}(I)$; and if $p = 1$ as well as $\lambda_k \equiv 1$ then one gets the class $\mathrm{BV}(I)$. For functions of $\Lambda\mathrm{BV}^{(p)}(\mathbb{T})$, the following result is well known [57].

Theorem 1.1.10. *$\hat{f}(n) = O\left(1 \Big/ \left(\sum_{j=1}^{|n|} \frac{1}{\lambda_j}\right)^{1/p}\right)$ as $|n| \to \infty$ for $f \in \Lambda\mathrm{BV}^{(p)}(\mathbb{T})$.*

The class $\phi\Lambda\mathrm{BV}$ of functions of ϕ-Λ-bounded variation over I was introduced by Schramm and Waterman [56] in 1982 as follows.

Definition 1.1.11. Given a function ϕ as in Definition 1.1.5 and a sequence Λ as in Definition 1.1.7, we say that $f \in \phi\Lambda\mathrm{BV}(I)$ (that is, f is of ϕ-Λ-*bounded variation* over I) if

$$V_{\phi\Lambda}(f; I) = \sup_{\{I_k\}} \left\{ \sum_k \frac{\phi(|f(b_k) - f(a_k)|)}{\lambda_k} \right\} < \infty,$$

where $\{I_k\}$ is as in Definition 1.1.7.

Note that if $\phi(u) = u^p$ one gets from this the class $\Lambda\mathrm{BV}^{(p)}(I)$. For functions of $\phi\Lambda\mathrm{BV}(\mathbb{T})$, Schramm and Waterman [57] proved the following result.

Theorem 1.1.12. *$\hat{f}(n) = O\left[\phi^{-1}\left(1/\sum_{j=1}^{|n|} \frac{1}{\lambda_j}\right)\right]$ as $|n| \to \infty$ for $f \in \phi\Lambda\mathrm{BV}(\mathbb{T})$.*

Further, considering the differences of order $r \geq 2$, the class r-BV of the functions of bounded r^{th} variation is also one of the interesting generalizations of the Jordan class. This is given as follows.

Definition 1.1.13. Given a positive integer r, we say that $f \in r\text{-BV}(I)$ (that is, f is of *bounded r^{th} variation* over I) if

$$V^{(r)}(f, I) = \sup_n \left\{ \sum_{i=0}^{n-r} |\Delta^r f(x_i)| \right\} < \infty,$$

where the supremum is taken over arbitrary $(n+1)$ points $x_0 < x_1 < ... < x_n$ in I in an arithmetic progression,

$$\Delta f(x_i) = f(x_{i+1}) - f(x_i)$$

and for $k \geq 2$

$$\Delta^k f(x_i) = \Delta^{k-1} f(x_{i+1}) - \Delta^{k-1} f(x_i)$$

so that

$$\Delta^r f(x_i) = \sum_{m=0}^{r} (-1)^m \binom{r}{m} f(x_{i+r-m}).$$

Clearly, $\text{BV}(I) \subset r\text{-BV}(I)$ but $\text{BV}(I) \neq r\text{-BV}(I)$. For example, it is well known that everywhere continuous but nowhere differentiable function of Weierstrass [21], given by

$$f(x) = \sum_{n=1}^{\infty} b^{-n} \cos\left(b^n x\right), \quad b \text{ an integer} > 1,$$

satisfies the condition

$$|f(x+h) + f(x-h) - 2f(x)| = O(|h|) \text{ as } h \to 0$$

uniformly in x in $I = [0, 2\pi]$ and therefore $f \in 2\text{-BV}(I)$ [79]; however, f being a nowhere differentiable function, $f \notin \text{BV}(I)$.

Yet another generalization of the concept of a function of bounded variation is due to Chanturia [5] who introduced the following class $\text{V}(h)$ in 1974.

Definition 1.1.14. Given a non-decreasing concave downward function $h(n)$ defined on the set of positive integers, we say that $f \in \mathrm{V}(h)(I)$ if there is a constant C such that

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| \leq C\, h(n), \quad \forall n \in \mathbb{N},$$

for every sequence $\{I_k\}$ as in Definition 1.1.5.

For functions of $\mathrm{V}[n^\alpha](\mathbb{T})$, the following result is well known [72, Theorem 3, p. 34].

Theorem 1.1.15. *$\hat{f}(n) = O\left(1/|n|^{1-\alpha}\right)$ as $|n| \to \infty$ for $f \in \mathrm{V}[n^\alpha](\mathbb{T})$.*

Interestingly, many authors have generalized above results where in it is shown that same rate of order of magnitude of Fourier coefficients holds even if these hypotheses are satisfied only locally provided the Fourier series is lacunary with certain gaps. For example, following results are known.

Theorem 1.1.16 (Noble [36] or [3, Vol. II, p. 269])**.** *$\hat{f}(n) = O(1/|n|^\alpha)$ as $|n| \to \infty$ for $f \in \mathrm{Lip}\,\alpha(I)$ and $\hat{f}(n) = O(1/|n|)$ as $|n| \to \infty$ for $f \in \mathrm{BV}(I)$ if the Fourier series of f is lacunary of the form*

$$\sum_{k\in\mathbb{Z}} \hat{f}(n_k) \mathrm{e}^{\mathrm{i n_k x}} \tag{1.3}$$

where $\{n_k\}_1^\infty$ is a strictly increasing sequence of natural numbers satisfying the gap condition

$$\lim_{k\to\infty} \frac{N_k}{\log N_k} = \infty \tag{1.4}$$

in which $N_k = \min\{n_{k+1} - n_k, n_k - n_{k-1}\}$ and $n_{-k} = -n_k$ for all k; and I is any subinterval of $[0, 2\pi]$.

Theorem 1.1.17 (Kennedy [27])**.** *Noble's Theorem 1.1.16 hold under the weaker gap condition*

$$(n_{k+1} - n_k) \to \infty \;\; \text{as} \;\; k \to \infty. \tag{1.5}$$

Theorem 1.1.18 (Mazhar [32])**.** *Noble's second result in Theorem 1.1.16 holds if we replace the condition '$f \in \mathrm{BV}(I)$' by the weaker condition '$f \in r\text{-}\mathrm{BV}(I) \cap \mathrm{L}^2(I)$'.*

Theorem 1.1.19 (Tomić [65])**.** *If* $f \in \operatorname{Lip}\alpha$ $(0 < \alpha < 1)$ *and* $\{n_k\}$ *satisfies the Hadamard lacunarity condition*

$$\liminf_{n\to\infty} \frac{n_{k+1}}{n_k} > 1 \tag{1.6}$$

then for the lacunary Fourier series (1.3) *of* f*, we have*

$$\hat{f}(n_k) = O\left(n_k^{-\beta}\right), \quad \beta = \alpha/(2+\alpha). \tag{1.7}$$

This result is related to earlier results of Noble [36] and Kennedy [27, 29]. Kennedy [28] then sharpened this estimation (1.7) and proved following result.

Theorem 1.1.20. *Under the hypothesis of Theorem 1.1.19 we have*

$$\hat{f}(n_k) = O(1)\left((\log n_k)/n_k\right)^{\alpha}. \tag{1.8}$$

Further, he posed the question whether one can possibly suppress the factor $(\log n_k)^{\alpha}$ in (1.8). The affirmative answer was given by J. P. Kahane [26, p. 210], M. Izumi and S. I. Izumi [25]. Tomić [64] then proved the following more general result.

Theorem 1.1.21. *If* f *has the Fourier series* (1.3) *with* $\{n_k\}$ *satisfying the Hadamard lacunary condition* (1.6) *and*

$$f(x_0 \pm t) - f(x_0) = O(1)\omega^*(t) \quad \text{as} \quad t \to +0, \tag{1.9}$$

where $\omega^*(t)$ *is a function such that*

(a) $\omega^*(0) = 0$, $\omega^*(t) > 0$ *as* $t \to +0$,

(b) $\omega^*(t_1) \leq A\,\omega^*(t_2)$ *as* $0 < t_1 < t_2 \to +0$ *and*

(c) *there exists* $\alpha > 0$ *such that*

$$\omega^*(t_1)t_1^{-\alpha} \geq B\omega^*(t_2)t_2^{-\alpha} \quad \text{when} \quad 0 < t_1 < t_2$$

in which A *and* B *are constants, then*

$$\hat{f}(n_k) = \begin{cases} O(1)\omega^*\left(1/n_k\right) & \text{if } 0 < \alpha < 1, \\ O(1)\omega^*\left(1/n_k\right)\log n_k & \text{if } \alpha = 1. \end{cases}$$

Patadia [45, Theorem 1] further generalized above result of Tomić by considering a more general lacunarity condition

$$\min\{n_{k+1}-n_k, n_k-n_{k-1}\} \geq C\,F(n_k), \tag{1.10}$$

where $F(n_k)$ increases to ∞ as $k\to\infty$, $F(n_k)\leq n_k$ for all $k\in\mathbb{N}$ and $C>0$ is a constant and by suppressing the factor $\log n_k$ in case $\alpha=1$ (observe that with $F(n_k)=n_k$ this condition (1.10) gives rise to the Hadamard gap condition (1.6)). In fact, he proved the following result.

Theorem 1.1.22. *If f satisfies* (1.9) *and $\{n_k\}$ satisfies* (1.10) *then*

$$\hat{f}(n_k) = O(1)\omega^*(1/F(n_k)).$$

This result also generalizes the result due to Chao [6, Theorem 1]. Patel and Shah [49] generalized Mazhar's Theorem 1.1.18 and proved the following result.

Theorem 1.1.23. *Mazhar's Theorem 1.1.18 holds if the gap condition* (1.4) *is replaced the weaker gap condition* (1.5).

Patadia and Vyas [41] later proved the following result.

Theorem 1.1.24. *If $f\in \mathrm{L}^1[-\pi,\pi]$ possesses a lacunary Fourier series* (1.3) *with small gaps*

$$(n_{k+1}-n_k)\geq q\geq 1, \quad \textit{for } k=0,1,2,..., \tag{1.11}$$

and I is a subinterval of length $|I|>2\pi/q$, then $f\in \Lambda\mathrm{BV}^{(p)}(I)$ implies $\hat{f}(n) = O\left(1\Big/\left(\sum_{j=1}^{|n|}\frac{1}{\lambda_j}\right)^{1/p}\right)$ and $f\in r\text{-}\mathrm{BV}(I)\cap \mathrm{L}^2(I)$ implies $\hat{f}(n)=O\left(\frac{1}{|n|}\right)$.

Observing this vast study, in the case of circle group, of determination of the definite rate at which the Fourier coefficients of functions of various subclasses of integrable functions tend to zero, it is interesting to know about similar study carried out in the setting of other groups. We describe the development in this regard in the setting of m-dimensional torus $\mathbb{T}^m$, $m>1$, the Walsh group and the Vilenkin group, along with the brief indication of what we have done in each setting, in the following.

(B). In the case of the **m-dimensional torus** $\mathbb{T}^m$, $m>1$, again the Riemann-Lebesgue lemma holds: $f\in \mathrm{L}^1(\mathbb{T}^m)$ implies $\hat{f}(\mathbf{n})=o(1)$ ($\mathbf{n}=(n^{(1)},...,n^{(m)})$) as

$|\mathbf{n}| = \sqrt{|n^{(1)}|^2 + ... + |n^{(m)}|^2} \to \infty$. Also, it is a fact that there is no definite rate at which Fourier coefficients tend to zero; and the study of definite rate at which Fourier coefficients tend to zero has been carried out for functions of certain subspaces of $\mathrm{L}^1(\mathbb{T}^m)$. Statements of some known results follows the following terminology and definitions.

Let m be a positive integer, let $\mathbb{T}^m$ be the m-dimensional torus identified with $\mathbf{Q} = [-\pi, \pi]^m$ and let its dual be identified with $\mathbb{Z}^m$. The points $(x_1, ..., x_m)$ of $\mathbf{Q}$ and $(n^{(1)}, ..., n^{(m)})$ of $\mathbb{Z}^m$ are denoted by $\mathbf{x}$ and $\mathbf{n}$ respectively; $\mathbf{n} \cdot \mathbf{x}$ denotes the scalar product given by $\mathbf{n} \cdot \mathbf{x} = n^{(1)}x_1 + ... + n^{(m)}x_m$ and $|\mathbf{x}|$ denotes the number $\sqrt{|x_1|^2 + ... + |x_m|^2}$. For $f \in \mathrm{L}^1(\mathbb{T}^m)$ its formal Fourier series is given by

$$f(\mathbf{x}) \sim \sum_{\mathbf{n} \in \mathbb{Z}^m} \hat{f}(\mathbf{n}) e^{i(\mathbf{n} \cdot \mathbf{x})}, \tag{1.12}$$

where $\hat{f}(\mathbf{n})$ denotes the $\mathbf{n}^{th}$ Fourier coefficient of $f(\mathbf{x})$ given by

$$\hat{f}(\mathbf{n}) = \frac{1}{(2\pi)^m} \int_{\mathbf{Q}} f(\mathbf{x}) e^{-i(\mathbf{n} \cdot \mathbf{x})} d\mathbf{x}. \tag{1.13}$$

Let $\mathbf{x_0} = (x_{01}, ..., x_{0m})$ denote an arbitrary point of $\mathbf{Q}$, δ is any arbitrary real number such that $0 < \delta \leq \pi$ and $I = I(\mathbf{x_0}, \delta)$ denote the m-dimensional subrectangle of $\mathbf{Q}$ given by

$$I(\mathbf{x_0}, \delta) = \{\mathbf{x} := (x_1, ..., x_m) \in \mathbf{Q} : |x_j - x_{0j}| \leq \delta \;\text{ for }\; j = 1, ..., m\}. \tag{1.14}$$

Definition 1.1.25. For $\alpha > 0$, $\alpha = l + a$ with $0 < a \leq 1$ and l a non-negative integer, we say that $f \in \operatorname{Lip}(\alpha, I)$ (or, f satisfies $\operatorname{Lip}\alpha$ condition on I) if f has continuous partial derivatives

$$D^\theta f(x) = \frac{\partial^{\theta_1 + ... + \theta_m}}{\partial x_1^{\theta_1} ... \partial x_m^{\theta_m}} f(x) \;\text{ for }\; \theta_1 + ... + \theta_m \leq l,$$

and

$$\sup_{\substack{x, y \in I, \\ |x-y| \leq \delta}} \left| D^\theta f(x) - D^\theta f(y) \right| = O\left(\delta^a\right) \;\text{ whenever }\; \theta_1 + ... + \theta_m = l.$$

In case $0 < \alpha \leq 1$, we get $l = 0$, $a = \alpha$; so that $D^\theta f(x) \equiv D^0 f(x)$ which is taken to be $f(x)$; this is the case particularly when the dimension m is 1. When $I = \mathbf{Q}$, we simply write $f \in \operatorname{Lip}\alpha$.

The following m-dimensional analogue of circle group result (Theorem 1.1.1) appears to be known though we are unable to give precise reference.

Theorem 1.1.26. *$f \in \mathrm{Lip}\,\alpha$ implies $\hat{f}(\mathbf{n}) = O\left(|\mathbf{n}|^{-\alpha}\right)$ as $|\mathbf{n}| \to \infty$.*

The following lacunary version of this result which is analogue of the corresponding circle group result (first part of Theorem 1.1.16) due to Noble has been proved in [44].

Theorem 1.1.27. *Let $\alpha > 0$ and $f \in \mathrm{L}^1(\mathbb{T}^m)$ with $\hat{f}(\mathbf{n}) = 0$ for $\mathbf{n} \in \mathbb{Z}^m \setminus E$, where $E \subset \mathbb{Z}^m$ is given by $E = \prod_{j=1}^m E^{(j)}$ in which*

$$E^{(j)} = \left\{..., n_{-2}^{(j)}, n_{-1}^{(j)}, n_0^{(j)}, n_1^{(j)}, n_2^{(j)}, ...\right\} \subset \mathbb{Z}$$

with $n_{-k}^{(j)} = -n_k^{(j)}$ for $k = 0, 1, 2, ...$ and with $\left\{n_k^{(j)}\right\}_{k=1}^{\infty}$ strictly increasing such that

$$\liminf_{k\to\infty} \frac{N_k^{(j)}}{\log n_k^{(j)}} = B^{(j)} > \frac{32e}{\delta}, \tag{1.15}$$

where $N_k^{(j)} = \min\left\{n_{k+1}^{(j)} - n_k^{(j)}, n_k^{(j)} - n_{k-1}^{(j)}\right\}$. If $\mathbf{n_s} = \left(n_{s_1}^{(1)}, ..., n_{s_m}^{(m)}\right)$ denotes any typical element of E then $f \in \mathrm{Lip}\,(\alpha, I)$ implies $\hat{f}\,(\mathbf{n_k}) = O\left(|\mathbf{n_k}|^{-\alpha}\right)$.

Recently in 2002, considering the two dimensional torus, Móricz [33] has obtained certain definite rate of order of magnitude of double Fourier coefficients of functions of bounded variation in the sense of Vitali and Hardy and Krause and in 2004 Fülöp and Móricz [10] have obtained such rate of multiple Fourier coefficients of functions of bounded variation in the sense of Vitali and Hardy and Krause. To state these results, we need the following definitions.

Definition 1.1.28. Let R be the rectangle $R = [a_1, b_1] \times ... \times [a_m, b_m]$ in $\mathbb{R}^m$ with sides parallel to the cooordinate axes, that is,

$$R = \{(x_1, ..., x_m) \in \mathbb{R}^m : a_j \leq x_j \leq b_j;\ j = 1, ..., m\},$$

where $-\infty < a_j < b_j < +\infty$ for each j. By a (finite) grid $\mathcal{P}$ of R we mean $\mathcal{P} = \mathcal{P}_1 \times ... \times \mathcal{P}_m$, where

$$\mathcal{P}_j : a_j = x_j^0 < x_j^1 < ... < x_j^{s_j} = b_j, \quad s_j \geq 1; \quad j = 1, 2, ..., m.$$

Definition 1.1.29. Let $f = f(x_1, ..., x_m)$ be a real or complex-valued function on R. For any subrectangle $R' = [\alpha_1, \beta_1] \times ... \times [\alpha_m, \beta_m]$ of R with $a_i \le \alpha_i < \beta_i \le b_i$ for all $i = 1, 2, ..., m$, we define $\Delta f(R')$ as follows: When $m = 2$ we put

$$\Delta f(R') := \Delta f([\alpha_1, \beta_1] \times [\alpha_2, \beta_2]) = f(\beta_1, \beta_2) - f(\beta_1, \alpha_2) - f(\alpha_1, \beta_2) + f(\alpha_1, \alpha_2);$$

for $m = 3$

$$\begin{aligned} \Delta f(R') :&= \Delta f([\alpha_1, \beta_1] \times ... \times [\alpha_3, \beta_3]) \\ &= [f(\beta_1, \beta_2, \beta_3) - f(\beta_1, \alpha_2, \beta_3) - f(\alpha_1, \beta_2, \beta_3) + f(\alpha_1, \alpha_2, \beta_3)] \\ &\quad\quad - [f(\beta_1, \beta_2, \alpha_3) - f(\beta_1, \alpha_2, \alpha_3) - f(\alpha_1, \beta_2, \alpha_3) + f(\alpha_1, \alpha_2, \alpha_3)] \\ &= \Delta_{[\alpha_3, \beta_3]} \Delta f([\alpha_1, \beta_1] \times [\alpha_2, \beta_2]), \mathit{say}; \end{aligned}$$

and successively for any $m \ge 3$

$$\Delta f(R') := \Delta f([\alpha_1, \beta_1] \times ... \times [\alpha_m, \beta_m]) = \Delta_{[\alpha_m, \beta_m]} \Delta f([\alpha_1, \beta_1] \times ... \times [\alpha_{m-1}, \beta_{m-1}]).$$

Definition 1.1.30. A function f is said to be of *bounded variation over* R (written as $f \in \mathrm{BV_V}(R)$) in the sense of Vitali (the names of Lebesgue, Fréchet, and de la Vallée Poussin are also indicated sometimes in the literature, see, for example [8]) if $V(f; R)$, the *total variation* of f over R, is finite, where

$$V(f; R) := \sup_{\mathcal{P}} \sum_{i_1=1}^{s_1} ... \sum_{i_m=1}^{s_m} \left| \Delta f\left([x_1^{i_1-1}, x_1^{i_1}] \times ... \times [x_m^{i_m-1}, x_m^{i_m}]\right) \right|, \tag{1.16}$$

in which the supremum is extended over all grids $\mathcal{P}$ of R.

As noted in [10], in case $m \ge 2$, a function f in the class $\mathrm{BV_V}(R)$ is not necessarily measurable in the sense of Lebesgue. This is a consequence of the trivial fact that if a function $f = f(x_1, ..., x_m)$ does not depend on at least one of the $x_1, ..., x_m$, then for any grid $\mathcal{P}$, we have

$$\Delta f\left([x_1^{i_1-1}, x_1^{i_1}] \times ... \times [x_m^{i_m-1}, x_m^{i_m}]\right) = 0, \quad i_j = 1, ..., s_j; \quad j = 1, ..., m.$$

Consequently, the class $\mathrm{BV_V}(R)$ contains functions for which the m-dimensional Lebesgue integral over R fails to exist. The following notion of bounded variation is motivated by this fact.

Definition 1.1.31. In case $m = 2$, we say that a function $f = f(x_1, x_2)$ is of *bounded variation* over $R := [a_1, b_1] \times [a_2, b_2]$ in the sense of Hardy or sometimes Hardy and Krause (see, for example, [8]), in symbol: $f \in \mathrm{BV_H}(R)$, if it is in the class $\mathrm{BV_V}(R)$ and if the marginal functions $f(x_1, a_2)$ and $f(a_1, x_2)$ are of bounded variation on the intervals $I_1 := [a_1, b_1]$ and $I_2 := [a_2, b_2]$, respectively in the ordinary sense.

In case $m \geq 3$, the notion of bounded variation in the sense of Hardy over a rectangle R can naturally be defined by the following recurrence: $f \in \mathrm{BV_H}(R)$ if $f \in \mathrm{BV_V}(R)$ and each of the marginal functions $f(x_1, ..., a_k, ..., x_m)$ is in the class $\mathrm{BV_H}(R(a_k))$, where $k = 1, ..., m$ and

$$R(a_k) = \{(x_1, ..., x_{k-1}, x_{k+1}, ..., x_m) \in \mathbb{R}^{m-1} : \; a_j \leq x_j \leq b_j \quad \text{for } j = 1, ..., k-1, k+1, ..., m\}. \tag{1.17}$$

This definition can be equivalently reformulated as follows: $f \in \mathrm{BV_H}(R)$ if and only if $f \in \mathrm{BV_V}(R)$ and for any choice of $(1 \leq) j_1 < ... < j_n (\leq m)$, $1 \leq n < m$, the function $f(x_1, ..., a_{j_1}, ..., a_{j_n}, ..., x_m)$ is in the class $\mathrm{BV_V}(R(a_{j_1}, ..., a_{j_n}))$, where

$$R(a_{j_1}, ..., a_{j_n}) := \{(x_{\ell_1}, ..., x_{\ell_{m-n}}) \in \mathbb{R}^{m-n} : \; a_j \leq x_j \leq b_j \text{ for } j = \ell_1, ..., \ell_{m-n}\} \tag{1.18}$$

in which $\{\ell_1, ..., \ell_{m-n}\}$ is the complementary set of $\{j_1, ..., j_n\}$ with respect to the set $\{1, ..., m\}$.

Móricz [33] proved the following result.

Theorem 1.1.32. *If $f \in \mathrm{BV_H}(\mathbb{T}^2)$ then*

$$\hat{f}(n^{(1)}, n^{(2)}) = \begin{cases} O\left(\frac{1}{|n^{(1)}n^{(2)}|}\right) & \text{if } n^{(1)}n^{(2)} \neq 0 \\ O\left(\frac{1}{|n^{(1)}|}\right) & \text{if } n^{(1)} \neq 0, n^{(2)} = 0 \\ O\left(\frac{1}{|n^{(2)}|}\right) & \text{if } n^{(1)} = 0, n^{(2)} \neq 0. \end{cases}$$

Fülöp and Móricz [10] proved the following two results.

Theorem 1.1.33. *If $f \in \mathrm{BV_V}\left([0, 2\pi]^m\right) \cap \mathrm{L}^1\left(\mathbb{T}^m\right)$ and $(n^{(1)}, ..., n^{(m)}) \in \mathbb{Z}^m$ is such that $n^{(j)} \neq 0$ for each j, then*

$$\left|\hat{f}(n^{(1)}, ..., n^{(m)})\right| \leq \frac{V\left(f; [0, 2\pi]^m\right)}{(2\pi)^m \prod_{j=1}^m n^{(j)}} \tag{1.19}$$

where $V\left(f; [0, 2\pi]^m\right)$ is defined in (1.16).

Theorem 1.1.34. *If* $f \in \mathrm{BV_H}([0,2\pi]^m)$ *and* $(n^{(1)},...,n^{(m)}) \in \mathbb{Z}^m$ *is such that* $n^{(j)} \neq 0$ *for* $j \in \{j_1,...,j_n\}$*;* $(1 \leq) j_1 < ... < j_n (\leq m)$ *and* $n^{(j)} = 0$ *for* $j \in \{\ell_1,...,\ell_{m-n}\}$*;* $(1 \leq) \ell_1 < ... < \ell_{m-n} (\leq m)$*, where* $\{\ell_1,...,\ell_{m-n}\}$ *is the complementary set of* $\{j_1,...,j_n\}$ *with respect to* $\{1,...,m\}$*, then*

$$\left|\hat{f}(n^{(1)},...,n^{(m)})\right| \leq \frac{V\left(\tilde{f};[0,2\pi]^n\right)}{(2\pi)^n \prod_{j=1,n^{(j)}\neq 0}^{m} n^{(j)}}$$

where

$$\tilde{f} = \tilde{f}(x_{j_1},...,x_{j_n}) = (2\pi)^{-m+n} \int_{\mathbb{T}^{m-n}} f(x_1,...,x_m)\, dx_{\ell_1}...dx_{\ell_{m-n}}.$$

In Chapter 2, Section 2.1, we define the concept of *bounded p-variation* ($p \geq 1$) for a function of several variables in two different ways and study the order of magnitude of trigonometric Fourier coefficients for functions of these classes. Our results will generalize those of [33] and [10] (see Theorems 1.1.32, 1.1.33 and 1.1.34) in the sense that for $p = 1$, our definitions will coincide with the definitions of Vitali and Hardy and Krause and our results with those of [33] and [10], except for the exact constant. In the same Chapter 2, in Section 2.2, carrying out the study further, we will prove lacunary analogues of results obtained in Section 2.1.

(C). We now take up the case of **Walsh** and **Vilenkin groups**. First of all observe that the Riemann-Lebesgue lemma is true for Fourier coefficients with respect the characters of any compact abelian group G, because such characters form a uniformly bounded orthonormal system on G [80, p. 45, (4.4)]. (It is interesting to note here that the Riemann-Lebesgue lemma is not necessarily true in general for any orthonormal system if it is not uniformly bounded, for example, it is not true for Haar's system [73, p. 16].) However, in general, there is no definite rate at which these Fourier coefficients tend to zero. In fact, it is known that the Fourier coefficients can tend to zero as slow as possible — in the sense that given a sequence $\{a_n\}$ of positive numbers tending to zero, one can find integrable functions f on compact abelian groups in general, including the Walsh and the Vilenkin group, such that $|\hat{f}(n)| \geq a_n$, for all n (see, for example, [23, 32.47 (b)]). Therefore, in these cases also, it is interesting to know for which subspaces of the space of integrable functions on G, there is some definite rate at which the Fourier coefficients tend to zero.

We begin with giving preliminaries for the Walsh group. Let $\{\varphi_n\}$ $(n = 0,1,2,...)$ denote the complete orthonormal Walsh system [73], where the subscript denotes the number of zeros (that is, sign-changes) in the interior of the interval $[0, 1]$. For a 1-periodic f in $L^1[0, 1]$ its Walsh Fourier series is given by

$$f(x) \sim \sum_{n=0}^{\infty} \hat{f}(n)\varphi_n(x), \tag{1.20}$$

where the n^{th} Walsh Fourier coefficient $\hat{f}(n)$ is given by

$$\hat{f}(n) = \int_0^1 f(x)\varphi_n(x)dx \quad (n = 0, 1, 2, ...). \tag{1.21}$$

Let us also describe the Payley enumeration of Walsh functions [50]. Let $\{r_n\}$, $n = 0, 1, 2, ...$, denote the class of Rademacher functions defined by

$$r_0(x) = 1 \ \ (0 \le x < 1/2), \ \ r_0(x) = -1 \ \ (1/2 \le x < 1),$$

$$r_0(x + 1) = r_0(x), \ \ r_n(x) = r_0(2^n x) \quad (n = 1, 2, 3, ...).$$

The complete orthonormal Walsh system [73], say $\{w_n\}$, $n = 0, 1, 2, ...$, as ordered by Payley [50], is then given by

$$w_0(x) \equiv 1, \quad w_n(x) = r_{n_1}(x)r_{n_2}(x) \cdots r_{n_k}(x)$$

if $n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_k}$, in which $n_1 > n_2 > ... > n_k \ge 0$. Observe that $w_0 = \varphi_0$ and for each $k \in \mathbb{N} \cup \{0\}$, the set $\{w_{2^k}, w_{2^k+1}, ..., w_{2^{k+1}-1}\}$ is a permutation of $\{\varphi_{2^k}, \varphi_{2^k+1}, ..., \varphi_{2^{k+1}-1}\}$. For the functions w_n, $\deg w_n$ denotes the degree of w_n defined by : $\deg w_0 = 0$ and $\deg w_n = n_1 + 1$, if w_n is represented as the product of Rademacher characters as in preceding lines. Accordingly, for each $j \in \mathbb{N}$ we have $w_{2^{j-1}} = r_{j-1}$ and $\deg w_{2^{j-1}} = \deg r_{j-1} = j$. The degree of any real linear combination of finitely many elements w_n $(n = 0, 1, 2, ...)$ (that is, a polynomial in functions w_n on $[0, 1]$) is the maximum of the degree of the elements w_n appearing in it.

In this case (setting of Walsh group), the study of certain definite rate of tending to zero of Fourier coefficients for the functions of the subspace $\text{Lip}\,\alpha[0, 1]$ and functions of the subspace $BV[0, 1]$ of $L^1[0, 1]$ was carried out by N. J. Fine [11, Theorems V and VI] and obtained following analogous results to the trigonometric case.

Theorem 1.1.35. *If* $f \in \text{Lip}\,\alpha[0, 1]$, $0 < \alpha \le 1$, *then* $\hat{f}(n) = O(1/n^{\alpha})$.

Theorem 1.1.36. *If* $f \in BV[0, 1]$ *then* $|\hat{f}(n)| = O(1/n)$.

However, it appears that such a study for functions of generalized bounded variation is not yet carried out for the Walsh group. In Chapter 3, Section 3.1, we carry out this study and prove Walsh group analogues of Theorems 1.1.4, 1.1.6, 1.1.8, 1.1.10 and 1.1.12. Carrying out the study further, in the following Section 3.2 of this Chapter 3, we show that if the Walsh Fourier series is lacunary having small gaps, then similar results for order of magnitude of Walsh Fourier coefficients holds if function is assumed to be of generalized bounded variation only locally.

Finally, in this chapter in Sections 3.3 and 3.4, we consider multiple Walsh Fourier coefficients and prove multidimensional analogues of our results of Sections 3.1 and 3.2.

(D). In Chapter 4, we consider the case of a **Vilenkin group**, that is, of a compact metrizable 0-dimensional and abelian group. In fact, in 1947, N. Ja. Vilenkin developed part of the Fourier Theory on these groups; the details of which are as follows. Let G be a Vilenkin group. Then the dual group X of G is a discrete, countable, torsion, abelian group [22, Theorems 24.15 and 24.26]. Vilenkin [68, Sections 1.1 and 1.2] proved the existence of a sequence $\{X_n\}$ of finite subgroups of X and of a sequence $\{\varphi_n\}$ in X such that the following hold:

(i) $X_0 = \{\chi_0\}$, where χ_0 is the identity character on G.

(ii) $X_0 \subset X_1 \subset X_2 \subset \dots$.

(iii) For each $n \geq 1$, the quotient group X_n/X_{n-1} is of prime order p_n.

(iv) $X = \cup_{n=0}^{\infty} X_n$.

(v) $\varphi_n \in X_{n+1} \setminus X_n$ for all $n \geq 0$.

(v) $\varphi_n^{p_{n+1}} \in X_n$ for all $n \geq 0$.

The group G is termed *bounded*, if

$$p_0 = \sup_{i=1,2,\dots} p_i < \infty;$$

otherwise G is said to be unbounded. Further if $p_i = p_0$ for all i, where p_0 is a fixed prime, then G is known as *primary*. Using the φ_n's, we can enumerate X as follows: Let $m_0 = 1$ and $m_n = \Pi_{i=1}^{n} p_i$ for $n = 1, 2, \dots$. Then each $k \in \mathbb{N}$ can be uniquely represented as $k = \sum_{i=0}^{s} a_i m_i$, with $0 \leq a_i < p_{i+1}$ for $0 \leq i \leq s$; and we define χ_k by the formula $\chi_k = \varphi_0^{a_0} \cdot \dots \cdot \varphi_s^{a_s}$. Observe that $\chi_{m_n} = \varphi_n$ for each $n \geq 0$. For $\chi \in X$ the degree of χ is defined by: deg $\chi_0 = 0$ and deg $\chi_k = s + 1$, if χ_k is written as the product of φ_n's as described in the preceding lines. Any complex linear combination of finitely many elements of X is called a Vilenkin polynomial on G, and the degree of such a polynomial is the maximum of the degree of elements of X appearing in the polynomial.

$G = \prod_{n=1}^{\infty} \mathbb{Z}_{p_n}$, $\{p_n\}$ — a sequence of prime numbers, is a standard example. If $p_n = 2$ for all n, X is the group of Walsh functions ψ_n, $n = 0, 1, 2, ...,$ and $X_n = \{\psi_0, \psi_1, ..., \psi_{2^n-1}\}$ (using Payley enumeration [50]) described by N. J. Fine [11]. If $p_n = p$ for all n, X is the group of generalized Walsh functions [7].

Let dx or m denote the normalized Haar measure on G. For $f \in \mathrm{L}^1(G)$ the Vilenkin Fourier series of f is defined as

$$S[f](x) = \sum_{n=0}^{\infty} \hat{f}(n)\chi_n(x), \tag{1.22}$$

where the n^{th} Vilenkin Fourier coefficient of f is given by

$$\hat{f}(n) = \int_G f(x)\bar{\chi}_n(x)dx. \tag{1.23}$$

Observe that for each n, $X_n = \{\chi_k : 0 \leq k < m_n\}$. Let G_n be the annihilator of X_n, that is,

$$G_n = \{x \in G : \chi(x) = 1, \chi \in X_n\} = \{x \in G : \chi_k(x) = 1,\ 0 \leq k < m_n\}.$$

Then obviously, we have the proper inclusions

$$G = G_0 \supset G_1 \supset G_2 \supset ... \supset G_n \supset G_{n+1} \supset ... \supset \{0\},\ \cap_{n=0}^{\infty} G_n = \{0\}, \tag{1.24}$$

and the G_n's form a fundamental system of neighborhoods of zero in G which are compact, open and closed subgroups of G. Further, the index of G_n in G is m_n, and since the Haar measure is translation invariant with $m(G) = 1$, one has $m(G_n) = 1/m_n$. The metric on G is then given by

$$d(x, y) = |x - y| \text{ for } x, y \in G,$$

where $|x| = 0$ if $x = 0$, and $|x| = 1/m_{n+1}$ if $x \in G_n \setminus G_{n+1}$ for $n = 0, 1, 2, ...$.

In [68, Section 3.2] Vilenkin proved that for each $n \geq 0$ there exists $x_n \in G_n \setminus G_{n+1}$ such that $\chi_{m_n}(x_n) = \exp(2\pi i/p_{n+1})$ and observed that each $x \in G$ has a unique representation $x = \sum_{i=0}^{\infty} b_i x_i$, with $0 \leq b_i < p_{i+1}$ for all $i \geq 0$. This representation of the elements of G enables one to order them by means of the lexicographic ordering of the corresponding sequence $\{b_n\}$ and one observes that for each $n = 0, 1, 2, \ldots$,

$$G_n = \left\{ x \in G : x = \sum_{i=0}^{\infty} b_i x_i,\ b_0 = \cdots = b_{n-1} = 0 \right\}.$$

Consequently, each coset of G_n in G has a representation of the form $z + G_n$, where $z = \sum_{i=0}^{n-1} b_i x_i$ for some choice of the b_i with $0 \leq b_i < p_{i+1}$. These z, ordered lexicographically, are denoted by $\{z_\alpha^{(n)}\}$ $(0 \leq \alpha < m_n)$.

It may be noted that the choice of $\varphi_n \in X_{n+1} \setminus X_n$ and of $x_n \in G_n \setminus G_{n+1}$ is not uniquely determined by the groups X and G. In the following, it is assumed that a particular choice has been made.

Note that the Riemann-Lebesgue lemma is true in this case also; and, like trigonometric and Walsh Fourier series, it is well known that if f belongs to $\mathrm{L}^1(G)$, there is no definite rate at which the Fourier coefficients tend to zero; in fact, the Vilenkin Fourier coefficients can tend to zero as slow as desired [23, 32.47 (b)]). Analogous to the Jordan class BV on an interval, using the ordering of G as defined above, Vilenkin [68] introduced the class $\mathrm{BV}(G)$ of functions of *bounded variation* on a Vilenkin group G defined by

$$\mathrm{BV}(G) = \left\{ f : G \to \mathbb{C} : V(f; G) = \sup \sum_{i=1}^{n} |f(z_i) - f(z_{i+1})| < \infty \right\} \tag{1.25}$$

where the sup refers to all systems $z_1, z_2, ..., z_{n+1}$ such that $z_k < z_{k+1}$; and studied the definite rate of tending to zero of Vilenkin Fourier coefficients of functions of this class — proving the following result [68, 3.22].

Theorem 1.1.37. *If $f \in \mathrm{BV}(G)$ and $m_k \leq n < m_{k+1}$ then $|\hat{f}(n)| \leq \frac{V(f;G)}{m_k}$.*

Later, considering the class $\mathrm{Lip}\,(\alpha, p, G)$ of functions satisfying Lipschitz condition of order α, $0 < \alpha \leq 1$, in the mean of order p, $1 \leq p < \infty$, defined by

$$\mathrm{Lip}\,(\alpha, p, G) = \left\{ f \in \mathrm{L}^p(G) : \omega^{(p)}(f, n) = O\left(m_n^{-\alpha}\right) \right\} \tag{1.26}$$

where $\omega^{(p)}(f, n)$ is the *n-th integral modulus of continuity of order p* for a function f in $\mathrm{L}^p(G)$ is defined as

$$\omega^{(p)}(f, n) = \sup\{||T_h f - f||_p : h \in G_n\};\quad (T_h f)(x) = f(x + h), \tag{1.27}$$

C. W. Onneweer carried out the study further in [40, Lemma 1] and proved the following result.

Theorem 1.1.38. *If* $f \in \mathrm{Lip}\,(\alpha, p, G)$, $1 \le p < \infty$, $0 < \alpha \le 1$, *then* $\hat{f}(n) = O\left(\frac{1}{n^\alpha}\right)$.

Later on, in [37, 38, 39], Onneweer and Waterman themselves introduced various classes of generalized bounded fluctuation and studied the problem of convergence of Vilenkin Fourier series of functions of these classes. However, it appears that for functions of these classes, the problem of determination of the order of magnitude of Fourier coefficients is not studied. In Chapter 4, Section 4.1, we carry out this study. Further, in Section 4.2 of the same Chapter 4, we give the definitions of generalized bounded fluctuation locally and study the order of magnitude of Vilenkin Fourier coefficients of functions having lacunary Vilenkin Fourier series with small gaps.

1.2 Absolute Convergence

The study of absolute convergence of Fourier series is one of the most important problems of Fourier Analysis and the problem has been studied intensively by many researchers in the setting of circle group in particular and classical groups in general. In the last Chapter 5, we study the absolute convergence of Vilenkin Fourier series. Naturally our efforts is to prove, in Vilenkin group setting, analogues of some of the known results of circle group. Therefore, let us first review, in **(A)**, the results on the absolute convergence of Fourier series in the circle group case and then in **(B)**, review the known results on the absolute convergence of Vilenkin Fourier series. At the end, we shall indicate the work done by us.

