CHAPTER 3

MODELS FOR DISCRETE VARIATE TIME SERIES

3.1 INTRODUCTION :

Frequently in practice time series is observed on a discrete variate. Often such processes arise as aggregated point processes. For example, daily sales of a product, daily record of number of phone calls from an office, yearly record of fatal accidents etc. Modelling of such time series is important for the further analysis of such data. Many times the models originally developed for continuous variate time series are adopted in such situations as an approximation. However this may not be always possible. Also wherever possible, the approximate nature of models may lead to unsatisfactory results.

To overcome this difficulty, one should look for models specifically developed for discrete valued time series. There are very few models available for discrete variate processes in the early literature, especially before 80's. Markov chains are extensively studied in the literature and they do provide models for discrete variate processes. However, Markov chains are over parameterized for practical applicability in a time series context. Jacobs and Lewis (1978 a, b), (1983) proposed the DARMA processes. These form a very general family of processes but they can not take advantage of the special structure of particular distributions.

Some simple models that are analytically more tractable are the INAR(1) models introduced independently by McKenzie (1985) and Al-Osh and Alzaid (1987). These models are similar in structure to AR(1) models. More about INAR models is discussed in Section 3.3. In Section 3.4 we describe discrete analogue of MINAR process called discrete minification process introduced by Littlejohn (1992). The models mentioned above accommodate many well-known distributions as stationary marginals.

We investigate whether classical minification (MINAR) processes accommodate any discrete distributions and find that the answer is affirmative. We give necessary and sufficient conditions for a discrete distribution to be a marginal of a stationary MINAR process. Many well-known discrete distributions satisfy these conditions. A general theory is discussed in Section 3.5, whereas a geometric MINAR process is discussed at length in Section 3.6. It is interesting to observe that a search for similar models in the scheme of maxima does not lead to success. Hence we propose the discrete analogues of MAXAR (extremal) processes. Two different models are introduced, and we show that both the models accommodate well-known discrete distributions. This work is contained in Section 3.7 of the present Chapter.

Before we describe the INAR processes, we describe in Section 3.2, a new function called alternative probability generating function. This function can be used in place of pgf while dealing with discrete distributions and its use, in our opinion, is more appealing than the use of pgf.

3.2 ALTERNATE PROBABILITY GENERATING FUNCTION :

Definition 3.2.1 : The alternate probability generating function of a nonnegative integer valued rv is defined as

$$P(s) = Q(1-s),\ 0 \le s \le 2; \qquad \ldots(3.2.1)$$

where $Q(\cdot)$ is the pgf of the random variable.

APGF determines the parent distribution uniquely, and it also enjoys all important properties of probability generating functions (pgf's), suggesting that it can be used as an alternative to pgf. Generally it is considered convenient to work with pgf's while dealing with discrete distributions. However, it will be shown that the use of alternate pgf (APGF) is more appealing.

Bondesson (1979) has drawn attention to the use of such functions. The term alternate pgf was coined by McKenzie (1986). Following is an important characterization of APGF.

Theorem 3.2.1 : *A real valued function* P(·) *defined on* [0, 2] *is an* APGF *of a probability distribution if and only if*

(1) P *is completely monotone on* (0, 1), *and*

(ii) P(0) = 1.

The result follows immediately from the definition of APGF and the similar characterization of pgf contained in Feller (1965, Lemma in VII.2).

In order to point out the analogy of APGF with Laplace transforms, we state below a well-known characterization of Laplace transforms. (See for example, Feller (1965)).

Theorem 3.2.2 : *A function* L *on* (0, ∞) *is the Laplace transform of a probability distribution if and only if*

(i) L *is completely monotone.*

(ii) L(0) = 1.

The analogy between the results of Theorem 3.2.1 and Theorem 3.2.2 is striking. Thus an important implication of Theorem 3.2.1 is that the Laplace transform of any nonnegative rv is an APGF of some nonnegative integer valued rv. For example, the Laplace transform of Gamma distribution is the APGF of Negative binomial distribution.

It would be interesting to find out what other pairs of distributions can be obtained in the above sense. In the next Theorem we answer this question in terms of compound distributions.

Theorem 3.2.3 : *Let Λ be a nonnegative random variable having distribution $F_\Lambda(\cdot)$ and Laplace transform $L_\Lambda(\cdot)$. Let the distribution of a rv X, conditional on $\Lambda = \lambda$, be Poisson(λ). Then the APGF of the unconditional distribution of X is $L_\Lambda(\cdot)$.*

Proof : Note that the APGF of the conditional distribution of X, conditional on $\Lambda = \lambda$, is given by

$$P_{X|\Lambda}(s) = E\{(1-s)^X|\Lambda = \lambda\} = e^{-\lambda s}$$

Hence the APGF of the unconditional distribution of X is

$$P_X(s) = \int_0^\infty e^{-\lambda s}\, dF_\Lambda(\lambda) = L_\Lambda(s)$$

This completes the proof ∎

Remark 3.2.1: The result of Theorem 3.2.3 was pointed out by Professor S. Dasgupta in a personal communication ∎

Remark 3.2.2 : It may be noted that every APGF need not be a Laplace transform, since there may be a function which is completely monotone on (0, 1) but not on (0, ∞), as required by Laplace transforms. Consider for example $P(s) = (1-s)$, which is an APGF but not a Laplace transform ∎

Next we note an important property of APGF in connection with the thinning operator defined in Section 1.3. If P_X is an APGF of X then the APGF of $\rho * X$ is given by

$$P_{\rho*X}(s) = E((1-s)^{\rho*X}) = E(E\{(1-s)^{\rho*X}|X=m\})$$

$$= \sum_{n=0}^{\infty} (1-s)^n \sum_{m=n}^{\infty} \binom{m}{n} \rho^n (1-\rho)^{m-n} P(X = m)$$

$$= \sum_{m=0}^{\infty} P(X = m) \sum_{n=0}^{m} \binom{m}{n} \{\rho(1-s)\}^n (1-\rho)^{m-n}$$

$$= \sum_{m=0}^{\infty} P(X = m) (1-\rho s)^m$$

$$= P_X(\rho s)$$

Thus we have

$$P_{\rho*X}(s) = P_X(\rho s). \qquad ...(3.2.2)$$

This result is analogous to

$$L_{\rho X}(s) = L_X(\rho s),$$

where L_X and $L_{\rho X}$ are Laplace transforms of rvs X and ρX respectively.

Next, Recall the definition of discrete self-decomposable laws given in Section 1.3. This definition can be stated in terms of APGF by the identity

$$P(s) = P(\rho s)\ P_{(\rho)}(s),$$

where $P_{(\rho)}$ is an APGF. Note that a similar identity in terms of cf's defines classical self-decomposable laws.

The analogy observed above between APGFs and Laplace transforms clearly suggests that the use of APGF's is more appealing than the use of pgf's.

3.3 INTEGER VALUED AR PROCESSES (INAR) :

McKenzie (1985), (1986) suggested some simple models for discrete variate time series. These models are analogous to AR(1) processes for continuous variate time series. The model proposed by McKenzie is of the form

$$X_n = \rho * X_{n-1} + Y_n \; ; \; n = 1, 2, \ldots, \qquad \ldots(3.3.1)$$

where $\rho \in (0, 1)$, $\rho*$ is a thinning operator described in Chapter 2, and $\{Y_n\}$ is a sequence of iid nonnegative integer valued random variables independent of X_0.

Al-Osh and Alzaid (1987) also introduced the process with structure at (3.3.1). Their work came out independently of McKenzie (1985). Al-Osh and Alzaid referred to their model as integer valued AR (INAR) processes.

The INAR process is similar in structure to AR process defined at (1.2.2), except that INAR process uses the thinning operation $\rho*X$ instead of scalar multiplication ρX. The similarity between the processes does not end here. Recall from Section 1.2 that only self-decomposable distributions can be the marginal distributions of stationary AR process. Interestingly, it turns out that the class of marginal distributions for stationary INAR process is same as the class of discrete self-decomposable laws introduced by Steutel and Van Harn (1979).

While constructing the INAR models McKenzie (1986) exploited the fact that the thinning operator $\rho*$, in case of discrete distributions, plays a role analogous to the scalar multiplication for continuous distributions, and that APGF of Negative binomial distribution and Laplace transform of Gamma distribution have same algebraic form. This similarity makes the construction of Negative binomial INAR process completely parallel to the construction of Gamma AR process (GAR(1)) of Gaver and Lewis (1980). Using the same technique McKenzie (1986) further gave the analogues of exponential MA process EMA(1), exponential ARMA process EARMA(p, q), a new exponential AR process NEAR(1) of Lawrence and Lewis (1977, 1980 and 1981), and the Gamma beta AR(1) process GBAR(1) of Lewis (1982). In all these analogues the Gamma variables are replaced by Negative binomial ones and the operation of scalar multiplication by the thinning operation.

Al-Osh and Alzaid (1987) presented INAR process with Poisson and geometric marginals. The autocorrelation structure of INAR processes is similar to that of AR processes as noted by McKenzie (1986) and Al-Osh and Alzaid (1987). Al-zaid and Al-Osh (1990) also proposed an INAR(p) process which is an obvious generalization of INAR(1) process and is given by

$$X_n = \alpha_1 * X_{n-1} + \alpha_2 * X_{n-2} + \ldots + \alpha_p * X_{n-p} + Y_n;$$

where $\{Y_n\}$ is a sequence of iid nonnegative integer valued random variables, Y_n being independent of $(X_{n-1}, X_{n-2}, \ldots, X_{n-p})$, and $\alpha_i \in (0, 1)$ such that $\sum_{i=1}^{p} \alpha_i < 1$.

The models of above type can be useful for a counting process $\{X_n\}$ in which a specific realization is attributed not only to the immediate past but also to previous realizations of the process.