(A). In the case of **circle group** if $f \in \mathrm{AC}$ and $f' \in \mathrm{L}^2$ (that is, if f is "sufficiently good"), then the Fourier series of f is trivially absolutely convergent [3, Chapter I, Section 26], that is, $f \in \mathrm{A}(1)$, where for $0 < \beta \le 2$, $\mathrm{A}(\beta)$ is defined by

$$\mathrm{A}(\beta) = \left\{ f \in \mathrm{L}^1[-\pi, \pi] : \sum_{n=1}^{\infty} (|a_n|^\beta + |b_n|^\beta) < \infty \ \text{ or } \ \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^\beta < \infty \right\},$$

in which $a_0, a_1, a_2, ...; b_1, b_2, ...$ are the usual trigonometric Fourier coefficients of f defined as

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)dx, \ a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx; \ b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx. \tag{1.28}$$

Of course, the classical results of Bernstein "$\mathrm{Lip}\,\alpha \subset \mathrm{A}(1)$ for $\alpha > \frac{1}{2}$" (for $\alpha = \frac{1}{2}$ not true) [3, Vol. II, p. 154] and Zygmund "$\mathrm{BV} \cap \mathrm{Lip}\,\alpha \subset \mathrm{A}(1)$ for $\alpha > 0$" [3, Vol. II, p. 161] are fundamental and historical. Bernstein and Zygmund later proved following slight extended versions of these results, namely "if $\sum_{n=1}^{\infty} (\omega(1/n, f)/\sqrt{n}) < \infty$ then $f \in \mathrm{A}(1)$" and "if $f \in \mathrm{BV}$ and $\sum_{n=1}^{\infty} \left(\sqrt{\omega(1/n, f)}/n\right) < \infty$ then $f \in \mathrm{A}(1)$", where

$$\omega\left(\frac{1}{n}, f\right) = \sup\left\{|f(x+h) - f(x)| : x \in [0, 2\pi], 0 < h \leq \frac{1}{n}\right\}.$$

All these results are generalized in the following result by Szász [3, Vol. II, p. 155].

Theorem 1.2.1 (Szász). *If $\omega^{(2)}(\delta, f)$ is the quadratic modulus of continuity of $f(x)$, that is,*

$$\omega^{(2)}(\delta, f) = \sup_{0 \leq h \leq \delta} \left\{\int_{-\pi}^{\pi} [f(x+h) - f(x-h)]^2 dx\right\}^{\frac{1}{2}},$$

then

$$\sum_{n=1}^{\infty} \frac{\omega^{(2)}(\frac{1}{n}, f)}{\sqrt{n}} < \infty \quad \textit{implies} \quad f \in \mathrm{A}(1).$$

Refer [3, Vol. II, p. 155 - 156] for the equivalent result due to Stečhkin in terms of $E_n^{(2)}(f)$, the best approximation to f in L^2 by trigonometric polynomials of degree not higher than n. It may be noted that Stečhkin [3, Vol. II, p. 196] has proved interesting extensions of these results providing sufficiency conditions in terms of quadratic modulus of continuity or L^2-best approximations to f by trigonometric polynomials of degree not higher than n for function belong to $\mathrm{A}^*(1)$, where for $0 < \beta \leq 1$ and a sequence $\{n_k\}$ of positive integers,

$$\mathrm{A}^*(\beta) = \left\{f \in \mathrm{L}^1 : \sum_{k=1}^{\infty} |\hat{f}(n_k)| < \infty\right\}.$$

Further, for lacunary Fourier series with various lacunary conditions, Patadia [45, 46, 47, 48] has proved interesting results showing that Szász and Stečhkin type sufficiency conditions hold even if the hypothesis on the generating function is satisfied only locally — the localness of hypothesis being dependent on the type of lacunarity in the Fourier series.

Observe the Bernstein and Zygmund theorems stated above. They interestingly illustrate that if function f is of bounded variation, then satisfying a $\mathrm{Lip}\,\alpha$ condition just for any $\alpha > 0$ is sufficient for its Fourier series to converge absolutely — which otherwise requires it to satisfy $\mathrm{Lip}\,\alpha$ condition for $\alpha > \frac{1}{2}$. Similar is the impact of function of bounded variation on the absolute convergence of its Fourier series when one considers the hypothesis in terms of modulus of continuity — Bernstein, in his result requires $\sum_{n=1}^{\infty}(\omega(1/n, f)/\sqrt{n}) < \infty$ when f is not necessarily of bounded variation, while Zygmund, in his result proves that just $\sum_{n=1}^{\infty}\left(\sqrt{\omega(1/n, f)}/n\right) < \infty$ is sufficient when f is of bounded variation. It is therefore interesting to know the impact on the absolute convergence of Fourier series of f when f belongs to various classes of functions of different generalized bounded variations.

After introducing various classes of functions of generalized bounded variations, Waterman [76], Shiba [58], Chanturia [4] and Schramm and Waterman [56] have proved following sufficiency conditions in terms of modulus of continuity for the function to be in $A(1)$ (in these results as well as in what follows, for any number p such that $1 \leq p \leq \infty$, we denote the conjugate of p by p', so that $1/p + 1/p' = 1$).

Theorem 1.2.2 (**Waterman**). *If* $f \in \mathrm{C}[-\pi, \pi] \cap \Lambda\mathrm{BV}[-\pi, \pi]$ *and*

$$\sum_{n=1}^{\infty} \frac{\sqrt{\lambda_n \omega(\frac{2\pi}{n}, f)}}{n} < \infty \quad \textit{then} \quad f \in \mathrm{A}(1).$$

Theorem 1.2.3 (**Shiba**). *If* $f \in \Lambda\mathrm{BV}^{(p)}[-\pi, \pi]$, $1 \leq p < 2r$, $1 < r < \infty$,

$$\textit{and} \quad \sum_{n=1}^{\infty} \frac{\lambda_n^{1/2r}(\omega^{(p+(2-p)r')}(\frac{\pi}{n}, f))^{1-p/2r}}{n^{1-1/2r'}} < \infty \quad \textit{then} \quad f \in \mathrm{A}(1).$$

Theorem 1.2.4 (**Chanturia**). *If* $f \in \mathrm{V}[n^{\alpha}][-\pi, \pi]$ $\left(0 \leq \alpha < \frac{1}{2}\right)$ *and*

$$\sum_{n=1}^{\infty} \frac{(\omega(\frac{1}{n}, f))^{(1-2\alpha)/(1-\alpha)}}{n} < \infty \quad \textit{then} \quad f \in \mathrm{A}(1).$$

Theorem 1.2.5 (**Schramm and Waterman**). *If* $f \in \Lambda\mathrm{BV}^{(p)}[-\pi, \pi]$, $1 \leq p < 2r$, $1 \leq r < \infty$, *and*

$$\sum_{n=1}^{\infty} \frac{\left(\sum_{k=1}^{n} \frac{1}{\lambda_k}\right)^{-1/2r} \left(\omega^{(p+(2-p)r')}\left(\frac{\pi}{n}, f\right)\right)^{1-p/2r}}{n^{1/2}} < \infty \quad \textit{then} \quad f \in \mathrm{A}(1).$$

Theorem 1.2.6 (**Schramm and Waterman**). *If* ϕ *is* Δ_2, $f \in \phi\Lambda\mathrm{BV}[-\pi, \pi]$, $1 \leq p < 2r$, $1 \leq r < \infty$, *and*

$$\sum_{n=1}^{\infty} \frac{\left[\phi^{-1}\left(\left(\sum_{k=1}^{n} \frac{1}{\lambda_k}\right)^{-1} \left(\omega^{(p+(2-p)r')}\left(\frac{\pi}{n}, f\right)\right)^{2r-p}\right)\right]^{1/2r}}{n^{1/2}} < \infty \quad \textit{then} \quad f \in \mathrm{A}(1).$$

Later Patadia and Vyas [41, Theorems 1 to 4 and Theorem 6] studied sufficiency conditions for the absolute convergence of Fourier series when it is lacunary having small gaps and obtained the lacunary analogues of the above results. We need the following definitions to state these results.

Definition 1.2.7. For $p \geq 1$, the p-integral modulus continuity $\omega_r^{(p)}(\delta, f, I)$ of f over I of higher differences of order $r \geq 1$ is defined by

$$\omega_r^{(p)}(\delta, f, I) = \sup_{0 \leq h \leq \delta} \left\| \sum_{m=0}^{r} \binom{r}{m} T_{(r-2m)h} f \right\|_{p,I},$$

where $T_h f(x) = f(x+h)$ for all x and $\|(\cdot)\|_{p,I} = \|(\cdot)\chi_I\|_p$ in which χ_I is the characteristic function of I and $\|(\cdot)\|_p$ denotes the L^p-norm. $p = \infty$ gives the modulus of continuity $\omega_r(\delta, f, I)$. If $r = 1$, we omit writing r.

Theorem 1.2.8. *Let $f \in \mathrm{L}^1[-\pi, \pi]$ possess a lacunary Fourier series with small gaps* (1.11) *and I, a subinterval of $[-\pi, \pi]$ of length $\delta_1 > 2\pi/q$. If $f \in \mathrm{L}^2(I)$ and*

$$\sum_{n=1}^{\infty} \left[\left(\omega_r^{(2)}(1/n, f, I)\right)^{\beta} N(n)^{1-\beta/2} n^{-1} \right] < \infty \quad (0 < \beta \leq 2), \tag{1.29}$$

then

$$\sum_{k=-\infty}^{\infty} |\hat{f}(n_k)|^{\beta}, \tag{1.30}$$

where $N(p) = \sum_{|n_k| \leq p} 1$.

Theorem 1.2.9. *Theorem 1.2.8 holds if* (1.29) *is replaced by*

$$\sum_{k=1}^{\infty} \left[\left(\omega_r^{(2)}(1/n_k, f, I)\right)^{\beta} / k^{\beta/2} \right] < \infty. \tag{1.31}$$

Theorem 1.2.10. *Let f and I be as in Theorem 1.2.8. If $f \in \Lambda\mathrm{BV}(I)$ and*

$$\sum_{T=1}^{\infty} [\lambda_{n_T} \omega(1/n_T, f, I)/T n_T]^{\beta/2} < \infty, \tag{1.32}$$

then (1.30) *holds.*

Theorem 1.2.11. *Let f and I be as in Theorem 1.2.8. If $f \in \mathrm{V}[n^{\alpha}](I)$, $0 \leq \alpha < \frac{1}{2}$, and*

$$\sum_{k=1}^{\infty} \left[\frac{(\omega(1/n_k, f, I))^{\beta(1-2\alpha)/(2(1-\alpha))}}{(k n_k)^{\beta/2}} \right] < \infty, \tag{1.33}$$

then (1.30) *holds.*

Theorem 1.2.12. *Let f and I be as in Theorem 1.2.8. If $f \in \mathrm{Lip}\,(\alpha, p, I) \cap r\text{-BV}(I)$ for $0 < \alpha \leq \frac{1}{2}$, $p > 2$ and $\alpha p > 1$, then its Fourier series converges absolutely, where for any $p \geq 1$, $0 < \alpha \leq 1$ and a subinterval I of $[-\pi, \pi]$ the class $\mathrm{Lip}\,(\alpha, p, I)$ is defined as*

$$\mathrm{Lip}\,(\alpha, p, I) = \left\{ f \in \mathrm{L}^1 : \|T_h f - T_{-h} f\|_{p,I} = O(|h|^\alpha) \ \text{ as } \ h \to 0 \right\}. \tag{1.34}$$

Continuing the study further, Vyas proved the following theorems [70, Theorem 1.1 and 1.2], [71, Theorem 1.1].

Theorem 1.2.13. *Let f and I be as in Theorem 1.2.8. If $f \in \Lambda\mathrm{BV}(I)$ and*

$$\sum_{k=1}^{\infty} \left(\frac{\omega(\frac{1}{n_k}, f, I)}{k \left(\sum_{j=1}^{n_k} \frac{1}{\lambda_j} \right)} \right)^{\beta/2} < \infty, \tag{1.35}$$

then (1.30) *holds.*

Theorem 1.2.14. *Let f and I be as in Theorem 1.2.8. If $f \in \Lambda\mathrm{BV}^{(p)}(I)$, $1 \leq p < 2r$, $1 < r < \infty$ and*

$$\sum_{k=1}^{\infty} \left(\frac{\left(\omega^{((2-p)r'+p)} \left(\frac{1}{n_k}, f, I \right) \right)^{2-p/r}}{k \left(\sum_{j=1}^{n_k} \frac{1}{\lambda_j} \right)^{1/r}} \right)^{\beta/2} < \infty, \tag{1.36}$$

then (1.30) *holds.*

Theorem 1.2.15. *Let f and I be as in Theorem 1.2.8. If $f \in \varphi\Lambda\mathrm{BV}(I)$, $1 \leq p < 2r$, $1 \leq r < \infty$ and*

$$\sum_{k=1}^{\infty} \left(\left[\varphi^{-1} \left(\frac{\left(\omega^{((2-p)r'+p)} \left(\frac{1}{n_k}, f, I \right) \right)^{2r-p}}{\sum_{j=1}^{n_k} \frac{1}{\lambda_j}} \right) \right]^{1/r} \Big/ k \right)^{\beta/2} < \infty, \tag{1.37}$$

then (1.30) *holds.*

(B). Let us now review the results about absolute convergence of Fourier series on **Vilenkin groups**. In the following, unless stated otherwise, we assume that G

is a bounded Vilenkin group as described in Section 1.1 (D) and for $\beta > 0$ we shall denote the set of all functions f in $\mathrm{L}^1(G)$ for which $\sum_{n=0}^{\infty} |\hat{f}(n)|^\beta < \infty$ by $\mathrm{A}(\beta)$.

Vilenkin himself proved the following analogue of Bernstein's Theorem concerning absolute convergence [68, Theorem 5] for a primary group G.

Theorem 1.2.16. $\operatorname{Lip}\alpha(G) \subset \mathrm{A}(1)$ *if* $\alpha > \frac{1}{2}$ *and* G *is primary, where for* $0 < \alpha \leq 1$,

$$\operatorname{Lip}\alpha(G) = \left\{ f : G \to \mathbb{C} : \omega_n(f) = O\left(m_n^{-\alpha}\right) \right\} \tag{1.38}$$

in which $\omega_n(f)$ *is the* n*-th modulus of continuity [39, Definition 2] of* f *given by*

$$\omega_n(f) = \sup\{|(T_h f - f)(x)| : x \in G, h \in G_n\}. \tag{1.39}$$

Continuing the study further, Onneweer [39] proved following interesting results.

Theorem 1.2.17. *If* $f \in \mathrm{L}^p(G)$, $1 \leq p \leq 2$ *and if* f *satisfies the condition*

$$\sum_{n=0}^{\infty} \left(\sum_{\alpha=0}^{m_n - 1} \left(\operatorname{osc}\left(f; z_\alpha^{(n)} + G_n\right)\right)^p \right)^{1/p} < \infty,$$

then $f \in A(1)$.

Corollary 1.2.18. *If* $\sum_{n=0}^{\infty} m_n^{\frac{1}{2}} \omega_n(f) < \infty$ *then* $f \in A(1)$.

Corollary 1.2.19. $\operatorname{Lip}\alpha(G) \subset \mathrm{A}(1)$ *if* $\alpha > \frac{1}{2}$.

Corollary 1.2.20. $\mathrm{GBF}^{(p)}(G) \cap \operatorname{Lip}\alpha(G) \subset \mathrm{A}(1)$ *for* $1 \leq p < 2$ *and* $\alpha > 0$, *where* $\mathrm{GBF}^{(p)}(G)$ *is defined as in Definition 4.1.4.*

Observe that Corollary 1.2.20 is analogue of Zygmund's theorem for trigonometric Fourier series. In order to obtain more precise information for functions belonging to class $\operatorname{Lip}\alpha(G)$, $0 < \alpha \leq \frac{1}{2}$, than is given in Corollaries 1.2.19 and 1.2.20, one may either consider a different form of bounded fluctuation or else one must consider classes of functions different from $\mathrm{A}(1)$. In this connection Onneweer proved the following result for functions of Λ-generalized bounded fluctuation (see Definition 4.1.5).

Corollary 1.2.21. *If* $f \in \operatorname{Lip}\alpha(G)$ *and if* $f \in \Lambda\mathrm{GBF}(G)$ *for some sequence* $\Lambda = \{\lambda_n\}$ *such that* $\lambda_{m_n} = O\left(m_n^\beta\right)$, *with* $0 \leq \beta < \alpha$, *then* $f \in \mathrm{A}(1)$.

Modifying the requirement that $f \in \mathrm{A}(1)$, Onneweer proved the following two theorems. Theorem 1.2.22 is the analogue on G of a well-known theorem of Szász [80, Chapter VI, (3.10)].

Theorem 1.2.22. *If $f \in \mathrm{Lip}\,\alpha(G)$, $0 < \alpha \leq 1$, then for all $\beta > 2/(2\alpha+1)$ we have $f \in \mathrm{A}(\beta)$.*

Theorem 1.2.23. *If $f \in \mathrm{Lip}\,\alpha(G)$ for $0 < \alpha \leq 1$, then $\sum_{n=0}^{\infty} n^{-\beta}|\hat{f}(n)| < \infty$ for all β with $\beta > \frac{1}{2} - \alpha$.*

Onneweer has also given existential proofs of the following two theorems; Theorem 1.2.25 says that Theorem 1.2.24 is best possible in some sense. Further, when G is primary, he constructed functions satisfying conditions of these theorems and hence proved Theorem 1.2.26.

Theorem 1.2.24. *There exists an $f \in \mathrm{Lip}\,\frac{1}{2}(G)$ such that $f \notin \mathrm{A}(1)$.*

Theorem 1.2.25. *For each α, $\frac{1}{2} \leq \alpha < 1$, and for each β, $\beta < \alpha$, there exists a function $f \in \mathrm{Lip}\,\beta(G)$ such that $\sum_{n=0}^{\infty} |\hat{f}(n)|^{2/(1+2\alpha)} = \infty$.*

Theorem 1.2.26. *If G is a primary group, then for each α, $0 < \alpha \leq 1$, there is a function $g_\alpha \in \mathrm{Lip}\,\alpha(G)$ such that $g_\alpha \notin \mathrm{Lip}\,\gamma(G)$ for any $\gamma > \alpha$; also $g_\alpha \in \mathrm{GBF}^{(\alpha^{-1})}$. Furthermore, $\sum_{n=0}^{\infty} |\hat{g}_\alpha(n)|^{\beta} = \infty$, where $\beta = 2/(2\alpha+1)$, and $\sum_{n=1}^{\infty} n^{\alpha-\frac{1}{2}}|\hat{g}_\alpha(n)| = \infty$. In particular, if $0 < \alpha \leq \frac{1}{2}$, then $g_\alpha \notin \mathrm{A}(1)$.*

Onneweer continued study further and proved more results in his second paper [40] for a bounded Vilenkin group G.

Theorem 1.2.27. *Let $1 \leq p \leq 2$ and $0 < \beta \leq p'$. If $f \in \mathrm{L}^p(G)$ and if*

$$\sum_{n=0}^{\infty} (m_n)^{(p'-\beta)/p'} \left(\omega^{(p)}(f,n)\right)^{\beta} < \infty$$

then $f \in \mathrm{A}(\beta)$, $\omega^{(p)}(f,n)$ being defined by (1.27).

Choosing $\beta = 1$ in Theorem 1.2.27 we obtain

Corollary 1.2.28. *Let $1 \leq p \leq 2$. If $f \in \mathrm{L}^p(G)$ and if $\sum_{n=0}^{\infty}(m_n)^{1/p}\omega^{(p)}(f,n) < \infty$, then $f \in \mathrm{A}(1)$.*

The next corollary is the analogue on G of a well-known result of trigonometric series due to Szász [62].

Corollary 1.2.29. *Let $0 < \alpha \leq 1$, $1 \leq p \leq 2$ and $\beta > p/(\alpha p + p - 1)$. Then $\mathrm{Lip}\,(\alpha, p, G) \subset \mathrm{A}(\beta)$, where $\mathrm{Lip}\,(\alpha, p, G)$ is as defined in (1.26).*

In case we choose $\beta = 1$ in Corollary 1.2.29 we obtain the following result.

Corollary 1.2.30. *Let $1 \leq p \leq 2$ and $1 < \alpha p \leq p$. Then $\mathrm{Lip}\,(\alpha, p, G) \subset \mathrm{A}(1)$.*

Onneweer has also proved that Corollaries 1.2.29 and 1.2.30 are best possible in the following sense.

Theorem 1.2.31. *Let $1 < p \leq 2$ and $0 < \alpha \leq 1$. Then there exists a function in $\mathrm{Lip}\,(\alpha, p, G)$ which does not belong to $\mathrm{A}(p/(\alpha p + p - 1))$. In particular, if $1 < p \leq 2$ and $\alpha p = 1$, then there exists a function in $\mathrm{Lip}\,(\alpha, p, G)$ whose Fourier series is not absolutely convergent.*

Onneweer also proved the following theorem.

Theorem 1.2.32. *Let $1 < p \leq 2$ and $0 < \alpha \leq 1$.*

(a) *If $\{c_n\}$ is a sequence for which $\sum_{n=1}^{\infty} |c_n n^{\alpha}|^p < \infty$, then there exists an f in $\mathrm{Lip}\,(\alpha, p', G)$ such that $\hat{f}(n) = c_n$ for all $n \in \mathbb{N}$.*

(b) *If $f \in \mathrm{Lip}\,(\alpha, p, G)$ and if $0 < \beta < \alpha$, then $\sum_{n=1}^{\infty} |\hat{f}(n) n^{\beta}|^{p'} < \infty$.*

(c) *There exists an $f \in \mathrm{Lip}\,(\alpha, p, G)$ such that $\sum_{n=1}^{\infty} |\hat{f}(n) n^{\alpha}|^{p'} = \infty$.*

Vilenkin and Rubinšteĭn [67] proved the following theorem, which is an analogue of a known theorem of Stečhkin [61].

Theorem 1.2.33. *If $f \in \mathrm{L}^2(G)$ then $\sum_{k=m_n}^{\infty} |\hat{f}(k)|^2 \leq \frac{1}{2} \left(\omega^{(2)}(f, n)\right)^2$.*

Quek and Yap [54] extended above results of Onneweer to general Vilenkin groups. They have considered the class $\mathrm{Lip}\,(\alpha, p, G)$ as in (1.26) and defined a similar class of functions $\mathfrak{Lip}\,(\alpha, p, G)$ for an arbitrary Vilenkin group G, $0 < \alpha \leq 1$ and $1 \leq p < \infty$ as

$$\mathfrak{Lip}\,(\alpha, p, G) = \left\{ f \in \mathrm{L}^p(G) : \omega^{(p)}(f, n) = O\left(m_{n+1}^{-\alpha}\right) \right\}. \tag{1.40}$$

Clearly $\mathfrak{Lip}\,(\alpha, p, G) \subset \mathrm{Lip}\,(\alpha, p, G)$ for all Vilenkin groups G, but the two classes need not be the same (see [54, Remark 5.9]). However, for bounded Vilenkin groups G, obviously $\mathfrak{Lip}\,(\alpha, p, G) = \mathrm{Lip}\,(\alpha, p, G)$, and we shall use the notation $\mathrm{Lip}\,(\alpha, p, G)$ in this case.

Here we state results of Quek and Yap. In these results the notation Δ_n denotes the difference $\Delta_n = D_{m_{n+1}} - D_{m_n}$ where D_k denotes the Dirichlet kernel of order k defined as $D_k(x) = \sum_{i=0}^{k-1} \chi_i(x),\ x \in G$.

Theorem 1.2.34. *Let* $1 \le p \le 2$ *and* $0 < \beta \le p'$. *If* $f \in \mathrm{L}^1(G)$ *and*

$$\sum_{n=0}^{\infty} (m_{n+1})^{(p'-\beta)/p'} (\|\Delta_n * f\|_p)^{\beta} < \infty,$$

then $f \in \mathrm{A}(\beta)$.

Lemma 1.2.35. *If* $1 \le p < \infty$ *and if* $f \in \mathrm{L}^p(G)$, *then*

$$\|D_{m_n} * f - f\|_p \le \omega^{(p)}(f, n)$$

for $n = 0, 1, 2, \ldots$.

Corollary 1.2.36. *Let* $1 \le p \le 2$ *and* $0 < \beta \le p'$. *If* $f \in \mathrm{L}^p(G)$ *and*

$$\sum_{n=0}^{\infty} (m_{n+1})^{(p'-\beta)/p'} (\omega^{(p)}(f, n))^{\beta} < \infty,$$

then $f \in \mathrm{A}(\beta)$.

Now Onneweer's Theorem 1.2.27 for bounded Vilenkin groups becomes a special case of Corollary 1.2.36 because if G is bounded, we can replace m_{n+1} by m_n in Corollary 1.2.36. If we take $\beta = 1$ in Theorem 1.2.34 and Corollary 1.2.36 we obtain

Corollary 1.2.37. *For* $1 \le p \le 2$, *we have*

(a) *if* $f \in \mathrm{L}^1(G)$ *and* $\sum_{n=0}^{\infty} (m_{n+1})^{1/p} \|\Delta_n * f\|_p < \infty$, *then* $f \in \mathrm{A}(1)$; *and*

(b) *if* $f \in \mathrm{L}^p(G)$ *and* $\sum_{n=0}^{\infty} (m_{n+1})^{1/p} \omega^{(p)}(f, n) < \infty$, *then* $f \in \mathrm{A}(1)$.

The next corollary is the extension of Onneweer result (Corollary 1.2.29) to arbitrary Vilenkin groups.

Corollary 1.2.38. *Let* $0 < \alpha \leq 1$, $1 \leq p \leq 2$ *and* $\beta > p/(\alpha p + p - 1)$. *Then* $\mathfrak{L}\mathrm{ip}\,(\alpha, p, G) \subset \mathrm{A}(\beta)$.

If we take $\beta = 1$ in the preceding corollary we obtain

Corollary 1.2.39. *Let* $1 \leq p \leq 2$ *and* $1 < \alpha p \leq p$. *Then* $\mathfrak{L}\mathrm{ip}\,(\alpha, p, G) \subset \mathrm{A}(1)$.

As noted above, if G is bounded, then the number m_{n+1} in Corollary 1.2.36 can be replaced by m_n to obtain Onneweer's Theorem 1.2.27. Quek and Yap's following theorem shows that Onneweer's theorem holds *only if* G is bounded.

Theorem 1.2.40. *Let* G *be an unbounded Vilenkin group. Let* $1 \leq p \leq 2$ *and* $0 < \beta < p'$. *Then there exists a function* $f \in \mathrm{L}^p(G)$ *such that*

$$\sum_{n=0}^{\infty} (m_n)^{(p'-\beta)/p'} (\omega^{(p)}(f, n))^{\beta} < \infty,$$

but $f \notin \mathrm{A}(\beta)$.

Onneweer (see Corollary 1.2.28) showed that if G is bounded, $1 \leq p \leq 2$, and if $f \in \mathrm{L}^p(G)$ is such that $\sum_{n=0}^{\infty} (m_n)^{1/p} \omega^{(p)}(f, n) < \infty$, then $f \in \mathrm{A}(1)$. For unbounded Vilenkin groups Quek and Yap proved the following (see Corollary 1.2.37 for a positive result).

Corollary 1.2.41. *Let* G *be an unbounded Vilenkin group and* $1 \leq p \leq 2$. *Then there exists* $f \in \mathrm{L}^p(G)$ *such that* $\sum_{n=0}^{\infty} (m_n)^{1/p} \omega^{(p)}(f, n) < \infty$, *but* $f \notin \mathrm{A}(1)$.

Yoshikazu Uno [66] proved an analogue of trigonometric series result of Schramm and Waterman [56].

Theorem 1.2.42. *Let* $1 \leq r < \infty$ *and* $1 \leq p < 2r$. *If* $f \in \Lambda\mathrm{GBF}^{(p)}$ *satisfies*

$$\sum_{n=0}^{\infty} \frac{(m_{n+1})^{\frac{1}{2}} \left(\omega^{((2-p)r'+p)}(f, n)\right)^{1-\frac{p}{2r}}}{\left(\sum_{j=1}^{m_n} \frac{1}{\lambda_j}\right)^{\frac{1}{2r}}} < \infty,$$

then $f \in \mathrm{A}(1)$, *where the class* $\Lambda\mathrm{GBF}^{(p)}$ *is defined as in Definition 4.1.12.*

In Chapter 5, Section 5.1, we study the absolute convergence of Vilenkin Fourier series for functions of various classes of generalized bounded fluctuations and obtain a generalization of a result of Uno [66].

An important trigonometric inequality essentially due to Wiener but later on made precise by Ingham concerning the lacunary trigonometric sums $f(x) = \sum A_k e^{in_k x}$, where A_k's are complex numbers, $n_{-k} = -n_k$ and $\{n_k\}$ satisfies the small gap condition (1.11), says that if I is any subinterval of $[-\pi, \pi]$ of length $|I| = 2\pi(1+\delta)/q > 2\pi/q$ then $\sum |A_k|^2 \leq A_\delta |I|^{-1} \int_I |f|^2$, $|A_k| \leq A_\delta |I|^{-1} \int_I |f|$ wherein A_δ depends only on δ. Such an inequality is proved in Chapter 5, Section 5.2, in the setting of the totally disconnected compact abelian groups G. The inequality is then applied to generalize the Bernstein, Szász and Stečhkin type results concerning the absolute convergence of Fourier series on G.

Finally in the same Chapter 5, in Section 5.3, we prove small gap analogues of our results of Section 5.1 when functions are of (generalized) bounded fluctuation locally on certain cosets.

Chapter 2

Multiple Trigonometric Fourier Coefficients

2.1 Order of magnitude of multiple trigonometric Fourier coefficients of functions of bounded p-variation

For a function of two variables several definitions of bounded variation are given and various properties are studied (see, for example, [24, 1]). F. Móricz [33] studied the order of magnitude of double Fourier coefficients with the help of Riemann-Stieltjes integral of functions of two variables and V. Fülöp and F. Móricz [10] studied the order of magnitude of multiple Fourier coefficients of functions of bounded variation in the sense of Vitali and Hardy & Krause (see [8]) in a straightforward way without using Riemann-Stieltjes integral (see Theorems 1.1.32, 1.1.33 and 1.1.34). Here we define the concept of *bounded p-variation* ($p \geq 1$) for a function of several variables in two different ways as follows and study the order of magnitude of Fourier coefficients for functions of these classes. Our results generalize those of [33] and [10] in the sense that for $p = 1$, our definitions coincides with the definitions of Vitali and Hardy & Krause and our results with those of [33] and [10], except for the exact constant. Results of this section are published in the form of a paper in [17] (see also MR2670992).

Definition 2.1.1. Let R be the rectangle as in Definition 1.1.28. By a (finite) partition of R we mean the a set $\{R_1, ..., R_n\}$, in which R_i's are pairwise disjoint (no two have common interior) subrectangles of R having their sides (faces) parallel to the standard coordinate hyperplanes and whose union is R.

Definition 2.1.2. For $p \geq 1$ we say that f is of *bounded p-variation over R* (written as $f \in \mathrm{BV_V}^{(p)}(R)$) if $V_p(f; R)$, the *total p-variation* of f over R, is finite, where

$$V_p(f; R) := \sup \left\{ \sum_{i=1}^{n} |\Delta f(R_i)|^p \right\}^{1/p}, \tag{2.1}$$

in which the supremum is taken over all partitions $\{R_1, ..., R_n\}$ of R and $\Delta f(R_i)$'s are defined as in Definition 1.1.29.

Remark 2.1.3. Note that for $p = 1$ our definition is equivalent to that of Definition 1.1.30 of Vitali (see, for example, [8, 10]). This is because if we take any grid $\mathcal{P}$ of R (see Definition 1.1.28), then it will give a partition of R in terms of disjoint union of subrectangles of R and so the corresponding sum does not exceed $V_1(f; R)$. Conversely, if $\{R_1, ..., R_n\}$ is any partition of R, by inserting (hyper) planes parallel to the standard coordinate hyperplanes (if necessary) in some rectangles from $R_1, ..., R_n$, we can form a grid $\mathcal{P}$ of R and by triangle inequality the sum $\sum_{i=1}^{n} |\Delta f(R_i)|$ does not exceed the corresponding sum for the grid $\mathcal{P}$ and hence it does not exceed $V(f; R)$ (see (1.16)).

As noted by Fülöp and F. Móricz [10, p. 96], in this case also, when $m \geq 2$, a function f in the class $\mathrm{BV_V}^{(p)}(R)$ is not necessarily measurable in the sense of Lebesgue. This is a consequence of the simple fact that if a function $f = f(x_1, ..., x_m)$ does not depend on at least one of the $x_1, ..., x_m$, then for any subrectangle R' of R we have $\Delta f(R') = 0$, so that $V_p(f; R) = 0$. Consequently, the class $\mathrm{BV_V}^{(p)}(R)$ contains functions for which the m-dimensional Lebesgue integral over R fails to exist. The following definition is motivated by this fact.

Definition 2.1.4. In case $m = 2$, we say that a function $f = f(x_1, x_2)$ is of bounded p-variation over $R := [a_1, b_1] \times [a_2, b_2]$, in symbol: $f \in \mathrm{BV_H}^{(p)}(R)$, if it is in the class $\mathrm{BV_V}^{(p)}(R)$ and if the marginal functions $f(x_1, a_2)$ and $f(a_1, x_2)$ are of bounded p-variation on the intervals $I_1 := [a_1, b_1]$ and $I_2 := [a_2, b_2]$, respectively in the sense of Wiener [77].

In case $m \geq 3$, the notion of bounded p-variation over a rectangle R can naturally be defined by the following recurrence: $f \in \mathrm{BV_H}^{(p)}(R)$ if $f \in \mathrm{BV_V}^{(p)}(R)$ and each of the marginal functions $f(x_1, ..., a_k, ..., x_m)$ is in the class ${\rm BV_H}^{(p)}(R(a_k))$, where $k = 1, ..., m$ and $R(a_k)$ is as in (1.17).

This definition can be equivalently reformulated as follows: $f \in {\rm BV_H}^{(p)}(R)$ if and only if $f \in {\rm BV_V}^{(p)}(R)$ and for any choice of $(1 \le)j_1 < ... < j_n(\le m)$, $1 \le n < m$, the function $f(x_1, ..., a_{j_1}, ..., a_{j_n}, ..., x_m)$ is in ${\rm BV_V}^{(p)}(R(a_{j_1}, ..., a_{j_n}))$, where $R(a_{j_1}, ..., a_{j_n})$ is as in (1.18).

Remark 2.1.5. As argued in Remark 2.1.3, when $p = 1$ our Definition 2.1.4 is equivalent to the Definition 1.1.31 given by Hardy and Krause (see, for example, [8, 10]).

First we state and prove certain lemmas which we require to prove our main theorems (Theorems 2.1.13 and 2.1.14).

Lemma 2.1.6. *If $f \in {\rm BV_H}^{(p)}(R)$ then f is bounded over R.*

Proof. Observe that when $m = 2$, for any $(x_1, x_2) \in R = I_1 \times I_2$ we have

$$\begin{aligned}|f(x_1, x_2)|^p &= |\{f(x_1, x_2) - f(x_1, a_2) - f(a_1, x_2) + f(a_1, a_2)\} \\ &\qquad + \{f(x_1, a_2) - f(a_1, a_2)\} + \{f(a_1, x_2) - f(a_1, a_2)\} + f(a_1, a_2)|^p \\ &\le 4^p\{|f(x_1, x_2) - f(x_1, a_2) - f(a_1, x_2) + f(a_1, a_2)|^p \\ &\qquad + |f(x_1, a_2) - f(a_1, a_2)|^p + |f(a_1, x_2) - f(a_1, a_2)|^p + |f(a_1, a_2)|^p\} \\ &\le 4^p\{(V_p(f; R))^p + (V_p(f(\cdot, a_2); I_1))^p + (V_p(f(a_1, \cdot); I_2))^p + |f(a_1, a_2)|^p\}.\end{aligned}$$

Similarly when $m \ge 2$, for any $\mathbf{x} \in R = [a_1, b_1] \times ... \times [a_m, b_m]$ we have

$$\begin{aligned}|f(\mathbf{x})|^p \le 2^{mp}\Bigg\{&(V_p(f; R))^p \\ &+ \sum_{n=1}^{m-1} \sum_{1\le j_1<...<j_n\le m} (V_p(f(\cdot, ..., a_{j_1}, ..., a_{j_n}, ..., \cdot); R(a_{j_1}, ..., a_{j_n})))^p + |f(\mathbf{a})|^p\Bigg\}.\end{aligned}$$

This completes the proof. □

Lemma 2.1.7. *If $f \in {\rm BV_H}^{(p)}(R)$ then for any arbitrary fixed values $c_{j_1} \in [a_{j_1}, b_{j_1}]$, ..., $c_{j_n} \in [a_{j_n}, b_{j_n}]$, $(1 \le)j_1 < ... < j_n(\le m)$, and $1 \le n < m$, the function $f(\cdot, ..., c_{j_1}, ..., c_{j_n}, ..., \cdot)$ is in ${\rm BV_H}^{(p)}(R(a_{j_1}, ..., a_{j_n}))$ and that*

$$\begin{aligned}(V_p(f(\cdot, ..., c_{j_1}, ..., c_{j_n}, ..., \cdot); R(a_{j_1}, ..., a_{j_n})))^p \le 2^{np}\Bigg\{&(V_p(f; R))^p \\ &+ \sum_{k=1}^{n} \sum_{\substack{s_1<...<s_k,\\ s_1,...,s_k\in\{j_1,...,j_n\}}} (V_p(f(\cdot, ..., a_{s_1}, ..., a_{s_k}, ..., \cdot); R(a_{s_1}, ..., a_{s_k})))^p\Bigg\}.\end{aligned}$$

Proof. First we will prove the lemma for $m = 2$. We must show that for any $a_2 < c_2 \le b_2$ and $a_1 < c_1 \le b_1$ we have

$$(V_p(f(\cdot, c_2); I_1))^p \le 2^p\{(V_p(f; R))^p + (V_p(f(\cdot, a_2); I_1))^p\};$$

$$(V_p(f(c_1, \cdot); I_2))^p \le 2^p\{(V_p(f; R))^p + (V_p(f(a_1, \cdot); I_2))^p\}.$$

Fix $a_2 < c_2 \le b_2$. Then for any partition $a_1 = x_1^0,\ x_1^1,\ ...,\ x_1^n = b_1$ of I_1 we have

$$\begin{aligned}
&\sum_{i=1}^{n} |f(x_1^i, c_2) - f(x_1^{i-1}, c_2)|^p \\
&= \sum_{i=1}^{n} |\{f(x_1^i, c_2) - f(x_1^i, a_2) - f(x_1^{i-1}, c_2) + f(x_1^{i-1}, a_2)\} + \{f(x_1^i, a_2) - f(x_1^{i-1}, a_2)\}|^p \\
&\le 2^p \Bigg\{ \sum_{i=1}^{n} |f(x_1^i, c_2) - f(x_1^i, a_2) - f(x_1^{i-1}, c_2) + f(x_1^{i-1}, a_2)|^p \\
&\qquad\qquad + \sum_{i=1}^{n} |f(x_1^i, a_2) - f(x_1^{i-1}, a_2)|^p \Bigg\} \\
&\le 2^p\{(V_p(f; R))^p + (V_p(f(\cdot, a_2); I_1))^p\}.
\end{aligned}$$

Taking supremum over all partitions of I_1 we get

$$(V_p(f(\cdot, c_2); I_1))^p \le 2^p\{(V_p(f; R))^p + (V_p(f(\cdot, a_2); I_1))^p\}.$$

Similarly for $a_1 < c_1 \le b_1$ we have

$$(V_p(f(c_1, \cdot); I_2))^p \le 2^p\{(V_p(f; R))^p + (V_p(f(a_1, \cdot); I_2))^p\}.$$

Now we will show the lemma for $m = 3$. By symmetry in the variables x_1, x_2, x_3, it is enough to show the following:

(i) For any $a_3 < c_3 \le b_3$

$$(V_p(f(\cdot, \cdot, c_3); R(a_3)))^p \le 2^p\{(V_p(f; R))^p + (V_p(f(\cdot, \cdot, a_3); R(a_3)))^p\}.$$

(ii) For any $a_2 < c_2 \le b_2$ and $a_3 < c_3 \le b_3$

$$\begin{aligned}
(V_p(f(\cdot, c_2, c_3); R(a_2, a_3)))^p \le 2^{2p}\{&(V_p(f; R))^p + (V_p(f(\cdot, a_2, \cdot); R(a_2)))^p \\
&+ (V_p(f(\cdot, \cdot, a_3); R(a_3)))^p + (V_p(f(\cdot, a_2, a_3); R(a_2, a_3)))^p\}.
\end{aligned}$$

To prove (i), consider a partition $\{R_i\}_{i=1}^s$ of $R(a_3)$. Then $\{R_i \times [a_3, c_3]\}_{i=1}^s$ is a collection of disjoint subrectangles of R. Therefore

$$\begin{aligned}\sum_{i=1}^{s} |\Delta f(\cdot, \cdot, c_3)(R_i)|^p &= \sum_{i=1}^{s} |\Delta f(R_i \times [a_3, c_3]) + \Delta f(\cdot, \cdot, a_3)(R_i)|^p \\ &\leq 2^p \left\{ \sum_{i=1}^{s} |\Delta f(R_i \times [a_3, c_3])|^p + \sum_{i=1}^{s} |\Delta f(\cdot, \cdot, a_3)(R_i)|^p \right\} \\ &\leq 2^p \{(V_p(f; R))^p + (V_p(f(\cdot, \cdot, a_3); R(a_3)))^p\}.\end{aligned}$$

Taking supremum over all partitions of $R(a_3)$ we get (i).

Next, to prove (ii), consider a partition $\{a_1 = x_1^0, x_1^1, ..., x_1^s = b_1\}$ of $R(a_2, a_3)$. Then

$$\begin{aligned}&\sum_{i=1}^{s} |f(x_1^i, c_2, c_3) - f(x_1^{i-1}, c_2, c_3)|^p \\ &= \sum_{i=1}^{s} |\Delta f([x_1^{i-1}, x_1^i] \times [a_2, c_2] \times [a_3, c_3]) + \Delta f(\cdot, a_2, \cdot)([x_1^{i-1}, x_1^i] \times [a_3, c_3]) \\ &\qquad + \Delta f(\cdot, \cdot, a_3)([x_1^{i-1}, x_1^i] \times [a_2, c_2]) + \{f(x_1^i, a_2, a_3) - f(x_1^{i-1}, a_2, a_3)\}|^p \\ &\leq 4^p \sum_{i=1}^{s} \Big\{ |\Delta f([x_1^{i-1}, x_1^i] \times [a_2, c_2] \times [a_3, c_3])|^p + |\Delta f(\cdot, a_2, \cdot)([x_1^{i-1}, x_1^i] \times [a_3, c_3])|^p \\ &\qquad + |\Delta f(\cdot, \cdot, a_3)([x_1^{i-1}, x_1^i] \times [a_2, c_2])|^p + |f(x_1^i, a_2, a_3) - f(x_1^{i-1}, a_2, a_3)|^p \Big\} \\ &\leq 4^p \{(V_p(f; R))^p + (V_p(f(\cdot, a_2, \cdot); R(a_2)))^p + (V_p(f(\cdot, \cdot, a_3); R(a_3)))^p \\ &\qquad + (V_p(f(\cdot, a_2, a_3); R(a_2, a_3)))^p\}.\end{aligned}$$

This proves the lemma for $m = 3$. A similar argument proves the lemma for any m. □

Lemma 2.1.8. *Let $f \in \mathrm{BV_V}^{(p)}(R)$, where $R = [a_1, b_1] \times ... \times [a_m, b_m]$. Let $\{R_1, ..., R_n\}$ be a partition of R. Then $f \in \mathrm{BV_V}^{(p)}(R_i)$ for each $i = 1, ..., n$, and that*

$$\sum_{i=1}^{n} (V_p(f; R_i))^p \leq (V_p(f; R))^p.$$

Proof. Let $\{R_{ij} : j = 1, ..., m_i\}$ be any partition of R_i, for each $i = 1, ..., n$. Then $\{R_{ij} : j = 1, ..., m_i; i = 1, ..., n\}$ is clearly a partition of R and since $f \in \mathrm{BV_V}^{(p)}(R)$,

$$\sum_{i=1}^{n} \sum_{j=1}^{m_i} |\Delta f(R_{ij})|^p \leq (V_p(f; R))^p.$$

Taking supremum over all partitions $\{R_{1j} : j = 1, ..., m_1\}$ of R_1 (keeping the partitions of $R_2, ..., R_n$ fixed) we get

$$(V_p(f; R_1))^p + \sum_{i=2}^{n} \sum_{j=1}^{m_i} |\Delta f(R_{ij})|^p \leq (V_p(f; R))^p.$$

Similarly taking supremum over all partitions of R_2 (keeping the partitions of $R_3, ..., R_n$ fixed), and continuing in this way for $R_3, ..., R_n$ we get the lemma. □

For the proofs of Lemmas 2.1.9, 2.1.10 and 2.1.11, for functions of bounded variation refer [1, p. 721-722]. We prove the results for functions of bounded p-variation.

Lemma 2.1.9. *Let $f \in \mathrm{BV_V}^{(p)}(R)$, where $R = [a_1, b_1] \times [a_2, b_2]$. Suppose $f(x, \bar{y})$ (respectively $f(\bar{x}, y)$) has no discontinuities of second kind for any fixed $\bar{y} \in [a_2, b_2]$ (respectively $\bar{x} \in [a_1, b_1]$). If $f(x, \bar{y})$ (respectively $f(\bar{x}, y)$) for some $\bar{y}$ (respectively $\bar{x}$) has only a denumerable number of discontinuities in x (respectively y), the discontinuities in x (respectively y) of $f(x, y)$ are located on a denumerable number of parallels to the y-axis (respectively x-axis).*

Proof. Let $E = \{(x, y) \in R : f$ has a discontinuity in $x\}$ and $E_{\bar{y}} = \{(\bar{x}, \bar{y}) \in R : f(x, \bar{y})$ is discontinuous at $\bar{x}\}$. Then $E_{\bar{y}} \subset E$ and by our assumption $E_{\bar{y}}$ is denumerable.

If possible suppose there is a non-denumerable set S of vertical lines each containing at least one point of E. Since $E_{\bar{y}}$ is denumerable, clearly only a denumerable subset of S made up wholly of points of $E_{\bar{y}}$. Let the remaining lines of S constitute the subset S_1; then each line of S_1 contains at least one point of E and no point of $E_{\bar{y}}$, and S_1 is non-denumerable. On each line of S_1 (which lie interior to R) choose a point of E; at this point the saltus of f in x is positive and hence its p^{th} power. This non-denumerable set of p^{th} powers of saltuses contains a subset whose elements are the terms of a divergent series. Thus there is a sequence $\{(x_i, y_i)\}_{i=1}^{\infty}$ of distinct points in E, which lie interior to R and on different lines in S_1, such that

$$\sum_{i=1}^{\infty} s_i^p = \infty,$$

where $s_i =$ the saltus in x at $(x_i, y_i) = |f(x_i+, y_i) - f(x_i-, y_i)|$. By the definition of $f(x_i+, y_i)$ and $f(x_i-, y_i)$, for every $\varepsilon_i = \frac{s_i}{4}$ there is a $\delta_i > 0$ such that

$$x_i - \delta_i < x < x_i \Rightarrow |f(x, y_i) - f(x_i-, y_i)| < \varepsilon_i$$

and

$$x_i < x < x_i + \delta_i \Rightarrow |f(x, y_i) - f(x_i+, y_i)| < \varepsilon_i.$$

Since the point $(x_i, \bar{y})$ lies on a line in S_1, $f(x, \bar{y})$ is continuous at x_i for each i. Thus, for each i, there is a $\delta_i' > 0$ such that

$$|x - x_i| < \delta_i' \Rightarrow |f(x, \bar{y}) - f(x_i, \bar{y})| < \frac{\varepsilon_i}{2}.$$

Put $\delta_i'' = \min\{\delta_i, \delta_i'\}$ and choose x_i', x_i'' such that $x_i - \delta_i'' < x_i' < x_i$, $x_i < x_i'' < x_i + \delta_i''$ for each i and the intervals $\{[x_i',\, x_i'']\}_{i=1}^{\infty}$ are pairwise disjoint. Then by above inequalities we have

$$|f(x_i', y_i) - f(x_i-, y_i)| < \varepsilon_i, \;\; |f(x_i'', y_i) - f(x_i+, y_i)| < \varepsilon_i;$$

and

$$|f(x_i', \bar{y}) - f(x_i, \bar{y})| < \frac{\varepsilon_i}{2}, \;\; |f(x_i'', \bar{y}) - f(x_i, \bar{y})| < \frac{\varepsilon_i}{2}.$$

Therefore

$$\begin{aligned}
&|f(x_i'', y_i) - f(x_i', y_i)| \\
&\quad = |f(x_i'', y_i) - f(x_i+, y_i) + f(x_i+, y_i) - f(x_i-, y_i) + f(x_i-, y_i) - f(x_i', y_i)| \\
&\quad \geq |f(x_i+, y_i) - f(x_i-, y_i)| - |f(x_i'', y_i) - f(x_i+, y_i) + f(x_i-, y_i) - f(x_i', y_i)| \\
&\quad \geq 4\varepsilon_i - 2\varepsilon_i = 2\varepsilon_i
\end{aligned}$$

and

$$|f(x_i'', \bar{y}) - f(x_i', \bar{y})| \leq |f(x_i'', \bar{y}) - f(x_i, \bar{y})| + |f(x_i, \bar{y}) - f(x_i', \bar{y})| < \frac{\varepsilon_i}{2} + \frac{\varepsilon_i}{2} = \varepsilon_i.$$

Hence

$$\begin{aligned}
|f(x_i'', y_i) - f(x_i', y_i) - f(x_i'', \bar{y}) + f(x_i', \bar{y})| & \\
&\geq |f(x_i'', y_i) - f(x_i', y_i)| - |f(x_i'', \bar{y}) - f(x_i', \bar{y})| \\
&\geq 2\varepsilon_i - \varepsilon_i = \varepsilon_i.
\end{aligned}$$

Thus if R_i denotes the rectangle with vertices (x_i'', y_i), (x_i', y_i), $(x_i'', \bar{y})$ and $(x_i', \bar{y})$ for each i, then

$$\sum_{i=1}^{\infty} |\Delta f(R_i)|^p \geq \sum_{i=1}^{\infty} \varepsilon_i^p = \frac{1}{4^p} \sum_{i=1}^{\infty} s_i^p = \infty.$$

This shows that $V_p(f; R) = \infty$; from this contradiction lemma follows. □

Lemma 2.1.10. *Let $f \in \mathrm{BV_V}^{(p)}(R)$, where $R = [a_1, b_1] \times [a_2, b_2]$. Then the set of all points $(\bar{x}, \bar{y}) \in R$ for which $f(x, y)$ is discontinuous at $(\bar{x}, \bar{y})$, but $f(x, \bar{y})$ is continuous at $\bar{x}$ and $f(\bar{x}, y)$ is continuous at $\bar{y}$, is denumerable.*

Proof. Let $(\bar{x}, \bar{y})$ be such a discontinuity. Then there exists $\varepsilon > 0$ such that for every $\delta > 0$ there is a point, say, (x', y') (depending on δ) such that

$$\sqrt{(x' - \bar{x})^2 + (y' - \bar{y})^2} < \delta \quad \text{but} \quad |f(x', y') - f(\bar{x}, \bar{y})| \geq \varepsilon. \tag{2.2}$$

Also, by the continuity of $f(\cdot, \bar{y})$ and $f(\bar{x}, \cdot)$ at $\bar{x}$ and $\bar{y}$ respectively, $\exists\, \delta > 0 \ni$

$$|x - \bar{x}| < \delta \Rightarrow |f(x, \bar{y}) - f(\bar{x}, \bar{y})| < \frac{\varepsilon}{4} \quad \text{and} \quad |y - \bar{y}| < \delta \Rightarrow |f(\bar{x}, y) - f(\bar{x}, \bar{y})| < \frac{\varepsilon}{4}.$$

For this δ, as above, there is a point (x', y') such that (2.2) holds. Since

$$\sqrt{(x' - \bar{x})^2 + (y' - \bar{y})^2} \geq |x' - \bar{x}| \quad \text{and} \quad \sqrt{(x' - \bar{x})^2 + (y' - \bar{y})^2} \geq |y' - \bar{y}|$$

we get

$$|f(x', \bar{y}) - f(\bar{x}, \bar{y})| < \frac{\varepsilon}{4}, \quad |f(\bar{x}, y') - f(\bar{x}, \bar{y})| < \frac{\varepsilon}{4};$$

which shows that

$$|f(x', \bar{y}) + f(\bar{x}, y') - 2f(\bar{x}, \bar{y})| < \frac{\varepsilon}{2}.$$

Thus for the rectangle R' with sides parallel to the axes and whose two vertices are $(\bar{x}, \bar{y})$ and (x', y'), we have

$$\begin{aligned} |\Delta f(R')|^p &= |f(x', y') - f(x', \bar{y}) - f(\bar{x}, y') + f(\bar{x}, \bar{y})|^p \\ &\geq (|f(x', y') - f(\bar{x}, \bar{y})| - |f(x', \bar{y}) + f(\bar{x}, y') - 2f(\bar{x}, \bar{y})|)^p \\ &> \left(\varepsilon - \frac{\varepsilon}{2}\right)^p = \left(\frac{\varepsilon}{2}\right)^p. \end{aligned}$$

The assumption that the set of such discontinuities is non-denumerable then leads to a contradiction just as in the case of Lemma 2.1.9. □

Lemma 2.1.11. *Let $f \in \mathrm{BV_H}^{(p)}(R)$, where $R = [a_1, b_1] \times [a_2, b_2]$. Then the discontinuities of $f(x, y)$ are located on a countable number of parallels to the axes.*

Proof. Since $f \in \mathrm{BV_H}^{(p)}(R)$, $f \in \mathrm{BV_V}^{(p)}(R)$ and the marginal functions $f(x, a_2)$ and $f(a_1, y)$ are of bounded p-variation on I_1 and I_2 respectively. Thus $f(x, a_2)$ has a denumerable number of discontinuities in x and $f(a_1, y)$ has a denumerable number of discontinuities in y. So, in view of Lemma 2.1.9, the discontinuities in x or y of $f(x, y)$ are located on a countable number of parallels to the coordinate axes. Now the lemma follows from Lemma 2.1.10. □

Lemma 2.1.12. *Let $f \in \mathrm{BV_H}^{(p)}(R)$, where $R = [a_1, b_1] \times ... \times [a_m, b_m]$. Then the discontinuities of f are located on a countable number of $(m-1)$-dimensional hyperplanes parallel to some of the coordinate hyperplanes.*

Proof. We will prove the lemma by using induction on m. In view of Lemma 2.1.11, it is true for $m = 2$. Suppose it is true when m is replaced by $m-1$. As $f \in \mathrm{BV_H}^{(p)}(R)$, the marginal function $f(x_1, ..., a_k, ..., x_m)$ is in the class $\mathrm{BV_H}^{(p)}(R(a_k))$ for each $k = 1, ..., m$. By induction hypothesis, for each k, the discontinuities of $f(x_1, ..., a_k, ..., x_m)$ are located on a countable number of $(m-2)$-dimensional hyperplanes of $R(a_k)$ parallel to some of the coordinate hyperplanes. Thus, arguing as in the proof of Lemma 2.1.9, we can see that the discontinuities of $f(x_1, ..., x_n)$ in $(x_1, ..., x_{k-1}, x_{k+1}, ..., x_m)$ are located on a denumerable number of $(m-2)$-dimensional planes of $R(a_k)$ parallel to some of the coordinate hyperplanes, for all $k = 1, ..., m$. Since each such plane is clearly embedded in an $(m-1)$-dimensional hyperplane in R parallel to some of the coordinate hyperplanes, such discontinuities of f are located on a countable number of $(m-1)$-dimensional hyperplanes parallel to some of the coordinate hyperplanes in R.

Further, as $f \in \mathrm{BV_V}^{(p)}(R)$, as arguing in Lemma 2.1.10, here we can see that the set of all points $(\bar{x}_1, ..., \bar{x}_m) \in R$ for which $f(x_1, ..., x_m)$ is discontinuous at $(\bar{x}_1, ..., \bar{x}_m)$ but each marginal function $f(x_1, ..., \bar{x}_k, ..., x_m)$ is continuous at $(\bar{x}_1, ..., \bar{x}_{k-1}, \bar{x}_{k+1}, ..., \bar{x}_m)$, is denumerable. Thus the lemma follows. $\square$

Theorem 2.1.13. *Let $f : \mathbb{R}^m \to \mathbb{C}$ be 2π-periodic in each variable. If f belongs to $\mathrm{BV_V}^{(p)}([0, 2\pi]^m) \cap \mathrm{L}^p(\mathbb{T}^m)$ $(p \geq 1)$ and $\mathbf{n} = (n^{(1)}, ..., n^{(m)}) \in \mathbb{Z}^m$ is such that $n^{(j)} \neq 0$ for each j, then*

$$\hat{f}(\mathbf{n}) = O\left(\frac{1}{|\prod_{j=1}^{m} n^{(j)}|^{1/p}}\right).$$

Proof. For the sake of simplicity in writing, we carry out the proof for $m = 2$, and we write (x, y) and (k, ℓ) in place of (x_1, x_2) and $(n^{(1)}, n^{(2)})$ respectively.

Let $\mathbf{n} = (k, \ell) \in \mathbb{Z}^2$ be such that $k \neq 0$, $\ell \neq 0$. Then the functions e^{-ikx} and $e^{-i\ell y}$ are periodic functions of periods $\frac{2\pi}{|k|}$ and $\frac{2\pi}{|\ell|}$ respectively. Thus by putting

$$a_r = r \cdot \frac{2\pi}{|k|} \ \ (r = 0, 1, ..., |k|); \quad b_s = s \cdot \frac{2\pi}{|\ell|} \ \ (s = 0, 1, ..., |\ell|)$$

we get

$$\int_{a_{r-1}}^{a_r} e^{-ikx} dx = 0 \ \ (r = 1, 2, ..., |k|); \quad \int_{b_{s-1}}^{b_s} e^{-i\ell y} dy = 0 \ \ (s = 1, 2, ..., |\ell|). \qquad (2.3)$$

In view of Fubini's theorem and (2.3), we have

$$\int_{a_{r-1}}^{a_r}\int_{b_{s-1}}^{b_s} f(x, b_{s-1}))e^{-ikx}e^{-i\ell y}dxdy = \int_{a_{r-1}}^{a_r} f(x, b_{s-1})\Big[\int_{b_{s-1}}^{b_s} e^{-i\ell y}dy\Big]e^{-ikx}dx = 0,$$

$$\int_{a_{r-1}}^{a_r}\int_{b_{s-1}}^{b_s} f(a_{r-1}, y)e^{-ikx}e^{-i\ell y}dxdy = \int_{b_{s-1}}^{b_s} f(a_{r-1}, y)\Big[\int_{a_{r-1}}^{a_r} e^{-ikx}dx\Big]e^{-i\ell y}dy = 0$$

and

$$\int_{a_{r-1}}^{a_r}\int_{b_{s-1}}^{b_s} f(a_{r-1}, b_{s-1})e^{-ikx}e^{-i\ell y}dxdy = f(a_{r-1}, b_{s-1})\int_{a_{r-1}}^{a_r} e^{-ikx}dx\int_{b_{s-1}}^{b_s} e^{-i\ell y}dy = 0$$

for all $r = 1, ..., |k|$ and $s = 1, ..., |\ell|$. Define three functions f_1, f_2, f_3 on $\mathbb{T}^2$ by setting

$$f_1(x, y) = f(x, b_{s-1}) \ \ (0 \le x < 2\pi;\ b_{s-1} \le y < b_s) \text{ for } s = 1, ..., |\ell|;$$

$$f_2(x, y) = f(a_{r-1}, y) \ \ (a_{r-1} \le x < a_r;\ 0 \le y < 2\pi) \text{ for } r = 1, ..., |k|;$$

and

$$f_3(x, y) = f(a_{r-1}, b_{s-1}) \ \ (a_{r-1} \le x < a_r;\ b_{s-1} \le y < b_s)$$

for $r = 1, ..., |k|;\ s = 1, ..., |\ell|$. Since $f \in \mathrm{BV_V}^{(p)}([0, 2\pi]^2) \cap \mathrm{L}^p(\mathbb{T}^2)$, each $f_i \in \mathrm{BV_V}^{(p)}([0, 2\pi]^2) \cap \mathrm{L}^p(\mathbb{T}^2)$ and hence $f - f_1 - f_2 + f_3 \in \mathrm{BV_V}^{(p)}([0, 2\pi]^2) \cap \mathrm{L}^p(\mathbb{T}^2)$. Further in view of Fubini's theorem and above relations we have

$$\begin{aligned}\int_0^{2\pi}\int_0^{2\pi} f_1(x, y)e^{-ikx}e^{-i\ell y}dxdy &= \int_0^{2\pi}\Big[\sum_{s=1}^{|\ell|}\int_{b_{s-1}}^{b_s} f(x, b_{s-1})e^{-i\ell y}dy\Big]e^{-ikx}dx \\ &= 0,\end{aligned}$$

$$\begin{aligned}\int_0^{2\pi}\int_0^{2\pi} f_2(x, y)e^{-ikx}e^{-i\ell y}dxdy &= \int_0^{2\pi}\Big[\sum_{r=1}^{|k|}\int_{a_{r-1}}^{a_r} f(a_{r-1}, y)e^{-ikx}dx\Big]e^{-i\ell y}dy \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\int_0^{2\pi}\int_0^{2\pi} f_3(x, y)e^{-ikx}e^{-i\ell y}dxdy &= \sum_{r=1}^{|k|}\sum_{s=1}^{|\ell|}\int_{a_{r-1}}^{a_r}\int_{b_{s-1}}^{b_s} f(a_{r-1}, b_{s-1})e^{-ikx}e^{-i\ell y}dxdy \\ &= 0.\end{aligned}$$

Using these equations in the definition of $\hat{f}(\mathbf{n})$ we get

$$\begin{aligned}
(2\pi)^2|\hat{f}(\mathbf{n})| &= \left|\int_0^{2\pi}\int_0^{2\pi} f(x,y)e^{-ikx}e^{-i\ell y}dxdy\right| \\
&= \left|\int_0^{2\pi}\int_0^{2\pi} [f(x,y)-f_1(x,y)-f_2(x,y)+f_3(x,y)]e^{-ikx}e^{-i\ell y}dxdy\right| \\
&\leq \int_0^{2\pi}\int_0^{2\pi} |f(x,y)-f_1(x,y)-f_2(x,y)+f_3(x,y)|dxdy \\
&\leq \left(\int_0^{2\pi}\int_0^{2\pi} |f(x,y)-f_1(x,y)-f_2(x,y)+f_3(x,y)|^p dxdy\right)^{1/p}(2\pi)^{2/q},
\end{aligned}$$

in view of the Hölder's inequality (when $p > 1$) since $f - f_1 - f_2 + f_3 \in \mathrm{L}^p(\mathbb{T}^2)$, where q is such that $1/p + 1/q = 1$. Observe that when $p = 1$, we don't use Hölder's inequality and in that case we consider the inequality except last step. In any case, it follows that

$$\begin{aligned}
&(2\pi)^2|\hat{f}(\mathbf{n})|^p \\
&\leq \int_0^{2\pi}\int_0^{2\pi} |f(x,y)-f_1(x,y)-f_2(x,y)+f_3(x,y)|^p dxdy \\
&= \sum_{r=1}^{|k|}\sum_{s=1}^{|\ell|}\int_{a_{r-1}}^{a_r}\int_{b_{s-1}}^{b_s} |f(x,y)-f_1(x,y)-f_2(x,y)+f_3(x,y)|^p dxdy \\
&= \sum_{r=1}^{|k|}\sum_{s=1}^{|\ell|}\int_{a_{r-1}}^{a_r}\int_{b_{s-1}}^{b_s} |f(x,y)-f(x,b_{s-1})-f(a_{r-1},y)+f(a_{r-1},b_{s-1})|^p dxdy \\
&\leq \sum_{r=1}^{|k|}\sum_{s=1}^{|\ell|}(V_p(f;[a_{r-1},a_r]\times[b_{s-1},b_s]))^p(a_r-a_{r-1})(b_s-b_{s-1}) \\
&\leq \frac{(2\pi)^2}{|k\ell|}(V_p(f;[0,2\pi]^2))^p,
\end{aligned}$$

in view of Lemma 2.1.8. Thus we get

$$|\hat{f}(\mathbf{n})| \leq \frac{V_p(f;[0,2\pi]^2)}{|k\ell|^{1/p}}. \tag{2.4}$$

This completes the proof. $\square$

Theorem 2.1.14. *Let $f : \mathbb{R}^m \to \mathbb{C}$ be 2π-periodic in each variable. If f belongs to $\mathrm{BV_H}^{(p)}([0,2\pi]^m)$ ($p \geq 1$) then for any $\mathbf{0} \neq \mathbf{n} = (n^{(1)}, ..., n^{(m)}) \in \mathbb{Z}^m$,*

$$\hat{f}(\mathbf{n}) = O\left(\frac{1}{|\prod_{j=1,n^{(j)}\neq 0}^{m} n^{(j)}|^{1/p}}\right).$$

Proof. Here also we will carry out the proof for $m = 2$ and use notations as in the proof of Theorem 2.1.13. Since $f \in {\rm BV_H}^{(p)}([0,2\pi]^2)$, in view of Lemma 2.1.11 (use Lemma 2.1.12 for general case), the discontinuities of f lie on countable number of parallels to the axes and hence f is measurable over $\mathbb{T}^2$ in the sense of Lebesgue. Further, by Lemma 2.1.6, f is bounded over $[0,2\pi]^2$ and hence $f \in {\rm L}^p(\mathbb{T}^2)$. As ${\rm BV_H}^{(p)}([0,2\pi]^2) \subset {\rm BV_V}^{(p)}([0,2\pi]^2)$, $f \in {\rm L}^p(\mathbb{T}^2) \cap {\rm BV_V}^{(p)}([0,2\pi]^2)$. Therefore if $\mathbf{n} = (k,\ell) \in \mathbb{Z}^2$ is such that $k \neq 0, \ell \neq 0$, by Theorem 2.1.13,

$$\hat{f}(\mathbf{n}) = O\bigg(\frac{1}{|k\ell|^{1/p}}\bigg).$$

Next, let $\mathbf{n} = (k,\ell) \in \mathbb{Z}^2$ be such that $k \neq 0$, $\ell = 0$ and let a_r's and f_2 be as defined in the proof of Theorem 2.1.13. Then we have

$$\int_0^{2\pi}\int_0^{2\pi} f_2(x,y)e^{-ikx}dxdy = \int_0^{2\pi}\bigg(\sum_{r=1}^{|k|} f(a_{r-1},y)\bigg[\int_{a_{r-1}}^{a_r} e^{-ikx}dx\bigg]\bigg)dy = 0,$$

in view of Fubini's theorem and (2.3); and,

$$\begin{aligned}(2\pi)^2|\hat{f}(\mathbf{n})| &= \bigg|\int_0^{2\pi}\int_0^{2\pi}[f(x,y) - f_2(x,y)]e^{-ikx}dxdy\bigg| \\ &\leq \bigg(\int_0^{2\pi}\int_0^{2\pi}|f(x,y) - f_2(x,y)|^p dxdy\bigg)^{1/p}(2\pi)^{2/q},\end{aligned}$$

in view of Hölder's inequality as in the proof of Theorem 2.1.13. Therefore

$$\begin{aligned}(2\pi)^2|\hat{f}(\mathbf{n})|^p &\leq \int_0^{2\pi}\bigg[\sum_{r=1}^{|k|}\int_{a_{r-1}}^{a_r}|f(x,y) - f(a_{r-1},y)|^p dx\bigg]dy \\ &\leq \int_0^{2\pi}\bigg[\sum_{r=1}^{|k|}(V_p(f(\cdot,y);[a_{r-1},a_r]))^p(a_r - a_{r-1})\bigg]dy \\ &\leq \frac{2\pi}{|k|}\int_0^{2\pi}(V_p(f(\cdot,y);[0,2\pi]))^p dy \\ &\leq \frac{2\pi}{|k|}\int_0^{2\pi} 2^p[(V_p(f;[0,2\pi]^2))^p + (V_p(f(\cdot,0);[0,2\pi]))^p]dy \\ &= \frac{(2\pi)^2 2^p[(V_p(f;[0,2\pi]^2))^p + (V_p(f(\cdot,0);[0,2\pi]))^p]}{|k|},\end{aligned}$$

in view of Lemma 2.1.8 (for a function of one variable) and Lemma 2.1.7. Thus we have

$$\hat{f}(\mathbf{n}) = \hat{f}(k,0) = O\left(\frac{1}{|k|^{1/p}}\right). \tag{2.5}$$

The case $k = 0$, $\ell \neq 0$, is similar to the above case and in this case we get

$$\hat{f}(0,\ell) = O\left(\frac{1}{|\ell|^{1/p}}\right). \tag{2.6}$$

This completes the proof. □

Remark 2.1.15. (2.4), (2.5) and (2.6) with $p = 1$ give the results of Móricz [33] and Fülöp and Móricz [10, for $n = 2$], except for the exact constant in their case. Thus, as far as the order of magnitude is concerned, our theorems generalize their results.

2.2 Order of magnitude of multiple trigonometric Fourier coefficients of lacunary series of functions of bounded p-variation

In Section 2.1, we have defined the notion of bounded p-variation ($p \geq 1$) for a function from a rectangle $[a_1, b_1] \times ... \times [a_m, b_m]$ to $\mathbb{C}$ and studied the order of magnitude of Fourier coefficients of such functions from $[0, 2\pi]^m$ to $\mathbb{C}$. J. R. Patadia [44, Theorem 3] studied the order of magnitude of Fourier coefficients of functions in $\mathrm{L}^1(\mathbb{T}^m)$ having lacunary Fourier series with certain gaps and which satisfy Lipschitz condition locally (that is, on certain smaller subsets of $[0, 2\pi]^m$). Here we study the order of magnitude of Fourier coefficients of functions in $\mathrm{L}^1(\mathbb{T}^m)$ having same type of lacunary Fourier series and are of bounded p-variation and prove result analogous to our earlier result (see Theorem 2.1.14). Results of this section are accepted for publication in the form of a paper in *Acta. Sci. Math. (Szeged)* [18].

Let $\mathbf{x_0} = (x_{01}, ..., x_{0m})$ denote an arbitrary point of $\mathbf{Q}$, δ any arbitrary real number such that $0 < \delta \leq \pi$ and $I = I(\mathbf{x_0}, \delta)$ denote the m-dimensional subrectangle of $\mathbf{Q}$ defined by (1.14).

Given a subset $E \subset \mathbb{Z}^m$, a function $f \in \mathrm{L}^1(\mathbb{T}^m)$ is said to be *E-spectral* (or, said to have spectrum E) if and only if $\hat{f}(\mathbf{n}) = 0$ for all $\mathbf{n}$ in $\mathbb{Z}^m \setminus E$. In what follows, we consider a set $E \subset \mathbb{Z}^m$ described in the following way: For each $j = 1, 2, ..., m$ consider sets $E^{(j)} = \{..., n_{-2}^{(j)}, n_{-1}^{(j)}, n_0^{(j)}, n_1^{(j)}, n_2^{(j)}, ...\} \subset \mathbb{Z}$ with $n_{-k}^{(j)} = -n_k^{(j)}$ for $k = 0, 1, 2, ...$ and with $\{n_k^{(j)}\}_{k=1}^{\infty}$ strictly increasing such that

$$\liminf_{k\to\infty} \frac{N_k^{(j)}}{\ln n_k^{(j)}} = B^{(j)} > \frac{8\pi e}{\delta}, \tag{2.7}$$

where $N_k^{(j)} = \min\left\{n_{k+1}^{(j)} - n_k^{(j)}, n_k^{(j)} - n_{k-1}^{(j)}\right\}$; and then put $E = \prod_{j=1}^m E^{(j)}$. $\mathbf{n_s} = (n_{s_1}^{(1)}, n_{s_2}^{(2)}, ..., n_{s_m}^{(m)})$ denotes the typical element of E. When $m = 1$, E will be taken to be $E^{(1)}$ with upper suffix in $n_k^{(1)}$'s and $N_k^{(1)}$'s omitted.

We need the following lemmas. Lemma 2.2.1 follows from a result due to Noble [36], or [3, Vol. II, p. 270] and Lemma 2.2.2 is its m-dimensional analogue by Patadia [44].

Lemma 2.2.1. *Let $0 < \delta \le \pi$. Then for sufficiently large n there exists a trigonometric polynomial $T_n(x)$ of degree $\le n$, with constant term 1, such that*

(a) $|T_n(x)| \le A_1\delta^{-1}$ *for all* $x \in [-\pi, \pi]$,

(b) $|T_n(x)| \le A_2 \exp(-n\delta/(8\pi e))$ *for all x such that* $\delta \le |x| \le \pi$,

where A_1 and A_2 are absolute constants.

Lemma 2.2.2. *Let $0 < \delta \le \pi$. Then for $\mathbf{n} = (n^{(1)}, ..., n^{(m)})$ such that each $n^{(j)}$ is sufficiently large, there exists a trigonometric polynomial*

$$T_{\mathbf{n}}(\mathbf{x}) = \sum_{\substack{|k^{(j)}| \le n^{(j)} \\ j=1,\dots,m}} c_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{x})},$$

with constant term 1, such that

(a) $|T_{\mathbf{n}}(\mathbf{x})| \le A_1\delta^{-m}$ *for all* $\mathbf{x} \in \mathbf{Q}$,

(b) $|T_{\mathbf{n}}(\mathbf{x})| \le A_2 \exp(-\delta(\mathbf{1}\cdot\mathbf{n})/(8\pi e))$ *for all* $\mathbf{x} \in \mathbf{Q} \setminus I(\mathbf{0}, \delta)$,

where $\mathbf{1} = (1, ..., 1)$ and A_1, A_2 are constants depending only on m.

The main theorem of this section is the following (in the proof C denotes a positive constant which may not have the same value at all places where it appear).

Theorem 2.2.3. *Let $E \subset \mathbb{Z}^m$ be described as above and $f : \mathbb{R}^m \to \mathbb{C}$ be 2π-periodic in each variable. If $f \in \mathrm{BV_H}^{(p)}(I)$ ($I = I(\mathbf{x_0}, \delta)$, $p \geq 1$), f is E-spectral and $\mathbf{n_k} = (n_{k_1}^{(1)}, ..., n_{k_m}^{(m)}) \in \mathbb{Z}^m$ is such that $|n_{k_j}^{(j)}|$ is sufficiently large for each j, then*

$$\hat{f}(\mathbf{n_k}) = O\left(\frac{1}{\left|\prod_{j=1}^{m} n_{k_j}^{(j)}\right|^{1/p}}\right).$$

Proof. We may assume without loss of generality that $\mathbf{x_0} = \mathbf{0}$. For, suppose the theorem is true when $\mathbf{x_0} = \mathbf{0}$ and consider the function $g(\mathbf{x}) = f(\mathbf{x} + \mathbf{x_0}) = (T_{\mathbf{x_0}} f)(\mathbf{x})$. Then

$$\mathbf{x} \in I(\mathbf{0}, \delta) \Leftrightarrow |x_j| \leq \delta \ \ \forall j \Leftrightarrow |x_i + x_{0j} - x_{0j}| \leq \delta \ \ \forall j \Leftrightarrow \mathbf{x} + \mathbf{x_0} \in I(\mathbf{x_0}, \delta).$$

Since $f \in \mathrm{BV_H}^{(p)}(I(\mathbf{x_0}, \delta))$, it follows that $g \in \mathrm{BV_H}^{(p)}(I(\mathbf{0}, \delta))$. Also,

$$g = T_{\mathbf{x_0}} f \Rightarrow \hat{g}(\mathbf{n}) = e^{i(\mathbf{n}\cdot\mathbf{x_0})} \hat{f}(\mathbf{n}) \ \ \forall \mathbf{n} \in \mathbb{Z}^m.$$

Since f is E-spectral, so is g and as the theorem is true when $\mathbf{x_0} = \mathbf{0}$, $\hat{g}(\mathbf{n_k}) = O(1/|\prod_{j=1}^{m} n_{k_j}^{(j)}|^{1/p})$. It follows now that $\hat{f}(\mathbf{n_k}) = O(1/|\prod_{j=1}^{m} n_{k_j}^{(j)}|^{1/p})$ in view of $|e^{i(\mathbf{n}\cdot\mathbf{x_0})}| = 1$.

For the sake of simplicity in writing, now onwards, we carry out the proof for $m = 2$, and we write (x, y) in place of (x_1, x_2). Since $f \in \mathrm{BV_H}^{(p)}([0, 2\pi]^2)$, in view of Lemma 2.1.11, the discontinuities of f lie on countable number of parallels to the axes and hence f is measurable over $\mathbb{T}^2$ in the sense of Lebesgue. Further, by Lemma 2.1.6, f is bounded over $[0, 2\pi]^2$ and hence $f \in \mathrm{L}^p(\mathbb{T}^2)$. As $\mathrm{BV_H}^{(p)}([0, 2\pi]^2) \subset \mathrm{BV_V}^{(p)}([0, 2\pi]^2)$, $f \in \mathrm{L}^p(\mathbb{T}^2) \cap \mathrm{BV_V}^{(p)}([0, 2\pi]^2)$.

For a given $\mathbf{n_k} = (n_{k_1}^{(1)}, n_{k_2}^{(2)})$, we take $\mathbf{M_k} = (M_{k_1}^{(1)}, M_{k_2}^{(2)})$, where for each $j = 1, 2$, $M_{k_j}^{(j)} = \min\{N_{k_j}^{(j)} - 1, |n_{k_j}^{(j)}|^{1/2}\}$. In view of the symmetry of the set $E^{(j)}$ and (2.7) we have

$$\liminf_{|k_j| \to \infty} \frac{N_{k_j}^{(j)} - 1}{\ln |n_{k_j}^{(j)}|} = \liminf_{|k_j| \to \infty} \frac{N_{k_j}^{(j)}}{\ln |n_{k_j}^{(j)}|} = B^{(j)} > \frac{8\pi e}{\delta},$$

for each $j = 1, 2$. Thus there is a positive integer K_0 such that $(N_{k_j}^{(j)} - 1)/(\ln |n_{k_j}^{(j)}|) > (8\pi e/\delta)$ for all $k_j \geq K_0$ and each $j = 1, 2$. Since

$$\lim_{k_j \to \infty} \frac{|n_{k_j}^{(j)}|^{1/2}}{\ln |n_{k_j}^{(j)}|} = \infty$$

for each j, there is a $K_1 \in \mathbb{N}$ such that $(|n_{k_j}^{(j)}|^{1/2})/(\ln |n_{k_j}^{(j)}|) > (8\pi e/\delta)$ for all $k_j \geq K_1$ and each $j = 1, 2$. Taking $K_2 = \max\{K_0, K_1\}$ we see that

$$M_{k_j}^{(j)} > \left(\frac{8\pi e}{\delta}\right) \ln |n_{k_j}^{(j)}| \tag{2.8}$$

for all $k_j \geq K_2$ and each $j = 1, 2$. Thus for $\mathbf{n_k}$ such that each $|n_{k_j}^{(j)}|$ is sufficiently large (2.8) holds.