It was noted by Alzaid and Al-Osh that the similarity between AR(p) and INAR(p) processes does not go beyond the representation of the process. It is shown that the autocorrelation function of the INAR(p) process is similar to that of the standard ARMA(p, p-1) process rather than to AR(p) process.

Al-Osh and Aly (1992) proposed an AR model with Negative binomial marginals. This process is a discrete analogue of the Gamma AR model of Sim (1990). The process is defined by

$$X_n = \alpha \odot X_{n-1} + \varepsilon_n.$$

Here $\alpha \in (0, 1)$ and the operator $\alpha\odot$ is defined as

$$\alpha \odot X = \sum_{i=0}^{N(X)} W_i,$$

where

(i) W_i are iid Geom $(\frac{\alpha}{1+\alpha})$ random variables, $W_0 = 0$,

(ii) For each fixed nonnegative integer value x of X, N(x) is a B(x, λ) random variable, $\lambda = \alpha p$ and $0 \le p \le 1$, and

(iii) ε_n are iid $NB(\frac{\alpha}{1+\alpha}, \nu)$ random variables with $\nu > 0$, and ε_n is independent of $\alpha \odot X_{n-1}$.

If X_0 has $NB(\frac{\alpha(1-p)}{1+\alpha(1-p)}, \nu)$ distribution, then the process is stationary, whereas for the arbitrary rv X_0,

$NB(\frac{\alpha(1-p)}{1+\alpha(1-p)}, \nu)$ is the limiting distribution of X_n as n tends to infinity.

It was shown that the properties analogous to the Gamma process of Sim (1990) are possessed by the above process. In particular, the autocorrelation of lag j is given by $\rho_j = \rho_1^j$, $\rho_1 = p$ being the autocorrelation of lag 1, and that the process is time reversible with both forward and backward regression being linear.

3.4 A DISCRETE MINIFICATION PROCESS :

A brief introduction to Minification processes was given in Section 1.2.2. A Minification process is defined by

$$X_n = \theta \min (X_{n-1}, Y_n); \; n \geq 1, \qquad \ldots(3.4.1)$$

where $\theta > 1$, and $\{Y_n\}$ is a sequence of iid random variables independent of X_0.

Minification processes were introduced by Tavares (1980) and studied later by Sim(1986), Chernick et al. (1988), Yeh et al. (1988) and Lewis and McKenzie (1991). All these authors considered minification processes with continuous marginals.

Littlejohn (1992) introduced a discrete analogue of minification process, and called it a discrete minification process. This process is defined by

$$X_n = \rho \backslash \min (X_{n-1}, Y_n); \qquad \ldots(3.4.2)$$

where $\rho \in (0, 1)$, $\{Y_n\}$ is a sequence of iid random variables with a common distribution G defined on $\mathbb{N}_0$.

Here $\rho\backslash$ represents an operation on nonnegative integer valued random variable which replaces the operation of scalar division by ρ for continuous random variables.

Recalling the operation of $\rho *$, note that conditional distribution of $\rho * X$ conditional on X is given by

$$P(\rho * X = n \mid X = m) = \binom{m}{n} (1-\rho)^{m-n} \rho^{n},$$

for n = 0, 1, ..., m; m = 0, 1, ...

Using Bayes' Theorem, distribution of X conditional on $\rho * X$ is obtained as

$$P(X = n \mid \rho * X = m) = \frac{\binom{n}{m} (1-\rho)^{n} P(X=n)}{\sum_{k=m}^{\infty} \binom{k}{m} (1-\rho)^{k} P(X=k)},$$

for n = m, m+1, ...

This leads to the definition of $\rho\backslash$, in terms of the conditional distribution of $\rho\backslash Z$ given Z, as follows

$$P(\rho\backslash Z = n \mid Z = m) = \frac{\binom{n}{m} (1-\rho)^{n} P(X=n)}{\sum_{k=m}^{\infty} \binom{k}{m} (1-\rho)^{k} P(X=k)},$$

for n = m, m+1, ...; m = 0, 1, ...

It is disturbing to note that the definition of $\rho\backslash Z$ operation depends on the probability distribution of X. This makes the definition of operator $\rho\backslash$ confusing. Also $\rho\backslash$ works as a left inverse of $\rho *$ only when it is operated on $\rho * X$.

The necessary and sufficient condition for a distribution F to be the marginal of Littlejohn's process is that

$$\frac{P(X = x)}{P(X \geq x)} \leq \frac{P(\rho * X = x)}{P(\rho * X \geq x)}.$$

It is shown that Binomial, Poisson and Negative binomial distributions belong to the class of stationary distributions of a discrete minification process.

The operator $\rho\backslash$ can be represented as

$$\rho\backslash Z = Z + \sum_{i=1}^{Z+r} G_i ,$$

or $$\rho\backslash Z = Z + Z_\rho ,$$

or $$\rho\backslash Z = Z + B ,$$

when the distribution of X is taken as NB(r, p) or Poisson(λ) or B(n, p) respectively, where G_i are iid Geom $(1-q(1-\rho))$ rvs, Z_ρ is a Poisson($(1-\rho)\lambda$) rv and B is a $B(n - Z, \frac{p(1-\rho)}{1-\rho p})$ rv.

It may be noted that in all the three cases the operator $\rho\backslash$ turns out to be a particular form of thickening operator T_4^+ defined in Chapter 2.

3.5 MINAR PROCESSES WITH DISCRETE MARGINALS :

To distinguish a minification process defined by (3.4.1) from a discrete minification process defined by (3.4.2), only the former will be called a MINAR process. The constant θ will be called the parameter of the MINAR process. Having noted the introduction of discrete minification process by Littlejohn (1992) and the fact that MINAR processes have been discussed only with continuous marginals, following question arises.

"Is it that discrete distributions can not be the marginals of a MINAR process ? "

To answer this question, we need to investigate the necessary condition that the marginal distribution of a stationary MINAR process must satisfy.

Let $\mathbb{F}_\theta$ be the class of all distributions F of X_0 such that $\{X_n\}$ is a stationary MINAR process, with parameter θ, θ >1. Let $\mathbb{F} = \bigcup_{\theta>1} \mathbb{F}_\theta$. and $\mathbb{F}^* = \bigcap_{\theta>1} \mathbb{F}_\theta$. Then it is interesting to identify the classes $\mathbb{F}$ and $\mathbb{F}^*$. Lewis and McKenzie (1991) have specified a necessary and sufficient condition for F to belong to $\mathbb{F}_\theta$, namely that

$$\frac{\bar{F}(\theta x)}{\bar{F}(x)}, \text{ say } \bar{G}_\theta(x) \text{ is a survival function,} \qquad \ldots(3.5.1)$$

where $\bar{F}(\cdot)$ is the survival function of X. This condition follows immediately from the definition of MINAR process.

Thus a necessary and sufficient condition for F to belong to $\mathbb{F}$ is that it satisfies (3.5.1) for some θ > 1, whereas a necessary and sufficient condition for F to belong to $\mathbb{F}^*$ is that (3.5.1) holds for every θ > 1. It is clear that $\mathbb{F}^*$ is same as the class of min-SD distributions (See Definition 1.2.1). Also we will show that the class $\mathbb{F}$ contains $\mathbb{F}^*$ as a proper subset.

We prove in the following Theorem that min-SD distributions are necessarily continuous.

THEOREM 3.5.1 : *Suppose* $F \in \mathbb{F}^*$. *Then* F *must be continuous at every* $x > 0$.

PROOF : If possible, let there exist a discontinuity point $x_0 > 0$ of F, Then since $\bar{F}$ is a monotonically decreasing function, it follows that

$$\bar{F}(x_0+) < \bar{F}(x_0-). \qquad ...(3.5.2)$$

Now, since $F \in \mathbb{F}^*$, for every $\theta > 1$ there exists a distribution G_θ such that,

$$\bar{F}(\theta x) = \bar{F}(x)\ \bar{G}_\theta(x).$$

Since $\bar{G}_\theta$ *is a survival function, we have*

$$\bar{G}_\theta(x_0+) \le \bar{G}_\theta(x_0-) \qquad \forall\ \theta > 1.$$

Then from (3.5.2) we get

$$\bar{F}(x_0+)\ \bar{G}_\theta(x_0+) < \bar{F}(x_0-)\ \bar{G}_\theta(x_0-),$$

which further implies that

$$\bar{F}\ ((\theta x_0)+) < \bar{F}\ ((\theta x_0)-) \qquad \forall\ \theta > 1.$$

Thus θx_0 is a discontinuity point of F for every $\theta > 1$. This implies that F is discontinuous at every $x > x_0$. This contradicts the fact that a monotonic function can have at the most countable number of discontinuity points. Hence we conclude that F is continuous at every $x > 0$. ∎

Thus the possibility of discrete distributions is ruled out for the class $\mathbb{F}^*$. It will be shown that the class $\mathbb{F}$ contains most of the well-known discrete distributions.

All important distributions F have a support $S_F \subset \mathbb{N}_0$. Before we obtain conditions for a distribution

F, with $S_F \subset \mathbb{N}_0$, to belong to $\mathbb{F}$, we prove an important result which shows that the parameter of a stationary MINAR process must be integer when the marginal distribution F has $S_F \subset \mathbb{N}_0$.

THEOREM 3.5.2 : *Let* F *with* $S_F \subset \mathbb{N}_0$ *be a distribution function such that* $F(x) < 1$ *for every real* x. *Suppose* $F \in \mathbb{F}_\theta$ *for some* $\theta > 1$. *Then* θ *must be an integer.*

PROOF : Since $F \in \mathbb{F}_\theta$, there exists a distribution function G_θ satisfying

$$\bar{F}(\theta x) = \bar{F}(x)\ \bar{G}_\theta(x).$$

Since $F(x) < 1$ for all real x, it follows that $G_\theta(x) < 1$ for all real x; for otherwise, $\bar{F}(\theta x)$ becomes zero for some x (or that $F(\theta x)$ becomes one for some x).

Let X and Y be independent random variables with distributions F and G_θ respectively. Then since Y is unbounded, it follows that for $r \in S_F$,

$$\begin{aligned} P(\min(X, Y) = r) &\geq P(X = r, Y > r) \\ &= P(X = r)\ P(Y > r) \\ &> 0 \end{aligned}$$

or equivalently $P(X = \theta r) > 0 \qquad \forall\ r \in S_F$.

Thus, the fact that $x_r = \theta r$ is an integer belonging to S_F, $\forall\ r \in S_F$ implies that $\theta = x_r/r$ is a rational number.

Therefore let θ be represented as $\theta = K/n$, where K and n are relatively prime. Then we have

$$r = \frac{n}{K} x_r \ \forall\ r \in S_F.$$

Now since r is an integer, it follows that x_r is a multiple of K; that is x_r is of the form $x_r = Kx'_r$, where x'_r is an integer, so that we have

$$r = nx'_r \quad \forall\ r \in S_F. \qquad ...(3.5.3)$$

Thus every r in S_F is a multiple of n. Then it follows that x_r is also a multiple of n, and hence x'_r is also a multiple of n. Then from (3.5.3) we get

$$r = n^2 x''_r \quad \forall\ r \in S_F.$$

By the repetition of this argument it follows that every integer in S_F is a multiple of n, $n^2, n^3, \ldots$. This can happen only when n = 1. Hence we conclude that θ must be an integer. ■

Following example shows that the condition $F(x) < 1$ in above Theorem is necessary for the result.

Example 3.5.1: Let X be a random variable taking values 1,2, ...,m. Let $\theta > m$ be arbitrary. Consider a random variable Y, independent of X, such that

$$P(\ Y = j/\theta) = P(X = j) \text{ for } j = 1,\ldots,m.$$

Since $(m/\theta) < 1$, the support of Y lies entirely to the left of the support of X, so that min(X, Y) = Y with probability 1. Also from the definition of X it is clear that $Y \overset{d}{=} \frac{1}{\theta} X$, so that we have $\min(X, Y) \overset{d}{=} \frac{1}{\theta} X$ ■

Henceforth, we will use the notation K, which is more convenient for an integer valued parameter, in place of θ in $\{X_n\}$ defined at (3.4.1). It is clear that the class $\mathbb{F}_K$ contains discrete as well as continuous distributions. We denote by $\mathbb{F}^{I}_{K}$ the class of all distributions F, with $S_F \subset \mathbb{N}_0$, which belong to $\mathbb{F}_K$. We also use the notation $\mathbb{F}^{I*}$ for the class $\bigcap_{K\geq 2} \mathbb{F}^{I}_{K}$.

If F belongs to $\mathbb{F}^{I}_{K}$ then from (3.5.1) it is clear that $G_K(x)$ has jumps only at the points of the form m/K, where m is a positive integer. It therefore looks more appropriate to express the process $\{X_n\}$ at (3.4.1) as

$$X_n = \min(\ KX_{n-1},\ Y_n)\ ,\quad n = 1,2,... \qquad ...(3.5.4)$$

for some integer $K \geq 2$.

With this new form, the necessary and sufficient condition for F to belong to $\mathbb{F}^{I}_{K}$ becomes,

$$\bar{G}_K(x) = \frac{\bar{F}(x)}{\bar{F}(x/K)} \text{ is monotonically increasing for } x \geq 0, \qquad ...(3.5.5)$$

where $\bar{F}(x) = P(X > x)$ is the discrete survival function of X. With this new expression, now $G_K(x)$ has jumps only at integer points. Next we give an alternative necessary and sufficient condition for F to belong to $\mathbb{F}^{I}_{K}$. Denote by $f(\cdot)$ the probability mass function (pmf) of F. Now since $G_K(x)$ has jumps only at integer points, (3.5.5) is satisfied if and only if

$$g_K(m) = \bar{G}_k(m-1) - \bar{G}_K(m) \geq 0 \text{ for } m = 1,2,...$$

From (3.5.5) it can be easily verified that,

$$g_K(m)=\begin{cases} \dfrac{\bar{F}(r)\bar{F}(rK-1) - \bar{F}(rK)\bar{F}(r-(1/K))}{\bar{F}(r-(1/K))\ \bar{F}(r)} & ;m = rK, \text{ for } r = 0, 1,\dots \\ \dfrac{\bar{F}(r+(j/K)\bar{F}(m-1)-\bar{F}(m)\bar{F}(r+(j-1)/K}{\bar{F}(r+(j-1)/K)\ \bar{F}(r+(j/K))} & ;m=rK+j,\ r=0,1,\dots\ j=1,\ 2,\ \dots,K-1 \end{cases}$$

That is,

$$g_K(m) = \begin{cases} \dfrac{\bar{F}(r)\bar{F}(rK-1) - \bar{F}(rK)\bar{F}(r-1)}{\bar{F}(r-1)\ \bar{F}(r)} & ;\ m = rK, \text{ for } r = 0,\ 1,\dots \\ \dfrac{f(m)}{\bar{F}(r)} & ;m = rK+j, \text{ for } r = 0,\ 1,\dots\ j = 1,\ 2,\ \dots,K-1 \end{cases} \qquad \dots(3.5.6)$$

Above equation follows by observing that $\bar{F}(r-(1/K)) = \bar{F}(r-1)$ and $\bar{F}(r+(j/K)) = \bar{F}(r+(j-1)/K) = \bar{F}(r)$.

Thus $g_K(m) \geq 0$ if and only if

$$\frac{\bar{F}(rK-1)}{\bar{F}(rK)} \geq \frac{\bar{F}(r-1)}{\bar{F}(r)}, \qquad \text{for } r = 0,1,\dots \qquad \dots(3.5.7)$$

Subtracting 1 from both the sides, we get

$$\frac{f(rK)}{\bar{F}(rK)} \geq \frac{f(r)}{\bar{F}(r)},$$

or equivalently

$$\frac{f(rK)}{\bar{F}(rK)+f(rK)} \geq \frac{f(r)}{\bar{F}(r)+f(r)}.$$

That is,

$$\lambda(rK) \geq \lambda(r)\ , \quad \text{for } r = 0,1,\dots \qquad \dots(3.5.7)'$$

where $\lambda(x) = \dfrac{P(X = x)}{P(X \geq x)}$ is a discrete hazard function corresponding to F.

Thus (3.5.7) (or equivalently $(3.5.7)'$ is a necessary and sufficient condition for F to belong to $\mathbb{F}_K^{I}$.

Remark 3.5.1 : In case of continuous random variables with df F and pdf f, the hazard function is defined as

$$\lambda(x) = \lim_{\delta x \to 0} P(x \le X \le x+\delta x \mid X \ge x) = f(x)/\bar{F}(x).$$

If we just adopt later expression after replacing f(x) by P(X=x) for defining a discrete hazard function, then there will be a fallacy in the definition, as for given X > x how can we have X = x ?. However the original definition (in terms of limit) leads to the correct expression for discrete hazard functions ■

Remark 3.5.2 : When F satisfies (3.5.7), $g_K(m)$ given by (3.5.6) is the probability mass function of the common distribution of the innovation process for a stationary MINAR process at (3.5.4), with parameter K, whose marginal distribution is F ■

Remark 3.5.3: If F is an absolutely continuous distribution, with $S_F \subset \mathbb{R}^+$, then differentiating $\bar{G}_\theta(x)$ in (3.5.1), we get a necessary and sufficient condition for F to belong to $\mathbb{F}_\theta$ in terms of survival function as

$$\theta\, \lambda(\theta x) \ge \lambda(x),$$

where $\lambda(x)$ is the hazard function of the distribution F. Lewis and McKenzie (1991) also obtained this condition using a different approach. This condition is analogous to the condition $(3.5.7)'$ ■

Remark 3.5.4 : If an absolutely continuous distribution F, with $S_F \subset \mathbb{R}^+$, belongs to $\mathbb{F}_\theta$ then the common pdf of the corresponding innovation process is given by

$$g(y) = \frac{\theta f(\theta y)\ \bar{F}(y) - f(y)\ \bar{F}(\theta y)}{(\bar{F}(y))^2},$$

where $f(\cdot)$ is the pdf of F. Above pdf is analogous to the pmf given by (3.5.6) ∎

Remark 3.5.5 : It is necessary to distinguish between the classes $\mathbb{F}_K^{\prime}$ and $\mathbb{F}^{\prime *}$ because the latter is a proper subclass of the former. To show this we give an example of a distribution F which belongs to $\mathbb{F}_2^{\prime}$ but does not belong to $\mathbb{F}^{\prime *}$

Let $\alpha \neq \beta \in (0,1)$. Consider a distribution F for which

$$P(X \geq x) = \begin{cases} (1-\alpha)^{x-1} & ;\ x \leq 3 \\ (1-\beta)^r (1-\alpha)^{x-1-r} & ;3(2)^{r-1} < x \leq 3(2)^r,\ r=1,\ 2, \ldots \end{cases}$$

It is easy to verify that

$$\lambda(x) = \begin{cases} \beta & ;\ x = 3(2)^r,\ r = 0,\ 1,\ \ldots \\ \alpha & ;\ \text{otherwise} \end{cases}.$$

Note that $\lambda(x) = \lambda(2x)$ for $x = 0,1,\ldots$

Thus from condition $(3.5.7)'$, F belongs to $\mathbb{F}_2^{\prime}$. But since $(3.5.7)'$ is not satisfied, for example, for K = 3, F does not belong to $\mathbb{F}^{\prime *}$ ∎

Next observe that (3.5.7) is satisfied for every $K \geq 2$, whenever

$$\frac{\bar{F}(x-1)}{\bar{F}(x)} \text{ is a nondecreasing function of } x, \quad \ldots(3.5.8)$$

or equivalently whenever

$$\lambda(x) \text{ is a nondecreasing function of } x \quad \ldots(3.5.8)'$$

Thus (3.5.8) (or (3.5.8)$'$) becomes a sufficient condition for F to belong to $\mathbb{F}^{l*}$. (3.5.8) (or (3.5.8)$'$) is generally easy to check as compared to (3.5.7) (or (3.5.7)$'$).