Now consider the trigonometric polynomial $T_{\mathbf{M_k}}(\mathbf{x})$ satisfying conditions of Lemma 2.2.2 corresponding to this $\mathbf{M_k}$ and δ. Since f is E-spectral, the choice of $\mathbf{M_k}$ and $T_{\mathbf{M_k}}(\mathbf{x})$ gives us

$$\begin{aligned}
\hat{f}(\mathbf{n_k}) &= \frac{1}{(2\pi)^2} \int_{\mathbf{Q}} f(\mathbf{x}) T_{\mathbf{M_k}}(\mathbf{x}) e^{-i(\mathbf{n_k}\cdot\mathbf{x})} d\mathbf{x} \\
&= \frac{1}{(2\pi)^2} \left(\int_{I(\mathbf{0},\delta)} + \int_{\mathbf{Q}\setminus I(\mathbf{0},\delta)} \right) f(\mathbf{x}) T_{\mathbf{M_k}}(\mathbf{x}) e^{-i(\mathbf{n_k}\cdot\mathbf{x})} d\mathbf{x} \\
&= I_1 + I_2, \quad \text{say.}
\end{aligned} \tag{2.9}$$

Now

$$\begin{aligned}
|I_2| &= \frac{1}{(2\pi)^2} \left| \int_{\mathbf{Q}\setminus I(\mathbf{0},\delta)} f(\mathbf{x}) T_{\mathbf{M_k}}(\mathbf{x}) e^{-i(\mathbf{n_k}\cdot\mathbf{x})} d\mathbf{x} \right| \\
&\leq \frac{1}{(2\pi)^2} A_2 e^{(-\delta(\mathbf{1}\cdot\mathbf{M_k})/(8\pi e))} \int_{\mathbf{Q}\setminus I(\mathbf{0},\delta)} |f(\mathbf{x})| d\mathbf{x} \\
&\leq \frac{1}{(2\pi)^2} A_2 e^{(-\delta(\mathbf{1}\cdot\mathbf{M_k})/(8\pi e))} ||f||_1.
\end{aligned} \tag{2.10}$$

In view of (2.8), for each $j = 1, 2$, we have

$$-\frac{\delta}{8\pi e} \cdot M_{k_j}^{(j)} < -\frac{\delta}{8\pi e} \cdot \frac{8\pi e}{\delta} \cdot \ln |n_{k_j}^{(j)}| = -\ln |n_{k_j}^{(j)}|,$$

and therefore

$$e^{-\frac{\delta}{8\pi e}(\mathbf{1}\cdot\mathbf{M_k})} = e^{-\frac{\delta}{8\pi e}(M_{k_1}^{(1)} + M_{k_2}^{(2)})} < e^{-\ln |n_{k_1}^{(1)}|} e^{-\ln |n_{k_2}^{(2)}|} = \frac{1}{|n_{k_1}^{(1)} n_{k_2}^{(2)}|}.$$

Using this in (2.10) we get

$$I_2 = O\left(\frac{1}{|n_{k_1}^{(1)} n_{k_2}^{(2)}|}\right). \tag{2.11}$$

Now we estimate I_1. Again, for simplicity, we put $n_{k_1}^{(1)} = u$ and $n_{k_2}^{(2)} = v$. Then there are unique non-negative integers α and β such that

$$\alpha \frac{2\pi}{|u|} \leq \delta < (\alpha+1) \frac{2\pi}{|u|}; \quad \beta \frac{2\pi}{|v|} \leq \delta < (\beta+1) \frac{2\pi}{|v|}.$$

Therefore

$$0 \leq \delta - \alpha \frac{2\pi}{|u|} < \frac{2\pi}{|u|}; \quad 0 \leq \delta - \beta \frac{2\pi}{|v|} < \frac{2\pi}{|v|}. \tag{2.12}$$

Since $0 < \alpha \frac{2\pi}{|u|}, \beta \frac{2\pi}{|v|} \leq \delta$, say, $J = \left[-\alpha \frac{2\pi}{|u|}, \alpha \frac{2\pi}{|u|}\right] \times \left[-\beta \frac{2\pi}{|v|}, \beta \frac{2\pi}{|v|}\right] \subset I(\mathbf{0}, \delta)$. Therefore we can write I_1 as

$$\begin{aligned} I_1 &= \frac{1}{(2\pi)^2} \int_{I(\mathbf{0},\delta)} (fT_{\mathbf{M_k}})(\mathbf{x}) e^{-i(\mathbf{n_k}\cdot\mathbf{x})} d\mathbf{X} \\ &= \frac{1}{(2\pi)^2} \left(\int_J + \int_{I(\mathbf{0},\delta)\setminus J} \right) (fT_{\mathbf{M_k}})(\mathbf{x}) e^{-i(\mathbf{n_k}\cdot\mathbf{x})} d\mathbf{X} \\ &= I_{11} + I_{12}, \text{ say.} \end{aligned} \tag{2.13}$$

Next we estimate I_{11}. Note that e^{-iux} and e^{-ivy} are periodic functions of periods $\frac{2\pi}{|u|}$ and $\frac{2\pi}{|v|}$ respectively. Thus by putting

$$a_r = r \frac{2\pi}{|u|} \ \ (r = -\alpha, -\alpha+1, ..., \alpha); \quad b_s = s \frac{2\pi}{|v|} \ \ (s = -\beta, -\beta+1, ..., \beta)$$

we get

$$\int_{a_{r-1}}^{a_r} e^{-iux} dx = 0 \quad (r = -\alpha+1, -\alpha+2, ..., \alpha) \tag{2.14}$$

and

$$\int_{b_{s-1}}^{b_s} e^{-ivy} dy = 0 \quad (s = -\beta+1, -\beta+2, ..., \beta). \tag{2.15}$$

Define three functions f_1, f_2, f_3 on J by setting

$$f_1(x,y) = (fT_{\mathbf{M_k}})(x, b_{s-1}) \ \ (a_{-\alpha} \leq x < a_\alpha; \ b_{s-1} \leq y < b_s)$$

for $s = -\beta+1, -\beta+2, ..., \beta$;

$$f_2(x,y) = (fT_{\mathbf{M_k}})(a_{r-1}, y) \ \ (a_{r-1} \leq x < a_r; \ b_{-\beta} \leq y < b_\beta)$$

for $r = -\alpha+1, -\alpha+2, ..., \alpha$; and

$$f_3(x,y) = (fT_{\mathbf{M_k}})(a_{r-1}, b_{s-1}) \ \ (a_{r-1} \leq x < a_r; \ b_{s-1} \leq y < b_s)$$

for $r=-\alpha+1,-\alpha+2,...,\alpha;\ s=-\beta+1,-\beta+2,...,\beta$.

Since $f\in \mathrm{BV_H}^{(p)}(I(\mathbf{0},\delta)),\ J\subset I(\mathbf{0},\delta)$ and $T_{\mathbf{M_k}}$ is a trigonometric polynomial, each $f_i\in \mathrm{BV_H}^{(p)}(J)$ and hence $(fT_{\mathbf{M_k}}-f_1-f_2+f_3)\in \mathrm{BV_H}^{(p)}(J)\subset \mathrm{L}^p(J)$. Further in view of Fubini's theorem and the relations (2.14) and (2.15) we have

$$\begin{aligned}\int_J f_1(\mathbf{x})e^{-i(\mathbf{n_k}\cdot\mathbf{x})}d\mathbf{x} &= \int_{a_{-\alpha}}^{a_\alpha}\int_{b_{-\beta}}^{b_\beta} f_1(x,y)e^{-iux}e^{-ivy}dxdy\\ &= \int_{a_{-\alpha}}^{a_\alpha}\left[\sum_{s=-\beta+1}^{\beta}(fT_{\mathbf{M_k}})(x,b_{s-1})\int_{b_{s-1}}^{b_s}e^{-ivy}dy\right]e^{-iux}dx=0,\end{aligned}$$

$$\begin{aligned}\int_J f_2(\mathbf{x})e^{-i(\mathbf{n_k}\cdot\mathbf{x})}d\mathbf{x} &= \int_{a_{-\alpha}}^{a_\alpha}\int_{b_{-\beta}}^{b_\beta} f_2(x,y)e^{-iux}e^{-ivy}dxdy\\ &= \int_{b_{-\beta}}^{b_\beta}\left[\sum_{r=-\alpha+1}^{\alpha}(fT_{\mathbf{M_k}})(a_{r-1},y)\int_{a_{r-1}}^{a_r}e^{-iux}dx\right]e^{-ivy}dy=0\end{aligned}$$

and

$$\begin{aligned}&\int_J f_3(\mathbf{x})e^{-i(\mathbf{n_k}\cdot\mathbf{x})}d\mathbf{x} = \int_{a_{-\alpha}}^{a_\alpha}\int_{b_{-\beta}}^{b_\beta} f_3(x,y)e^{-iux}e^{-ivy}dxdy\\ &= \sum_{r=-\alpha+1}^{\alpha}\sum_{s=-\beta+1}^{\beta}(fT_{\mathbf{M_k}})(a_{r-1},b_{s-1})\left[\int_{a_{r-1}}^{a_r}e^{-iux}dx\right]\left[\int_{b_{s-1}}^{b_s}e^{-ivy}dy\right]=0.\end{aligned}$$

Using these equations in the expressions for I_{11} we get

$$\begin{aligned}(2\pi)^2|I_{11}| &= \left|\int_J (fT_{\mathbf{M_k}})(\mathbf{x})e^{-i(\mathbf{n_k}\cdot\mathbf{x})}d\mathbf{x}\right|\\ &= \left|\int_{a_{-\alpha}}^{a_\alpha}\int_{b_{-\beta}}^{b_\beta}(fT_{\mathbf{M_k}}-f_1-f_2+f_3)(x,y)e^{-iux}e^{-ivy}dxdy\right|\\ &\le \int_{a_{-\alpha}}^{a_\alpha}\int_{b_{-\beta}}^{b_\beta}|(fT_{\mathbf{M_k}}-f_1-f_2+f_3)(x,y)|dxdy\\ &\le \left(\int_{a_{-\alpha}}^{a_\alpha}\int_{b_{-\beta}}^{b_\beta}|(fT_{\mathbf{M_k}}-f_1-f_2+f_3)(x,y)|^p dxdy\right)^{1/p}(2a_\alpha\cdot 2b_\beta)^{1/q}\\ &\le \left(\int_{a_{-\alpha}}^{a_\alpha}\int_{b_{-\beta}}^{b_\beta}|(fT_{\mathbf{M_k}}-f_1-f_2+f_3)(x,y)|^p dxdy\right)^{1/p}\left(4\delta^2\right)^{1/q},\end{aligned}$$

in view of the Hölder's inequality (when $p>1$) since $(fT_{\mathbf{M_k}} - f_1 - f_2 + f_3) \in \mathrm{L}^p(J)$, where q is such that $1/p+1/q=1$. Observe that when $p=1$, we don't use Hölder's inequality and in that case we consider the inequality except last two steps.

In any case, it follows that

$$
\begin{aligned}
|I_{11}|^p &\le C \int_{a_{-\alpha}}^{a_\alpha} \int_{b_{-\beta}}^{b_\beta} |(fT_{\mathbf{M_k}} - f_1 - f_2 + f_3)(x,y)|^p dxdy \\
&= C \sum_{r=-\alpha+1}^{\alpha} \sum_{s=-\beta+1}^{\beta} \int_{a_{r-1}}^{a_r} \int_{b_{s-1}}^{b_s} |(fT_{\mathbf{M_k}} - f_1 - f_2 + f_3)(x,y)|^p dxdy \\
&= C \sum_{r=-\alpha+1}^{\alpha} \sum_{s=-\beta+1}^{\beta} \int_{a_{r-1}}^{a_r} \int_{b_{s-1}}^{b_s} |(fT_{\mathbf{M_k}})(x,y) - (fT_{\mathbf{M_k}})(x,b_{s-1}) \\
&\qquad\qquad - (fT_{\mathbf{M_k}})(a_{r-1},y) + (fT_{\mathbf{M_k}})(a_{r-1},b_{s-1})|^p dxdy \\
&\le C \sum_{r=-\alpha+1}^{\alpha} \sum_{s=-\beta+1}^{\beta} \left(V_p(fT_{\mathbf{M_k}}; [a_{r-1},a_r]\times[b_{s-1},b_s])\right)^p (a_r - a_{r-1})(b_s - b_{s-1}) \\
&\le C \frac{(2\pi)^2}{|uv|} (V_p(fT_{\mathbf{M_k}}; J))^p \\
&\le C \frac{(2\pi)^2}{|uv|} (V_p(fT_{\mathbf{M_k}}; I(\mathbf{0},\delta)))^p,
\end{aligned}
$$

in view of Lemma 2.1.8. Thus we get

$$I_{11} = O\left(\frac{1}{|uv|^{1/p}}\right). \tag{2.16}$$

Finally, we have

$$I_{12} = I_{121} + I_{122} + I_{123} + I_{124} + I_{125} + I_{126} + I_{127} + I_{128}, \tag{2.17}$$

where $I_{121}, ..., I_{128}$ are integrals of the function $(1/(2\pi)^2)\,(fT_{\mathbf{M_k}})(\mathbf{x})e^{-i(\mathbf{n_k}\cdot\mathbf{x})}$ over the rectangles $[-\delta, a_{-\alpha}]\times[-\delta, b_{-\beta}]$, $[-\delta, a_{-\alpha}]\times[b_\beta,\delta]$, $[a_\alpha,\delta]\times[-\delta,b_{-\beta}]$, $[a_\alpha,\delta]\times[b_\beta,\delta]$, $[a_{-\alpha},a_\alpha]\times[-\delta,b_{-\beta}]$, $[a_{-\alpha},a_\alpha]\times[b_\beta,\delta]$, $[-\delta,a_{-\alpha}]\times[b_{-\beta},b_\beta]$ and $[a_\alpha,\delta]\times[b_{-\beta},b_\beta]$ respectively.

Since $f \in \mathrm{BV_H}^{(p)}(I(\mathbf{0},\delta))$, it is bounded on $I(\mathbf{0},\delta)$ and as $T_{\mathbf{M_k}}$ is a trigonometric polynomial, there is a constant $M \ge 0$ such that $|(fT_{\mathbf{M_k}})(\mathbf{x})| \le M$ for all $\mathbf{x} \in I(\mathbf{0},\delta)$.

Therefore we have

$$\begin{aligned}|I_{121}| &\leq \frac{M}{(2\pi)^2}\int_{-\delta}^{a_{-\alpha}}\int_{-\delta}^{b_{-\beta}} dxdy \\ &= \frac{M}{(2\pi)^2}(a_{-\alpha}+\delta)(b_{-\beta}+\delta) \\ &\leq \frac{M}{(2\pi)^2}\left(\frac{2\pi}{|u|}\right)\left(\frac{2\pi}{|v|}\right),\end{aligned}$$

showing that $I_{121} = O\left(\frac{1}{|uv|}\right)$.

Similarly, we have $I_{122}, I_{123}, I_{124} = O\left(\frac{1}{|uv|}\right)$.

To estimate I_{125}, we define a function h on $[a_{-\alpha}, a_\alpha] \times [-\delta, b_{-\beta}] = J'$, say, by setting

$$h(x,y) = (fT_{\mathbf{M_k}})(a_{r-1}, y) \ \ (a_{r-1} \leq x < a_r;\ -\delta \leq y < b_{-\beta})$$

for $r = -\alpha+1, -\alpha+2, ..., \alpha$. Since $f \in \mathrm{BV_H}^{(p)}(I(\mathbf{0},\delta))$, $J' \subset I(\mathbf{0},\delta)$ and $T_{\mathbf{M_k}}$ is a trigonometric polynomial, $h \in \mathrm{BV_H}^{(p)}(J')$ and hence $(fT_{\mathbf{M_k}} - h) \in \mathrm{BV_H}^{(p)}(J') \subset \mathrm{L}^p(J')$. Further in view of Fubini's theorem and (2.14) we have

$$\begin{aligned}&\int_{a_{-\alpha}}^{a_\alpha}\int_{-\delta}^{b_{-\beta}} h(x,y)e^{-iux}e^{-ivy}dxdy \\ &\quad = \sum_{r=-\alpha+1}^{\alpha}\int_{a_{r-1}}^{a_r}\int_{-\delta}^{b_{-\beta}} h(x,y)e^{-iux}e^{-ivy}dxdy \\ &\quad = \sum_{r=-\alpha+1}^{\alpha}\int_{-\delta}^{b_{-\beta}}\left[(fT_{\mathbf{M_k}})(a_{r-1},y)\left\{\int_{a_{r-1}}^{a_r} e^{-iux}dx\right\}e^{-ivy}\right]dy = 0.\end{aligned}$$

Thus

$$\begin{aligned}(2\pi)^2|I_{125}| &= \left|\int_{a_{-\alpha}}^{a_\alpha}\int_{-\delta}^{b_{-\beta}}(fT_{\mathbf{M_k}})(x,y)e^{-iux}e^{-ivy}dxdy\right| \\ &= \left|\int_{a_{-\alpha}}^{a_\alpha}\int_{-\delta}^{b_{-\beta}}(fT_{\mathbf{M_k}} - h)(x,y)e^{-iux}e^{-ivy}dxdy\right| \\ &\leq \left(\int_{a_{-\alpha}}^{a_\alpha}\int_{-\delta}^{b_{-\beta}}|(fT_{\mathbf{M_k}} - h)(x,y)|^p dxdy\right)^{1/p}(2a_\alpha(b_{-\beta}+\delta))^{1/q} \\ &\leq \left(\int_{a_{-\alpha}}^{a_\alpha}\int_{-\delta}^{b_{-\beta}}|(fT_{\mathbf{M_k}} - h)(x,y)|^p dxdy\right)^{1/p}\left(2\delta^2\right)^{1/q}\end{aligned}$$

in view of Hölder's inequality as in the case of the estimate of I_{11}.

Therefore

$$
\begin{aligned}
|I_{125}|^p &\le C \int_{a_{-\alpha}}^{a_\alpha} \int_{-\delta}^{b_{-\beta}} |(fT_{\mathbf{M_k}} - h)(x,y)|^p dx dy \\
&= C \int_{-\delta}^{b_{-\beta}} \left[\sum_{r=-\alpha+1}^{\alpha} \int_{a_{r-1}}^{a_r} |(fT_{\mathbf{M_k}})(x,y) - (fT_{\mathbf{M_k}})(a_{r-1},y)|^p dx \right] dy \\
&\le C \int_{-\delta}^{b_{-\beta}} \left[\sum_{r=-\alpha+1}^{\alpha} (V_p((fT_{\mathbf{M_k}})(\cdot,y);[a_{r-1},a_r]))^p \, (a_r - a_{r-1}) \right] dy \\
&\le C \, \frac{2\pi}{|u|} \int_{-\delta}^{b_{-\beta}} (V_p((fT_{\mathbf{M_k}})(\cdot,y);[a_{-\alpha},a_\alpha]))^p \, dy \\
&\le C \, \frac{2\pi}{|u|} \int_{-\delta}^{b_{-\beta}} 2^p [(V_p((fT_{\mathbf{M_k}});[a_{-\alpha},a_\alpha] \times [-\delta, b_{-\beta}]))^p \\
&\qquad\qquad\qquad + (V_p((fT_{\mathbf{M_k}})(\cdot,-\delta);[a_{-\alpha},a_\alpha]))^p] dy \\
&= C \, \frac{1}{|u|} (b_{-\beta} + \delta) \le C \left(\frac{1}{|uv|} \right),
\end{aligned}
$$

in view of Lemma 2.1.8 (for a function of one variable) and Lemma 2.1.7. Thus we have

$$I_{125} = O\left(\frac{1}{|uv|^{1/p}} \right).$$

Similar arguments shows that

$$I_{126}, I_{127}, I_{128} = O\left(\frac{1}{|uv|^{1/p}} \right).$$

Using estimates of $I_{121}, ..., I_{128}$ in (2.17) and observing that $\frac{1}{|uv|} \le \frac{1}{|uv|^{1/p}}$ we obtain

$$I_{12} = O\left(\frac{1}{|uv|^{1/p}} \right). \tag{2.18}$$

The proof of theorem is now completed in view of (2.9), (2.11), (2.13), (2.16) and (2.18). □

Remark 2.2.4. This theorem gives lacunary analogue of our earlier result (see Theorem 2.1.14) for any $p \ge 1$ and hence that of the results of Móricz [33] and Fülöp and Móricz [10, for $n = 2$] for $p = 1$, except for the exact constant in their case.

Chapter 3

Walsh Fourier Coefficients

3.1 Order of magnitude of Walsh Fourier coefficients of functions of generalized bounded variation

It appears that while the study of the order of magnitude of the trigonometric Fourier coefficients for the functions of various classes of functions of generalized bounded variation such as $\mathrm{BV}^{(p)}$ $(p \geq 1)$ [77], $\phi\mathrm{BV}$ [78], $\Lambda\mathrm{BV}$ [76], $\Lambda\mathrm{BV}^{(p)}$ $(p \geq 1)$ [58], $\phi\Lambda\mathrm{BV}$ [56] etc. (refer also pages 2 to 6 for definitions of these classes) has been carried out, such a study for the Walsh Fourier coefficients has not yet been done. The only results available are due to N. J. Fine [11], who proved Theorems 1.1.35 and 1.1.36, where, in proving Theorem 1.1.36 he used second mean value theorem. In this section we carry out this study. Interestingly, here no use of the second mean value theorem is made. For the classes $\mathrm{BV}^{(p)}$ and $\phi\mathrm{BV}$, Taibleson-like technique [63] for Walsh Fourier coefficients is developed. However, for the classes $\Lambda\mathrm{BV}^{(p)}$ and $\phi\Lambda\mathrm{BV}$ this technique seems to be not working and hence classical technique [57] is applied. In the case of $\Lambda\mathrm{BV}$, it is also shown that the result is best possible in a certain sense. Results of this section are published in the form of a paper in [13] (see also MR2417326).

Theorem 3.1.1. *For $p \geq 1$, if $f \in \mathrm{BV}^{(p)}[0,1]$ then $\hat{f}(n) = O(1/(n^{\frac{1}{p}}))$.*

Proof. Let $n \in \mathbb{N}$. Let $k \in \mathbb{N} \cup \{0\}$ be such that $2^k \leq n < 2^{k+1}$ and put $a_i = (i/2^k)$ for $i = 0, 1, 2, 3, ..., 2^k$. Since φ_n takes the value 1 on one half of each of the intervals

$(a_{i-1},\ a_i)$ and the value -1 on the other half, we have

$$\int_{a_{i-1}}^{a_i} \varphi_n(x)dx = 0, \quad \text{for all} \ \ i = 1, 2, 3, ..., 2^k.$$

Define a step function g by $g(x) = f(a_{i-1})$ on $[a_{i-1},\ a_i)$, $i = 1, 2, 3, ..., 2^k$. Then

$$\int_0^1 g(x)\varphi_n(x)dx = \sum_{i=1}^{2^k} f(a_{i-1}) \int_{a_{i-1}}^{a_i} \varphi_n(x)dx = 0.$$

Therefore,

$$\begin{aligned} |\hat{f}(n)| &= \left| \int_0^1 [f(x) - g(x)]\varphi_n(x)dx \right| \\ &\leq \int_0^1 |f(x) - g(x)|dx \qquad (3.1) \\ &\leq ||f - g||_p ||1||_q \\ &= \left(\sum_{i=1}^{2^k} \int_{a_{i-1}}^{a_i} |f(x) - f(a_{i-1})|^p dx \right)^{\frac{1}{p}}, \end{aligned}$$

by Hölder's inequality as $f,\ g \in \mathrm{BV}^{(p)}[0, 1]$ and $\mathrm{BV}^{(p)}[0, 1] \subset \mathrm{L}^p[0, 1]$. Hence,

$$\begin{aligned} |\hat{f}(n)|^p &\leq \sum_{i=1}^{2^k} \int_{a_{i-1}}^{a_i} |f(x) - f(a_{i-1})|^p dx \\ &\leq \sum_{i=1}^{2^k} \int_{a_{i-1}}^{a_i} (V_p(f; [a_{i-1}, a_i]))^p dx \\ &= \sum_{i=1}^{2^k} (V_p(f; [a_{i-1}, a_i]))^p \left(\frac{1}{2^k}\right) \\ &\leq \left(\frac{1}{2^k}\right) (V_p(f; [0, 1]))^p \\ &\leq \left(\frac{2}{n}\right) (V_p(f; [0, 1]))^p, \end{aligned}$$

which completes the proof. □

Remark 3.1.2. Theorem 3.1.1 with $p = 1$ gives Theorem 1.1.36 of Fine [11, Theorem VI].

Theorem 3.1.3. *If $f \in \phi\mathrm{BV}[0,1]$ then $\hat{f}(n) = O(\phi^{-1}(1/n))$.*

Proof. Let $c > 0$. Using Jensen's inequality and proceeding as in Theorem 3.1.1, we get

$$\begin{aligned}
\phi\left(c\int_0^1 |f(x)-g(x)|dx\right) &\leq \int_0^1 \phi(c|f(x)-g(x)|)dx \\
&= \sum_{i=1}^{2^k}\int_{a_{i-1}}^{a_i} \phi(c|f(x)-f(a_{i-1})|)dx \\
&\leq \sum_{i=1}^{2^k}\int_{a_{i-1}}^{a_i} V_\phi(cf;[a_{i-1},a_i])dx \\
&= \sum_{i=1}^{2^k} V_\phi(cf;[a_{i-1},a_i])\left(\frac{1}{2^k}\right) \\
&\leq \left(\frac{2}{n}\right) V_\phi(cf;[0,1]).
\end{aligned}$$

Since ϕ is convex and $\phi(0)=0$, for sufficiently small $c \in (0,1)$, $V_\phi(cf;[0,1]) < 1/2$. This completes the proof in view of (3.1). □

Remark 3.1.4. If $\phi(x) = x^p,\ p \geq 1$, then the class $\phi\mathrm{BV}$ coincides with the class $\mathrm{BV}^{(p)}$ and Theorem 3.1.3 with Theorem 3.1.1.

Remark 3.1.5. Note that in the proof of Theorems 3.1.1 and 3.1.3, we have used the fact that if $a = a_0 < a_1 < ... < a_n = b$, then

$$\sum_{i=1}^{n} (V_p(f;[a_{i-1},a_i]))^p \leq (V_p(f;[a,b]))^p$$

and

$$\sum_{i=1}^{n} V_\phi(f;[a_{i-1},a_i]) \leq V_\phi(f;[a,b]),$$

for any $n \geq 2$ (see [35, 1.17, p. 15]). Such inequalities for functions of the class $\Lambda\mathrm{BV}^{(p)}$ $(p \geq 1)$ (resp., $\phi\Lambda\mathrm{BV}$), which contains $\mathrm{BV}^{(p)}$ (resp., $\phi\mathrm{BV}$) properly, do not hold true.

In fact, the following proposition shows that the validity of such inequality for the class $\Lambda\mathrm{BV}^{(p)}$ (resp., $\phi\Lambda\mathrm{BV}$) virtually reduces the class to $\mathrm{BV}^{(p)}$ (resp., $\phi\mathrm{BV}$) and hence we prove Theorem 3.1.9 and Theorem 3.1.10 applying a technique different from Taibleson-like technique [63] which we have applied in proving Theorem 3.1.1 and Theorem 3.1.3.

Proposition 3.1.6. *Let $f \in \phi\Lambda\mathrm{BV}[a,b]$. If there is a constant C such that*

$$\sum_{i=1}^{n} V_{\phi\Lambda}(f;[a_{i-1},a_i]) \le CV_{\phi\Lambda}(f;[a,b])),$$

for any sequence of points $\{a_i\}_{i=0}^{n}$ with $a = a_0 < a_1 < ... < a_n = b$, then $f \in \phi\mathrm{BV}[a,b]$.

Proof. For any partition $a = x_0 < x_1 < ... < x_n = b$ of $[a,b]$, we have

$$\begin{aligned}\sum_{i=1}^{n} \phi(|f(x_i) - f(x_{i-1})|) &= \lambda_1 \sum_{i=1}^{n} \frac{\phi(|f(x_i) - f(x_{i-1})|)}{\lambda_1} \\ &\le \lambda_1 \sum_{i=1}^{n} V_{\phi\Lambda}(f;[x_{i-1},x_i]) \\ &\le \lambda_1 CV_{\phi\Lambda}(f;[a,b]),\end{aligned}$$

which shows that $f \in \phi\mathrm{BV}[a,b]$. □

Remark 3.1.7. $\phi(x) = x^p$ $(p \ge 1)$ in this proposition will give analogous result for $\Lambda\mathrm{BV}^{(p)}$.

To prove Theorem 3.1.9 and Theorem 3.1.10, we need the following lemma.

Lemma 3.1.8. *For any $n \in \mathbb{N}$, $|\hat{f}(n)| \le \omega_p(1/n;f)$, where $\omega_p(\delta;f)$ $(\delta > 0,\ p \ge 1)$ denotes the integral modulus of continuity of order p of f given by*

$$\omega_p(\delta;f) = \sup_{|h|\le\delta} \left(\int_0^1 |f(x+h) - f(x)|^p dx \right)^{1/p}.$$

Proof. The inequality [11, Theorem IV, p. 382] $|\hat{f}(n)| \le \omega_1(1/n;f)$ and the fact that $\omega_1(1/n;f) \le \omega_p(1/n;f)$ for $p \ge 1$ immediately proves the lemma. □

Theorem 3.1.9. *If $1-$periodic $f \in \Lambda\mathrm{BV}^{(p)}[0,1]$ $(p \ge 1)$ then*

$$\hat{f}(n) = O\left(1 \Big/ \left(\sum_{j=1}^{n} \frac{1}{\lambda_j} \right)^{\frac{1}{p}} \right).$$

Proof. For any $n \in \mathbb{N}$, put $\theta_n = \sum_{j=1}^{n} 1/\lambda_j$. Let $f \in \Lambda\text{BV}^{(p)}[0,1]$. For $0 < h \le 1/n$, put $k = [1/h]$. Then for a given $x \in \mathbb{R}$ all the points $x + jh,\ j = 0, 1, ..., k$ lies in the interval $[x, x+1]$ of length 1 and

$$\int_0^1 |f(x) - f(x+h)|^p dx = \int_0^1 |f_j(x)|^p dx, \quad j = 1, 2, ..., k,$$

where $f_j(x) = f(x + (j-1)h) - f(x + jh)$, for all $j = 1, 2, ..., k$. Since the left hand side of this equation is independent of j, multiplying both sides by $1/(\lambda_j \theta_k)$ and summing over $j = 1, 2, ..., k$, we get

$$\begin{aligned}
\int_0^1 |f(x) - f(x+h)|^p dx &\le \left(\frac{1}{\theta_k}\right) \int_0^1 \sum_{j=1}^{k} \left(\frac{|f_j(x)|^p}{\lambda_j}\right) dx \\
&\le \frac{(V_{p\Lambda}(f; [0,1]))^p}{\theta_k} \\
&\le \frac{(V_{p\Lambda}(f; [0,1]))^p}{\theta_n},
\end{aligned}$$

because $\{\lambda_j\}$ is non-decreasing and $0 < h \le 1/n$. The case $-1/n \le h < 0$ is similar and using Lemma 3.1.8 we get

$$|\hat{f}(n)|^p \le (\omega_p(1/n; f))^p \le \frac{(V_{p\Lambda}(f; [0,1]))^p}{\theta_n}.$$

This completes the proof. $\square$

Theorem 3.1.10. *If* $1-$*periodic* $f \in \phi\Lambda\text{BV}[0,1]$ *then*

$$\hat{f}(n) = O\left(\phi^{-1}\left(1 \Big/ \left(\sum_{j=1}^{n} \frac{1}{\lambda_j}\right)\right)\right).$$

Proof. Let $f \in \phi\Lambda\text{BV}[0,1]$. Then for h, k and $f_j(x)$ as in the proof of Theorem 3.1.9 and for $c > 0$ by Jensen's inequality,

$$\begin{aligned}
\phi\left(c \int_0^1 |f(x) - f(x+h)|) dx\right) &\le \int_0^1 \phi(c|f(x) - f(x+h)|) dx \\
&= \int_0^1 \phi(c|f_j(x)|) dx, \quad j = 1, 2, ..., k.
\end{aligned}$$

Multiplying both sides by $1/(\lambda_j\theta_k)$ and summing over $j = 1, 2, ..., k$, we get

$$\begin{aligned}\phi\left(c\int_0^1 |f(x) - f(x+h)|)dx\right) &\leq \left(\frac{1}{\theta_k}\right)\int_0^1 \sum_{j=1}^{k}\left(\frac{\phi(c|f_j(x)|)}{\lambda_j}\right)dx \\ &\leq \frac{V_{\phi\Lambda}(cf;[0,1])}{\theta_k} \\ &\leq \frac{V_{,\phi\Lambda}(cf;[0,1])}{\theta_n}.\end{aligned}$$

Since ϕ is convex and $\phi(0) = 0$, $\phi(\alpha x) \leq \alpha\phi(x)$, for $0 < \alpha < 1$. So we may choose c sufficiently small so that $V_{\phi\Lambda}(cf;[0,1]) \leq 1$. But then we have

$$\int_0^1 |f(x) - f(x+h)|)dx \leq \frac{1}{c}\phi^{-1}\left(\frac{1}{\theta_n}\right).$$

Thus it follows in view of Lemma 3.1.8 that

$$|\hat{f}(n)| \leq \omega_1(1/n; f) \leq \frac{1}{c}\phi^{-1}\left(\frac{1}{\theta_n}\right),$$

which completes the proof. □

Following theorem shows that Theorem 3.1.9 with $p = 1$ is best possible in a certain sense.

Theorem 3.1.11. *If* $\Gamma\mathrm{BV}[0,1] \supseteq \Lambda\mathrm{BV}[0,1]$ *properly, then there exists* $f \in \Gamma\mathrm{BV}[0,1]$ *such that*

$$\hat{f}(n) \neq O\left(1\Big/\left(\sum_{j=1}^{n}\frac{1}{\lambda_j}\right)\right).$$

Proof. It is known [52] that if ΓBV contains ΛBV properly with $\Gamma = \{\gamma_n\}$ then $\theta_n \neq O(\rho_n)$, where in $\rho_n = \sum_{j=1}^{n}\frac{1}{\gamma_j}$ for each n. Also, if $c_0 = 0$, $c_{n+1} = 1$ and $c_1 < c_2 < ... < c_n$ denote all the n points of $(0,1)$ where the function φ_n changes its sign in $(0,1)$, $n_0 \in \mathbb{N}$ is such that $\rho_n \geq \frac{1}{2}$ for all $n \geq n_0$ and $E = \{n \in \mathbb{N} : n \geq n_0$ is even$\}$, then for each $n \in E$, for the function

$$f_n = \sum_{k=1}^{n+1}\frac{(-1)^{k-1}}{4\rho_n}\chi_{[c_{k-1},c_k)}$$

extended 1-periodically on $\mathbb{R}$,

$$V_\Gamma(f_n,[0,1]) = \sum_{k=1}^{n+1}\frac{|f_n(c_k) - f_n(c_{k-1})|}{\gamma_k} = \sum_{k=1}^{n}\frac{1}{\gamma_k}\cdot\frac{1}{2\rho_n} = \frac{1}{2}$$

because

$$f_n(c_{n+1}) = f_n(1) = f_n(0) = \frac{1}{4\rho_n} = f_n(c_n)$$

as $\varphi_n \equiv 1$ on $[c_0, c_1)$. Hence $||f_n|| = \frac{1}{4\rho_n} + \frac{1}{2} \leq 1$ for each $n \in E$ in the Banach space $\Gamma\text{BV}[0,1]$ with $||f|| = |f(0)| + V_\Gamma(f, [0,1])$. Observe that for $f \in \Gamma\text{BV}[0,1]$

$$||f||_1 \leq \int_0^1 \left(\frac{|f(x) - f(0)|}{\gamma_1} \gamma_1 + |f(0)| \right) dx \leq C||f||, \quad C = \max\{1, \gamma_1\},$$

and hence, for each $n \in \mathbb{N}$ the linear map $T_n : \Gamma\text{BV}[0,1] \to \mathbb{R}$ defined by $T_n(f) = \theta_n \hat{f}(n)$ is bounded as

$$|T_n(f)| = \theta_n |\hat{f}(n)| \leq \theta_n ||f||_1 \leq \theta_n C||f||, \ \forall f \in \Gamma\text{BV}[0,1].$$

Next, for each $n \in E$ since $f_n \cdot \varphi_n = \frac{1}{4\rho_n}$ on $[0,1)$, we see that

$$T_n(f_n) = \theta_n \hat{f}_n(n) = \theta_n \int_0^1 f_n(x)\varphi_n(x)dx = \frac{1}{4}\left(\frac{\theta_n}{\rho_n}\right) \neq O(1)$$

and hence

$$\sup\{||T_n|| : n \in \mathbb{N}\} \geq \sup\{||T_n|| : n \in E\} \geq \sup\{|T_n(f_n)| : n \in E\} = \infty.$$

Therefore, an application of the Banach-Steinhaus theorem gives an $f \in \Gamma\text{BV}[0,1]$ such that $\sup\{|T_n(f)| : n \in \mathbb{N}\} = \infty$. It follows that $\theta_n \hat{f}(n) = T_n(f) \neq O(1)$ and hence the theorem is proved. □

3.2 Order of magnitude of Walsh Fourier coefficients of series with small gaps for functions of generalized bounded variation

In Section 3.1 we have studied the order of magnitude of Walsh Fourier coefficients of the functions of various classes of generalized bounded variation. Here we continue this study further and obtain the analogous results for the lacunary Walsh Fourier series with small gaps. Interestingly, here also we use the technique which we have developed in Section 3.1 and prove the corresponding results. We also use the results for non-lacunary Walsh Fourier series to prove the results for lacunary Walsh Fourier series in an elegant way. The results of this section are published in [15].

Definition 3.2.1. Let $\{n_k\}_{k=1}^{\infty}$ be an increasing sequence of positive integers. A function $f \in \mathrm{L}^1[0,1]$ is said to have a lacunary Walsh Fourier series with *small gaps* if its Walsh Fourier coefficients $\hat{f}(n)$ vanish for $n \neq n_k,\ k \in \mathbb{N}$, where $\{n_k\}$ satisfies the small gap condition (1.11) or, in particular, more stringent small gap condition (1.5).

Theorem 3.2.2. *Let a 1-periodic $f \in \mathrm{L}^1[0,1]$ possess a lacunary Walsh Fourier series*

$$\sum_{k=1}^{\infty} \hat{f}(n_k)\varphi_{n_k}(x) \tag{3.2}$$

with small gaps (1.11) and $I = [0,\ 2^{-N})$ be an interval of length $|I| = 2^{-N} \geq 1/q$. Then $f \in \mathrm{BV}^{(p)}(I)$ $(p \geq 1)$ implies $\hat{f}(n_k) = O(1/(n_k)^{\frac{1}{p}})$.

Proof. Consider the polynomail $P_N(x)$ (this is essentially the same polynomial as considered by Patadia (see [43, p. 20]) defined as follows: If $N = 0$, put $P_N \equiv 1$ and if $N \in \mathbb{N}$ then put $P_N(x) = \prod_{k=0}^{N-1}(1 + r_k(x))$. Then

$$\begin{aligned} x \in I = [0, 2^{-N}) &\Rightarrow 1 + r_k(x) = 1 + \varphi_{2^k}(x) = 1 + 1 = 2,\ \forall k = 0, 1, ..., N-1 \\ &\Rightarrow P_N(x) = 2^N. \end{aligned}$$

On the other hand, if $x \in [0,1) \setminus I$ then exactly one of the following holds:

$$x \in [1/2, 1),\ x \in [1/2^2, 1/2), ..., x \in [1/2^N, 1/2^{N-1}).$$

Thus at least one of the following holds:

$$r_0(x) = -1,\ r_1(x) = -1, ..., r_{N-1}(x) = -1.$$

This shows that at least one of $1 + r_k(x)$ is 0 and hence $P_N(x) = 0$. We claim that if $k \in \mathbb{N}$ is such that $\hat{f}(n_k) \neq 0$ then

$$(fP_N)\hat{}(n_k) = \hat{f}(n_k) \quad (k = 1, 2, 3, ...). \tag{3.3}$$

Let $k \in \mathbb{N}$ be such that $\hat{f}(n_k) \neq 0$. Then

$$\begin{aligned} (fP_N)\hat{}(n_k) &= \int_0^1 f(x)P_N(x)\varphi_{n_k}(x)dx \\ &= \hat{f}(n_k) + \sum_{i=0}^{N-1} \hat{f}(r_i\varphi_{n_k}) + \sum_{i,j=0}^{N-1} \hat{f}(r_ir_j\varphi_{n_k}) + ... + \hat{f}(r_0r_1...r_{N-1}\varphi_{n_k}). \end{aligned} \tag{3.4}$$

By our assumption the first term in the right hand side of (3.4) is nonzero. The characters appearing in the other terms in right hand side of (3.4) are of the form $\varphi\varphi_{n_k}$ wherein φ is such that deg φ is positive and $\leq N$. In view of the Payley ordering of Walsh characters, for each $j \in \mathbb{N}$ there are totally 2^{j-1} characters of degree j, namely $\varphi_{2^{j-1}} \equiv r_{j-1}$, $\varphi_{2^{j-1}+1} \equiv r_{j-1}\varphi_1$, $\varphi_{2^{j-1}+2} \equiv r_{j-1}\varphi_2$, ..., $\varphi_{2^j-1} \equiv r_{j-1}\varphi_{2^{j-1}-1} \equiv r_{j-1}r_{j-2}...r_1r_0$. Consequently, total number of characters of positive degree $\leq N$ is given by $2^0+2^1+2^2+...+2^{N-1} = 2^N - 1$; they are from φ_1 to φ_{2^N-1}. It follows that when φ_{n_k} is multipied by any character of positive degree $\leq N$ the resulting character φ_m is such that

$$n_k < m \leq n_k + 2^N - 1 < n_k + 2^N \leq n_k + q \leq n_{k+1},$$

since the lacunary Walsh Fourier series (3.2) of f has gaps (1.11) with $q \geq 2^N$. Since $\hat{f}(n_k) \neq 0$, all the terms of the right hand side of (3.4) vanish except the first. This means that (3.3) holds.

Let k be large enough and $m \in \mathbb{N} \cup \{0\}$ be such that $\hat{f}(n_k) \neq 0$, $2^m \leq n_k < 2^{m+1}$ and $m > N$. Then

$$\hat{f}(n_k) = (fP_N)\hat{}(n_k) = 2^N \int_0^{1/2^N} f(x)\varphi_{n_k}(x)dx, \tag{3.5}$$

since $P_N = 2^N$ on $I = [0, 2^{-N})$ and $P_N = 0$ on $[0,1) \setminus I$. Put $a_i = (i/2^m)$ for $i = 0,1,2,3,...,2^m$. Then, since $2^m \leq n_k < 2^{m+1}$, φ_{n_k} takes the value 1 on one half of each of the intervals $(a_{i-1},\ a_i)$ and the value -1 on the other half. Therefore we have

$$\int_{a_{i-1}}^{a_i} \varphi_{n_k}(x)dx = 0, \quad \text{for all} \;\; i = 1,2,3,...,2^m. \tag{3.6}$$

Define a step function g by $g(x) = f(a_{i-1})$ on $[a_{i-1},\ a_i)$, $i = 1,2,3,...,2^{m-N}$. Then in view of (3.6) we have

$$\int_0^{1/2^N} g(x)\varphi_{n_k}(x)dx = \sum_{i=1}^{2^{m-N}} f(a_{i-1}) \int_{a_{i-1}}^{a_i} \varphi_{n_k}(x)dx = 0.$$

Thus in view of (3.5) we have

$$|\hat{f}(n_k)| = 2^N \left| \int_0^{1/2^N} [f(x) - g(x)]\varphi_{n_k}(x)dx \right| \leq 2^N \int_0^{1/2^N} |f(x) - g(x)|dx. \tag{3.7}$$

Therefore

$$|\hat{f}(n_k)| \le 2^N ||f-g||_{p,I} ||1||_{q,I} = 2^{N/p} \left(\sum_{i=1}^{2^{m-N}} \int_{a_{i-1}}^{a_i} |f(x) - f(a_{i-1})|^p dx \right)^{\frac{1}{p}},$$

by Hölder's inequality as f, $g \in \mathrm{BV}^{(p)}(I)$ and $\mathrm{BV}^{(p)}(I) \subset \mathrm{L}^p(I)$. Hence,

$$\begin{aligned}
|\hat{f}(n_k)|^p &\le 2^N \sum_{i=1}^{2^{m-N}} \int_{a_{i-1}}^{a_i} |f(x) - f(a_{i-1})|^p dx \\
&\le 2^N \sum_{i=1}^{2^{m-N}} \int_{a_{i-1}}^{a_i} (V_p(f;[a_{i-1},a_i]))^p dx \\
&= 2^N \sum_{i=1}^{2^{m-N}} (V_p(f;[a_{i-1},a_i]))^p \left(\frac{1}{2^m}\right) \\
&\le \left(\frac{2^N}{2^m}\right) (V_p(f;I))^p \\
&\le \left(\frac{2 \cdot 2^N}{n_k}\right) (V_p(f;I))^p,
\end{aligned}$$

which completes the proof. □

Theorem 3.2.3. *Let f and I be as in Theorem 3.2.2. Then $f \in \phi\mathrm{BV}(I)$ implies $\hat{f}(n_k) = O(\phi^{-1}(1/n_k))$.*

Proof. Proceeding as in the proof of Theorem 3.2.2 we get (3.7). For $c > 0$ using Jensen's inequality, we have

$$\begin{aligned}
\phi\left(2^N c \int_0^{1/2^N} |f(x) - g(x)| dx\right) &\le 2^N \int_0^{1/2^N} \phi(c|f(x) - g(x)|) dx \\
&= 2^N \sum_{i=1}^{2^{m-N}} \int_{a_{i-1}}^{a_i} \phi(c|f(x) - f(a_{i-1})|) dx \\
&\le 2^N \sum_{i=1}^{2^{m-N}} \int_{a_{i-1}}^{a_i} V_\phi(cf;[a_{i-1},a_i]) dx \\
&= 2^N \sum_{i=1}^{2^{m-N}} V_\phi(cf;[a_{i-1},a_i]) \left(\frac{1}{2^m}\right) \\
&\le 2^N \left(\frac{2}{n_k}\right) V_\phi(cf;I).
\end{aligned}$$

Since ϕ is convex and $\phi(0) = 0$, for sufficiently small $c \in (0, 1)$, $V_\phi(cf; I) \le 1/2^{N+1}$. This completes the proof in view of (3.7). □

Remark 3.2.4. If $\phi(x) = x^p$, $p \ge 1$, then the class $\phi\mathrm{BV}(I)$ coincides with the class $\mathrm{BV}^{(p)}(I)$ and Theorem 3.2.3 with Theorem 3.2.2.

Theorem 3.2.5. *Let f and I be as in Theorem 3.2.2. Then $f \in \Lambda\mathrm{BV}^{(p)}(I)$ $(p \ge 1)$ implies*

$$\hat{f}(n_k) = O\left(1 \Big/ \left(\sum_{j=1}^{n_k} \frac{1}{\lambda_j}\right)^{\frac{1}{p}}\right).$$

Proof. Proceeding as in the proof of Theorem 3.2.2 we get (3.6). Define h on $[0, 1)$ by $h = f$ on $I = [0, 1/2^N)$, $h(x) = f(1/2^N)$, $\forall x \in [1/2^N, 1)$; and extend h, 1-periodically on $\mathbb{R}$. We claim that $h \in \Lambda\mathrm{BV}^{(p)}[0, 1]$. Let $\{I_n\}$ be a sequence of non-overlapping intervals in $[0, 1]$ and consider the sum $S = \sum_n |h(I_n)|^p/\lambda_n$ where $h(I_n) = h(b_n) - h(a_n)$ if $I_n = [a_n, b_n]$. If $I_n \subset [1/2^N, 1)$ then by definition of h, $h(I_n) = 0$. Thus, $S = \sum_k |h(I_{n_k})|^p/\lambda_{n_k}$, where no I_{n_k} is contained in $[1/2^N, 1)$. Since the sequence $\{I_n\}$ is non-overlapping, there can be at most one interval, say, I_{n_j} which intersects $(1/2^N, 1)$. If $I_{n_j} = [a, b]$, let $I'_{n_j} = I_{n_j} \cap [0, 1/2^N]$, $I''_{n_j} = I_{n_j} \cap [1/2^N, b]$. Then again by the definition of h, $h(I''_{n_j}) = 0$ and hence $h(I_{n_j}) = h(I'_{n_j}) + h(I''_{n_j}) = h(I'_{n_j})$. Thus

$$S = \sum_{k,\, k \ne j} \frac{|h(I_{n_k})|^p}{\lambda_{n_k}} + \frac{|h(I'_{n_j})|^p}{\lambda_{n_j}}.$$

Also, in $\{I_n\}$, there can be at most one interval, say, I_{n_t} of the form $[c, 1]$ where $c \in (1/2^N, 1]$. But then

$$\begin{aligned} S &= \sum_{\substack{k \\ k \ne j,\, k \ne t}} \frac{|h(I_{n_k})|^p}{\lambda_{n_k}} + \frac{|h(I'_{n_j})|^p}{\lambda_{n_j}} + \frac{|h(1) - h(c)|^p}{\lambda_{n_t}} \\ &= \sum_{\substack{k \\ k \ne j,\, k \ne t}} \frac{|f(I_{n_k})|^p}{\lambda_{n_k}} + \frac{|f(I'_{n_j})|^p}{\lambda_{n_j}} + \frac{|f(0) - f(1/2^N)|^p}{\lambda_{n_t}} \\ &\le V_{p\Lambda}(f; I) + \frac{|f(0) - f(1/2^N)|^p}{\lambda_1}. \end{aligned}$$

It follows that $V_{p\Lambda}(h; [0, 1]) \le V_{p\Lambda}(f; I) + (|f(0) - f(1/2^N)|^p/\lambda_1)$. Since $f \in \Lambda\mathrm{BV}^{(p)}(I)$, we have $h \in \Lambda\mathrm{BV}^{(p)}[0, 1]$ and hence by Theorem 3.1.9

$$\hat{h}(n) = O(1/(\theta_n)^{\frac{1}{p}}), \tag{3.8}$$

where $\theta_n = \sum_{j=1}^{n} 1/\lambda_j,\ \forall n \in \mathbb{N}$. But

$$\begin{aligned}
\hat{h}(n_k) &= \int_0^1 h(x)\varphi_{n_k}(x)dx \\
&= \int_0^{1/2^N} f(x)\varphi_{n_k}(x)dx + f(1/2^N)\int_{1/2^N}^1 \varphi_{n_k}(x)dx \\
&= \int_0^{1/2^N} f(x)\varphi_{n_k}(x)dx + f(1/2^N)\sum_{i=2^{m-N}+1}^{2^m}\int_{a_{i-1}}^{a_i} \varphi_{n_k}(x)dx \\
&= \int_0^{1/2^N} f(x)\varphi_{n_k}(x)dx,
\end{aligned}$$

in view of (3.6). Thus by (3.5) and (3.8), $\hat{f}(n_k) = 2^N\hat{h}(n_k) = O(1/(\theta_{n_k})^{\frac{1}{p}})$ and hence the theorem is proved. □

Theorem 3.2.6. *Let f and I be as in Theorem 3.2.2. Then $f \in \phi\Lambda\mathrm{BV}(I)$ implies*

$$\hat{f}(n_k) = O\biggl(\phi^{-1}\Bigl(1\Big/\Bigl(\sum_{j=1}^{n_k}\frac{1}{\lambda_j}\Bigr)\Bigr)\biggr).$$

Proof. Proceeding as in the proof of Theorem 3.2.2 we get (3.6). Now if $f \in \phi\Lambda\mathrm{BV}(I)$, we can see, in a similar way, that the function h considered in the proof of Theorem 3.2.5 is in $\phi\Lambda\mathrm{BV}[0,1]$. Thus by Theorem 3.1.10, $\hat{h}(n) = O(\phi^{-1}(1/\theta_n))$ and hence in view of (3.5),

$$\hat{f}(n_k) = 2^N\hat{h}(n_k) = O(\phi^{-1}(1/\theta_{n_k})).$$

This completes the proof. □

3.3 Order of magnitude of multiple Walsh Fourier coefficients of functions of bounded p-variation

In 1949, N. J. Fine [11] proved using the second mean value theorem that if f is of bounded variation on $[0,1]$ and if $\hat{f}(n)$ denotes its (one dimensional) Walsh Fourier coefficient, then $\hat{f}(n) = O(\frac{1}{n})$, for all $n \neq 0$. In Section 3.1 we have studied the order of magnitude of Walsh Fourier coefficients of functions of various classes of generalized bounded variation and extended the result of Fine to these classes.

Further in Section 2.1, we have defined the notion of bounded p-variation ($p \geq 1$) for a complex-valued function on the rectangle $[a_1, b_1] \times ... \times [a_m, b_m]$ (m being a positive integer) and studied the order of magnitude of trigonometric Fourier coefficients of such functions on $[0, 2\pi]^m$. Here we study the order of magnitude of Walsh Fourier coefficients for a function of bounded p-variation from $[0, 1]^m$ to $\mathbb{C}$ and obtain analogous results. For $m = 1$, our new results give our earlier result (see Theorem 3.1.1). Also, for $p = 1$, our results give the Walsh analogue of the results of Móricz [33] and Fülöp and Móricz [10] (see Theorems 1.1.32, 1.1.33 and 1.1.34), except possibly for the exact constant in their case. Multiple Walsh Fourier coefficient is defined as follows (refer, for example, [12]).

Definition 3.3.1. For a periodic $f = f(x_1, ..., x_m)$ with period 1 in each variable and Lebesgue integrable over the m-dimensional torus $\mathbb{I}^m := [0, 1)^m$, in symbol $f \in \mathrm{L}^1(\mathbb{I}^m)$, its formal Walsh Fourier series is given by

$$f(x_1, ..., x_m) \sim \sum_{(n^{(1)},...,n^{(m)}) \in (Z^+)^m} \hat{f}(n^{(1)}, ..., n^{(m)}) w_{n^{(1)}}(x_1)...w_{n^{(m)}}(x_m)$$

where $\hat{f}(n^{(1)}, ..., n^{(m)}) \equiv \hat{f}(\mathbf{n})$ is the $\mathbf{n}^{th}$ multiple Walsh Fourier coefficient of f defined by

$$\hat{f}(\mathbf{n}) = \int_{\mathbb{I}^m} f(x_1, ..., x_m) w_{n^{(1)}}(x_1)...w_{n^{(m)}}(x_m) dx_1...dx_m. \tag{3.9}$$

Theorem 3.3.2. *Let $f : \mathbb{R}^m \to \mathbb{C}$ be 1-periodic in each variable. If f belongs to $\mathrm{BV_V}^{(p)}([0, 1]^m) \cap \mathrm{L}^p(\mathbb{I}^m)$ ($p \geq 1$) and $\mathbf{n} = (n^{(1)}, ..., n^{(m)}) \in \mathbb{N}^m$, then*

$$\hat{f}(\mathbf{n}) = O\left(\frac{1}{\left(\prod_{j=1}^m n^{(j)}\right)^{1/p}}\right).$$

Proof. For the sake of simplicity in writing, we carry out the proof for $m = 2$, and we write (x, y) and (k, ℓ) in place of (x_1, x_2) and $(n^{(1)}, n^{(2)})$ respectively.

Let $\mathbf{n} = (k, \ell) \in \mathbb{N}^2$. Let $s, t \in \mathbb{Z}^+$ be such that $2^s \leq k < 2^{s+1}$ and $2^t \leq \ell < 2^{t+1}$. For each $i = 0, 1, 2, 3, ..., 2^s$ and $j = 0, 1, 2, 3, ..., 2^t$ put $a_i = (i/2^s)$, $b_j = (j/2^t)$. Then by definition of Walsh functions, φ_k takes the value 1 on one half of each of the intervals (a_{i-1}, a_i) and the value -1 on the other half, and hence

$$\int_{a_{i-1}}^{a_i} \varphi_k(x) dx = 0, \qquad (i = 1, 2, 3, ..., 2^s). \tag{3.10}$$

Similarly, the function φ_ℓ takes the value 1 on one half of each of the intervals $(b_{j-1},\ b_j)$ and the value -1 on the other half, and hence

$$\int_{b_{j-1}}^{b_j} \varphi_\ell(y)dy = 0, \qquad (j = 1, 2, 3, ..., 2^t). \tag{3.11}$$

Define three functions f_1, f_2, f_3 on $\mathbb{I}^2$ by setting

$$f_1(x,y) = f(a_{i-1}, y) \ \ (a_{i-1} \le x < a_i;\ 0 \le y < 1) \text{ for } i = 1, 2, 3, ..., 2^s;$$
$$f_2(x,y) = f(x, b_{j-1}) \ \ (0 \le x < 1;\ b_{j-1} \le y < b_j) \text{ for } j = 1, 2, 3, ..., 2^t;$$

and

$$f_3(x,y) = f(a_{i-1}, b_{j-1}) \ \ (a_{i-1} \le x < a_i;\ b_{j-1} \le y < b_j)$$

for $i = 1, 2, 3, ..., 2^s;\ j = 1, 2, 3, ..., 2^t$. Then in view of Fubini's theorem and relations (3.10) and (3.11) we have

$$\int_0^1 \int_0^1 f_1(x,y)\varphi_k(x)\varphi_\ell(y)dxdy = \int_0^1 \left[\sum_{i=1}^{2^s} f(a_{i-1}, y) \int_{a_{i-1}}^{a_i} \varphi_k(x)dx \right] \varphi_\ell(y)dy = 0,$$

$$\int_0^1 \int_0^1 f_2(x,y)\varphi_k(x)\varphi_\ell(y)dxdy = \int_0^1 \left[\sum_{j=1}^{2^t} f(x, b_{j-1}) \int_{b_{j-1}}^{b_j} \varphi_\ell(y)dy \right] \varphi_k(x)dx = 0$$

and

$$\int_0^1 \int_0^1 f_3(x,y)\varphi_k(x)\varphi_\ell(y)dxdy$$
$$= \sum_{i=1}^{2^s} \sum_{j=1}^{2^t} f(a_{i-1}, b_{j-1}) \Big[\int_{a_{i-1}}^{a_i} \varphi_k(x)dx \Big] \left[\int_{b_{j-1}}^{b_j} \varphi_\ell(y)dy \right] = 0.$$

Using these equations in the definition of $\hat{f}(\mathbf{n})$ (see (3.9)) we get

$$\begin{aligned}
|\hat{f}(\mathbf{n})| &= \left| \int_0^1 \int_0^1 f(x,y)\varphi_k(x)\varphi_\ell(y)dxdy \right| \\
&= \left| \int_0^1 \int_0^1 [f(x,y) - f_1(x,y) - f_2(x,y) + f_3(x,y)]\varphi_k(x)\varphi_\ell(y)dxdy \right| \\
&\le \int_0^1 \int_0^1 |f(x,y) - f_1(x,y) - f_2(x,y) + f_3(x,y)|dxdy \\
&\le \left(\int_0^1 \int_0^1 |f(x,y) - f_1(x,y) - f_2(x,y) + f_3(x,y)|^p dxdy \right)^{1/p} (1)^{2/q},
\end{aligned}$$

in view of the Hölder's inequality (when $p > 1$) since $f - f_1 - f_2 + f_3 \in \mathrm{L}^p(\mathbb{I}^2)$, where q is such that $1/p + 1/q = 1$. Observe that when $p = 1$, we don't use Hölder's inequality and in that case we consider the inequality except last step. In any case, it follows that

$$\begin{aligned}
|\hat{f}(\mathbf{n})|^p &\leq \int_0^1 \int_0^1 |f(x,y) - f_1(x,y) - f_2(x,y) + f_3(x,y)|^p dxdy \\
&= \sum_{i=1}^{2^s} \sum_{j=1}^{2^t} \int_{a_{i-1}}^{a_i} \int_{b_{j-1}}^{b_j} |f(x,y) - f_1(x,y) - f_2(x,y) + f_3(x,y)|^p dxdy \\
&= \sum_{i=1}^{2^s} \sum_{j=1}^{2^t} \int_{a_{i-1}}^{a_i} \int_{b_{j-1}}^{b_j} |f(x,y) - f(a_{i-1},y) - f(x,b_{j-1}) + f(a_{i-1},b_{j-1})|^p dxdy \\
&\leq \sum_{i=1}^{2^s} \sum_{j=1}^{2^t} (V_p(f;[a_{i-1},a_i] \times [b_{j-1},b_j]))^p (a_i - a_{i-1})(b_j - b_{j-1}) \\
&\leq \frac{1}{2^s 2^t} (V_p(f;[0,1]^2))^p \leq \frac{2^2}{k\ell} (V_p(f;[0,1]^2))^p,
\end{aligned}$$

in view of Lemma 2.1.8. Thus we get

$$|\hat{f}(\mathbf{n})| \leq \frac{4^{1/p} \cdot V_p(f;[0,1]^2)}{(k\ell)^{1/p}}. \tag{3.12}$$

This completes the proof. □

Theorem 3.3.3. *Let $f : \mathbb{R}^m \to \mathbb{C}$ be 1-periodic in each variable. If f belongs to $\mathrm{BV_H}^{(p)}([0,1]^m)$ $(p \geq 1)$ then for any $\mathbf{0} \neq \mathbf{n} = (n^{(1)}, ..., n^{(m)}) \in (\mathbb{Z}^+)^m$,*

$$\hat{f}(\mathbf{n}) = O\left(\frac{1}{\left(\prod_{j=1, n^{(j)} \neq 0}^{m} n^{(j)} \right)^{1/p}} \right).$$

Proof. Here also we will carry out the proof for $m = 2$ and use notations as in the proof of Theorem 3.3.2. Since $f \in \mathrm{BV_H}^{(p)}([0,1]^2)$, in view of Lemma 2.1.11 (use Lemma 2.1.12 for general case), the discontinuities of f lie on countable number of parallels to the axes and hence f is measurable over $\mathbb{I}^2$ in the sense of Lebesgue. Further, by Lemma 2.1.6, f is bounded over $[0,1]^2$ and hence $f \in \mathrm{L}^p(\mathbb{I}^2)$. As $\mathrm{BV_H}^{(p)}([0,1]^2) \subset \mathrm{BV_V}^{(p)}([0,1]^2)$, $f \in \mathrm{L}^p(\mathbb{I}^2) \cap \mathrm{BV_V}^{(p)}([0,1]^2)$. Therefore if $\mathbf{n} = (k,\ell) \in \mathbb{N}^2$, by Theorem 3.3.2,

$$\hat{f}(\mathbf{n}) = O\left(\frac{1}{(k\ell)^{1/p}} \right).$$

Next, let $\mathbf{n} = (k, \ell) \in (\mathbb{Z}^+)^2$ be such that $k \neq 0$, $\ell = 0$ and let a_i's and f_1 be as defined in the proof of Theorem 3.3.2. Then we have

$$\int_0^1 \int_0^1 f_1(x,y)\varphi_k(x)dxdy = \int_0^1 \left(\sum_{i=1}^{2^s} f(a_{i-1}, y) \left[\int_{a_{i-1}}^{a_i} \varphi_k(x)dx \right] \right) dy = 0,$$

in view of Fubini's theorem and (3.10); and,

$$\begin{aligned} |\hat{f}(\mathbf{n})| &= \left| \int_0^1 \int_0^1 [f(x,y) - f_1(x,y)]\varphi_k(x)dxdy \right| \\ &\leq \left(\int_0^1 \int_0^1 |f(x,y) - f_1(x,y)|^p dxdy \right)^{1/p} (1)^{2/q}, \end{aligned}$$

in view of Hölder's inequality as in the proof of Theorem 3.3.2. Therefore

$$\begin{aligned} |\hat{f}(\mathbf{n})|^p &\leq \int_0^1 \left[\sum_{i=1}^{2^s} \int_{a_{i-1}}^{a_i} |f(x,y) - f(a_{i-1}, y)|^p dx \right] dy \\ &\leq \int_0^1 \left[\sum_{i=1}^{2^s} (V_p(f(\cdot, y); [a_{i-1}, a_i]))^p (a_i - a_{i-1}) \right] dy \\ &\leq \frac{1}{2^s} \int_0^1 (V_p(f(\cdot, y); [0,1]))^p dy \\ &\leq \frac{2}{k} \int_0^1 2^p [(V_p(f; [0,1]^2))^p + (V_p(f(\cdot, 0); [0,1]))^p] dy \\ &= \frac{2^{p+1}[(V_p(f; [0,1]^2))^p + (V_p(f(\cdot, 0); [0,1]))^p]}{k}, \end{aligned}$$

in view of Lemma 2.1.8 (for a function of one variable) and Lemma 2.1.7. Thus we have

$$\hat{f}(\mathbf{n}) = \hat{f}(k, 0) = O\left(\frac{1}{k^{1/p}} \right). \tag{3.13}$$

The case $k = 0$, $\ell \neq 0$, is similar to the above case and in this case we get

$$\hat{f}(0, \ell) = O\left(\frac{1}{\ell^{1/p}} \right). \tag{3.14}$$

This completes the proof. $\square$

Remark 3.3.4. Theorem 3.3.2 or Theorem 3.3.3 with $m = 1$ gives our earlier result (see Theorem 3.1.1). (3.12), (3.13) and (3.14) with $p = 1$ give Walsh analogues of the results of Móricz [33] (see Theorem 1.1.32) and Fülöp and Móricz [10, for $n = 2$] (see Theorem 1.1.34), except possibly for the exact constant in their case.

3.4 Order of magnitude of multiple Walsh Fourier coefficients of series with small gaps for functions of bounded p-variation

In 1949, N. J. Fine [11] proved using the second mean value theorem that if f is of bounded variation on $[0, 1]$ and if $\hat{f}(n)$ denotes its (one dimensional) Walsh Fourier coefficient, then $\hat{f}(n) = O(\frac{1}{n})$, for all $n \neq 0$. In Section 3.1 we have studied the order of magnitude of Walsh Fourier coefficients of functions of various classes of generalized bounded variation and extended the result of Fine to these classes. The small gap analogue of results of Section 3.1 is given in Section 3.2. Further in Section 2.1, we have defined the notion of bounded p-variation ($p \geq 1$) for a function from a rectangle $[a_1, b_1] \times ... \times [a_m, b_m]$ to $\mathbb{C}$ and studied the order of magnitude of trigonometric Fourier coefficients of such functions from $[0, 2\pi]^m$ to $\mathbb{C}$. We have also studied the order of magnitude of trigonometric Fourier coefficients of functions from $[0, 2\pi]^m$ to $\mathbb{C}$ having lacunary Fourier series with certain gaps and are of bounded p-variation only locally in Section 2.2. In Section 3.3 we have studied the order of magnitude of Walsh Fourier coefficients for a function of bounded p-variation from $[0, 1]^m$ to $\mathbb{C}$ having non-lacunary Fourier series. Here we study the order of magnitude of Walsh Fourier coefficients of functions from $[0, 1]^m$ to $\mathbb{C}$ which are of bounded p-variation locally and having lacunary Walsh Fourier series having small gaps. Our new result generalizes and gives lacunary analogue of our earlier result (Theorem 3.3.2). For $m = 1$, our new result give our earlier result (Theorem 3.1.1).

Given a subset $E \subset (\mathbb{Z}^+)^m$, a function $f \in \mathrm{L}^1(\mathbb{I}^m)$ is said to be *E-spectral* (or, said to have spectrum E) if and only if $\hat{f}(\mathbf{n}) = 0$ for all $\mathbf{n}$ in $(\mathbb{Z}^+)^m \setminus E$. In what follows, we consider a set $E \subset (\mathbb{Z}^+)^m$ described in the following way: For each $j = 1, 2, ..., m$ consider sets $E^{(j)} = \{n_0^{(j)}, n_1^{(j)}, n_2^{(j)}, ...\} \subset \mathbb{Z}^+$ with $\{n_k^{(j)}\}_{k=1}^{\infty}$ strictly increasing for each j and satisfying the small gap conditions

$$(n_{k+1}^{(j)} - n_k^{(j)}) \geq q \geq 1, \quad (k = 1, 2, ...;\ j = 1, 2, ..., m); \tag{3.15}$$

and then put $E = \prod_{j=1}^m E^{(j)}$. $\mathbf{n_s} = (n_{s_1}^{(1)}, n_{s_2}^{(2)}, ..., n_{s_m}^{(m)})$ denotes the typical element of E. When $m = 1$, E will be taken to be $E^{(1)}$ with the superscript in $n_k^{(1)}$'s omitted.

Theorem 3.4.1. *Let $E \subset (\mathbb{Z}^+)^m$ be described as above and $f : \mathbb{R}^m \to \mathbb{C}$ be 1-periodic in each variable. If $f \in \mathrm{BV_V}^{(p)}(I) \cap \mathrm{L}^p(I)$ $(p \geq 1)$, where I is the rectangle $I = [0, 2^{-N_1}] \times ... \times [0, 2^{-N_m}]$ in which $2^{-N_j} \geq 1/q$ for each j; f is E-spectral and $\mathbf{n_k} = (n_{k_1}^{(1)}, ..., n_{k_m}^{(m)}) \in (\mathbb{Z}^+)^m$ is such that $n_{k_j}^{(j)}$ is sufficiently large for each j, then*

$$\hat{f}(\mathbf{n_k}) = O\left(\frac{1}{|\prod_{j=1}^m n_{k_j}^{(j)}|^{1/p}}\right).$$

Proof. For the sake of simplicity in writing, we carry out the proof for $m = 2$. For each $j = 1, 2$, consider the polynomial $P_{N_j}(x_j)$ defined as follows: If $N_j = 0$, put $P_{N_j} \equiv 1$ and if $N_j \in \mathbb{N}$ then put $P_{N_j}(x_j) = \prod_{i=0}^{N_j - 1}(1 + r_i(x_j))$. Then as in the proof of Theorem 3.2.2, we have

$$P_{N_j}(x_j) = \begin{cases} 2^{N_j} & \text{if } x_j \in [0, 2^{-N_j}), \\ 0 & \text{if } x_j \in [0, 1) \setminus [0, 2^{-N_j}). \end{cases}$$

Consider $\mathbf{N} = (N_1, N_2)$ and put $P_{\mathbf{N}}(x_1, x_2) = P_{N_1}(x_1)P_{N_2}(x_2)$. Then by the above property of P_{N_j} $(j = 1, 2)$, we have

$$P_{\mathbf{N}}(x_1, x_2) = \begin{cases} 2^{N_1 + N_2} & \text{if } (x_1, x_2) \in I, \\ 0 & \text{if } (x_1, x_2) \in \mathbb{I}^2 \setminus I. \end{cases} \tag{3.16}$$

We claim that if $\mathbf{n_k} = (n_{k_1}^{(1)}, n_{k_2}^{(2)}) \in (\mathbb{Z}^+)^2$ is such that $\hat{f}(\mathbf{n_k}) \neq 0$ then $(fP_{\mathbf{N}})\hat{}(\mathbf{n_k}) = \hat{f}(\mathbf{n_k})$. In fact, writing (x, y) in place of (x_1, x_2), we have

$$\begin{aligned}
(fP_{\mathbf{N}})\hat{}(\mathbf{n_k}) &= \int_{\mathbb{I}^2} f(x, y)P_{N_1}(x)P_{N_2}(y)\varphi_{n_{k_1}^{(1)}}(x)\varphi_{n_{k_2}^{(2)}}(y)dxdy \\
&= \int_{\mathbb{I}^2} f(x, y)\left(\prod_{i=0}^{N_1 - 1}(1 + r_i(x))\right)\left(\prod_{j=0}^{N_2 - 1}(1 + r_j(y))\right)\varphi_{n_{k_1}^{(1)}}(x)\varphi_{n_{k_2}^{(2)}}(y)dxdy \\
&= \hat{f}(\mathbf{n_k}) + \sum_{i=0}^{N_1 - 1}\hat{f}(r_i\varphi_{\mathbf{n_k}}) + \sum_{j=0}^{N_2 - 1}\hat{f}(r_j\varphi_{\mathbf{n_k}}) + \sum_{i,j=0}^{N_1 - 1}\hat{f}(r_ir_j\varphi_{\mathbf{n_k}}) + \sum_{i,j=0}^{N_2 - 1}\hat{f}(r_ir_j\varphi_{\mathbf{n_k}}) \\
&\quad + \sum_{i=0}^{N_1 - 1}\sum_{j=0}^{N_2 - 1}\hat{f}(r_ir_j\varphi_{\mathbf{n_k}}) + ... + \hat{f}(r_0...r_{N_1 - 1}r_0...r_{N_2 - 1}\varphi_{\mathbf{n_k}}).
\end{aligned} \tag{3.17}$$

By our assumption the first term in the right hand side of (3.17) is nonzero. The characters appearing in the other terms in the right hand side of (3.17) are of the form $(\varphi\varphi_{n^{(1)}_{k_1}})(\psi\varphi_{n^{(2)}_{k_2}})$ where φ is (a function of x alone) such that deg $\varphi \leq N_1$ and ψ is (a function of y alone) such that deg $\psi \leq N_2$ and the degree of at least one of φ and ψ is nonzero. In view of the Payley ordering of Walsh characters, for each $j \in \mathbb{N}$ there are totally 2^{j-1} characters of degree j, namely $\varphi_{2^{j-1}} \equiv r_{j-1}$, $\varphi_{2^{j-1}+1} \equiv r^{j-1}\varphi_1$, $\varphi^{2^{j-1}+2} \equiv r_{j-1}\varphi_2$, ..., $\varphi_{2^j-1} \equiv r_{j-1}\varphi_{2^{j-1}-1} \equiv r_{j-1}r_{j-2}\cdots r_1r_0$. Consequently, total number of characters of positive degree $\leq N$ is given by $2^0+2^1+2^2+...+2^{N-1} = 2^N - 1$; they are from φ_1 to φ_{2^N-1}. It follows that when $\varphi_{n^{(j)}_{k_j}}$ is multiplied by any character of positive degree $\leq N_j$ the resulting character φ_{m_j} is such that

$$n^{(j)}_{k_j} < m_j \leq n^{(j)}_{k_j} + 2^{N_j} - 1 < n^{(j)}_{k_j} + 2^{N_j} \leq n^{(j)}_{k_j} + q \leq n^{(j)}_{k_j+1},$$

in view of (3.15) and the fact that $q \geq 2^{N_j}$. Since either deg $\varphi > 0$ or deg $\psi > 0$, either $m_1 \notin E_1$ or $m_2 \notin E_2$. Therefore $(m_1, m_2) \notin E$. Since f is E-spectral, $\hat{f}\left((\varphi\varphi_{n^{(1)}_{k_1}})(\psi\varphi_{n^{(2)}_{k_2}})\right) = \hat{f}(\varphi_{m_1}\varphi_{m_2}) \equiv \hat{f}(m_1, m_2) = 0$. Thus all the terms of the right hand side of (3.17) vanish except the first. This means that

$$(fP_{\mathbf{N}})\hat{\ }(\mathbf{n_k}) = \hat{f}(\mathbf{n_k}) \text{ if } \hat{f}(\mathbf{n_k}) \neq 0. \tag{3.18}$$

Now, let $\mathbf{n_k} = (n^{(1)}_{k_1}, n^{(2)}_{k_2})$ be such that $n^{(j)}_{k_j}$ are large enough with $\hat{f}(\mathbf{n_k}) \neq 0$ and let $m_j \in \mathbb{N}$ be such that $2^{m_j} \leq n^{(j)}_{k_j} < 2^{m_j+1}$ with $m_j > N_j$ for each $j = 1, 2$. For simplicity in notation, let us write k, ℓ, s and t for $n^{(1)}_{k_1}$, $n^{(2)}_{k_2}$, m_1 and m_2 respectively. Then $2^s \leq k < 2^{s+1}$, $2^t \leq \ell < 2^{t+1}$ and in view of (3.18) and (3.16) we have

$$\hat{f}(\mathbf{n_k}) = (fP_{\mathbf{N}})\hat{\ }(\mathbf{n_k}) = 2^{N_1+N_2}\int_0^{2^{-N_2}}\int_0^{2^{-N_1}} f(x,y)\varphi_k(x)\varphi_\ell(y)dxdy. \tag{3.19}$$

Putting $a_i = (i/2^s)$ for each $i = 0, 1, 2, 3, ..., 2^s$ and $b_j = (j/2^t)$ for each $j = 0, 1, 2, 3, ..., 2^t$, as in the proof of Theorem 3.3.2, we get (3.10) and (3.11).

Next, define three functions f_1, f_2, f_3 on $I = [0, 2^{-N_1}) \times [0, 2^{-N_2})$ by setting

$$f_1(x,y) = f(a_{i-1}, y) \;\; (a_{i-1} \leq x < a_i;\; 0 \leq y < 2^{-N_2}) \;\text{ for }\; i = 1, 2, 3, ..., 2^{s-N_1};$$
$$f_2(x,y) = f(x, b_{j-1}) \;\; (0 \leq x < 2^{-N_1};\; b_{j-1} \leq y < b_j) \;\text{ for }\; j = 1, 2, 3, ..., 2^{t-N_2};$$

and

$$f_3(x,y) = f(a_{i-1}, b_{j-1}) \;\; (a_{i-1} \leq x < a_i;\; b_{j-1} \leq y < b_j)$$

for $i = 1, 2, 3, ..., 2^{s-N_1};\ j = 1, 2, 3, ..., 2^{t-N_2}$.

Then in view of Fubini's theorem and relations (3.10) and (3.11) we have

$$\int_0^{2^{-N_2}} \int_0^{2^{-N_1}} f_1(x,y)\varphi_k(x)\varphi_\ell(y)dxdy$$
$$= \int_0^{2^{-N_2}} \left[\sum_{i=1}^{2^{s-N_1}} f(a_{i-1}, y) \int_{a_{i-1}}^{a_i} \varphi_k(x)dx \right] \varphi_\ell(y)dy = 0,$$

$$\int_0^{2^{-N_2}} \int_0^{2^{-N_1}} f_2(x,y)\varphi_k(x)\varphi_\ell(y)dxdy$$
$$= \int_0^{2^{-N_1}} \left[\sum_{j=1}^{2^{t-N_2}} f(x, b_{j-1}) \int_{b_{j-1}}^{b_j} \varphi_\ell(y)dy \right] \varphi_k(x)dx = 0$$

and

$$\int_0^{2^{-N_2}} \int_0^{2^{-N_1}} f_3(x,y)\varphi_k(x)\varphi_\ell(y)dxdy$$
$$= \sum_{i=1}^{2^{s-N_1}} \sum_{j=1}^{2^{t-N_2}} f(a_{i-1}, b_{j-1}) \left[\int_{a_{i-1}}^{a_i} \varphi_k(x)dx \right] \left[\int_{b_{j-1}}^{b_j} \varphi_\ell(y)dy \right] = 0.$$

Using these equations in (3.19) we get

$$|\hat{f}(\mathbf{n_k})| = 2^{N_1+N_2} \left| \int_0^{2^{-N_2}} \int_0^{2^{-N_1}} f(x,y)\varphi_k(x)\varphi_\ell(y)dxdy \right|$$
$$= 2^{N_1+N_2} \left| \int_0^{2^{-N_2}} \int_0^{2^{-N_1}} (f - f_1 - f_2 + f_3)(x,y)\varphi_k(x)\varphi_\ell(y)dxdy \right|$$
$$\leq 2^{N_1+N_2} \int_0^{2^{-N_2}} \int_0^{2^{-N_1}} |(f - f_1 - f_2 + f_3)(x,y)|dxdy$$
$$\leq 2^{N_1+N_2} \left(\int_0^{2^{-N_2}} \int_0^{2^{-N_1}} |(f - f_1 - f_2 + f_3)(x,y)|^p dxdy \right)^{1/p} \left(2^{-(N_1+N_2)}\right)^{1/q},$$

in view of the Hölder's inequality (when $p > 1$) since $f - f_1 - f_2 + f_3 \in \mathrm{L}^p(I)$, where q is such that $1/p + 1/q = 1$. Observe that when $p = 1$, we don't use Hölder's inequality and in that case we consider the inequality except last step.

In any case, it follows that

$$
\begin{aligned}
|\hat{f}(\mathbf{n_k})|^p &\le 2^{N_1+N_2}\int_0^{2^{-N_2}}\int_0^{2^{-N_1}}|f(x,y)-f_1(x,y)-f_2(x,y)+f_3(x,y)|^p dxdy\\
&= 2^{N_1+N_2}\sum_{i=1}^{2^{s-N_1}}\sum_{j=1}^{2^{t-N_2}}\int_{b_{j-1}}^{b_j}\int_{a_{i-1}}^{a_i}|f(x,y)-f_1(x,y)-f_2(x,y)+f_3(x,y)|^p dxdy\\
&= 2^{N_1+N_2}\sum_{i=1}^{2^{s-N_1}}\sum_{j=1}^{2^{t-N_2}}\int_{b_{j-1}}^{b_j}\int_{a_{i-1}}^{a_i}|f(x,y)-f(a_{i-1},y)-f(x,b_{j-1})+f(a_{i-1},b_{j-1})|^p dxdy\\
&\le 2^{N_1+N_2}\sum_{i=1}^{2^{s-N_1}}\sum_{j=1}^{2^{t-N_2}}(V_p(f;[a_{i-1},a_i]\times[b_{j-1},b_j]))^p(a_i-a_{i-1})(b_j-b_{j-1})\\
&\le \frac{2^{N_1+N_2}}{2^s2^t}(V_p(f;I))^p \le \frac{2^{N_1+N_2+2}}{k\ell}(V_p(f;I))^p,
\end{aligned}
$$

in view of Lemma 2.1.8. Thus we get

$$
|\hat{f}(\mathbf{n_k})| \le \frac{2^{(N_1+N_2+2)/p}\cdot V_p(f;I)}{(k\ell)^{1/p}}. \tag{3.20}
$$

This completes the proof. □

Remark 3.4.2. Theorem 3.4.1 gives lacunary analogue of our earlier result Theorem 3.3.2. Since ${\rm BV_H}^{(p)}(I) \subset {\rm BV_V}^{(p)}(I)\cap {\rm L}^p(I)$ (in view of Lemma 2.1.12), Theorem 3.4.1 is true if we replace the assumption "$f \in {\rm BV_V}^{(p)}(I)\cap {\rm L}^p(I)$" by "$f \in {\rm BV_H}^{(p)}(I)$". In that case it gives lacunary analogue of our earlier result Theorem 3.3.3 and simultaneously Walsh analogue of our earlier result Theorem 2.2.3.