Remark 3.5.6 : Note that condition (3.5.8) is not necessary for F to belong to $\mathbb{F}^{l*}$. To see this, consider the distribution F defined by

$$P(X \geq x) = \begin{cases} (1-\alpha)^{x-1} & ;x = 1,2 \\ (1-\alpha)(1-\beta) & ;x = 3 \\ (1-\alpha)^2(1-\beta)^{x-3} & ;x = 4,5,\ldots \end{cases} \qquad \alpha < \beta \in (0, 1).$$

Then we have $\lambda(1) = \lambda(3) = \alpha$, and $\lambda(x) = \beta$ for $x = 2,4,5,\ldots$. Clearly F satisfies the condition (3.5.7)$'$ for $K = 2,3,\ldots$, and hence belongs to $\mathbb{F}^{l*}$, without satisfying (3.5.8)$'$ (which is equivalent to (3.5.8)) ■

Theorem 3.5.3 : *Let* F *be a probability distribution with* $S_F \subset \mathbb{N}_0$, *and let* f(x) *be its* pmf. *If*

$$\frac{f(x)}{f(x+1)} \text{ is a nondecreasing function of } x \quad \ldots(3.5.9)$$

then $F \in \mathbb{F}^{l*}$.

Proof : Consider

$$\frac{\bar{F}(x-1)}{\bar{F}(x)} = \frac{\sum_{k=x}^{\infty} f(k)}{\sum_{k=x+1}^{\infty} f(k)} = \frac{\sum_{k=x}^{\infty}\left[\frac{f(k)}{f(k+1)}\right] f(k+1)}{\sum_{k=x+1}^{\infty} f(k)}$$

$\geq \frac{f(x)}{f(x+1)}$ if $\frac{f(k)}{f(k+1)}$ is a nondecreasing function of k.

Thus the discrete hazard function $\lambda(\cdot)$ corresponding to F satisfies $\lambda(x) \leq \lambda(x+1)$, which is a sufficient condition for F to belong to $\mathbb{F}^{1*}$ ∎

The sufficient condition (3.5.9) for (3.5.8)′ to hold was inspired by the sufficient condition

$$\frac{a(x)}{a(x+1)} \text{ is nondecreasing in } x$$

for the power series distributions given by

$$f(x;\theta) = a(x)\theta^{x}, \; x = 0, 1, \ldots$$

The latter condition was suggested by Professor Y.S. Sathe, in a personal communication, The technique used by Professor Sathe is interesting and is shown below for obtaining the sufficient condition at (3.5.9).

Let X be a random variable with distribution F and pmf f(x). Let Y be a random variable with pmf

$$P(Y=j) = \frac{f(j)}{P(X \geq x)} \; ; \; j = x, x+1, \ldots.$$

Since $Y \geq x$ with probability one, it follows that whenever $f(k)/f(k+1)$ is a nondecreasing function of k, we have $\frac{f(Y)}{f(Y+1)} \geq \frac{f(x)}{f(x+1)}$ with probability one. Hence,

$$\frac{f(x+1)}{f(x)} \geq E\left(\frac{f(Y+1)}{f(Y)}\right) = \sum_{j=x}^{\infty} \frac{f(j+1)}{f(j)} \frac{f(j)}{P(X \geq x)}$$

$$= \sum_{j=x}^{\infty} \frac{f(j+1)}{P(X \geq x)} = \frac{P(X \geq x+1)}{P(X \geq x)}$$

Thus we have $\frac{f(x+1)}{P(X \geq x+1)} \geq \frac{f(x)}{P(X \geq x)}$.

That is, $\lambda(x) \leq \lambda(x+1)$ and hence $F \in \mathbb{F}^{I*}$.

Using the approach considered above (either ours or that of Professor Sathe), we get the following result.

Theorem 3.5.4 : *Let* F *be a probability distribution with* $S_F \subset \mathbb{N}_0$, *and let* f(x) *be its* pmf. *If*

$$\frac{f(r(K-1)+x)}{f(x)} \text{ is a nonincreasing function of } x \qquad \ldots(3.5.10)$$

for every nonnegative integer r, then $F \in \mathbb{F}^{I*}$.

Remark 3.5.7 : It may be noted that the conditions (3.5.9) and (3.5.10) are considerably easier to check as compared to (3.5.8) and (3.5.7) respectively, although (3.5.10) is not a necessary condition whereas (3.5.7) is. Replacing the roles of summation and pmf in above approach by integration and pdf respectively, we can obtain a condition that is sufficient for an absolutely continuous distribution F to belong to $\mathbb{F}_\theta$. The condition is

$$\frac{f(x\theta)}{f(x)} \text{ is a nonincreasing function of } x,$$

where f is the pdf of F (See Remark 3.5.3) ∎

Next we show that the Poisson and negative binomial families belong to $\mathbb{F}'^{*}$.

Poisson distribution : Consider a Poisson (λ) distribution. Then we have $\frac{f(x)}{f(x+1)} = \frac{x+1}{\lambda}$, which is an increasing function of x for $\forall\, \lambda > 0$.

Thus condition (3.5.9) is satisfied. Hence it follows that Poisson family belongs to $\mathbb{F}'^{*}$.

A more circuitous argument was made in Kalamkar (1995) to prove this fact while verifying (3.5.8)′. For completeness we discuss that proof.

Let s and t be nonnegative integers such that $s \geq t$. Then

$$\binom{x+s}{x} \geq \binom{x+t}{x} \qquad \text{for all integers } x \geq 0.$$

so that $$\frac{(x+s)!}{s!} \geq \frac{(x+t)!}{t!} \qquad \text{for all } x \geq 0.$$

Taking reciprocal and multiplying by $e^{-\lambda}\lambda^{x}$, followed by summation over x, we get

$$\frac{t!}{s!} \geq \frac{\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^{x}}{(x+s)!}}{\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^{x}}{(x+t)!}} \qquad \text{for all } \lambda > 0.$$

That is, $\lambda^{s-t} \frac{t!}{s!} \geq \frac{\sum_{x=s}^{\infty} \frac{e^{-\lambda} \lambda^x}{x!}}{\sum_{x=t}^{\infty} \frac{e^{-\lambda} \lambda^x}{x!}}$ for all $\lambda > 0$.

or $$\frac{f(s)}{f(t)} \geq \frac{\bar{F}(s-1)}{\bar{F}(t-1)} .$$

Thus $(3.5.8)'$ is satisfied. This proves that the Poisson family belongs to $\mathbb{F}^{\prime *}$.

Negative binomial distribution :

Consider a negative binomial distribution with probability mass function

$$P(X = x) = \binom{x+r-1}{r-1} p^r q^x \qquad x = 0,1,.. \quad r \text{ positive integer},$$

$0 < p < 1$; $q = 1-p$.

Then we have

$$\frac{f(x)}{f(x+1)} = \frac{\binom{x+r-1}{r-1}}{\binom{x+r}{r-1}} \frac{1}{q}$$, which is an increasing function of x for every r and p. This follows from the fact that for all non negative integers a and b,

$$\frac{\binom{y}{b}}{\binom{y+a}{b}}$$ is an increasing function of y ∎

It may be noted that both the examples considered here have support $\mathbb{N}_0$. In Remark 1.2.2, we noted that the minification process is more naturally defined on $(0, \infty)$

rather than on $[0, \infty)$. Then a question arises whether the examples given above are worth mentioning. This question is answered by the following Theorem.

Theorem 3.5.5 : *If* F *concentrated on* $\mathbb{R}^+$ *belongs to* $\mathbb{F}_\theta$, *for some* $\theta > 1$, *then* $F_{(a)}$, *the distribution* F *when truncated below at a, also belongs to* $\mathbb{F}_\theta$.

Proof : Observe that

$$\bar{F}_{(a)}(x) = \frac{\bar{F}(x)}{\bar{F}(a)} \quad \forall\ x \geq a.$$

so that
$$\frac{\bar{F}_{(a)}(\theta x)}{\bar{F}_{(a)}(x)} = \frac{\bar{F}(\theta x)}{\bar{F}(x)}.$$

Hence $F_{(a)}$ satisfies (3.5.1) whenever F satisfies (3.5.1) and vice versa ■

3.6. A GEOMETRIC MINAR PROCESS

In this Section we study the MINAR process with Geom^+ marginals in detail.

3.6.1 THE PROCESS

We have seen in the previous Section that the negative binomial family belongs to $\mathbb{F}^{1^*}$; the geometric distribution, which is a particular case of the negative binomial, also belongs to $\mathbb{F}^{1^*}$. Also it follows from Theorem 3.5.5 that the Geometric distribution conventrated on $\mathbb{N}$, which we denote by $\text{Geom}^+(p)$, with pmf

$$P(X = x) = p(1-p)^{x-1} \ ; \ x=1,.... \ ; \ 0 < p < 1.$$

also belongs to $\mathbb{F}^{1^*}$.

The MINAR process at (3.5.4) with geometric marginals will be referred to as a geometric MINAR process. For the geometric MINAR process, with parameter K, the survival function of the innovation process is given by

$$\bar{G}_K(x) = \frac{P(X > x)}{P(X > x/K)} = q^{[x]-[x/K]} \; ; \; x \geq 0 \text{ and } q = 1-p, \qquad \ldots(3.6.1)$$

where [x] denotes the largest integer not exceeding x.

Thus the geometric MINAR process with parameter K is given by

$$X_n = \min(KX_{n-1}, Y_n) \; , \quad n = 1,2,... \qquad \ldots(3.6.2)$$

for some integer $K \geq 2$, Where $\{Y_n\}$ is a sequence of iid random variables, independent of X_0, with the common distribution G_K specified by (3.6.1).

If X_0 is distributed as $Geom^+(p)$ then the process is strictly stationary with $Geom^+(p)$ marginals, whereas for an arbitrary random variable X_0, X_n converges in distribution to a $Geom^+(p)$ random variable.

It can be shown that G_K defined by (3.6.1) is in fact the distribution of a random variable Y where,

$$Y \stackrel{d}{=} X + \left[\frac{X-1}{K-1} \right] \qquad \ldots(3.6.3)$$

and X is a $Geom^+(p)$ random variable. This can be shown as follows.

Since X is integer valued, for r = 0, 1,...; j = 0, 1,..., if $X > (K-1)r+j$ then $X \geq (K-1)r+j+1$, which implies that

$$\frac{X-1}{K-1} \geq r + \frac{j}{K-1} .$$

That is, $X + \left[\frac{X-1}{K-1}\right] > rK+j.$

Conversely, if $X \leq (K-1)r+j$ then $X < (K-1)r+j+1$, which implies that

$$\frac{X-1}{K-1} < r + \frac{j}{K-1}$$

That is, $X + \left[\frac{X-1}{K-1}\right] \leq rK+j.$

Thus $\{X > (K-1)r + j)\} \equiv \{X + \left[\frac{X-1}{K-1}\right] > rK+j\}$. Therefore

$$P\{X + \left[\frac{X-1}{K-1}\right] > rK+j\} = P\{X > (K-1)r + j)\}$$
$$= q^{[K-1]r+j} = P(Y > rK+j)$$

This observation implies that in the case of a Geometric distribution on ℕ, Y does not take values of the form rK. This property of the innovation process in fact characterizes a geometric scale minification process, as will be shown in the Chapter 4.

The joint distribution of X_n and X_{n-1} is obtained in the next subsection. The process is also simulated to study the behaviour of sample paths. The Simulation results are mentioned in Subsection 3.6.3.

3.6.2 THE JOINT DISTRIBUTION OF X_n and X_{n-j}

For the stationary MINAR processes defined at (3.4.1), Lewis and McKenzie (1991) obtained the expression for the joint distribution of contiguous terms X_n and X_{n-1}. Using their approach, the joint survival function for the process $\{X_n\}$ defined at (3.6.2) is given by

$$\bar{H}(x, y; n, n-j, K) = P(X_n > x, X_{n-j} > y)$$
$$= P(X_{n-1} > x/K, Y_n > x, X_{n-j} > y)$$

By repeating the use of (3.6.2) we get

$$\bar{H}(x, y; n, n-j, K) = \bar{F}(\max\{x/K^j, y\})\, \bar{G}(x/K^{j-1}) \ldots \bar{G}(x)$$
$$= \frac{\bar{F}(\max\{x/K^j, y\})\, \bar{F}(x)}{\bar{F}(x/K^j)}$$

where $\bar{F}(x) = q^{[x]}$ is the survival function of $\text{Geom}^+(p)$.

Remark 3.6.1 : It is interesting to note that

$$\bar{H}(x, y; n, n-j, K) = \bar{H}(x, y; n, n-1, K^j);$$

so that the joint distribution of X_n and X_{n-j} can be obtained by replacing K by K^j in the joint distribution of X_n and X_{n-1}. As a consequence, it is sufficient to study the joint distribution of X_n and X_{n-1} ■

Next we find correlation function for the proposed geometric MINAR process. In the light of Remark 3.6.1, it is sufficient to obtain the correlation coefficient between X_n and X_{n-1}.

Using the joint survival function $\bar{H}$, we can compute

$$E(X_n X_{n-1}) = \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} \bar{H}(x, y; n, n-1, K).$$

$$= \sum_{y=0}^{\infty} \sum_{x=0}^{Ky-1} \bar{H}(x, y; n, n-1, K) + \sum_{y=0}^{\infty} \sum_{x=Ky}^{\infty} \bar{H}(x, y; n, n-1, K)$$

$$= \sum_{y=0}^{\infty} \sum_{x=0}^{Ky-1} \frac{\bar{F}(y)\bar{F}(x)}{\bar{F}(x/K)} + \sum_{y=0}^{\infty} \sum_{x=Ky}^{\infty} \bar{F}(x)$$

$$= A + B \text{ , Say.} \qquad \ldots(3.6.4)$$

where $A = \sum_{y=0}^{\infty} \bar{F}(y) \sum_{x=0}^{Ky-1} q^{x - [x/K]}$

and $B = \sum_{y=0}^{\infty} \sum_{x=Ky}^{\infty} \bar{F}(x)$

Denoting q^{K-1} by α, we find

$$\sum_{x=0}^{Ky-1} q^{x-[x/K]} = \sum_{x=0}^{K-1} q^x + \sum_{x=k}^{2K-1} q^{x-1} + \ldots\ldots + \sum_{x=K(y-1)}^{Ky-1} q^{x-(y-1)}$$

$$= \sum_{x=0}^{K-1} q^x + \sum_{x=k-1}^{2K-2} q^x + \ldots\ldots + \sum_{x=(K-1)(y-1)}^{Ky-y} q^x$$

$$= \left(\sum_{x=0}^{K-1} q^x \right) (1 + \alpha + \alpha^2 + \ldots + \alpha^{y-1})$$

Thus, $A = \sum_{y=0}^{\infty} q^y \frac{(1 - q\alpha)\ (1 - \alpha^y)}{(1 - q)\ (1 - \alpha)}$

$$= \frac{(1-q\alpha)}{(1-q)(1-\alpha)} \left[\frac{1}{1-q} - \frac{1}{1-q\alpha}\right]$$

$$= q/p^2.$$

Further,

$$B = \sum_{y=0}^{\infty} \sum_{x=Ky}^{\infty} q^x = \sum_{y=0}^{\infty} \frac{q^{ky}}{1-q} = \frac{1}{p(1- q\alpha)} .$$

Substituting the values of A and B in (3.6.4), we get

$$E\ (X_n X_{n-1}) = \frac{q}{p^2} + \frac{1}{p(1-q\alpha)} ,$$

Also Recall that $E(X_n) = E(X_{n-1}) = \frac{1}{p}$ and $V(X_n) = V(X_{n-1}) = \frac{q}{p^2}$.

Thus the autocorrelation of Lag 1 is given by

$$\rho(1) = r_{X_n, X_{n-1}} = \frac{p}{q} \left(\frac{q^K}{1-q^K} \right).$$

From the Remark 3.6.1 it is clear that the autocorrelation of Lag j is given by,

$$\rho(j) = r_{X_n, X_{n-j}} = \frac{p}{q} \left(\frac{q^{K^j}}{1 - q^{K^j}} \right).$$

Note that the autocorrelation function is not similar to that of the classical AR processes, unlike in the exponential process of Tavares (1980) or the non negative geometric process of Littlejohn (1992), where $\rho(j)$ is same as $\rho(1)^j$. The $Geom^+$ process of Littlejohn does not have the similarity mentioned above, as We note in the next Section.

3.6.3 A SIMULATION STUDY

A simulation study of our $Geom^+$ MINAR process and the discrete minification process of Littlejohn (1992) with $Geom^+$ marginals was carried out in order (i) to study the characteristics of the processes, and (ii) to investigate the discrimination power of sample paths.

Littlejohn's $Geom^+$ process is defined as

$$X_n = \rho\backslash \min(X_{n-1}, Y_n) \quad , n = 1,2,...$$

where $\rho \in (0, 1)$, X_0 follows $Geom^+(p)$. $\{Y_n\}$ is a sequence of iid random variables, independent of X_0, having $Geom(\frac{\bar{\rho}p}{1-\bar{\rho}q})$ distribution, where $\bar{\rho} = (1-\rho)$. The operation $\rho\backslash Z$ is defined as

$$\rho\backslash Z = \sum_{i=0}^{Z} N_i - I(Z > 0),$$

where the N_i are iid $Geom^+(1-\bar{\rho}q)$ variates, and I(.) is an indicator random variable.

The joint pgf of X_n and X_{n-1} for this process is given by,

$$Q(s_1, s_2) = \frac{\bar{\rho}p(1-\bar{\rho}q)(s_2-1)}{(1-s_2q)\ (1-s_1q\bar{\rho})} + \frac{s_1p(1-q(\bar{\rho}+\rho s_2))(\bar{\rho}+\rho s_2)}{(1-s_2q)\ (1-s_1q(\bar{\rho}+\rho s_2))},$$

and the correlation coefficient between X_n and X_{n-1} is

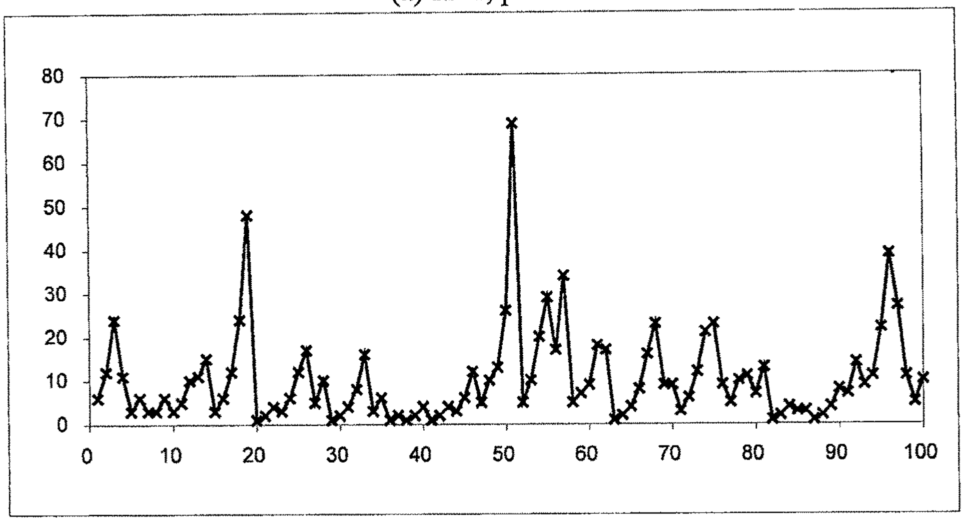
$$\text{corr}\,(X_n, X_{n-1}) = \frac{\rho^2}{p+\rho q}\ .$$

(For details See the Appendix at the end of this Chapter).