Chapter 4

Vilenkin Fourier Coefficients

4.1 Order of magnitude of Vilenkin Fourier coefficients of functions of generalized bounded fluctuation

Onneweer and Waterman introduced various classes of functions of bounded fluctuation and studied the convergence problems for functions of these classes [37, 38, 39]. However, it appears that the order of magnitude of Vilenkin Fourier coefficients for functions of such classes has not yet been studied. The only results available, in this case, seems to be Theorems 1.1.37 and 1.1.38 for functions of the classes $\mathrm{BV}(G)$ and $\mathrm{Lip}\,(\alpha, p, G)$ respectively. In this section, we carry out this study for non-lacunary Vilenkin Fourier series. Results of this section are published in [16] (see also MR2662990).

First we give definitions of various classes of functions of bounded and generalized bounded fluctuation on a Vilenkin group G (see Chapter 1, Section 1.1 (D) for terminology) as below. In these definitions f denotes a complex-valued function defined on G.

Definition 4.1.1. For $H \subset G$, we define the *oscilation* of f on H by

$$\mathrm{osc}(f; H) = \sup\{|f(x_1) - f(x_2)| : x_1, x_2 \in H\}.$$

Onneweer and Waterman [38, Definitions 4, 5] generalized the concept of bounded variation (see (1.25) for definition of class $\mathrm{BV}(G)$) in two ways as follows.

Definition 4.1.2. The function f is said to be of *bounded fluctuation* ($f \in \mathrm{BF}$) if the *total fluctuation* of f on G, given by

$$F(f;G) = \sup\left\{\sum_{n=1}^{N} \operatorname{osc}(f;I_n)\right\}$$

is finite, where the supremum is taken over all finite disjoint collections $\{I_1, I_2, ..., I_N\}$, in which each I_n is a coset of some $G_{m(n)}$ and $\cup_{n=1}^{N} I_n = G$.

Definition 4.1.3. The function f is said to be of *generalized bounded fluctuation* ($f \in \mathrm{GBF}$) if the *total generalized fluctuation* of f on G, given by

$$GF(f;G) = \sup_n\left\{\sum_{\alpha=0}^{m_n-1} \operatorname{osc}\left(f; z_\alpha^{(n)} + G_n\right)\right\}$$

is finite.

Further, they have observed that for continuous functions f on G, $f \in \mathrm{BV}$ implies $f \in \mathrm{BF}$ and $f \in \mathrm{BF}$ implies $f \in \mathrm{GBF}$. They have also shown that the function f defined on $G = \mathbb{Z}_2^\infty$ as

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x \in z_{2^n-2}^{(n)} + G_n \ \ (n \text{ odd}), \\ 0 & \text{if } x \in z_{2^n-2}^{(n)} + G_n \ \ (n \text{ even}), \\ 0 & \text{if } x = (1,1,...), \end{cases}$$

is continuous and of generalized bounded fluctuation but not of bounded fluctuation on G.

Later, Onneweer extended the notion of generalized bounded fluctuation to p-generalized bounded fluctuation [39, Definition 4] as follows.

Definition 4.1.4. Let $p \geq 1$ be a real number. The function f is said to be of *p-generalized bounded fluctuation* ($f \in \mathrm{GBF}^{(p)}$) if the *total generalized p-fluctuation* of f on G, given by

$$GF_p(f;G) = \sup_n\left\{\left(\sum_{\alpha=0}^{m_n-1} \left(\operatorname{osc}\left(f; z_\alpha^{(n)} + G_n\right)\right)^p\right)^{1/p}\right\}$$

is finite.

In [39, Definition 5] Onneweer defined the following class of functions of Λ-generalized bounded fluctuation.

Definition 4.1.5. Let Λ be a sequence as in Definition 1.1.7. The function f is said to be of Λ-*generalized bounded fluctuation* ($f \in \Lambda\mathrm{GBF}$) if the *total generalized* Λ-*fluctuation* of f on G, given by

$$\Lambda GF(f;G) = \sup_n \left\{ \sum_{\alpha=0}^{m_n-1} \frac{\operatorname{osc}\left(f; z_\alpha^{(n)} + G_n\right)}{\lambda_{\alpha+1}} \right\}$$

is finite.

Here we give following definition.

Definition 4.1.6. Let $p \geq 1$ be a real number. The function f is said to be of p-*bounded fluctuation* ($f \in \mathrm{BF}^{(p)}$) if the *total* p-*fluctuation* of f on G, given by

$$F_p(f;G) = \sup \left\{ \left(\sum_{n=1}^{N} (\operatorname{osc}(f; I_n))^p \right)^{1/p} \right\}$$

is finite, where the supremum is considered as in Definition 4.1.2.

Onneweer and Waterman have defined the following class of functions of ϕ-bounded fluctuation [37, Definition 3].

Definition 4.1.7. Let ϕ be a function as in Definition 1.1.5. The function f is said to be of ϕ-*bounded fluctuation* ($f \in \phi\mathrm{BF}(G)$) if the *total* ϕ-*fluctuation* of f on G, given by

$$F_\phi(f;G) = \sup \left\{ \sum_{n=1}^{N} \phi(\operatorname{osc}(f; I_n)) \right\}$$

is finite, where the supremum is considered as in Definition 4.1.2.

Onneweer and Waterman have defined the following class of functions of Λ-bounded fluctuation [37, Definition 2].

Definition 4.1.8. Let Λ be a sequence as in Definition 1.1.7. The function f is said to be of Λ-*bounded fluctuation* ($f \in \Lambda\mathrm{BF}$) if the *total* Λ-*fluctuation* of f on G, given by

$$F_\Lambda(f;G) = \sup \left\{ \sum_{n=1}^{\infty} \frac{\operatorname{osc}(f; I_n)}{\lambda_n} \right\}$$

is finite, where the supremum is taken over all sequences $\{I_n\}$ of disjoint cosets in G.

Here we define the class $\Lambda\mathrm{BF}^{(p)}$ of functions of p-Λ-bounded fluctuation and the class $\phi\Lambda\mathrm{BF}$ of functions ϕ-Λ-bounded fluctuation as follows.

Definition 4.1.9. Let $p \geq 1$ be a real number and Λ be a sequence as in Definition 1.1.7. The function f is said to be of p-Λ-*bounded fluctuation* ($f \in \Lambda\mathrm{BF}^{(p)}$) if the *total* p-Λ-*fluctuation* of f on G, given by

$$F_{p\Lambda}(f;G) = \sup\left\{\left(\sum_{n=1}^{\infty} \frac{(\mathrm{osc}(f;I_n))^p}{\lambda_n}\right)^{1/p}\right\}$$

is finite, where the supremum is taken over all sequences $\{I_n\}$ of disjoint cosets in G.

Definition 4.1.10. Let ϕ be a function as in Definition 1.1.5 and Λ be a sequence as in Definition 1.1.7. The function f is said to be of ϕ-Λ-*bounded fluctuation* ($f \in \phi\Lambda\mathrm{BF}$) if the *total* ϕ-Λ-*fluctuation* of f on G, given by

$$F_{\phi\Lambda}(f;G) = \sup\left\{\sum_{n=1}^{\infty} \frac{\phi(\mathrm{osc}(f;I_n))}{\lambda_n}\right\}$$

is finite, where the supremum is taken over all sequences $\{I_n\}$ of disjoint cosets in G.

Onneweer and Waterman have defined the following class of functions of ϕ-generalized bounded fluctuation [38, Definition 6].

Definition 4.1.11. Let ϕ be a function as in Definition 1.1.5. The function f is said to be of ϕ-*generalized bounded fluctuation* ($f \in \phi\mathrm{GBF}(G)$) if the *total generalized* ϕ-*fluctuation* of f on G, given by

$$GF_{\phi}(f;G) = \sup_n\left\{\sum_{\alpha=0}^{m_n-1} \phi(\mathrm{osc}(f;z_\alpha^{(n)} + G_n))\right\}$$

is finite.

Uno [66] defined the concept of p-Λ-generalized bounded fluctuation which is defined as follows.

Definition 4.1.12. The function f is said to be of *p-Λ-generalized bounded fluctuation* ($f \in \Lambda\mathrm{GBF}^{(p)}$) if the *total generalized p-Λ-fluctuation* of f on G, given by

$$GF_{p\Lambda}(f;G) = \sup_n \sup_\alpha \left(\sum_{j=0}^{m_n-1} \frac{\left(\operatorname{osc}(f; z_\alpha^{(n)} + G_n)\right)^p}{\lambda_{j+1}} \right)^{1/p}$$

is finite, where $\sup_\alpha$ denotes that the supremum taken over all permutations of the set $\{0, 1, ..., m_n - 1\}$.

Here we define the concept of ϕ-Λ-generalized bounded fluctuation as follows.

Definition 4.1.13. Let ϕ be a function as in Definition 1.1.5 and Λ be a sequence as in Definition 1.1.7. The function f is said to be of *ϕ-Λ-generalized bounded fluctuation* ($f \in \phi\Lambda\mathrm{GBF}$) if the *total generalized ϕ-Λ-fluctuation* of f on G, given by

$$GF_{\phi\Lambda}(f;G) = \sup_n \sup_\alpha \left\{ \sum_{j=0}^{m_n-1} \frac{\phi(\operatorname{osc}(f; z_\alpha^{(n)} + G_n))}{\lambda_{j+1}} \right\}$$

is finite, where $\sup_\alpha$ has the same meaning as in Definition 4.1.12.

We observe that if $p = 1$, $\mathrm{BF}^{(p)} = \mathrm{BF}$, $\Lambda\mathrm{BF}^{(p)} = \Lambda\mathrm{BF}$ and $\mathrm{GBF}^{(p)} = \mathrm{GBF}$; if $\lambda_n \equiv 1$, $\Lambda\mathrm{BF}^{(p)} = \mathrm{BF}^{(p)}$; and if $\phi(x) = x^p$, then $\phi\mathrm{BF} = \mathrm{BF}^{(p)}$, $\phi\Lambda\mathrm{BF} = \Lambda\mathrm{BF}^{(p)}$, $\phi\mathrm{GBF} = \mathrm{GBF}^{(p)}$ and $\phi\Lambda\mathrm{GBF} = \Lambda\mathrm{GBF}^{(p)}$. Also from the definitions it is clear that $\mathrm{BF} \subset \mathrm{GBF}$, $\mathrm{BF}^{(p)} \subset \mathrm{GBF}^{(p)}$, $\Lambda\mathrm{BF}^{(p)} \subset \Lambda\mathrm{GBF}^{(p)}$, $\phi\mathrm{BF} \subset \phi\mathrm{GBF}$, $\phi\Lambda\mathrm{BF} \subset \phi\Lambda\mathrm{GBF}$. Further if $\lambda_n = n, \forall n \in \mathbb{N}$, then $\Lambda\mathrm{BF} = \mathrm{HBF}$— the class of functions of harmonic bounded fluctuation, etc.

Theorem 4.1.14. *If $f \in \mathrm{GBF}^{(p)}$ then $\hat{f}(n) = O(1/(m_k)^{1/p})$, where $m_k \le n < m_{k+1}$.*

Proof. Since $n \ge m_k$ and the Haar measure is translation invariant, it follows (see, for example, [51, p. 114, Eq. (15)]) that

$$\int_{z_\alpha^{(k)} + G_k} \chi_n(x)dx = 0$$

for all $\alpha = 0, 1, ..., m_k - 1$; and hence

$$\int_{z_\alpha^{(k)} + G_k} \bar{\chi}_n(x)dx = 0, \quad (\alpha = 0, 1, ..., m_k - 1).$$

If we now define a step function g on G by $g(x) = f(z_\alpha^{(k)})$ on $z_\alpha^{(k)} + G_k,\ \alpha = 0, 1, ..., m_k - 1$, then

$$\int_G g(x)\bar{\chi}_n(x)dx = \sum_{\alpha=0}^{m_k-1} f(z_\alpha^{(k)}) \int_{z_\alpha^{(k)}+G_k} \bar{\chi}_n(x)dx = 0.$$

Therefore

$$|\hat{f}(n)| = \left| \int_G [f(x) - g(x)]\bar{\chi}_n(x)dx \right| \leq \int_G |f(x) - g(x)|dx \tag{4.1}$$

and hence

$$|\hat{f}(n)| \leq ||f - g||_p ||1||_q = \left(\sum_{\alpha=0}^{m_k-1} \int_{z_\alpha^{(k)}+G_k} |f(x) - f(z_\alpha^{(k)})|^p dx \right)^{\frac{1}{p}},$$

by Hölder's inequality as $f,\ g \in \mathrm{GBF}^{(p)}$ and $\mathrm{GBF}^{(p)} \subset \mathrm{L}^p$. Thus

$$\begin{aligned} |\hat{f}(n)|^p &\leq \sum_{\alpha=0}^{m_k-1} \int_{z_\alpha^{(k)}+G_k} |f(x) - f(z_\alpha^{(k)})|^p dx \\ &\leq \sum_{\alpha=0}^{m_k-1} \int_{z_\alpha^{(k)}+G_k} (\mathrm{osc}(f; z_\alpha^{(k)} + G_k))^p dx \\ &= \sum_{\alpha=0}^{m_k-1} (\mathrm{osc}(f; z_\alpha^{(k)} + G_k))^p \frac{1}{m_k} \\ &\leq \frac{1}{m_k} (GF_p(f; G))^p. \end{aligned} \tag{4.2}$$

This completes the proof. □

Corollary 4.1.15. *If G is bounded, then $f \in \mathrm{GBF}^{(p)}$ implies $\hat{f}(n) = O(1/n^{1/p})$.*

Proof. Since G is a bounded Vilenkin group, there exists p_0 such that $p_k \leq p_0$ for all k. Hence $m_{k+1} = m_k \cdot p_{k+1} \leq m_k \cdot p_0$ and therefore, from $m_k \leq n < m_{k+1}$ one gets $1/m_k \leq p_0/n$, completing the proof. □

Remark 4.1.16. Since $\mathrm{BF}^{(p)} \subset \mathrm{GBF}^{(p)}$, Theorem 4.1.14 and Corollary 4.1.15 obviously holds true for functions in $\mathrm{BF}^{(p)}$ also.

Theorem 4.1.17. *If $f \in \phi\mathrm{GBF}$ then $\hat{f}(n) = O(\phi^{-1}(1/m_k))$, where $m_k \leq n < m_{k+1}$.*

Proof. Proceeding as in the proof of Theorem 4.1.14 we get (4.1). Now, by Jensen's inequality, for $c > 0$

$$\begin{aligned}
\phi\left(c\int_G |f(x)-g(x)|dx\right) &\leq \int_G \phi(c|f(x)-g(x)|)dx \\
&= \sum_{\alpha=0}^{m_k-1} \int_{z_\alpha^{(k)}+G_k} \phi(c|f(x)-f(z_\alpha^{(k)})|)dx \\
&\leq \sum_{\alpha=0}^{m_k-1} \int_{z_\alpha^{(k)}+G_k} \phi(\text{osc}(cf; z_\alpha^{(k)}+G_k))dx \\
&= \sum_{\alpha=0}^{m_k-1} \phi(\text{osc}(cf; z_\alpha^{(k)}+G_k))\frac{1}{m_k}.
\end{aligned}$$

Therefore we have

$$\phi\left(c\int_G |f(x)-g(x)|dx\right) \leq \frac{1}{m_k} GF_\phi(cf; G). \tag{4.3}$$

Since ϕ is convex and $\phi(0) = 0$, we have $\phi(ax) \leq a\phi(x)$ for $0 < a < 1$ and for all $x \geq 0$. Therefore, choosing c in $(0, 1)$ so small that $GF_\phi(cf; G) \leq 1$, one gets

$$|\hat{f}(n)| \leq \int_G |f(x)-g(x)|dx \leq \frac{1}{c}\phi^{-1}(1/m_k)$$

in view of (4.3) and (4.1). This completes the proof. □

Corollary 4.1.18. *If G is bounded, then $f \in \phi\text{GBF}$ implies $\hat{f}(n) = O(\phi^{-1}(\frac{1}{n}))$.*

Proof. As in the proof of Corollary 4.1.15 we have $1/m_k \leq p_0/n$, so by (4.3)

$$\phi\left(c\int_G |f(x)-g(x)|dx\right) \leq \frac{p_0}{n} GF_\phi(cf; G).$$

Now choosing $c \in (0, 1)$ small enough so that $p_0 \cdot GF_\phi(cf; G) \leq 1$, we get

$$|\hat{f}(n)| \leq \int_G |f(x)-g(x)|dx \leq \frac{1}{c}\phi^{-1}\left(\frac{1}{n}\right),$$

in view of (4.3) and (4.1). This completes the proof. □

Remark 4.1.19. Since $\phi\text{BF} \subset \phi\text{GBF}$, Theorem 4.1.17 and Corollary 4.1.18 hold for functions in ϕBF also.

To prove Theorems 4.1.21 and 4.1.24, the following lemma due to Schramm and Waterman [56] is needed.

Lemma 4.1.20. *If* $a_1 \geq a_2 \geq ... \geq a_n > 0$, $\sum_{i=1}^{n} a_i = 1$ *and* $b_1 \geq b_2 \geq ... \geq b_n$, *then*

$$\sum_{i=1}^{n} b_i \leq n \sum_{i=1}^{n} a_i b_i.$$

Theorem 4.1.21. *If* $f \in \Lambda\mathrm{GBF}^{(p)}$ *then*

$$\hat{f}(n) = O\left(1 \Big/ \left(\sum_{j=1}^{m_k} \frac{1}{\lambda_j}\right)^{1/p}\right),$$

where $m_k \leq n < m_{k+1}$.

Proof. Since $\Lambda\mathrm{GBF}^{(p)} \subset \mathrm{L}^p$, proceeding as in the proof of Theorem 4.1.14 we get (4.2). Let α_i, $i = 0, 1, ..., m_k - 1$, denote a rearrangement of $0, 1, ..., m_k - 1$ such that $\{b_i\}_{i=0}^{m_k-1}$ is non-increasing, where

$$b_i = \int_{z_{\alpha_i}^{(k)} + G_k} |f(x) - f(\alpha_i)^{(k)})| dx$$

for all i. For each $n \in \mathbb{N}$ put $\theta_n = \sum_{j=1}^{n} \frac{1}{\lambda_j} = \sum_{i=0}^{n-1} \frac{1}{\lambda_{i+1}}$ and for each $i = 0, 1, ..., m_k - 1$ put $a_i = 1/(\lambda_{i+1}\theta_{m_k})$. Then $\{a_i\}_{i=0}^{m_k-1}$ is non-increasing and $\sum_{i=0}^{m_k-1} a_i = 1$. Therefore by Lemma 4.1.20 we have

$$\begin{aligned}
\sum_{\alpha=0}^{m_k-1} \int_{z_{\alpha}^{(k)} + G_k} |f(x) - f(z_{\alpha}^{(k)})|^p dx &= \sum_{i=0}^{m_k-1} b_i \\
&\leq m_k \sum_{i=0}^{m_k-1} a_i b_i \\
&= \frac{m_k}{\theta_{m_k}} \sum_{i=0}^{m_k-1} \int_{z_{\alpha_i}^{(k)} + G_k} \left(\frac{|f(x) - f(z_{\alpha_i}^{(k)})|^p}{\lambda_{i+1}}\right) dx \\
&\leq \frac{m_k}{\theta_{m_k}} \sum_{i=0}^{m_k-1} \int_{z_{\alpha_i}^{(k)} + G_k} \left(\frac{(\mathrm{osc}(f; z_{\alpha_i}^{(k)} + G_k))^p}{\lambda_{i+1}}\right) dx \\
&\leq \frac{m_k}{\theta_{m_k}} \sum_{i=0}^{m_k-1} \frac{(\mathrm{osc}(f; z_{\alpha_i}^{(k)} + G_k))^p}{\lambda_{i+1}} \cdot \frac{1}{m_k} \\
&\leq \frac{(GF_{p\Lambda}(f; G))^p}{\theta_{m_k}},
\end{aligned}$$

and hence the theorem is proved in view of (4.2). $\square$

Corollary 4.1.22. *If G is bounded, then $f \in \Lambda\mathrm{GBF}^{(p)}$ implies*

$$\hat{f}(n) = O\left(1\Big/\left(\sum_{j=1}^{n}\frac{1}{\lambda_j}\right)^{1/p}\right).$$

Proof. Since G is a bounded Vilenkin group, there exists p_0 such that $p_k \leq p_0$ for all k. Since $m_k \leq n < m_{k+1}$ and $\{\lambda_n\}$ is increasing we have $\theta_n < \theta_{m_{k+1}} \leq p_0 \cdot \theta_{m_k}$ so that $1/(\theta_{m_k})^{1/p} = O(1/(\theta_n)^{1/p})$. Thus by Theorem 4.1.21 we have $\hat{f}(n) = O(1/(\theta_{m_k}))^{1/p} = O(1/(\theta_n^{1/p}))$ and hence the corollary is proved. □

Remark 4.1.23. Since $\Lambda\mathrm{BF}^{(p)} \subset \Lambda\mathrm{GBF}^{(p)}$, Theorem 4.1.21 and Corollary 4.1.22 hold for functions in $\Lambda\mathrm{BF}^{(p)}$ also. In fact, in [16], we have proved the results the functions of the class $\Lambda\mathrm{BF}^{(p)}$ only and we have now observed that the same proof works for functions of the class $\Lambda\mathrm{GBF}^{(p)}$.

Theorem 4.1.24. *If $f \in \phi\Lambda\mathrm{GBF}$ then*

$$\hat{f}(n) = O\left(\phi^{-1}\left(1\Big/\sum_{j=1}^{m_k}\frac{1}{\lambda_j}\right)\right),$$

where $m_k \leq n < m_{k+1}$.

Proof. Let $c > 0$. Taking now $b_i = \int_{z_{\alpha_i}^{(k)}+G_k} \phi(c|f(x) - f(\alpha_i)^{(k)})|)dx$ for all i, and proceeding as in Theorem 4.1.21, one gets by Jensen's inequality and Lemma 4.1.20,

$$\begin{aligned}
\phi\left(c\int_G |f(x) - g(x)|dx\right) &\leq \int_G \phi(c|f(x) - g(x)|)dx \\
&= \sum_{i=0}^{m_k-1} b_i \\
&\leq m_k \sum_{i=0}^{m_k-1} a_i b_i \\
&= \frac{m_k}{\theta_{m_k}} \sum_{i=0}^{m_k-1} \int_{z_{\alpha_i}^{(k)}+G_k} \left(\frac{\phi(c|f(x) - f(z_{\alpha_i}^{(k)})|)}{\lambda_{i+1}}\right)dx \\
&\leq \frac{m_k}{\theta_{m_k}} \sum_{i=0}^{m_k-1} \int_{z_{\alpha_i}^{(k)}+G_k} \left(\frac{\phi(\mathrm{osc}(cf; z_{\alpha_i}^{(k)} + G_k))}{\lambda_{i+1}}\right)dx \\
&= \frac{m_k}{\theta_{m_k}} \sum_{i=0}^{m_k-1} \frac{\phi(\mathrm{osc}(cf; z_{\alpha_i}^{(k)} + G_k))}{\lambda_{i+1}} \cdot \frac{1}{m_k}.
\end{aligned}$$

Therefore we have

$$\phi\left(c\int_G |f(x)-g(x)|dx\right) \le \frac{GF_{\phi\Lambda}(cf;G))}{\theta_{m_k}}. \tag{4.4}$$

Choose $c \in (0,1)$ so small that $GF_{\phi\Lambda}(cf;G)) \le 1$. Then by (4.4)

$$\int_G |f(x)-g(x)|dx \le \frac{1}{c}\phi^{-1}\left(\frac{1}{\theta_{m_k}}\right),$$

and hence the theorem is proved in view of (4.1). □

Corollary 4.1.25. *If G is bounded, then $f \in \phi\Lambda\text{GBF}$ implies*

$$\hat{f}(n) = O\left(\phi^{-1}\left(1\Big/\sum_{j=1}^{n}\frac{1}{\lambda_j}\right)\right).$$

Proof. Since G is bounded as in the proof of Corollary 4.1.22, we have $1/\theta_{m_k} \le p_0/\theta_n$, so by (4.4)

$$\phi\left(c\int_G |f(x)-g(x)|dx\right) \le \frac{p_0}{\theta_n}F_{\phi\Lambda}(cf;G).$$

Now choosing $c \in (0,1)$ small enough such that $p_0 \cdot F_{\phi\Lambda}(cf;G) \le 1$, we get

$$\int_G |f(x)-g(x)|dx \le \frac{1}{c}\phi^{-1}\left(\frac{1}{\theta_n}\right),$$

and hence the corollary is proved in view of (4.1). □

Remark 4.1.26. Since $\phi\Lambda\text{BF} \subset \phi\Lambda\text{GBF}$, Theorem 4.1.24 and Corollary 4.1.25 hold for functions in $\phi\Lambda$BF also. In fact, in [16], we have proved the results the functions of the class $\phi\Lambda$BF only and we have now observed that the same proof works for functions of the class $\phi\Lambda\text{GBF}^{(p)}$.

4.2 Order of magnitude of Vilenkin Fourier coefficients of series with small gaps for functions of generalized bounded fluctuation

In Section 4.1 we have studied the order of magnitude of Vilenkin Fourier coefficients of functions of certain classes of functions of bounded and generalized bounded fluctuation on a Vilenkin group G introduced by Onneweer and Waterman [37, 38, 39].

Here we define these concepts locally and study the order of magnitude of Vilenkin Fourier coefficients of functions of these classes, when the Vilenkin Fourier series is lacunary having small gaps and prove the Vilenkin group analogue (Corollary 4.2.10, below) of the results of Patadia and Vyas [41, Theorem 5]. As in the case of trigonometric Fourier series [41], here also we give an interconnection between the 'type of lacunarity' in Vilenkin Fourier series and the 'localness' of the hypothesis to be satisfied by the generic functions, which allow us to interpolate results concerning order of magnitude of Fourier coefficients of lacunary and non-lacunary Vilenkin Fourier series. Results of this section are accepted for publication in the form of a paper in *Kyoto Journal of Mathematics* [14].

Let G and X be as in Section 1.1 (D). Then we observe that for $l, N \in \mathbb{N}$ if $l > N$ then $G_l \subset G_N$ and therefore

$$G_l = \left\{ x \in G : x = \sum_{i=l}^{\infty} b_i x_i \right\} = \left\{ x \in G_N : x = \sum_{i=N}^{\infty} b_i x_i,\ b_N = \cdots = b_{l-1} = 0 \right\}.$$

Thus each coset of G_l in G_N has a representation of the form $z + G_l$, where $z = \sum_{i=N}^{l-1} b_i x_i$ for some choice of the b_i with $0 \leq b_i < p_{i+1}$. These $(m_l/m_N) = p_{N+1} p_{N+2} \cdot \cdot \cdot p_l = L$ (say) cosets of G_l in G_N are precisely the cosets $z_\alpha^{(l)} + G_l$, $\alpha = 0, 1, ..., L-1$, of G_l in G in that order. Also observe that for a given $y_0 = \sum_{i=0}^{\infty} c_i x_i$ in G and $N \in \mathbb{N}$, the coset $y_0 + G_N$ given by

$$y_0 + G_N = \left\{ x = \sum_{i=0}^{\infty} b_i x_i \in G : b_i = c_i,\ i = 0, 1, ..., N-1 \right\}$$

contains y_0 and is of Haar measure $1/m_N$. Since G_N is the disjoint union of the cosets $z_\alpha^{(l)} + G_l$, $\alpha = 0, 1, ..., L-1$, for $l > N$, the coset $y_0 + G_N$ is the disjoint union of the cosets $y_0 + z_\alpha^{(l)} + G_l$, $\alpha = 0, 1, ..., L-1$.

Definition 4.2.1. Let $\{n_k\}_{k=1}^{\infty}$ be an increasing sequence of positive integers. A function $f \in \mathrm{L}^1(G)$ is said to have a lacunary Vilenkin Fourier series with small gaps if its Vilenkin Fourier coefficients $\hat{f}(n)$ vanish for $n \neq n_k,\ k \in \mathbb{N}$, where $\{n_k\}$ satisfies the small gap condition (1.11) or, in particular, a more stringent small gap condition (1.5).

We define various classes of functions of bounded fluctuation over a coset of G as follows. In these definitions ϕ is a function as in Definition 1.1.5, Λ a sequence as in Definition 1.1.7 and f a complex-valued function defined on G.

Definition 4.2.2. We say f is of *ϕ-bounded fluctuation* over y_0+G_N ($f \in \phi\mathrm{BF}(y_0+G_N)$) if the *total ϕ-fluctuation* of f on y_0+G_N given by

$$F_\phi(f; y_0+G_N) = \sup\left\{\sum_{t=1}^{T} \phi(\mathrm{osc}(f; I_t))\right\}$$

is finite, where the supremum is taken over all finite disjoint collections $\{I_1, I_2, ..., I_T\}$ in which each I_t is a coset of some $G_{m(t)}$ and $\cup_{t=1}^{T} I_t = y_0 + G_N$.

Definition 4.2.3. We say f is of *ϕ-Λ-bounded fluctuation* over y_0+G_N ($f \in \phi\Lambda\mathrm{BF}(y_0+G_N)$) if the *total ϕ-Λ-fluctuation* of f on y_0+G_N given by

$$F_{\phi\Lambda}(f; y_0+G_N) = \sup_{\{I_n\}}\left\{\sum_{n} \frac{\phi\left(\mathrm{osc}(f; I_n)\right)}{\lambda_n}\right\}$$

is finite, where the supremum is taken over all sequences $\{I_n\}$ of disjoint cosets in y_0+G_N.

Definition 4.2.4. We say f is of *ϕ-generalized bounded fluctuation* over y_0+G_N ($f \in \phi\mathrm{GBF}(y_0+G_N)$) if the *total generalized ϕ-fluctuation* of f on y_0+G_N given by

$$GF_\phi(f; y_0+G_N) = \sup_{l \geq N} \sum_{\alpha=0}^{m_l/m_N-1} \phi\left(\mathrm{osc}\left(f; y_0 + z_\alpha^{(l)} + G_l\right)\right)$$

is finite.

Definition 4.2.5. We say f is of *ϕ-Λ-generalized bounded fluctuation* over y_0+G_N ($f \in \phi\Lambda\mathrm{GBF}(y_0+G_N)$) if the *total generalized ϕ-Λ-fluctuation* of f on y_0+G_N, given by

$$GF_{\phi\Lambda}(f; y_0+G_N) = \sup_{l \geq N}\sup_{\alpha} \sum_{j=0}^{m_n/m_N-1} \frac{\phi(\mathrm{osc}(f; y_0 + z_\alpha^{(l)} + G_N))}{\lambda_{j+1}}$$

is finite, where $\sup_\alpha$ denotes the supremum taken over all permutations of the set $\{0, 1, ..., m_n - 1\}$.

We observe that if $\lambda_n \equiv 1$, $\phi\Lambda\mathrm{BF} = \phi\mathrm{BF}$. If $\phi(x) = x^p$ ($p \geq 1$) then $\phi\mathrm{BF}$ (respectively, $\phi\mathrm{GBF}$, $\phi\Lambda\mathrm{GBF}$) is denoted as $\mathrm{BF}^{(p)}$ (respectively, $\mathrm{GBF}^{(p)}$, $\Lambda\mathrm{GBF}^{(p)}$) and functions of this class are called functions of *p-bounded fluctuation* (respectively, *p-generalized bounded fluctuation*, *p-Λ-generalized bounded fluctuation*). Also when

$p = 1$, the class $\mathrm{BF}^{(p)}$ (respectively, $\mathrm{GBF}^{(p)}$) is denoted as BF (respectively, GBF) and functions of this class are called functions of *bounded fluctuation* (respectively, *generalized bounded fluctuation*). Further, from Definitions 4.2.2 and 4.2.4, it is clear that $\phi\mathrm{BF} \subset \phi\mathrm{GBF}$ and from Definitions 4.2.3 and 4.2.5, it is clear that $\phi\Lambda\mathrm{BF} \subset \phi\Lambda\mathrm{GBF}$.

When $y_0 + G_N = G$, our Definitions 4.2.2, 4.2.3, 4.2.4 and 4.2.5 are same as the Definitions 4.1.7, 4.1.10, 4.1.11 and 4.1.13 respectively.

Theorem 4.2.6. *Let $f \in \mathrm{L}^1(G)$ possess a lacunary Vilenkin Fourier series*

$$\sum_{k=1}^{\infty} \hat{f}(n_k)\chi_{n_k}(x) \tag{4.5}$$

with small gaps (1.11) *and $I = y_0 + G_N$ be the coset with Haar measure $1/m_N \geq 1/q$. Then $f \in \phi\mathrm{GBF}(I)$ implies $\hat{f}(n_k) = O(\phi^{-1}(1/m_l))$, where $m_l \leq n_k < m_{l+1}$. If, in addition, G is bounded then $\hat{f}(n_k) = O(\phi^{-1}(1/n_k))$.*

Proof. We may assume without loss of generality that $x_0 = 0$; for, otherwise one works with $g = T_{y_0} f \in \phi\mathrm{GBF}(G_N)$ whose Fourier series also has gaps (1.11). Then $I = G_N$ and if we consider the polynomial $P_N(x)$ [42, Lemma 4] defined by

$$\begin{aligned} P_N(x) &= \prod_{k=0}^{N-1} (1 + \varphi_k(x) + \varphi_k^2(x) + \cdots + \varphi_k^{p_k-1}(x)) \\ &= 1 + \sum_{i=0}^{N-1} \varphi_i(x) + \sum_{i,j=0, i\neq j}^{N-1} \sum_{l=1}^{p_i-1} \sum_{m=1}^{p_j-1} \varphi_i^l(x) \cdot \varphi_j^m(x) + \cdots + \left(\prod_{i=0}^{N-1} \varphi_i^{p_i-1}(x) \right) \end{aligned}$$

having constant term 1 and with degree $\leq N$ then

$$P_N(x) = \begin{cases} m_N & \text{if } x \in I, \\ 0 & \text{if } x \in G \setminus I. \end{cases} \tag{4.6}$$

Note that if $k \in \mathbb{N}$ is such that $\hat{f}(n_k) \neq 0$ then $(f \cdot P_N)\hat{}(n_k) = \hat{f}(n_k)$. In fact,

$$\begin{aligned} (f \cdot P_N)\hat{}(n_k) &= \int_G f(x) P_N(x) \bar{\chi}_{n_k}(x) dx \\ &= \hat{f}(n_k) + \sum_{i=0}^{N-1} \hat{f}(\bar{\varphi}_i \chi_{n_k}) + \sum_{i,j=0, i\neq j}^{N-1} \sum_{l=1}^{p_i-1} \sum_{m=1}^{p_j-1} \hat{f}(\bar{\varphi}_i^l \bar{\varphi}_j^m \chi_{n_k}) + \ldots + \hat{f}\left(\prod_{i=0}^{N-1} \bar{\varphi}_i^{p_i-1} \chi_{n_k} \right). \end{aligned} \tag{4.7}$$

The characters appearing in the right hand side of (4.7) are of the form $\chi\chi_{n_k}$ wherein χ is such that deg χ is positive and $\leq N$. Observe that for each $j \in \mathbb{N}$ there are totally $m_{j-1}(p_j - 1) = m_j - m_{j-1}$ characters of degree j, namely $\chi_i \varphi_{j-1}^{a_{j-1}}$; $0 \leq i < m_{j-1}$; $1 \leq a_{j-1} \leq p_j - 1$ and they constitute $X_j - X_{j-1}$. Consequently, total number of characters of positive degree $\leq N$ is given by

$$(m_1 - m_0) + (m_2 - m_1) + \cdots + (m_N - m_{N-1}) = m_N - 1;$$

they are from χ_1 to $\chi_{m_N - 1}$ and they constitute $\cup_{j=1}^{m_N}(X_j - X_{j-1})$. It follows that when χ_{n_k} is multiplied by any character of positive degree $\leq N$ the resulting character χ_m is such that

$$n_k < m \leq n_k + m_N - 1 < n_k + m_N \leq n_k + q \leq n_{k+1},$$

because the lacunary Vilenkin Fourier series (4.5) of f has gaps (1.11) with $q \geq m_N$. Since $\hat{f}(n_k) \neq 0$, all the terms of the right hand side of (4.7) vanish except the first.

Now Let k be large enough and $l \in \mathbb{N} \cup \{0\}$ be such that $\hat{f}(n_k) \neq 0, m_l \leq n_k < m_{l+1}$ and $l > N$. Then, in view of (4.6)

$$\hat{f}(n_k) = (fP_N)\hat{\ }(n_k) = m_N \int_{G_N} f(x)\bar{\chi}_{n_k}(x)dx. \tag{4.8}$$

Since $n_k \geq m_l$ and the Haar measure is translation invariant, it follows (see, for example, [51, p. 114, Eq. (15)]) that

$$\int_{z_\alpha^{(l)} + G_l} \chi_{n_k}(x)dx = 0$$

for all $\alpha = 0, 1, ..., m_l - 1$; and hence

$$\int_{z_\alpha^{(l)} + G_l} \bar{\chi}_{n_k}(x)dx = 0, \quad (\alpha = 0, 1, ..., m_l - 1).$$

Now, put $L = \frac{m_l}{m_N} = (p_{N+1}p_{N+2} \cdots p_l)$ and define a step function g on G_N by $g(x) = f(z_\alpha^{(l)})$ for x in $z_\alpha^{(l)} + G_l$, $\alpha = 0, 1, ..., L - 1$. Then

$$\int_{G_N} g(x)\bar{\chi}_{n_k}(x)dx = \sum_{\alpha=0}^{L-1} f(z_\alpha^{(l)}) \int_{z_\alpha^{(l)} + G_l} \bar{\chi}_{n_k}(x)dx = 0.$$

Therefore in view of (4.8) we have

$$|\hat{f}(n_k)| = \left| m_N \int_{G_N} [f(x) - g(x)]\bar{\chi}_{n_k}(x)dx \right| \leq m_N \int_{G_N} |f(x) - g(x)|dx. \tag{4.9}$$

Now, by Jensen's inequality, for $c > 0$

$$\phi\left(m_N \cdot c \cdot \int_{G_N} |f(x) - g(x)|dx\right) \leq m_N \int_{G_N} \phi(c|f(x) - g(x)|)dx$$
$$= m_N \sum_{\alpha=0}^{L-1} \int_{z_\alpha^{(k)}+G_l} \phi(c|f(x) - f(z_\alpha^{(l)})|)dx. \tag{4.10}$$

Therefore

$$\phi\left(m_N \cdot c \cdot \int_{G_N} |f(x) - g(x)|dx\right) \leq m_N \sum_{\alpha=0}^{L-1} \int_{z_\alpha^{(l)}+G_l} \phi\left(\operatorname{osc}\left(cf; z_\alpha^{(l)} + G_l\right)\right) dx$$
$$= m_N \sum_{\alpha=0}^{L-1} \phi\left(\operatorname{osc}\left(cf; z_\alpha^{(l)} + G_l\right)\right) \frac{1}{m_l},$$

and hence

$$\phi\left(m_N \cdot c \cdot \int_{G_N} |f(x) - g(x)|dx\right) \leq \left(\frac{m_N}{m_l}\right) GF_\phi(cf; I). \tag{4.11}$$

Since ϕ is convex and $\phi(0) = 0$, we have $\phi(ax) \leq a\phi(x)$ for $0 < a < 1$ and for all $x \geq 0$. Therefore, choosing c in $(0, 1)$ so small that $(m_N \cdot GF_\phi(cf; I)) \leq 1$, one gets

$$|\hat{f}(n_k)| \leq m_N \int_{G_N} |f(x) - g(x)|dx \leq \left(\frac{m_N}{m_N \cdot c}\right) \phi^{-1}\left(\frac{1}{m_l}\right)$$

in view of (4.11) and (4.9). This shows that $\hat{f}(n_k) = O\left(\phi^{-1}(1/m_l)\right)$.