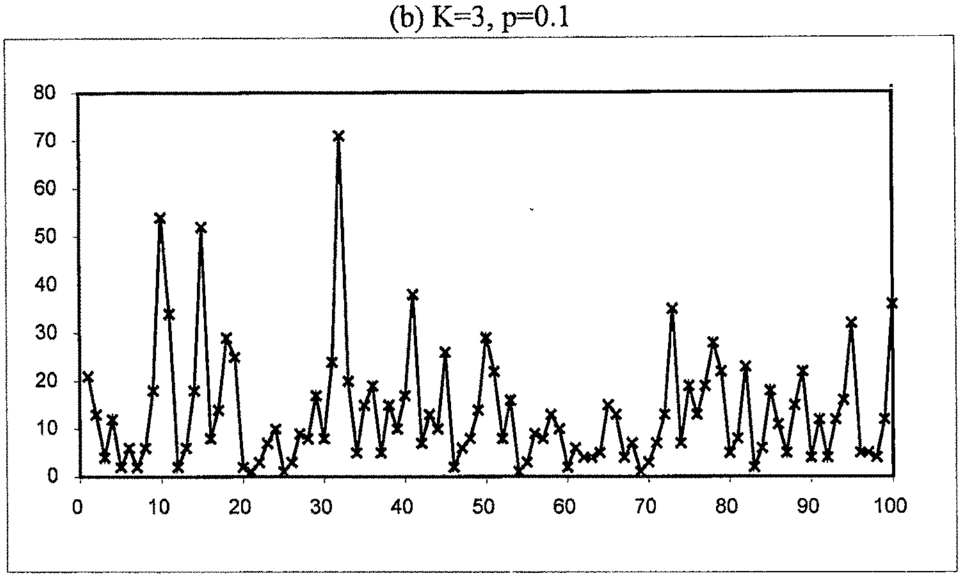
To make the sample paths of the two processes comparable, the values of ρ were selected in such a way that the auto correlation of lag 1 was the same in both.

Sample runs of size 100 were carried out for various values of p and K (ρ in the case of Littlejohn's process). It was found in both the processes that a trend is apparent when K is small (ρ is high) and p is also small. An increase in either K or p results in reduced autocorrelation. The sample paths also show no clear trend in such cases. The simulated paths of our process are shown in Figure 1 (a, b and c) and that of Littlejohn's process (with ρ chosen appropriately) in Figure 2 (a, b and c) for p = 0.1 and K = 2, 3, 4 respectively. For small values of K (2 and 3 here) the trend is clear: the increases are geometric followed by a sharp fall in both the processes. This trend tends to be less apparent for higher values of K (4 here). The geometric increase is clearer in our process than in Littlejohn's process. This is because of the thickening applied to X_{n-1}, which is deterministic (scalar multiplication) in our process and probabilistic in

(a) K=2, p=0.1



(b) K=3, p=0.1



(c) K=4, p=0.1

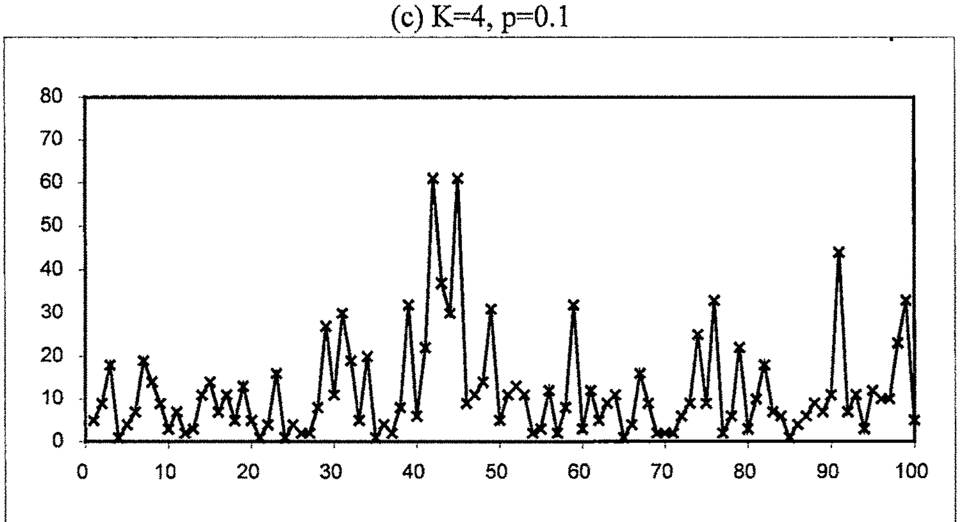


Figure 1 : Our Process

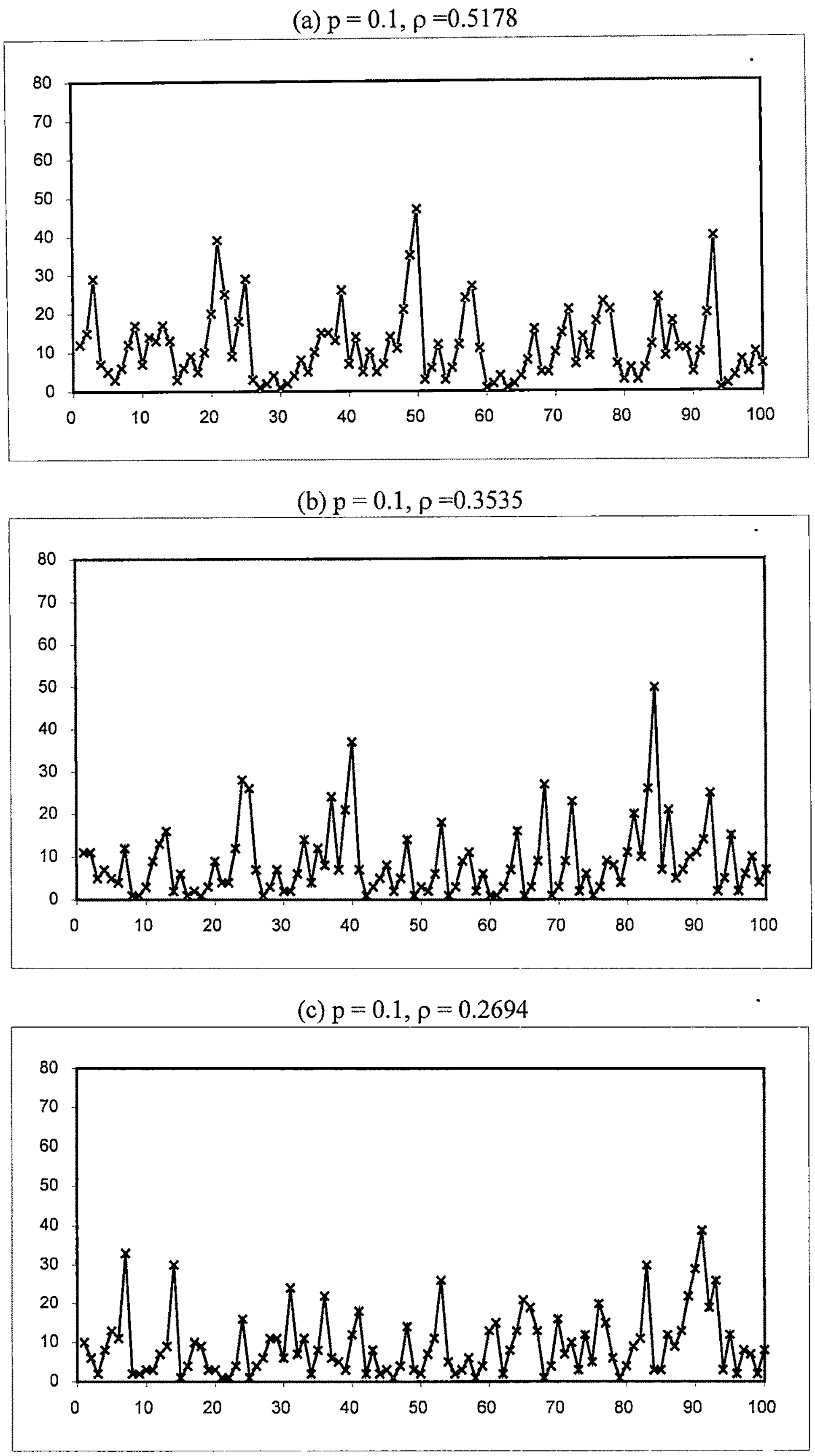


Figure 2 : Littlejohn's Process

Littlejohn's process also tends to an iid sequence as ρ decreases. The sample runs appear to discriminate between the processes only when autocorrelations of lag 1 are not small (i.e., greater than about 0.4). Clear geometric increases followed by a sharp fall seem characteristic of our process, while less clarity in the geometric increases seem characteristic of Littlejohn's.

3.7 DISCRETE MAXIMUM PROCESSES

3.7.1 MAXIMUM PROCESS OF ALPUIM

Alpuim (1989) introduced an extremal process, which may be called a MAXAR(1) process in the present context. This process is given by

$$X_n = \rho \max (X_{n-1}, Y_n), \; n = 1, 2, \ldots \qquad \ldots(3.7.1)$$

for some $\rho \in (0, 1)$, where $\{Y_n\}$ is a sequence of iid random variables, independent of a nonnegative random variable X_0.

Let $\mathbb{L}_\rho$ be the class of all distributions F of X_0 for which the process at (3.7.1) is stationary for some innovation process $\{Y_n\}$. Let $\mathbb{L}^* = \bigcap_{\rho\in(0,1)} \mathbb{L}_\rho$ and $\mathbb{L} = \bigcup_{\rho\in(0,1)} \mathbb{L}_\rho$. Alpuim (1989) showed that if $F \in \mathbb{L}^*$ then F can not be discrete. However, an F belonging to $\mathbb{L}$ can be discrete. Using different arguments we have shown earlier in Section 3.5 that a similar result holds for minification processes also.

It is of interest to know whether any well-known discrete distribution belong to $\mathbb{L}$, although it is not in $\mathbb{L}^*$. To answer this question consider a random variable X taking values $\{a, a+1, \ldots.b\}$, where $a \geq 0$ and $b \leq \infty$. Then the distribution F of X belongs to $\mathbb{L}$ if and only if there exist a random variable Y satisfying $\rho \max(X, Y) \overset{d}{=} X$. The distribution function G of Y is given by

$$G(x) = \frac{P(X \leq \rho x)}{P(X \leq x)} \quad \forall\ x \geq a. \qquad \ldots(3.7.2)$$

Let $x_0 = a/\rho$. Then from (3.7.2) it follows that $P(Y < x_0) = 0$. Further, if $F(x_0) < 1$ then (3.7.2) implies that $G(x_0) > G(x_0+1)$, which is a contradiction. Thus $F(x_0) = 1$. Hence we conclude that $\max(X, Y) = Y$ with probability 1, in which case $\{X_n\}$ is a sequence of iid random variables.

This shows that no well-known distribution belongs to $\mathbb{L}$. This is contrary to the case of minification processes, for which the class of stationary distributions contains well-known discrete distributions. Hence we do not further investigate the class of discrete distributions which may belong to $\mathbb{L}$.

We propose two models which are the discrete versions of maximum process and call them discrete maximum process-I and discrete maximum process-II respectively. The class of stationary distributions is obtained for both the processes. It is shown that all the well-known distributions belong to the class of stationary distributions for both the models.

3.7.2 MAXIMUM PROCESS - I

A unified approach for constructing the stationary autoregressive models was discussed in Chapter 2. Using this approach we proceed to treat max (X, Y) as a thickening operator on X, which when used in conjuction with thinning operator $\rho*$ would satisfy $\rho * \max(X, Y) \stackrel{d}{=} X$. If such a rv Y exists then the process defined below, which we call the discrete maximum process-I, becomes stationary.

$$X_n = \rho * \max(X_{n-1}, Y_n) , \quad n = 1, 2, \ldots \qquad \ldots(3.7.3)$$

Remark 3.7.1 : It may be noted that max(X, Y) serves as a right inverse of $\rho *$ in the same sense as $\rho\backslash$ operator of Littlejohn (1992) is a left inverse of $\rho *$. Further max (X, Y) is an unambiguously defined thickening operation as compared to the operation $\rho\backslash X$ ∎

Remark 3.7.2 : Littlejohn (1992, Page no. 86) commented that the operator $\rho *$ does not have a right inverse. Possibly what he means is that $\rho *$ does not have a right inverse for every value of ρ. Our result also supports this observation as our right inverse (that is max operator) also exists only for restricted values of ρ ∎

Let $\mathbb{M}_\rho$ be the class of all distributions which can be marginals of a stationary discrete maximum process-I. We now determine the conditions that a distribution F must satisfy in order to belong to $\mathbb{M}_\rho$. Note that for an integer valued random variable Z, $\rho * Z \stackrel{d}{=} X$ if and only if their probability generating functions (pgf's)are same.

That is,

$$Q_{\rho * Z}(s) = Q_X(s) \qquad \text{for all } s \in [-1, 1],$$

or equivalently

$$Q_Z(1-\rho+\rho s) = Q_X(s) \qquad \text{for all } s \in [-1, 1].$$

That is,

$$Q_Z(t) = Q_X((t+\rho-1)/\rho) \qquad \text{for all } 1-2\rho \le t \le 1,$$

or

$$\sum_{k=0}^{\infty} P(Z = k)\ t^k = \sum_{k=0}^{\infty} P(X = k)\ ((t+\rho-1)/\rho)^k.$$

The two power series agree on an interval in [0, 1] if and only if the coefficients of t^k are same in both the series for all $k \ge 0$ (Ref. Theorem 8.5, Rudin (1976)). That is

$$P(Z = k) = \rho^{-k} \sum_{n=k}^{\infty} \binom{n}{k} ((\rho-1)/\rho)^{n-k}\ P(X = n)$$

$$= \frac{Q_X^{(k)}((\rho-1)/\rho)}{k!\ \rho^k} = \theta_k \text{ say, for } k = 0, 1, \ldots;$$

...(3.7.4)

where $Q_X^{(k)}(\cdot)$ denotes the k^{th} derivative of $Q_X(\cdot)$. Thus $\rho * Z \stackrel{d}{=} X$ if and only if the distribution of Z is $\{\theta_k\}$, where θ_k is as defined in (3.7.4).

Remark 3.7.3 : A necessary and sufficient condition for the infinite series in the expression of P(Z=k) to be convergent is that $\rho > 1/(1+\alpha)$ when $Q_X(s)$ exists for $|s| < \alpha$. Recall that $\alpha \ge 1$. ■

Next we have to find a random variable Y such that $\max(X,Y) \stackrel{d}{=} Z$. Let $F(\cdot)$ be the distribution function of X and $G(\cdot)$ be the distribution function of Y. Then we require that for k = 0, 1, ...

$$F(k)\, G(k) = \sum_{j=0}^{k} \theta_j \qquad \ldots(3.7.5)$$

Thus a necessary and sufficient condition for F to belong to $\mathbb{M}_\rho$ is that G defined by (3.7.5) is a distribution function; in that case the common distribution of the associated innovation process is G.

First note that $\lim_{k\to\infty} G(k) = \sum_{j=0}^{\infty} \theta_j = 1$, being the Taylor series expansion of $Q_X(t)$ about the point $(\rho-1)/\rho$, evaluated at t=1. Thus the only requirement is that G given by (3.7.5) is monotonically nondecreasing.

G is nondecreasing if and only if for k = 1, 2, ...

$$\frac{F(k)}{F(k-1)} \le \frac{\sum_{j=0}^{k} \theta_j}{\sum_{j=0}^{k-1} \theta_j}$$

or equivalently,

$$\frac{P(X = k)}{F(k-1)} \le \frac{\theta_k}{\sum_{j=0}^{k-1} \theta_j} \qquad \ldots(3.7.6)$$

Now F belongs to $\mathbb{M}_\rho$ if and only if there exists a rv Z such that $\rho * Z \stackrel{d}{=} X$, and a rv Y such that $\max(X, Y) \stackrel{d}{=} Z$.

For the existence of Z, a necessary and sufficient condition is that $\{\theta_k\}$ is a probability distribution, where θ_k is given by (3.7.4); and for the existence of Y, (3.7.6) is a necessary and sufficient condition. However, note that (3.7.6) can be satisfied only if all θ_k's are nonnegative (for otherwise the condition is violated when θ_k changes its sign for the first time), hence ensuring the existence of Z. Thus (3.7.6), where θ_k is defined at (3.7.4), is a necessary and sufficient condition for a distribution F to belong to $\mathbb{M}_\rho$. Further if (3.7.6) is satisfied then the common distribution of innovation random variables is G given by (3.7.5).

Next we show that Binomial, Poisson and Negative binomial distributions belong to $\mathbb{M}_\rho$ for appropriate choices of $\rho \in (0,1)$.

<u>Binomial distribution</u> :

For the binomial distribution B(n,p), we have

$$\theta_k = \binom{n}{k} (p/\rho)^k \, (1-p/\rho)^{n-k}, \quad \rho \in [p,\ 1).$$

Thus Z also follows binomial distribution B(n, p/ρ). Now consider for B(n, p) rv X

$$\frac{P(X=k)}{F(k-1)} = \frac{\binom{n}{k} p^k (1-p)^{n-k}}{\sum_{x=0}^{k-1} \binom{n}{x} p^x (1-p)^{n-x}}$$

$$= \frac{\binom{n}{k}}{\sum_{x=0}^{k-1} \binom{n}{x} \left(\frac{1-p}{p}\right)^{k-x}} ,$$

which is a monotonically increasing function of p. Now Since $p/\rho > p$, it follows that condition (3.7.6) is satisfied. Thus binomial distribution B(n, p) belongs to $\mathbb{M}_\rho$ for $\rho \in [p,1)$.

<u>Poisson distribution</u> :

For the Poisson(λ) distribution, we have $\theta_k = \frac{e^{-\lambda/\rho}(\lambda/\rho)^k}{k!}$. Thus the variable Z also follows Poisson distribution Poisson(λ/ρ). For Poisson(λ) distribution it can be easily verified that P(X=k)/F(k-1) is a monotonically increasing function of λ. Since $\lambda/\rho > \lambda$, it follows that condition (3.7.6) is satisfied. Thus a Poisson distribution belongs to $\mathbb{M}_\rho$ for every $\rho \in (0, 1)$.