Finally, if G is bounded, there is a positive integer p_0 such that $p_l \leq p_0$ for all l. Thus $n_k < m_{l+1} = m_l \cdot p_{l+1} \leq m_l \cdot p_0$, which shows that $\frac{1}{m_l} \leq \frac{p_0}{n_k}$ and hence (4.11) gives

$$\phi\left(m_N \cdot c \cdot \int_{G_N} |f(x) - g(x)|dx\right) \leq \left(\frac{p_0 \cdot m_N}{n_k}\right) GF_\phi(cf; I). \tag{4.12}$$

Choosing now c in $(0, 1)$ so small that $(p_0 \cdot m_N \cdot GF_\phi(cf; I)) \leq 1$, one obtains

$$|\hat{f}(n_k)| \leq m_N \int_{G_N} |f(x) - g(x)|dx \leq \left(\frac{m_N}{m_N \cdot c}\right) \phi^{-1}\left(\frac{1}{n_k}\right)$$

in view of (4.12) and (4.9). □

Taking $\phi(x) = x^p$ $(p \geq 1)$ in Theorem 4.2.6, we get the following.

Corollary 4.2.7. *Let f and I be as in Theorem 4.2.6. Then $f \in \mathrm{GBF}^{(p)}(I)$ $(p \geq 1)$ implies $\hat{f}(n_k) = O(1/(m_l)^{\frac{1}{p}})$, where $m_l \leq n_k < m_{l+1}$. If, in addition, G is bounded then $\hat{f}(n_k) = O(1/(n_k)^{\frac{1}{p}})$.*

Remark 4.2.8. Since $\phi\mathrm{BF} \subset \phi\mathrm{GBF}$, Theorem 4.2.6 holds for functions in $\phi\mathrm{BF}$ also. Similarly, as $\mathrm{BF}^{(p)} \subset \mathrm{GBF}^{(p)}$, Corollary 4.2.7 holds for functions in $\mathrm{BF}^{(p)}$ also.

Theorem 4.2.9. *Let f and I be as in Theorem 4.2.6. Then $f \in \phi\Lambda GBF(I)$ implies*

$$\hat{f}(n_k) = O\left(\phi^{-1}\left(1\Big/\left(\sum_{j=1}^{m_l}\frac{1}{\lambda_j}\right)\right)\right),$$

where $m_l \leq n_k < m_{l+1}$. If, in addition, G is bounded then

$$\hat{f}(n_k) = O\left(\phi^{-1}\left(1\Big/\left(\sum_{j=1}^{n_k}\frac{1}{\lambda_j}\right)\right)\right).$$

Proof. Proceeding as in the proof of Theorem 4.2.6, for $c > 0$, we get (4.9) and (4.10). Let α_i, $i = 0, 1, ..., L-1$, denote a rearrangement of $0, 1, ..., L-1$ such that $\{b_i\}_{i=0}^{L-1}$ is non-increasing, where

$$b_i = \int_{z_{\alpha_i}^{(l)}+G_l} \phi\left(c|f(x) - f(z_{\alpha_i}^{(l)})|\right) dx$$

for all i. For each $i = 0, 1, ..., L-1$ put $a_i = \frac{1}{\lambda_{i+1}\theta_L}$, where $\theta_n = \sum_{j=1}^{n}\frac{1}{\lambda_j}$, for all $n \in \mathbb{N}$. Then $\{a_i\}_{i=0}^{L-1}$ is non-increasing and $\sum_{i=0}^{L-1} a_i = 1$. Therefore by Lemma 4.1.20

$$\begin{aligned}
\sum_{\alpha=0}^{L-1}\int_{z_{\alpha}^{(l)}+G_l}\phi(c|f(x) - f(z_{\alpha}^{(l)})|)dx &= \sum_{i=0}^{L-1} b_i \leq L\sum_{i=0}^{L-1} a_i b_i \\
&= \frac{L}{\theta_L}\sum_{i=0}^{L-1}\int_{z_{\alpha_i}^{(l)}+G_l}\left(\frac{\phi(c|f(x) - f(z_{\alpha_i}^{(l)})|)}{\lambda_{i+1}}\right)dx \\
&\leq \frac{L}{\theta_L}\sum_{i=0}^{L-1}\int_{z_{\alpha_i}^{(l)}+G_l}\left(\frac{\phi(\mathrm{osc}(cf; z_{\alpha_i}^{(l)} + G_l))}{\lambda_{i+1}}\right)dx \\
&= \frac{m_l}{m_N\theta_L}\sum_{i=0}^{L-1}\frac{\phi(\mathrm{osc}(cf; z_{\alpha_i}^{(l)} + G_l))}{\lambda_{i+1}}\cdot\frac{1}{m_l} \\
&\leq \frac{GF_{\phi\Lambda}(cf; I)}{m_N\theta_L}.
\end{aligned}$$

Therefore

$$\sum_{\alpha=0}^{L-1} \int_{z_\alpha^{(l)}+G_l} \phi(c|f(x)-f(z_\alpha^{(l)})|)dx \le \frac{GF_{\phi\Lambda}(cf;I)}{\theta_{m_l}}, \tag{4.13}$$

since $\{\lambda_i\}$ is non-decreasing. In view of (4.13) and (4.10) we get

$$\phi\left(m_N \cdot c \cdot \int_{G_N} |f(x)-g(x)|dx\right) \le \frac{m_N \cdot GF_{\phi\Lambda}(cf;I)}{\theta_{m_l}}. \tag{4.14}$$

Since ϕ is convex and $\phi(0)=0$, we can choose c in $(0,1)$ so small such that $(m_N \cdot GF_{\phi\Lambda}(cf;I)) \le 1$. This proves, in view of (4.14) and (4.9), that $\hat{f}(n_k) = O(\phi^{-1}(1/\theta_{m_l}))$.

Finally, if G is bounded, $\frac{1}{\theta_{m_l}} \le \frac{p_0}{\theta_{n_k}}$ and hence by (4.10) and (4.13)

$$\phi\left(m_N \cdot c \cdot \int_{G_N} |f(x)-g(x)|dx\right) \le \frac{m_N \cdot p_0 \cdot GF_{\phi\Lambda}(cf;I)}{\theta_{n_k}}.$$

Choosing now $c \in (0,1)$ small enough such that $(m_N \cdot p_0 \cdot GF_{\phi\Lambda}(cf;G)) \le 1$, we then get

$$\int_{G_N} |f(x)-g(x)|dx \le \left(\frac{1}{m_N \cdot c}\right)\phi^{-1}\left(\frac{1}{\theta_{n_k}}\right),$$

and hence the theorem in view of (4.9). □

Taking $\phi(x)=x^p$ $(p \ge 1)$ in Theorem 4.2.9, we get the following result, which is the Vilenkin group analogue of the result of Patadia and Vyas [41, Theorem 5].

Corollary 4.2.10. *Let f and I be as in Theorem 4.2.6. Then $f \in \Lambda\mathrm{GBF}^{(p)}(I)$ $(p \ge 1)$ implies*

$$\hat{f}(n_k) = O\left(1 \Big/ \left(\sum_{j=1}^{m_l} \frac{1}{\lambda_j}\right)^{\frac{1}{p}}\right),$$

where $m_l \le n_k < m_{l+1}$. If, in addition, G is bounded then

$$\hat{f}(n_k) = O\left(1 \Big/ \left(\sum_{j=1}^{n_k} \frac{1}{\lambda_j}\right)^{\frac{1}{p}}\right).$$

Remark 4.2.11. Since $\phi\Lambda\mathrm{BF} \subset \phi\Lambda\mathrm{GBF}$, Theorem 4.2.9 holds for functions in $\phi\Lambda\mathrm{BF}$ also. Similarly, as $\Lambda\mathrm{BF}^{(p)} \subset \Lambda\mathrm{GBF}^{(p)}$, Corollary 4.2.10 holds for functions in $\Lambda\mathrm{BF}^{(p)}$ also. In fact, in [14], we have proved results for the functions of the class $\phi\Lambda\mathrm{BF}$ and now observed that the same proof works for functions of the class $\phi\Lambda\mathrm{GBF}$.

Remark 4.2.12. Observe that $n_k = k$ for all $k \implies q = 1$ in (1.11) $\implies I$ is of Haar measure 1 in above theorems $\implies I = G$; and one gets corresponding results (of Section 4.1) for non-lacunary Vilenkin Fourier series [16]. On the other hand, if the Vilenkin Fourier series (4.5) of $f \in \mathrm{L}^1(G)$ has gaps (1.5) then above results hold if the coset I is just of positive measure. Because if $|I| > 0$, by the form of I, $|I| = 1/m_N$ where $N \in \mathbb{N}$ can be taken as large as required. In view of (1.5), one gets $(n_{k+1} - n_k) \geq m_N$ for all $k \geq k_0$ for a suitable $k_0 = k_0(N)$. Then adding to $f(x)$ the Vilenkin polynomial $\sum_{j=1}^{k_0}(-\hat{f}(n_j))\chi_{n_j}(x)$ one gets a function g whose Fourier series is lacunary of the form (4.5) having gaps (1.11) with $q = m_N$ and results are true for g. Since f and g differ by a polynomial, results are true for f as well. Our results thus interpolates lacunary and non-lacunary results concerning order of magnitude of Fourier coefficients—displaying beautiful interconnection between types of lacunarity (as determined by q in (1.11)) and localness of hypothesis to be satisfied by the generic function (as determined by the q-dependent length of I).

Chapter 5

Absolute Convergence Of Vilenkin Fourier Series

5.1 Absolute convergence of non-lacunary Fourier series of functions of generalized bounded fluctuation on Vilenkin groups

In Chapter 4 we have studied the order of magnitude of Vilenkin Fourier coefficients of functions of various classes of functions of generalized bounded fluctuation. Here we study the absolute convergence of Vilenkin Fourier series for functions of these classes. Our result (see Theorem 5.1.3) generalizes the earlier result Theorem 1.2.42 of Uno [66].

In what follows, it is assumed that G is a bounded Vilenkin Group and f is a complex-valued function on G. Vilenkin Fourier coefficients are defined by (1.23) in Chapter 1, Section 1.1 (D) and various concepts of generalized bounded fluctuation are defined in Section 4.1. Here following results are obtained. Main results of this section are Theorems 5.1.3 and 5.1.7. To prove them, following lemmas due to Vilenkin N. Ja. and Rubinstěin A. I. [67, p. 5] and Stečhkin [60, Lemma 2] are needed.

Lemma 5.1.1. *For each $N = 0, 1, 2, ...$ and $k \geq m_N$ we have*

(a) $\int_{G_N} \chi_k(h)dh = 0$;

(b) $\int_{G_N} |\chi_k(h) - 1|^2 dh = 2\int_{G_N}[1 - \mathrm{Re}\chi_k(h)]dh = 2|G_N| = \frac{2}{m_N}$.

Lemma 5.1.2. *If $u_n \geq 0$, for $n \in \mathbb{N}$, $u_n \not\equiv 0$ and a function $F(u)$ is concave, increasing, and $F(0) = 0$, then $\sum_1^\infty F(u_n) \leq 2\sum_1^\infty F(\frac{1}{n}\sum_{k=n}^\infty u_k)$.*

Theorem 5.1.3. *If $f \in \Lambda\mathrm{GBF}^{(p)}$, $1 \leq p < 2r$, $1 \leq r < \infty$ and*

$$\sum_{n=0}^{\infty}\left[\frac{(m_n)^{2/\beta-1}\left(\omega^{(p+(2-p)r')}(f,n)\right)^{2-p/r}}{\left(\sum_{j=1}^{m_n}\frac{1}{\lambda_j}\right)^{1/r}}\right]^{\beta/2} < \infty,$$

then

$$\sum_{k=0}^{\infty}|\hat{f}(k)|^\beta < \infty, \qquad (0 < \beta \leq 2). \tag{5.1}$$

Proof. Let $M \in \mathbb{N}$ be fixed and let $N \in \mathbb{N}$ be the integer such that $m_N \leq M < m_{N+1}$. For each $\alpha = 0, 1, ..., m_N - 1$ and $h \in G_N$ put

$$f_\alpha(x) = f\left(x + z_\alpha^{(N)} + h\right) - f\left(x + z_\alpha^{(N)}\right), \ \ \forall x \in G.$$

Then for each $n \geq 0$ we have

$$\hat{f}_\alpha(n) = \hat{f}(n)\chi_n\left(z_\alpha^{(N)} + h\right) - \hat{f}(n)\chi_n\left(z_\alpha^{(N)}\right) = \hat{f}(n)\chi_n\left(z_\alpha^{(N)}\right)(\chi_n(h) - 1).$$

Since $f \in \Lambda\mathrm{GBF}^{(p)}$ for any $x \in G = G_0$ we see that

$$\begin{aligned}
|f(x)|^p &= |f(0) + f(x) - f(0)|^p \\
&\leq 2^p|f(0)|^p + 2^p|f(x) - f(0)|^p \\
&= 2^p|f(0)|^p + 2^p\lambda_1\left(\frac{|f(x) - f(0)|^p}{\lambda_1}\right) \\
&\leq 2^p|f(0)|^p + 2^p\lambda_1\left(\mathrm{osc}\left(f; z_0^{(0)} + G_0\right)\right)^p \\
&\leq 2^p|f(0)|^p + 2^p\lambda_1\left(\Lambda GF_p(f;G)\right)^p.
\end{aligned}$$

Thus f is bounded on G and hence $f \in \mathrm{L}^2(G)$. As a result each $f_\alpha \in \mathrm{L}^2(G)$ and so by Parseval's equality (since $|\chi_n(z_\alpha^{(N)})| = 1$) we have

$$B(h) \equiv \sum_{n=0}^{\infty}|\hat{f}(n)|^2|\chi_n(h) - 1|^2 = ||f_\alpha||_2^2, \ \ \forall\alpha. \tag{5.2}$$

Now, suppose $r > 1$ and set $2 = \frac{p+(2-p)r'}{r'} + \frac{p}{r}$; then using the Hölder's inequality we get

$$\begin{aligned}
||f_\alpha||_2^2 &= \int_G |f_\alpha(x)|^2 dx \\
&= \int_G |f_\alpha(x)|^{(\frac{p+(2-p)r'}{r'}+\frac{p}{r})} dx \\
&= \int_G \left(|f_\alpha(x)|^{(p+(2-p)r')}\right)^{1/r'} \left(|f_\alpha(x)|^p\right)^{1/r} dx \\
&\leq \left\{\int_G |f_\alpha(x)|^{(p+(2-p)r')} dx\right\}^{1/r'} \left\{\int_G |f_\alpha(x)|^p dx\right\}^{1/r} \\
&\leq (\Omega_N)^{1/r} \left(\int_G |f_\alpha(x)|^p dx\right)^{1/r},
\end{aligned}$$

where $\Omega_N = \left(\omega^{(p+(2-p)r')}(f,N)\right)^{2r-p}$ since $h \in G_N$. This together with (5.2) implies

$$(B(h))^r \leq \Omega_N \int_G |f_\alpha(x)|^p dx, \tag{5.3}$$

for all $\alpha = 0, 1, ..., m_N - 1$. Since the left hand side of (5.3) is independent of α, multiplying both the sides of it by $(1/\lambda_{\alpha+1})$ and taking summation over α, we get

$$(B(h))^r \theta_{m_N} \leq \Omega_N \int_G \left(\sum_{\alpha=0}^{m_N-1} \frac{|f_\alpha(x)|^p}{\lambda_{\alpha+1}}\right) dx,$$

where $\theta_t = \sum_{j=1}^{t}(1/\lambda_j) = \sum_{j=0}^{t-1}(1/\lambda_{j+1})$, for all $t \in \mathbb{N}$; and hence

$$B(h) \leq \left(\frac{\Omega_N}{\theta_{m_N}}\right)^{1/r} \left\{\int_G \left(\sum_{\alpha=0}^{m_N-1} \frac{|f_\alpha(x)|^p}{\lambda_{\alpha+1}}\right) dx\right\}^{1/r}.$$

Integrating both sides of this inequality over G_N with respect to h we get

$$\int_{G_N} B(h) dh \leq \left(\frac{\Omega_N}{\theta_{m_N}}\right)^{1/r} \int_{G_N} \left\{\int_G \sum_{\alpha=0}^{m_N-1} \frac{|f_\alpha(x)|^p}{\lambda_{\alpha+1}} dx\right\}^{1/r} dh. \tag{5.4}$$

Now, for any $h \in G_N$ and any $x \in G$ the points $x + z_\alpha^{(N)} + h$ and $x + z_\alpha^{(N)}$ lie in the coset $x + z_\alpha^{(N)} + G_N$ of G_N in G and hence

$$|f_\alpha(x)| = |f(x + z_\alpha^{(N)} + h) - f(x + z_\alpha^{(N)})| \leq \text{osc}(f, x + z_\alpha^{(N)} + G_N). \tag{5.5}$$

Since $f \in \Lambda\mathrm{GBF}^{(p)}$, for any $h \in G_N$ and $x \in G$, in view of (5.5), we have

$$\sum_{\alpha=0}^{m_N-1} \frac{|f_\alpha(x)|^p}{\lambda_{\alpha+1}} \leq \sum_{\alpha=0}^{m_N-1} \frac{\left(\mathrm{osc}(f, x + z_\alpha^{(N)} + G_N)\right)^p}{\lambda_{\alpha+1}} \leq (\Lambda G F_p(f; G))^p\,, \tag{5.6}$$

because for any $x \in G$, the finite sequence of cosets $\{x + z_\alpha^{(N)} + G_N : \alpha = 0, 1, ..., m_N - 1\}$ is a rearrangement of the sequence $\{z_\alpha^{(N)} + G_N : \alpha = 0, 1, ..., m_N - 1\}$. Further, from (5.2),

$$\int_{G_N} B(h)dh \geq \sum_{k=M}^{\infty} |\hat{f}(k)|^2 \int_{G_N} |\chi_k(h) - 1|^2 dh = \left(\frac{2}{m_N}\right) \sum_{k=M}^{\infty} |\hat{f}(k)|^2, \tag{5.7}$$

in view of Lemma 5.1.1, because $k \geq M$ implies $k \geq m_N$. Using (5.6) and (5.7) in (5.4) we get

$$R_M \equiv \sum_{k=M}^{\infty} |\hat{f}(k)|^2 = O\left[\left(\frac{\Omega_N}{\theta_{m_N}}\right)^{1/r}\right]. \tag{5.8}$$

Applying Lemma 5.1.2 with $u_k = |\hat{f}(k)|^2$ and $F(u) = u^{\beta/2}$ we get

$$\sum_{k=1}^{\infty} |\hat{f}(k)|^\beta = \sum_{k=1}^{\infty} F(u_k) \leq 2 \sum_{k=1}^{\infty} F\left(\frac{1}{k} \sum_{j=k}^{\infty} |\hat{f}(j)|^2\right) = 2 \sum_{k=1}^{\infty} F\left(\frac{R_k}{k}\right). \tag{5.9}$$

Thus in view of (5.8) we get

$$\begin{aligned}
\sum_{k=1}^{\infty} |\hat{f}(k)|^\beta &= O(1) \sum_{k=1}^{\infty} \left(\frac{R_k}{k}\right)^{\beta/2} \\
&= O(1) \sum_{n=0}^{\infty} \sum_{k=m_n}^{m_{n+1}-1} \left(\frac{R_k}{k}\right)^{\beta/2} \\
&= O(1) \sum_{n=0}^{\infty} \sum_{k=m_n}^{m_{n+1}-1} \left[\frac{(\Omega_n)^{1/r}}{m_n(\theta_{m_n})^{1/r}}\right]^{\beta/2} \\
&= O(1) \sum_{n=0}^{\infty} \left[\frac{(\Omega_n)^{1/r}}{m_n(\theta_{m_n})^{1/r}}\right]^{\beta/2} (m_{n+1} - m_n) \\
&= O(1) \sum_{n=0}^{\infty} \left[\frac{(m_n)^{2/\beta-1}\left(\omega^{(p+(2-p)r')}(f, n)\right)^{2-p/r}}{\left(\sum_{j=1}^{m_n} \frac{1}{\lambda_j}\right)^{1/r}}\right]^{\beta/2} < \infty,
\end{aligned}$$

because G is bounded and by the assumption of theorem. This completes the proof of the theorem for $r > 1$.

For the case $r = 1$, $s = \infty$, simply note that

$$|f_\alpha(x)|^2 = |f_\alpha(x)|^{2-p}|f_\alpha(x)|^p \le (\omega_N(f))^{2-p}|f_\alpha(x)|^p,$$

because

$$|f_\alpha(x) = f(x + z_\alpha^{(N)} + h) - f(x + z_\alpha^{(N)})| \le \omega_N(f)$$

since $h \in G_N$; and proceed as above. □

Remark 5.1.4. Since $\Lambda\mathrm{BF}^{(p)} \subset \Lambda\mathrm{GBF}^{(p)}$, Theorem 5.1.3 obviously holds true for functions in $\Lambda\mathrm{BF}^{(p)}$ also.

Taking $\beta = 1$ in Theorem 5.1.3 we obtain

Corollary 5.1.5. *Let $1 \le r < \infty$ and $1 \le p < 2r$. If $f \in \Lambda\mathrm{GBF}^{(p)}$ satisfies*

$$\sum_{n=0}^{\infty} \frac{(m_n)^{1/2}\left(\omega^{(p+(2-p)r')}(f,n)\right)^{1-p/2r}}{\left(\sum_{j=1}^{m_n}\frac{1}{\lambda_j}\right)^{1/2r}} < \infty,$$

then (5.1) *holds for $\beta = 1$.*

Remark 5.1.6. Corollary 5.1.5 is a result due to Yoshikazu Uno [66].

Theorem 5.1.7. *If $f \in \phi\Lambda\mathrm{GBF}$, $1 \le p < 2r$, $1 \le r < \infty$ and*

$$\sum_{k=1}^{\infty}\left[(m_n)^{2/\beta-1}\left\{\phi^{-1}\left(\frac{\left(\omega^{(p+(2-p)r')}(f,n)\right)^{2r-p}}{\sum_{j=1}^{m_n}\frac{1}{\lambda_j}}\right)\right\}^{1/r}\right]^{\beta/2} < \infty,$$

then (5.1) *holds, in which ϕ is a Δ_2-function (that is, there is a constant $d \ge 2$ such that $\phi(2x) \le d\phi(x)$, $\forall x \ge 0$).*

Proof. Since $f \in \phi\Lambda\mathrm{GBF}$ for any $x \in G$ we have

$$|f(x)| \le |f(0)| + C\phi^{-1}(\Lambda GF_\phi(f;G)).$$

Thus f is bounded on G and hence $f \in \mathrm{L}^2(G)$. For $r > 1$, proceeding as in the proof of Theorem 5.1.3 we get (5.3). Since multiplying f by a positive constant alters $\omega^{(p)}(f,n)$ (see (1.27)) by the same constant, and ϕ is Δ_2, we may assume that $|f(x)| \leq \frac{1}{2}$ for all x. But then from (5.3) we get

$$(B(h))^r \leq \Omega_N \int_G |f_\alpha(x)|dx, \qquad (\alpha = 0, 1, ..., m_N - 1).$$

Since $\phi(2x) \leq d\phi(x), \forall x \geq 0$, we get $\phi(ax) \leq d^{\log_2 a+1}\phi(x), \forall x \geq 0, \forall a \geq 1$. For, using induction on n we get

$$\phi(2^n x) \leq d^n \phi(x), \forall x \geq 0, \forall n \in \mathbb{N}.$$

Next, if $a \geq 1$ is any real number, choosing $n \in \mathbb{N}$ such that $2^{n-1} \leq a < 2^n$ we get $0 < \frac{a}{2^n} < 1$. Therefore for all $x \geq 0$ we have

$$\phi(ax) = \phi\left(\frac{a}{2^n} \cdot 2^n x\right) \leq \frac{a}{2^n}\phi(2^n x) \leq \frac{a}{2^n} d^n \phi(x) < d^n \phi(x) \leq d^{\log_2 a+1}\phi(x).$$

Since $\Omega_N \geq 0$, if $\Omega_N < 1$ then we get

$$\phi((B(h))^r) \leq \phi\left(\Omega_N \int_G |f_\alpha(x)|dx\right) \leq \Omega_N \phi\left(\int_G |f_\alpha(x)|dx\right).$$

Further when $\Omega_N \geq 1$, as above

$$\begin{aligned}
\phi((B(h))^r) &\leq \phi\left(\Omega_N \int_G |f_\alpha(x)|dx\right) \\
&\leq d^{\log_2 \Omega_N + 1}\phi\left(\int_G |f_\alpha(x)|dx\right) \\
&= d(\Omega_N)^{\log_2 d}\phi\left(\int_G |f_\alpha(x)|dx\right) \\
&= d(\Omega_N)^{\log_2 d - 1}\Omega_N \phi\left(\int_G |f_\alpha(x)|dx\right) \\
&\leq d\Omega_N \phi\left(\int_G |f_\alpha(x)|dx\right),
\end{aligned}$$

in view of the fact that $(\Omega_N)^{\log_2 d-1} \leq 1$, as $|f(x)| \leq \frac{1}{2}, \forall x$ and $\log_2 d - 1 \geq 0$. Since $d \geq 2$, in either case

$$\phi\left((B(h))^r\right) \leq d\Omega_N \phi\left(\int_G (|f_\alpha(x)|)dx\right) \leq d\Omega_N \int_G \phi(|f_\alpha(x)|)dx,$$

in view of the Jensen's inequality. Now multiplying both the sides of this inequality by $(1/\lambda_{\alpha+1})$ and taking summation over $\alpha = 0, 1, ..., m_N - 1$ we get

$$\phi\left((B(h))^r\right) \leq d\left(\frac{\Omega_N}{\theta_{m_N}}\right) \int_G \left(\sum_{\alpha=0}^{m_N-1} \frac{\phi(|f_\alpha(x)|)}{\lambda_{\alpha+1}}\right) dx. \qquad (5.10)$$

Since $f \in \phi\Lambda\text{GBF}$ and ϕ is increasing, for all $h \in G_N$ and $x \in G$ we have

$$\sum_{\alpha=0}^{m_N-1} \frac{\phi(|f_\alpha(x)|)}{\lambda_{\alpha+1}} \leq \sum_{\alpha=0}^{m_N-1} \frac{\phi\left(\text{osc}(f; x + z_\alpha^{(N)} + G_N)\right)}{\lambda_{\alpha+1}} \leq \Lambda GF_\phi(f; G). \tag{5.11}$$

Using (5.11) in (5.10) we get $\phi\left((B(h))^r\right) \leq C\left(\frac{\Omega_N}{\theta_{m_N}}\right)$, where C is a constant such that $C \geq 1$. Thus $(B(h))^r \leq \phi^{-1}\left\{C\left(\frac{\Omega_N}{\theta_{m_N}}\right)\right\} \leq C\phi^{-1}\left(\frac{\Omega_N}{\theta_{m_N}}\right)$ and therefore

$$B(h) = O\left[\left\{\phi^{-1}\left(\frac{\Omega_N}{\theta_{m_N}}\right)\right\}^{1/r}\right].$$

Integrating both sides of this inequality over G_N with respect to h, in view of (5.7) we get

$$R_M \equiv \sum_{k=M}^{\infty} |\hat{f}(n_k)|^2 \leq \left(\frac{m_N}{2}\right) \int_{G_N} B(h)dh = O\left[\left\{\phi^{-1}\left(\frac{\Omega_N}{\theta_{m_N}}\right)\right\}^{1/r}\right].$$

Thus in view of (5.9) we get

$$\begin{aligned}
\sum_{k=1}^{\infty} |\hat{f}(k)|^\beta &= O(1) \sum_{k=1}^{\infty} \left(\frac{R_k}{k}\right)^{\beta/2} \\
&= O(1) \sum_{n=0}^{\infty} \sum_{k=m_n}^{m_{n+1}-1} \left(\frac{R_k}{k}\right)^{\beta/2} \\
&= O(1) \sum_{n=0}^{\infty} \sum_{k=m_n}^{m_{n+1}-1} \left[\frac{1}{m_n}\left\{\phi^{-1}\left(\frac{\Omega_n}{\theta_{m_n}}\right)\right\}^{1/r}\right]^{\beta/2} \\
&= O(1) \sum_{n=0}^{\infty} \left[\frac{1}{m_n}\left\{\phi^{-1}\left(\frac{\Omega_n}{\theta_{m_n}}\right)\right\}^{1/r}\right]^{\beta/2} (m_{n+1} - m_n) \\
&= O(1) \sum_{n=0}^{\infty} \left[(m_n)^{2/\beta-1}\left\{\phi^{-1}\left(\frac{\Omega_n}{\theta_{m_n}}\right)\right\}^{1/r}\right]^{\beta/2} < \infty,
\end{aligned}$$

since G is bounded and in view of the assumption of the theorem. This completes the proof of the theorem for $r > 1$. For the case $r = 1$, $s = \infty$, the proof is similar as that of Theorem 5.1.3. □

Remark 5.1.8. Since $\phi\Lambda\text{BF} \subset \phi\Lambda\text{GBF}$, Theorem 5.1.7 obviously holds true for functions in $\phi\Lambda\text{BF}$ also. With $\beta = 1$, Theorem 5.1.3 and Theorem 5.1.7 are bounded Vilenkin group analogue of the corresponding circle group results of Schramm and Waterman [56].

5.2 The Wiener-Ingham type inequality and its application to harmonic analysis

Wiener's investigations of properties of gap series in terms of separation condition on exponents using real and complex variable methods is classical. While attempting to prove Fabry's version of the classical Hadamard gap theorem concerning functions with natural boundary, Wiener proved an important trigonometric inequality concerning lacunary trigonometric sums. This inequality was later on made precise by Ingham and it is given below (see [80, Vol. I, p. 222]).

Theorem 5.2.1. *Consider a finite lacunary trigonometric sum*

$$f(x) = \sum_{-N}^{N} A_k \exp(in_k x) \quad (n_{-k} = -n_k) \tag{5.12}$$

where $\{A_k\}$ *is a sequence of complex numbers and* $\{n_k\}$ *is a sequence of integers satisfying the small gap condition* (1.11)*. If* I *is any subinterval of* $[-\pi, \pi]$ *of length* $|I| = 2\pi(1+\delta)/q > 2\pi/q$ *then*

$$\sum |A_k|^2 \leq A_\delta |I|^{-1} ||f||_{2,I}^2, \tag{5.13}$$

$$|A_k| \leq A_\delta |I|^{-1} ||f||_{1,I}, \tag{5.14}$$

where A_δ *depends only on* δ *and* $||f||_{p,I} = ||f\chi_I||_p$ *for* $p \geq 1$ *in which* χ_I *is the characteristic function of* I *and* $||\cdot||_p$ *is the usual* L^p *norm. The results hold for infinite trigonometric sums if the series* (5.12) *converges uniformly.*

This theorem is actually true for nonharmonic trigonometric sums (that is, when n_k's are real) and exhibits a beautiful interconnection between the 'type of lacunarity' in the trigonometric sums and the 'localness of the hypothesis' satisfied by f. It may be observed that for infinite sums the requirement of their uniform convergence to $f(x)$ can be replaced by a weaker hypothesis: "$\{A_k\}$ satisfies

$$\sum_{-\infty}^{\infty} |A_k| s^{|n_k|} < \infty \qquad (0 < s < 1) \tag{5.15}$$

and f is defined on I by $\lim_{s\to 1} ||f - f_s||_{2,I} = 0$ where $f_s(x) = \sum_{-\infty}^{\infty} A_k e^{in_k x} s^{|n_k|}$ " — in view of the facts that the symmetric partial sums of $f_s(x)$, to which the theorem can be applied, converges uniformly to $f(x)$ and $||f_s||_{2,I} \to ||f||_{2,I}$ as $s \to 1$. Hence, in particular, if the Fourier series of $f \in \mathrm{L}^1[-\pi,\pi]$ is lacunary of the form $\sum \hat{f}(n_k)e^{in_kx}$ with n_k satisfying (1.11) and $I \subset [-\pi,\pi]$ is an interval of length $|I| > 2\pi/q$ then Theorem 5.2.1 can be applied to the Fourier series of f if $f \in \mathrm{L}^2(I)$ — showing that the lacunary Fourier series with small gaps behave well on $[-\pi,\pi]$ if it behave well on any interval $I \subset [-\pi,\pi]$ of length $> 2\pi/q$. Since several results of Sidon, Zygmund, Stečhkin and others, showing that the lacunary Fourier series on the circle group $\mathbb{T}$ with Hadamard type big gaps behave well on $\mathbb{T}$ whenever they behave well on a subset E of $\mathbb{T}$ positive measure, are extended to the case of a general compact abelian group (refer [31]), it is natural to inquire whether the above phenomenon for the lacunary Fourier series with small gaps as illustrated by the Wiener-Ingham Theorem 5.2.1 can be extended to other compact abelian groups. It is shown here that type of Theorem 5.2.1 can be extended to the case of Vilenkin groups G. Ingham's idea in the case of $\mathbb{T}$ in proving Theorem 5.2.1 was to select a suitable function $P(x)$ vanishing outside a concerned subinterval I and possessing absolutely convergent Fourier series, and then to show that $||\ |f^2|P\ ||_{1,I}$ majorizes a fixed multiple of $\sum |A_k|^2$. However, the technique employed here is entirely different: one selects a trigonometric polynomial P vanishing outside a concerned subinterval I, having constant term 1 and of degree less than the gaps in the Fourier series of f; and then shows that $\hat{f}(\chi) = (fP)\hat{\ }(\chi)$ for concerned characters χ — the technique which works with remarkable ease in the present setting unlike in the case of $\mathbb{T}$. The Wiener-Ingham type inequality established in the setting of Vilenkin groups is then applied to estimate the tails of $\sum |\hat{f}(n_k)|^2$ for the Fourier series of $f \in \mathrm{L}^2$ with small gaps in terms of local mean modulus of continuity of f. Also, we get the analogues on G of Patadia's earlier results on $\mathbb{T}$ extending the results concerning the absolute convergence of *non-lacunary* Fourier series on G to the *lacunary* Fourier series with small gaps.

We prove the following analogue of the *Wiener-Ingham inequality* of Theorem 5.2.1.

Theorem 5.2.2. *Let f and I be as in Theorem 4.2.6. If $f \in \mathrm{L}^p(I)$, $1 < p \le 2$, then*

$$\left(\sum_{k=1}^{\infty} |\hat{f}(n_k)|^{p'}\right)^{1/p'} \le |I|^{-1}||f||_{p,I}, \tag{5.16}$$

$$|\hat{f}(n_k)| \le |I|^{-1}||f||_{1,I}. \tag{5.17}$$

Proof. By our assumption, we have $|I| = 1/m_N$ with $m_N \le q$ because $|I| \ge 1/q$. Consider the polynomial $P \equiv P_N$ of degree N and having the constant term 1 constructed in the following way:

$$P(x) = \prod_{k=0}^{N-1} \left(1 + \overline{\varphi_k(y_0)}\varphi_k(x) + \overline{(\varphi_k(y_0))^2}(\varphi_k(x))^2 + ... + \overline{(\varphi_k(y_0))^{p_{k+1}-1}}(\varphi_k(x))^{p_{k+1}-1}\right).$$

Put $y_0 = \sum_{i=0}^{\infty} c_i x_i$, $0 \le c_i < p_{i+1}$, $i = 0, 1, 2, ...$. Then for each $k = 0, 1, ..., N-1$,

$$\begin{aligned} x \in I \Rightarrow x = \sum_{i=0}^{\infty} b_i x_i \text{ with } 0 \le b_i < p_{i+1} \text{ for all } i, \text{ and } b_i = c_i \text{ for } i = 0, 1, ..., N-1 \\ \Rightarrow \varphi_k(x) = \varphi_k\left(\sum_{i=0}^{N-1} c_i x_i\right) \varphi_k\left(\sum_{i=N}^{\infty} b_i x_i\right) \\ \Rightarrow \varphi_k(x) = \varphi_k\left(\sum_{i=0}^{N-1} c_i x_i\right) \cdot 1 \\ \Rightarrow \varphi_k(x) = \varphi_k\left(\sum_{i=0}^{N-1} c_i x_i\right) \varphi_k\left(\sum_{i=N}^{\infty} c_i x_i\right) = \varphi_k(y_0) \end{aligned}$$

because $\varphi_k \in X_{k+1} \subset X_N$ for $k = 0, 1, ..., N-1$, and $\sum_{i=N}^{\infty} b_i x_i, \sum_{i=N}^{\infty} c_i x_i \in G_N$. Therefore,

$$x \in I \Rightarrow P(x) = P(y_0) = \prod_{k=0}^{N-1} p_{k+1} = m_N. \tag{5.18}$$

Also, $x \notin I$ implies $x = \sum_{i=0}^{\infty} b_i x_i$ with $0 \le b_i < p_{i+1}$ for all i and $b_j \ne c_j$ for some $j \in \{0, 1, ..., N-1\}$. If $k = \min\{j : b_j \ne c_j\}$ then

$$\overline{\varphi_k(y_0)}\varphi_k(x) = \overline{\varphi_k(c_k x_k)}\varphi_k(b_k x_k) = (\varphi_k(x_k))^{b_k} \left(\overline{\varphi_k(x_k)}\right)^{c_k} = \left(e^{2\pi i/p_{k+1}}\right)^{b_k - c_k} = t_k,$$

say; because $\sum_{i=0}^{k-1} c_i x_i = \sum_{i=0}^{k-1} b_i x_i$, $\varphi_k \in X_{k+1}$ and $\sum_{i=k+1}^{\infty} c_i x_i, \sum_{i=k+1}^{\infty} b_i x_i \in G_{k+1}$. It follows that the k^{th} factor of the product comprising $P(x)$ is $(1 + t_k + (t_k)^2 + ... + (t_k)^{p_{k+1}-1}) = 0$ because $b_k - c_k \ne 0$ implies $t_k \ne 1$ is the ${p_{k+1}}^{th}$ root of unity. Thus

$$x \notin I \text{ implies } P(x) = 0. \tag{5.19}$$

In view of (5.18) and (5.19) we have $fP \in \mathrm{L}^p(G)$ because $f \in \mathrm{L}^p(I)$; and hence an application of Hausdorff-Young inequality gives us

$$\left(\sum_{\chi \in X} |(fP)\hat{\ }(\chi)|^{p'}\right)^{1/p'} \le ||fP||_{p,G}.$$

Therefore

$$\left(\sum_{k=1}^{\infty}|(fP)\hat{\ }(n_k)|^{p'}\right)^{1/p'} \leq m_N||f||_{p,I}, \tag{5.20}$$

where for each $k = 1, 2, ...$ we have

$$\begin{aligned}(fP)\hat{\ }(n_k) &= \int_G f(x)P(x)\overline{\chi_{n_k}(x)}dx \\ &= \hat{f}(n_k) + \sum_{i=0}^{N-1}\overline{\varphi_i(y_0)}\hat{f}(\bar{\varphi}_i\chi_{n_k}) \\ &\quad + \sum_{\substack{i,j=0 \\ i\neq j}}^{N-1}\sum_{u=1}^{p_{i+1}-1}\sum_{m=1}^{p_{j+1}-1}\overline{(\varphi_i(y_0))^u(\varphi_j(y_0))^m}\hat{f}\left(\overline{(\varphi_i)^u(\varphi_j)^m}\chi_{n_k}\right) \\ &\quad + ... + \prod_{i=0}^{N-1}\overline{(\varphi_i(y_0))^{p_{i+1}-1}}\hat{f}\left(\left(\prod_{i=0}^{N-1}\overline{(\varphi_i)^{p_{i+1}-1}}\right)\chi_{n_k}\right). \end{aligned} \tag{5.21}$$

The first term in the right hand side of (5.21) is nonzero because the Fourier series of f is lacunary of the form (4.5), and the characters appearing in the other terms are of the form $\chi\chi_{n_k}$ wherein χ is such that deg χ is positive and $\leq N$. However, the total number of characters of positive degree $\leq N$ is given by $(m_1 - m_0) + (m_2 - m_1) + ... + (m_N - m_{N-1}) = m_N - 1$, they are from χ_1 to χ_{m_N-1} and they constitute $\cup_{j=1}^{N}(X_j - X_{j-1})$ — because for each $j = 1, 2, ...$ there are exactly $m_j - m_{j-1}$ characters of degree j, namely: $\chi_{m_{j-1}} \equiv \varphi_{j-1}$, $\chi_{m_{j-1}+1} \equiv \varphi_{j-1}\chi_1, ..., \chi_{m_j-1} \equiv \prod_{i=0}^{j-1}(\varphi_i)^{p_{i+1}-1}$, constituting $X_j - X_{j-1}$. Therefore when χ_{n_k} is multiplied by any character of positive degree $\leq N$ the resulting character χ_m is such that

$$n_k < m \leq n_k + m_N - 1 < n_k + m_N \leq n_k + q \leq n_{k+1}$$

because $q \geq m_N$. Therefore in view of the gaps (1.11) in the Fourier series (4.5) of f, all the terms on the right side of (5.21) vanish except the first. Thus

$$(fP)\hat{\ }(n_k) = \hat{f}(n_k) \ \text{ for } \ k = 1, 2, ... \, . \tag{5.22}$$

(5.20) and (5.22) obviously imply (5.16) because $|I| = 1/m_N$. Next, for each $k = 1, 2, ...$ we have

$$|\hat{f}(n_k)| = |(fP)\hat{\ }(n_k)| \leq \| \, |f| \, |P| \, |w_{n_k}| \, \|_{1,G} = m_N \| f \|_{1,I} = |I|^{-1} \| f \|_{1,I} \, .$$

This proves (5.17) and the theorem is proved. □

Remark 5.2.3. In particular, Theorem 5.2.2 obviously holds for any polynomial $f(x) = \sum_{i=1}^{M} d_k \chi_{n_k}(x)$ on G with $\{n_k\}$ satisfying (1.11) because then $\hat{f}(n_k) = d_k$ for $k = 1, 2, ..., M$ and $\hat{f}(n_k) = 0$ otherwise. Theorem 5.2.2 with $p = p' = 2$ is the analogue of the Wiener-Ingham inequality in the setting of Vilenkin groups.

Remark 5.2.4. The conclusion (5.16) of Theorem 5.2.2 with $p = p' = 2$ clearly gives the global L^2-integrability of f from its local L^2-integrability ($f \in \mathrm{L}^2(I)$ implies $f \in \mathrm{L}^2(G)$) if the Fourier series (4.5) of f possess gaps (1.11).

Corollary 5.2.5. *Let $f \in \mathrm{L}^1(G)$ possess a lacunary Fourier series of the form* (4.5) *with $\{n_k\}$ satisfying the gap condition* (1.5)*. Then for any coset $I = y_0 + G_N$ as in Theorem 5.2.2 but now just of positive measure (that is, N is arbitrary large) we have*

(a) (5.16) *and* (5.17) *hold with a suitable constant on the right side if $f \in \mathrm{L}^p(I)$, where* $1 < p \leq 2$;

(b) $f \in \mathrm{L}^2(I)$ *implies* $f \in \mathrm{L}^2(G)$.

Proof. The form of I implies $|I| = 1/m_N$, where $N \in \mathbb{N}$ may be taken very large because I is just of positive measure. In view of (1.5), we have $(n_{k+1} - n_k) \geq m_N$ for all $k \geq k_0$ for suitable $k_0 = k_0(N)$. Then adding to $f(x)$ the polynomial $\sum_{j=0}^{k_0}(-\hat{f}(n_j))\chi_{n_j}$ we get the function g whose Fourier series is lacunary of the form (4.5) having gaps (1.11) with $q = m_N$. Since $f \in \mathrm{L}^p(I)$ implies $g \in \mathrm{L}^p(I)$, Theorem 5.2.2 is applicable to g. But $\hat{f}(n) = \hat{g}(n)$ for all but finitely many n, hence the corollary is proved. □

Remark 5.2.6. Corollary 5.2.5 (b) is an analogue on G of a result due to Payley & Wiener quoted by Kennedy [29, Lemma 4].

Now we give an application of the Wiener-Ingham Inequality. Theorem 5.2.2 is a sort of an extension of the Hausdorff-Young inequality — in a sense that if there are no gaps in the Fourier series, that is, if equality holds throughout in (1.11) then one needs to take $I = G$ and (5.16) coincides with the Hausdorff-Young inequality. Consequently, the analogues on a Vilenkin group G of the well-known results of Bernstein, Zygmund, Szász and Stečhkin concerning the absolute convergence of Fourier series on G obtained by Vilenkin and Rubinstein [67], Onneweer [40] and

Quek and Yap [54, 55] can be extended for the lacunary Fourier series on G. We begin by estimating the tails of $\sum |\hat{f}(n_k)|^2$ in terms of the mean modulus of continuity / the best approximation of f considered only on a coset of G.

To prove Theorem 5.2.10, we need following definitions lemma.

Definition 5.2.7. For a function $f : G \to \mathbb{C}$ and $n \in \mathbb{N} \cup \{0\}$, we define the *$n$-th modulus of continuity* of f over the coset $I = y_0 + G_N$ by

$$\omega_n(f, I) = \sup\{|(T_h f - f)(x)| : x \in I, h \in G_n\},$$

where $(T_h f)(x) = f(x + h),\ \forall x \in G$.

Definition 5.2.8. For a function $f : G \to \mathbb{C}$, $n \in \mathbb{N} \cup \{0\}$ and $1 \le p < \infty$, we define the *n-th integral modulus of continuity of order p* of f over the coset $I = y_0 + G_N$ by

$$\omega^{(p)}(f, n, I) = \sup\{||T_h f - f||_{p,I} : h \in G_n\},$$

where $||(\cdot)||_{p,I} = ||(\cdot)\chi_I||_p$ in which χ_I is the characteristic function of I and $||(\cdot)||_p$ denotes the L^p norm on G.

When $p = \infty$ we put, $\omega^{(\infty)}(f, n, I) = \omega_n(f, I)$, where $\omega_n(f, I)$ is as in Definition 5.2.7. Also, when $I = G$, we omit writing I and in that case $\omega_n(f)$ is the n-th modulus of continuity on G as in Definition 2 of [39].

Lemma 5.2.9. *For each $N = 0, 1, 2, ...$ the following statements are true.*

(a) $G_N - \{0\} = \cup_{n=N}^{\infty}(G_n - G_{n+1})$*, the union being disjoint.*

(b) $h \in G_N$ *if and only if* $|h| < 1/m_N$.

(c) *If* $n \ge m_N$ *then* $\chi_n \in X \setminus X_n$.

(d) *For* $h \in G_N - \{0\}$ *we have* $\chi_n(h) = 1$ *if* $n < m_N$ *and* $\chi_n(h) \neq 1$ *if* $n \ge m_N$.

Proof. In view of (1.24), (a) is obvious. Observing that $|h| = 1/m_{n+1}$ if and only if $h \in G_n \setminus G_{n+1}$ for each $n = 0, 1, 2, ...$, (b) follows from (a). The fact that $X_n = \{\chi_0, \chi_1, \chi_2, ..., \chi_{m_n - 1}\}$ implies (c) which in turn clearly gives (d). □

Theorem 5.2.10. *Under the hypothesis of Theorem 5.2.2 with $p = p' = 2$, we have*

$$\sum_{n_k \geq m_n} |\hat{f}(n_k)|^2 \leq \left(1/2|I|^2\right) \left(\omega^{(2)}(f, n, I)\right)^2 \quad \textit{for} \quad n \geq N, \tag{5.23}$$

$$\sum_{n_k \geq m_N} |\hat{f}(n_k)|^2 \leq \left(1/2|I|^2\right) \left(E^{(2)}(f, N, I)\right)^2, \tag{5.24}$$

where $\omega^{(2)}(f, n, I)$ is as in Definition 5.2.8 and

$$E^{(2)}(f, N, I) = \inf_T \| f - T \|_{2,I},$$

in which T is a trigonometric polynomial on G of degree not exceeding N.

Proof. $f \in \mathrm{L}^2(I)$ implies $(T_h f - f) \in \mathrm{L}^2(I)$ for any $h \in G_N$. For, if $h \in G_N$ then

$$\begin{aligned} x \in I = y_0 + G_N &\Rightarrow x = y_0 + y, \text{ for some } y \in G_N \\ &\Rightarrow x + h = y_0 + y + h \in y_0 + G_N = I, \end{aligned}$$

and

$$\begin{aligned} x + h \in I = y_0 + G_N &\Rightarrow x + h = y_0 + z, \text{ for some } z \in G_N \\ &\Rightarrow x = y_0 + z - h \in y_0 + G_N = I, \end{aligned}$$

since G_N is a subgroup of G; which shows that

$$\begin{aligned} \int_I |T_h f(x)|^2 dx &= \int_{y_0 + G_N} |f(x + h)|^2 dx \\ &= \int_{y_0 + h + G_N} |f(x)|^2 dx \\ &= \int_{y_0 + G_N} |f(x)|^2 dx < \infty, \end{aligned}$$

since $h \in G_N$ and $f \in \mathrm{L}^2(I) = \mathrm{L}^2(y_0 + G_N)$. Since the Fourier series of f is lacunary of the form (4.5) with gaps (1.11) so is the Fourier series of $(T_h f - f)$. Therefore, Theorem 5.2.2 gives

$$\sum_{k=1}^{\infty} |(T_h f - f)\hat{}\,(n_k)|^2 \leq |I|^{-2} \| T_h f - f \|_{2,I}^2 .$$

Since $(T_h f - f)\hat{}\,(n_k) = (T_h f)\hat{}\,(n_k) - \hat{f}(n_k) = \hat{f}(n_k)(\chi_{n_k}(h) - 1)$, this gives

$$\sum_{k=1}^{\infty} |\hat{f}(n_k)|^2 |\chi_{n_k}(h) - 1|^2 \leq |I|^{-2} \| T_h f - f \|_{2,I}^2 . \tag{5.25}$$

Integrating both sides of (5.25) over G_n $(n \geq N)$ with respect to h and applying Lemma 5.2.9 (b) and Lemma 5.1.1, (5.23) is obtained.

Next, for any trigonometric polynomial T on G with degree not exceeding N, $f - T \in \mathrm{L}^2(I)$ and the Fourier coefficients of f and $f - T$ differ for, at most, first m_N terms. The proof of Theorem 5.2.2 shows that it can be applied to $f - T$ to get

$$\sum_{n_k \geq m_N} |\hat{f}(n_k)|^2 \leq |I|^{-2} \parallel f - T \parallel_{2,I}^2 . \tag{5.26}$$

Since (5.26) holds for an arbitrary trigonometric polynomial of degree not exceeding N, (5.24) clearly follows. □

Theorem 5.2.11. *Under the hypothesis of Theorem 5.2.2 we have*

$$\sum_{k=1}^{\infty} |\hat{f}(n_k)|^{\beta} < \infty \tag{5.27}$$

whenever

$$\sum_{k=0}^{\infty} (m_{k+1})^{1-\beta/p'} (\parallel \Delta_k * f \parallel_{p,I})^{\beta} < \infty, \tag{5.28}$$

where in $0 < \beta \leq p'$ *and* $\Delta_k * f$ *is the convolution of* $\Delta_k = D_{m_{k+1}} - D_{m_k}$ *with* f, $D_t = \sum_{i=0}^{t} \chi_i$ *being the Dirichlet kernel of order* t.

Proof. $f \in \mathrm{L}^p(I)$ implies $\Delta_N * f \in \mathrm{L}^p(I)$ for each $N = 0, 1, 2, \ldots$. Since the Fourier series of f is lacunary of the form (4.5) with gaps (1.11) so is the Fourier series of $\Delta_N * f$ as $(\Delta_N * f)\hat{}(\chi) = (\Delta_N)\hat{}(\chi)\hat{f}(\chi)$. Therefore by Theorem 5.2.2 , for each $N = 0, 1, 2, \ldots$, we have

$$\left(\sum{}' |\hat{f}(n_k)|^{p'}\right)^{1/p'} \leq |I|^{-1} \parallel \Delta_N * f \parallel_{p,I}, \tag{5.29}$$

where in $\sum'$ indicates that the summation is over those n_k for which n_k satisfy $m_N \leq n_k < m_{N+1}$. Hölder's inequality and (5.28) now give

$$\begin{aligned}
\sum_{k=1}^{\infty} |\hat{f}(n_k)|^{\beta} &= \sum_{N=0}^{\infty} \left(\sum{}' |\hat{f}(n_k)|^{\beta}\right) \\
&\leq \sum_{N=0}^{\infty} \left(\left(\sum{}' |\hat{f}(n_k)|^{p'}\right)^{\beta/p'} \left(\sum{}' 1\right)^{1-\beta/p'} \right) \\
&= O(1) \sum_{N=0}^{\infty} (\|\Delta_N * f\|_{p,I})^{\beta} (m_{N+1} - m_N)^{1-\beta/p'} \\
&= O(1)
\end{aligned}$$

which proves the theorem. □

Corollary 5.2.12. *Theorem 5.2.11 holds if* (5.28) *is replaced by the condition*

$$\sum_{N=0}^{\infty}(m_{N+1})^{1-\beta/p'}\left(\omega^{(p)}(f,N,I)\right)^{\beta}<\infty,$$

where $\omega^{(p)}(f,N,I)$ is defined as in Definition 5.2.8.

Proof. We have

$$\| \Delta_N * f \|_{p,I} \leq \| D_{m_{N+1}} * f - f \|_{p,I} + \| D_{m_N} * f - f \|_{p,I};$$

and

$$(D_{m_N} * f - f)(x) = \int_{G_N} (f(x-y) - f(x)) D_{m_N}(y) dy$$

because $1 = \| D_{m_N} \|_1 = \| D_{m_N} \|_{1,G_N}$ as $m(G_N) = 1/m_N$ and $D_{m_N}(y) = m_N$ for $y \in G_N$; $D_{m_N}(y) = 0$ for $y \in G \setminus G_N$. It follows that

$$\begin{aligned}
\| D_{m_N} * f - f \|_{p,I} &= \left(\int_I \left| \int_{G_N} (f(x-y) - f(x)) D_{m_N}(y) dy \right|^p dx \right)^{1/p} \\
&\leq \left(\int_I \left(\int_{G_N} |f(x-y) - f(x)| D_{m_N}(y) dy \right)^p dx \right)^{1/p} \\
&\leq \int_{G_N} \left(\int_I |f(x-y) - f(x)|^p (D_{m_N}(y))^p dx \right)^{1/p} dy \\
&\leq \int_{G_N} \omega^{(p)}(f,N,I) D_{m_N}(y) dy \\
&= \omega^{(p)}(f,N,I) \qquad (5.30)
\end{aligned}$$

in view of the Minkowski inequality for integrals. The corollary now obviously follows from Theorem 5.2.11. □

Corollary 5.2.13. *Under the hypothesis of Theorem 5.2.2 with $p = p' = 2$ the Fourier series of f converges absolutely if either*

$$\sum_{N=1}^{\infty}(m_{N+1})^{1/2}\omega^{(2)}(f,N,I)<\infty \quad \textit{or} \quad \sum_{N=1}^{\infty}(m_{N+1})^{1/2}\omega(f,N,I)<\infty.$$

Proof. Corollary 5.2.12 with $p = 2$, $\beta = 1$ and an argument of [67] gives the result. □

Corollary 5.2.14. *Let* $0 < p \leq 2$, $\alpha > \frac{1}{p} - \frac{1}{2}$. *If* $f \in \mathrm{Lip}\,(\alpha, p, I)$ *then*

$$\left(\sum_{n_k \geq m_L} |\hat{f}(n_k)|^p\right)^{1/p} = O\left((m_{L+1})^{1/p-\alpha-1/2}\right),$$

where $\mathrm{Lip}\,(\alpha, p, I) = \left\{f \in \mathrm{L}^p(I) : \omega^{(p)}(f, n, I) = O\left((m_{n+1})^{-\alpha}\right)\right\}$.

Proof. In view of the Hölder's inequality, (5.29) and (5.30) we have

$$\begin{aligned}\sum{}' |\hat{f}(n_k)|^p &\leq \left(\sum{}' |\hat{f}(n_k)|^2\right)^{p/2} (m_{N+1} - m_N)^{1-p/2} \\ &\leq |I|^{-p} \parallel \Delta_N * f \parallel_{2,I}^p (m_{N+1})^{1-p/2} \\ &= O\left(\omega^{(p)}(f, N, I)\right)^p (m_{N+1})^{1-p/2} \\ &= O(1)(m_{N+1})^{p(-\alpha+1/p-1/2)} \end{aligned} \tag{5.31}$$

for $N = 0, 1, 2, \ldots$. Since $m_{k+1} \geq 2m_k$ for all k and $p(1/p-\alpha-1/2) < 0$ the corollary follows from (5.31) upon taking summation on both sides from $N = L$ onwards. □

Remark 5.2.15. Theorem 5.2.10 is an analogue on G of Patadia's earlier result [47, Lemma 3]. As explained in Corollary 5.2.5, in case the Fourier series of f is lacunary of the form (4.5) with $\{n_k\}$ satisfying the gap condition (1.5) then Theorems 5.2.10 and 5.2.11 as well as Corollaries 5.2.12, 5.2.13 and 5.2.14 hold true even if $y_0 + G_N = I \subset G$ is just of positive measure. In this case, Corollary 5.2.12 with $p = p' = 2$ is analogue on G of Patadia's earlier result [47, Theorem 1].

Remark 5.2.16. Theorem 5.2.10 is also an extension of the Vilenkin and Rubinstěin result [67, Theorem 1, p. 2] because if there are no gaps in the Fourier series if f then one needs to take $I = G$ and Theorem 5.2.10 reduces to their result. Similarly, Theorem 5.2.11 and Corollary 5.2.12 extend the results of Quek and Yap [54, Theorem 3.1 and Corollary 3.3] while Corollaries 5.2.13 and 5.2.14 extend the Quek and Yap results of [55, Theorems 4.2, 4.3 and 4.5].

5.3 Absolute convergence of lacunary Fourier series of functions of generalized bounded fluctuation on Vilenkin groups

In Section 5.1, we have studied absolute convergence of non-lacunary Vilenkin Fourier series for the functions of various classes of generalized bounded fluctuation. Here we study the absolute convergence of lacunary Vilenkin Fourier series for functions of these classes. Our new results generalizes and gives lacunary analogues of our earlier results of Section 5.1.

Let G be bounded and $f: G \to \mathbb{C}$. Here the following results are obtained.

Theorem 5.3.1. *Let $f \in \mathrm{L}^1(G)$ possess a lacunary Vilenkin Fourier series* (4.5) *with small gaps* (1.11) *and $I = y_0 + G_{N_0}$ be a coset with Haar measure $1/m_{N_0} \geq 1/q$. If $f \in \Lambda\mathrm{GBF}^{(p)}(I)$, $1 \leq p < 2r$, $1 \leq r < \infty$ and*

$$\sum_{n=N_0}^{\infty} \left\{ \left[\frac{(\omega^{(p+(2-p)r')}(f,n,I))^{2-p/r}}{(\sum_{j=1}^{m_n/m_{N_0}} \frac{1}{\lambda_j})^{1/r}} \right]^{\frac{\beta}{2}} \sum_{\substack{k \\ m_n \leq n_k < m_{n+1}}} \frac{1}{k^{\beta/2}} \right\} < \infty,$$

then (5.27) *holds for $0 < \beta \leq 2$.*

Proof. We may assume without loss of generality that $y_0 = 0$ so that $I = G_{N_0}$; for, otherwise one works with $g = T_{y_0} f \in \Lambda\mathrm{GBF}^{(p)}(G_{N_0})$ whose Fourier series also has gaps (1.11).

Let $M \in \mathbb{N}$ be fixed such that $n_M \geq m_{N_0}$ and let $N \in \mathbb{N}$ be the integer such that $m_N \leq n_M < m_{N+1}$. Then clearly $N \geq N_0$. Put $t_N = m_N/m_{N_0}$ and for each $\alpha = 0, 1, ..., t_N - 1$, $h \in G_N$ define

$$f_\alpha(x) = f(x + z_\alpha^{(N)} + h) - f(x + z_\alpha^{(N)}), \ \ \forall x \in G.$$

Then

$$\hat{f}_\alpha(n) = \hat{f}(n)\chi_n(z_\alpha^{(N)} + h) - \hat{f}(n)\chi_n(z_\alpha^{(N)}) = \hat{f}(n)\chi_n(z_\alpha^{(N)})(\chi_n(h) - 1), \ \ \forall n \geq 0.$$

Since $f \in \Lambda\mathrm{GBF}^{(p)}(G_{N_0})$ for any $x \in G_{N_0}$ we have

$$\begin{aligned}
|f(x)|^p &= |f(0) + f(x) - f(0)|^p \\
&\leq 2^p|f(0)|^p + 2^p|f(x) - f(0)|^p \\
&= 2^p|f(0)|^p + 2^p\lambda_1 \left(\frac{|f(x) - f(0)|^p}{\lambda_1} \right) \\
&\leq 2^p|f(0)|^p + 2^p\lambda_1 \left(\frac{(\mathrm{osc}(f; z_0^{(0)} + G_{N_0}))^p}{\lambda_1} \right) \\
&\leq 2^p|f(0)|^p + 2^p\lambda_1(\Lambda GF_p(f; G_{N_0}))^p.
\end{aligned}$$

Thus f is bounded on $G_{N_0} = I$ and hence $f \in \mathrm{L}^2(I)$. In view of (5.16) for $p' = 2$, $f \in \mathrm{L}^2(G)$ and hence each $f_\alpha \in \mathrm{L}^2(I)$. Since the Fourier series of f_α also has gaps (1.11), again using the same inequality for f_α (since $|\chi_n(z_\alpha^{(N)})| = 1$) we get

$$B(h) \equiv \sum_{k=1}^{\infty} |\hat{f}(n_k)|^2 |\chi_{n_k}(h) - 1|^2 \leq |I|^{-2} ||f_\alpha||_{2,I}^2, \quad \forall \alpha. \tag{5.32}$$

Now, suppose $r > 1$ and set $2 = \frac{p+(2-p)r'}{r'} + \frac{p}{r}$; then using the Hölder's inequality we get

$$\begin{aligned} ||f_\alpha||_{2,I}^2 &= \int_I |f_\alpha(x)|^2 dx \\ &= \int_I |f_\alpha(x)|^{(\frac{p+(2-p)r'}{r'} + \frac{p}{r})} dx \\ &= \int_I \left(|f_\alpha(x)|^{(p+(2-p)r')}\right)^{1/r'} \left(|f_\alpha(x)|^p\right)^{1/r} dx \\ &\leq \left\{\int_I |f_\alpha(x)|^{(p+(2-p)r')} dx\right\}^{1/r'} \left\{\int_I |f_\alpha(x)|^p dx\right\}^{1/r} \\ &\leq (\Omega_N)^{1/r} \left(\int_I |f_\alpha(x)|^p dx\right)^{1/r}, \end{aligned}$$

where $\Omega_N = \left(\omega^{(p+(2-p)r')}(f, N, I)\right)^{2r-p}$ since $h \in G_N$. This together with (5.32) implies

$$(B(h))^r \leq |I|^{-2r} \Omega_N \int_I |f_\alpha(x)|^p dx, \tag{5.33}$$

for all $\alpha = 0, 1, ..., t_N - 1$. Since the left hand side of (5.33) is independent of α, multiplying both the sides of it by $(1/\lambda_{\alpha+1})$ and taking summation over α, we get

$$(B(h))^r \theta_{t_N} \leq |I|^{-2r} \Omega_N \int_I \left(\sum_{\alpha=0}^{t_N-1} \frac{|f_\alpha(x)|^p}{\lambda_{\alpha+1}}\right) dx,$$

where $\theta_t = \sum_{j=1}^{t} (1/\lambda_j) = \sum_{j=0}^{t-1} (1/\lambda_{j+1})$, for all $t \in \mathbb{N}$; and hence

$$B(h) \leq |I|^{-2} \left(\frac{\Omega_N}{\theta_{t_N}}\right)^{1/r} \left\{\int_I \left(\sum_{\alpha=0}^{t_N-1} \frac{|f_\alpha(x)|^p}{\lambda_{\alpha+1}}\right) dx\right\}^{1/r}.$$

Integrating both sides of this inequality over G_N with respect to h we get

$$\int_{G_N} B(h) dh \leq |I|^{-2} \left(\frac{\Omega_N}{\theta_{t_N}}\right)^{1/r} \int_{G_N} \left\{\int_I \sum_{\alpha=0}^{t_N-1} \frac{|f_\alpha(x)|^p}{\lambda_{\alpha+1}} dx\right\}^{1/r} dh. \tag{5.34}$$

Now, for any $h \in G_N$ and any $x \in I = G_{N_0}$ the points $x + z_\alpha^{(N)} + h$ and $x + z_\alpha^{(N)}$ lie in the coset $x + z_\alpha^{(N)} + G_N$ of G_N in G_{N_0} (since $N \geq N_0$) and hence

$$|f_\alpha(x)| = |f(x + z_\alpha^{(N)} + h) - f(x + z_\alpha^{(N)})| \leq \operatorname{osc}(f, x + z_\alpha^{(N)} + G_N). \tag{5.35}$$

Since $f \in \Lambda GBF^{(p)}(I)$, for $h \in G_N$, in view of (5.35),

$$\sum_{\alpha=0}^{t_N-1} \frac{|f_\alpha(x)|^p}{\lambda_{\alpha+1}} \leq \sum_{\alpha=0}^{t_N-1} \frac{(\operatorname{osc}(f, x + z_\alpha^{(N)} + G_N))^p}{\lambda_{\alpha+1}} \leq (\Lambda GF_p(f; I))^p, \tag{5.36}$$

for all $x \in I$; because for any $x \in I$, the finite sequence of cosets $\{x + z_\alpha^{(N)} + G_N : \alpha = 0, 1, ..., t_N - 1\}$ is a rearrangement of the sequence $\{z_\alpha^{(N)} + G_N : \alpha = 0, 1, ..., t_N - 1\}$ since this collection gives all the cosets of G_N in I. Further, since $m_N \leq n_M < m_{N+1}$ and $n_k \geq n_M$ for $k \geq M$, from (5.32),

$$\begin{aligned} \int_{G_N} B(h)dh &\geq \sum_{k=M}^{\infty} |\hat{f}(n_k)|^2 \int_{G_N} |\chi_{n_k}(h) - 1|^2 dh \\ &= \left(\frac{2}{m_N}\right) \sum_{k=M}^{\infty} |\hat{f}(n_k)|^2, \end{aligned} \tag{5.37}$$

for all α; in view of Lemma 5.1.1. Using (5.36) and (5.37) in (5.34) we get

$$\begin{aligned} R_{n_M} &\equiv \sum_{n=n_M}^{\infty} |\hat{f}(n)|^2 \\ &= \sum_{k=M}^{\infty} |\hat{f}(n_k)|^2 \\ &\leq |I|^{-2} \left(\frac{m_N}{2}\right) \left(\frac{\Omega_N}{\theta_{t_N}}\right)^{1/r} \int_{G_N} \left\{ \int_I (\Lambda GF_p(f; I))^p dx \right\}^{1/r} dh \\ &= |I|^{-2} \left(\frac{m_N}{2}\right) \left(\frac{\Omega_N}{\theta_{t_N}}\right)^{1/r} \left(\frac{(\Lambda GF_p(f; I))^p}{m_{N_0}}\right)^{1/r} \left(\frac{1}{m_N}\right) \\ &= O\left[\left(\frac{\Omega_N}{\theta_{t_N}}\right)^{1/r}\right]. \end{aligned} \tag{5.38}$$

Now, applying Lemma 5.1.2 with $u_k = |\hat{f}(n_k)|^2$ and $F(u) = u^{\beta/2}$ we get (5.9) in view of (5.38) we obtain

$$\begin{aligned}\sum_{k=1}^{\infty}|\hat{f}(n_k)|^{\beta} &= O(1)\sum_{k=1}^{\infty}\left(\frac{R_{n_k}}{k}\right)^{\beta/2}\\ &= O(1)\sum_{n=0}^{\infty}\sum_{\substack{k\\ m_n\le n_k<m_{n+1}}}\left(\frac{R_{n_k}}{k}\right)^{\beta/2}\\ &= O(1)\left\{1+\sum_{n=N_0}^{\infty}\sum_{\substack{k\\ m_n\le n_k<m_{n+1}}}\left[\frac{(\Omega_n)^{1/r}}{k(\theta_{t_n})^{1/r}}\right]^{\beta/2}\right\}\\ &= O(1)\left\{1+\sum_{n=N_0}^{\infty}\left[\frac{(\Omega_n)^{1/r}}{(\theta_{t_n})^{1/r}}\right]^{\beta/2}\sum_{\substack{k\\ m_n\le n_k<m_{n+1}}}\frac{1}{k^{\beta/2}}\right\}<\infty,\end{aligned}$$

by the assumption of theorem. Thus the theorem is proved for $r>1$.

For the case $r=1$, $r'=\infty$, simply note that

$$|f_\alpha(x)|^2=|f_\alpha(x)|^{2-p}|f_\alpha(x)|^p\le(\omega_N(f))^{2-p}|f_\alpha(x)|^p,$$

because

$$|f_\alpha(x)=f(x+z_\alpha^{(N)}+h)-f(x+z_\alpha^{(N)})|\le\omega_N(f)$$

since $h\in G_N$; and proceed as above. □

Remark 5.3.2. Since $\Lambda\mathrm{BF}^{(p)}(I)\subset\Lambda\mathrm{GBF}^{(p)}(I)$, Theorem 5.3.1 obviously holds for functions in $\Lambda\mathrm{BF}^{(p)}(I)$ also.

When the Fourier series is non-lacunary, taking $n_k=k$ for all k and $I=G$ in Theorem 5.3.1 we obtain

Corollary 5.3.3. *Let* $1\le r<\infty$ *and* $1\le p<2r$. *If* $f\in\Lambda\mathrm{GBF}^{(p)}(G)$ *satisfies*

$$\sum_{n=0}^{\infty}\left[\frac{(m_n)^{2/\beta-1}(\omega^{(p+(2-p)r')}(f,n))^{2-p/r}}{(\sum_{j=1}^{m_n}\frac{1}{\lambda_j})^{1/r}}\right]^{\frac{\beta}{2}}<\infty,$$

then (5.27) *holds for* $0<\beta\le 2$.

Proof. Taking $n_k = k$ for all k, $N_0 = 0$ and $I = G$ we have

$$\sum_{n=N_0}^{\infty}\left\{\left[\frac{(\omega^{(p+(2-p)r')}(f,n,I))^{2-p/r}}{\left(\sum_{j=1}^{m_n/m_{N_0}}\frac{1}{\lambda_j}\right)^{1/r}}\right]^{\frac{\beta}{2}}\sum_{\substack{k\\ m_n\le n_k<m_{n+1}}}\frac{1}{k^{\beta/2}}\right\}$$

$$=\sum_{n=0}^{\infty}\left\{\left[\frac{(\omega^{(p+(2-p)r')}(f,n))^{2-p/r}}{\left(\sum_{j=1}^{m_n}\frac{1}{\lambda_j}\right)^{1/r}}\right]^{\frac{\beta}{2}}\sum_{k=m_n}^{m_{n+1}}\frac{1}{k^{\beta/2}}\right\}$$

$$=\sum_{n=0}^{\infty}\left\{\left[\frac{(\omega^{(p+(2-p)r')}(f,n))^{2-p/r}}{\left(\sum_{j=1}^{m_n}\frac{1}{\lambda_j}\right)^{1/r}}\right]^{\frac{\beta}{2}}\frac{1}{(m_n)^{\beta/2}}(m_{n+1}-m_n)\right\}$$

$$=O(1)\sum_{n=0}^{\infty}\left[\frac{(m_n)^{1-2/\beta}(\omega^{(p+(2-p)r')}(f,n))^{2-p/r}}{\left(\sum_{j=1}^{m_n}\frac{1}{\lambda_j}\right)^{1/r}}\right]^{\frac{\beta}{2}}<\infty,$$

since G is bounded and by the assumption of the corollary. Thus Corollary 5.3.3 follows from Theorem 5.3.1. □

Theorem 5.3.4. *Let f and I be as in Theorem 5.3.1. If $f \in \phi\Lambda\mathrm{GBF}(I)$, $1 \le p < 2r$, $1 \le r < \infty$ and*

$$\sum_{n=N_0}^{\infty}\left\{\left[\left\{\phi^{-1}\left(\frac{(\omega^{(p+(2-p)r')}(f,n,I))^{2r-p}}{\sum_{j=1}^{m_n/m_{N_0}}\frac{1}{\lambda_j}}\right)\right\}^{1/r}\right]^{\frac{\beta}{2}}\sum_{\substack{k\\ m_n\le n_k<m_{n+1}}}\frac{1}{k^{\beta/2}}\right\}<\infty,$$

then (5.27) *holds, in which ϕ is a Δ_2-function.*

Proof. As in the proof of Theorem 5.3.1, here also we may assume that $y_0 = 0$. Since $f \in \phi\Lambda\mathrm{GBF}(I)$ for any $x \in I = G_{N_0}$, we have

$$|f(x)| \le |f(0)| + C\phi^{-1}(\Lambda GF_\phi(f;I)).$$

Thus f is bounded on I and hence $f \in \mathrm{L}^2(I)$. For $r > 1$, proceeding as in the proof of Theorem 5.3.1 we get (5.33). Since multiplying f by a positive constant alters $\omega^{(p)}(f,n,I)$ by the same constant, and ϕ is Δ_2, we may assume that $|f(x)| \le \frac{1}{2}$ for all $x \in I$. But then from (5.33) we get

$$(B(h))^r \le |I|^{-2r}\Omega_N\int_I |f_\alpha(x)|dx, \qquad (\alpha = 0,1,...,t_N-1).$$

Since $\phi(2x) \leq d\phi(x), \forall x \geq 0$, as in the proof of Theorem 5.1.7 we get $\phi(ax) \leq d^{\log_2 a+1}\phi(x), \forall x \geq 0, \forall a \geq 1$. Since $|I|^{-2r}\Omega_N \geq 0$, if $|I|^{-2r}\Omega_N < 1$ then we get

$$\phi\left(m_{N_0}(B(h))^r\right) \leq \phi\left(m_{N_0}|I|^{-2r}\Omega_N \int_I |f_\alpha(x)|dx\right) \leq |I|^{-2r}\Omega_N\phi\left(m_{N_0}\int_I |f_\alpha(x)|dx\right).$$

Further when $|I|^{-2}\Omega_N \geq 1$, as above

$$\begin{aligned}
\phi\left(m_{N_0}(B(h))^r\right) &\leq \phi\left(m_{N_0}|I|^{-2r}\Omega_N \int_I |f_\alpha(x)|dx\right)\\
&\leq d^{\log_2(|I|^{-2r}\Omega_N)+1}\cdot\phi\left(m_{N_0}\int_I |f_\alpha(x)|dx\right)\\
&= d\cdot(|I|^{-2r}\Omega_N)^{\log_2 d}\cdot\phi\left(m_{N_0}\int_I |f_\alpha(x)|dx\right)\\
&= d\cdot|I|^{-2r\log_2 d}\cdot(\Omega_N)^{\log_2 d-1}\cdot\Omega_N\cdot\phi\left(m_{N_0}\int_I |f_\alpha(x)|dx\right)\\
&\leq d\cdot|I|^{-2r\log_2 d}\cdot\Omega_N\cdot\phi\left(m_{N_0}\int_I |f_\alpha(x)|dx\right),
\end{aligned}$$

in view of the fact that $(\Omega_N)^{\log_2 d-1} \leq 1$, as $|f(x)| \leq \frac{1}{2}, \forall x \in I$, and $\log_2 d - 1 \geq 0$. Therefore in either case

$$\phi\left(m_{N_0}(B(h))^r\right) = O(1)\Omega_N\phi\left(m_{N_0}\int_I(|f_\alpha(x)|)dx\right) = O(1)\Omega_N m_{N_0}\int_I \phi(|f_\alpha(x)|)dx,$$

in view of the Jensen's inequality. Now multiplying both the sides of this inequality by $(1/\lambda_{\alpha+1})$ and taking summation over $\alpha = 0, 1, ..., t_N - 1$ we get

$$\phi\left(m_{N_0}(B(h))^r\right) = O(1)\left(\frac{\Omega_N}{\theta_{t_N}}\right)\int_I\left(\sum_{\alpha=0}^{t_N-1}\frac{\phi(|f_\alpha(x)|)}{\lambda_{\alpha+1}}\right)dx. \tag{5.39}$$

Since $f \in \phi\Lambda\text{GBF(I)}$ and ϕ is increasing, for all $h \in G_N$ and $x \in I$ we have

$$\sum_{\alpha=0}^{t_N-1}\frac{\phi(|f_\alpha(x)|)}{\lambda_{\alpha+1}} \leq \sum_{\alpha=0}^{t_N-1}\frac{\phi(\text{osc}(f; x + z_\alpha^{(N)} + G_N))}{\lambda_{\alpha+1}} \leq GF_{\phi\Lambda}(f; I). \tag{5.40}$$

Using (5.40) in (5.39) we get

$$\phi\left(m_{N_0}(B(h))^r\right) \leq C\left(\frac{\Omega_N}{\theta_{t_N}}\right),$$

where C is a constant such that $C \geq 1$. Thus

$$m_{N_0}(B(h))^r \leq \phi^{-1}\left\{C\left(\frac{\Omega_N}{\theta_{t_N}}\right)\right\} \leq C\phi^{-1}\left(\frac{\Omega_N}{\theta_{t_N}}\right)$$

and therefore

$$B(h) = O\left[\left\{\phi^{-1}\left(\frac{\Omega_N}{\theta_{t_N}}\right)\right\}^{1/r}\right].$$

Integrating both sides of this inequality over G_N with respect to h, in view of (5.37) we get

$$R_{n_M} \equiv \sum_{k=M}^{\infty} |\hat{f}(n_k)|^2 \leq \left(\frac{m_N}{2}\right)\int_{G_N} B(h)dh = O\left[\left\{\phi^{-1}\left(\frac{\Omega_N}{\theta_{t_N}}\right)\right\}^{1/r}\right].$$

Thus from (5.9) we get

$$\begin{aligned}\sum_{k=1}^{\infty} |\hat{f}(n_k)|^\beta &= O(1)\sum_{k=1}^{\infty}\left(\frac{R_{n_k}}{k}\right)^{\beta/2} = O(1)\sum_{n=0}^{\infty}\sum_{\substack{k\\ m_n\leq n_k<m_{n+1}}}\left(\frac{R_{n_k}}{k}\right)^{\beta/2}\\ &= O(1)\left\{1+\sum_{n=N_0}^{\infty}\sum_{\substack{k\\ m_n\leq n_k<m_{n+1}}}\left[\frac{1}{k}\left\{\phi^{-1}\left(\frac{\Omega_n}{\theta_{t_n}}\right)\right\}^{1/r}\right]^{\beta/2}\right\}\\ &= O(1)\left\{1+\sum_{n=N_0}^{\infty}\left[\left\{\phi^{-1}\left(\frac{\Omega_n}{\theta_{t_n}}\right)\right\}^{1/r}\right]^{\beta/2}\sum_{\substack{k\\ m_n\leq n_k<m_{n+1}}}\frac{1}{k^{\beta/2}}\right\} < \infty,\end{aligned}$$

in view of the assumption of the theorem. This completes the proof of the theorem for $r > 1$. For the case $r = 1$, $r' = \infty$, the proof is similar as that of Theorem 5.3.1. □

Corollary 5.3.5. *If $f \in \phi\Lambda\mathrm{GBF}(G)$, $1 \leq p < 2r$, $1 \leq r < \infty$ and*

$$\sum_{n=0}^{\infty}\left[(m_n)^{2/\beta-1}\left\{\phi^{-1}\left(\frac{(\omega^{(p+(2-p)r')}(f,n))^{2r-p}}{\sum_{j=1}^{m_n}\frac{1}{\lambda_j}}\right)\right\}^{1/r}\right]^{\frac{\beta}{2}} < \infty,$$

then (5.27) *holds, in which ϕ is a Δ_2-function.*

Proof. Similar as the proof of Corollary 5.3.3. □

Remark 5.3.6. Since $\phi\Lambda\mathrm{BF}(I) \subset \phi\Lambda\mathrm{GBF}(I)$, Theorem 5.3.4 obviously holds for functions in $\phi\Lambda\mathrm{BF}(I)$ also. Corollaries 5.3.3 and 5.3.5 are our earlier results (see Theorems 5.1.3 and 5.1.7). Thus Theorems 5.3.1 and 5.3.4 generalizes and gives lacunary analogues of our earlier results. Also, Theorems 5.3.1 and 5.3.4 are Vilenkin group analogue of the corresponding circle group results of Vyas [70, Theorem 1.1] and [71, Theorem 1.1] respectively.

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