<u>Negative binomial distribution</u> :

For the negative binomial distribution NB(r, p), with probability mass function

$$P(X = k) = \binom{k+r-1}{k} p^r (1-p)^k , \quad k = 0, 1, \ldots,$$

it turns out that the variable Z follows negative binomial distribution NB(r, $\frac{1-p}{1-p+p\rho}$), whenever $\rho \in (\frac{1-p}{2-p}, 1)$. Also for negative binomial distribution NB(r, p) it can be verified that P(X=k)/F(k-1) is a monotonically decreasing function of p. Since $\frac{1-p}{1-p+p\rho} > (1-p)$, it follows that the condition (3.7.6) is satisfied, Thus a negative binomial distribution NB(r, p) belongs to $\mathbb{M}_\rho$ for $\rho \in (\frac{1-p}{2-p}, 1)$.

A NECESSARY CONDITION :

It is observed that the following condition is necessary for F to belong to $\mathbb{M}_\rho$.

$$P(X=k) = 0 \text{ for some } k \geq 0 \text{ implies } P(X = k+1) = 0 \quad \ldots(3.7.7)$$

For if (3.7.7) is not satisfied then max (X, Y) takes value k+1 or more with positive probability and hence $\rho*\max(X, Y)$ takes value k with positive probability. In that case $\rho*\max(X, Y) \stackrel{d}{=} X$ does not hold. Note however, that the condition (3.7.7) is not sufficient for F to belong to $\mathbb{M}_\rho$. For example consider the distribution which gives equal mass 1/3 to each of the points in {0, 1, 2}. This distribution satisfies (3.7.7) but it does not belong to $\mathbb{M}_\rho$ for any value of ρ as $\frac{P(X=0)}{P(X=1)} > \frac{\theta_1}{\theta_2}$ for all ρ in (0, 1) and the necessary condition (3.7.6) is not satisfied.

Remark 3.7.4 : Since geometric distribution with support $\mathbb{N}$ does not satisfy condition (3.7.7), it follows that it does not belong to $\mathbb{M}_\rho$ for any $\rho \in (0, 1)$ ∎

3.7.3 MAXIMUM PROCESS - II

Next we propose another structure which also may be called a discrete maximum process. Define a process $\{X_n\}$ by

$$X_n = \max\ (\rho * X_{n-1},\ Y_n\}\ ,\ n = 1,\ 2,\ \ldots \qquad \ldots(3.7.8)$$

for $\rho \in (0, 1)$, where $\{Y_n\}$ is a sequence of iid random variables, independent of X_0, with a common distribution G defined on $\mathbb{N}_0$.

Remark 3.7.5 : Here the operation of taking maximum serves as a left inverse of operation $\rho *$. Another left inverse is an operator $\rho\backslash$ appearing in Littlejohn's (1992) discrete minification process. This also shows the non- uniqueness of left inverse of $\rho *$ operator ■

For an integer valued random variable X the variable $\rho *\max(X,\ Y)$ and $\max\ (\rho * X,\ Z)$ need not have the same distribution for any choice of Y and Z, an example of which will be provided later in this Section. Also when they have the same distribution, the operations themselves are different (that is $\rho *$ is not distributive over max operation). This makes it clear that the process with structure (3.7.8) needs to be studied separately.

Let M_ρ be the class of all distributions which can be marginals of a stationary process $\{X_n\}$ defined by (3.7.8). We now obtain a condition that is necessary and sufficient for F to belong to M_ρ.

Let X be a random variable with a distribution F having support $\mathbb{N}_0$. Then F belong to M_ρ if and only if for some nonnegative integer valued random variable Y, with distribution function G,

$$\max(\rho * X,\ Y) \stackrel{d}{=} X;$$

that is, if and only if for x = 0, 1, ...

$$F(x) = G(x) \sum_{k=0}^{\infty} P(X = k)\ P(Z_k \le x), \qquad \ldots(3.7.9)$$

where $Z_0 \equiv 0$ and Z_k is distributed as $B(k, \rho)$ for $k \ge 1$.

Thus a necessary and sufficient condition for F to belong to M_ρ is that $G(\cdot)$ defined by equation (3.7.9) is a distribution function. In that case G may be taken as a common distribution of the innovation process $\{Y_n\}$.

First note that $G(\infty) = 1$ and $G(0) \ge 0$. Thus we only require to ensure that G(x) defined by (3.7.9) is a nondecreasing function of x. That is, for x = 0, 1, ...

$$G(x) \le G(x+1). \qquad \ldots(3.7.10)$$

We first prove the following lemma which is useful for further discussion but the result may be of independent interest.

Lemma 3.7.1 : *Let $\{a_n\}$ and $\{b_n\}$ be the sequences of nonnegative real numbers. Define*

$$A_n = \sum_{k=0}^{n} a_k\ , \quad B_n = \sum_{k=0}^{n} b_k\ , \quad \text{and} \quad c_n = A_n/B_n.$$

Then $\{c_n\}$ is monotonically increasing whenever $\{a_n/b_n\}$ is monotonically increasing.

Proof : Define $A_{-1} = B_{-1} = 0$. Since $\{a_n/b_n\}$ is monotonically increasing, it follows that for fixed $n \geq 1$,

$$a_n/b_n - a_k/b_k \geq 0 \ , \ k = 0, 1, \ldots n-1.$$

That is, $\qquad a_n b_k - a_k b_n \geq 0$,

which further implies that

$$\sum_{k=0}^{n-1} (a_n b_k - a_k b_n) = a_n B_{n-1} - b_n A_{n-1} \geq 0.$$

Adding and subtracting $A_{n-1}B_{n-1}$, we get

$$(a_n + A_{n-1})B_{n-1} - A_{n-1}(b_n + B_{n-1}) = A_n B_{n-1} - A_{n-1}B_n \geq 0.$$

Since $n \geq 1$ is arbitrary, it follows that $\{c_n\}$ is a monotonically increasing sequence. ∎

Remark 3.7.6 : We believe that the result of above Lemma must be well-known. However, we proved it here for the ready reference ∎

From the Lemma, a sufficient condition for F to belong to M_ρ is that

$$\frac{P(X=k)}{P(\rho * X = k)} \text{ is a nondecreasing function of } k. \qquad \ldots(3.7.11)$$

or equivalently in terms of pgf of X,

$$\frac{Q_X^{(k)}(0)}{\rho^k Q_X^{(k)}(1-\rho)} \text{ is a nondecreasing function of } k,$$

where Q_X is the pgf of X and $Q_X^{(k)}(s)$ denotes the k^{th} derivative of Q_X, evaluated at s.

Next we show that binomial, Poisson, negative binomial and the $Geom^+$ distribution belong to the class M_ρ.

Binomial distribution :

The pgf of the binomial distribution B(n, p) is

$$Q(s) = (1-p+ps)^n.$$

Hence we have

$$Q^{(k)}(s) = \frac{n!}{(n-k)!}(1-p+ps)^{n-k}\, p^k,$$

and

$$\frac{Q^{(k)}(0)}{\rho^k Q^{(k)}(1-\rho)} = \left(\frac{1-p}{1-p\rho}\right)^n \left(\frac{1-p\rho}{(1-p)\rho}\right)^k.$$

Since $(1-p\rho)/((1-p)\rho) > 1$ for every $\rho \in (0, 1)$, it follows that condition (3.7.11) is satisfied. Thus the binomial distribution belongs to M_ρ for every $\rho \in (0, 1)$. It may be recalled that binomial distribution does not belong to $\mathbb{M}_\rho$ for every value of ρ in (0, 1).

Poisson distribution :

The pgf of the Poisson(λ) distribution is given by

$$Q(s) = e^{-\lambda(1-s)}.$$

Hence we have,

$$\frac{Q^{(k)}(0)}{\rho^k Q^{(k)}(1-\rho)} = e^{-\lambda(1-\rho)}\rho^{-k},$$

which is clearly an increasing function of k for every $\rho \in (0,1)$. Hence the Poisson distribution belongs to M_ρ for every ρ in (0, 1)

Negative binomial distribution :

The pgf of the negative binomial distribution NB(r, p) is given by

$$Q(s) = \left(\frac{p}{1-qs}\right)^r, \text{ where } q = 1-p.$$

Hence we have

$$\frac{Q^{(k)}(0)}{\rho^k Q^{(k)}(1-\rho)} = (p + q\rho)^r \left(\frac{p+q\rho}{\rho}\right)^k.$$

Since $(p+q\rho)/\rho > 1$ for every $\rho \in (0, 1)$, the condition (3.7.11) is satisfied. Hence a negative binomial distribution belongs to M_ρ for every $\rho \in (0, 1)$. Again it may be recalled that negative binomial distribution does not belong to $\mathbb{M}_\rho$ for every value of ρ in (0, 1).

Geom^+ distribution :

Consider a geometric distribution defined by the probability mass function $P(X = x) = p\,(1-p)^{x-1}$, $x = 1, 2, \ldots$ The pgf of this distribution is given by

$$Q(s) = ps/(1-qs).$$

Hence we have,

$$\frac{Q^{(k)}(0)}{\rho^k Q^{(k)}(1-\rho)} = \begin{cases} 0 & ; k = 0 \\ (p+q\rho)\left(\frac{p+q\rho}{\rho}\right)^k & ; k = 1, 2, \ldots \end{cases}$$

This is a strictly increasing function of x as $(p+q\rho)/\rho > 1$ for every $\rho \in (0, 1)$. Thus it follows that geometric distribution concentrated on $\mathbb{N}$ belongs to M_ρ for every $\rho \in (0, 1)$.

Remark 3.7.7 : It may be noted that the geometric distribution considered above does not belong to $\mathbb{M}_\rho$ for any value of ρ. This makes it clear that the two models proposed here are not equivalent.

APPENDIX

Construction of $Geom^{+}$ discrete minification process of Littlejohn :

The pmf of $Geom^{+}(p)$ is

$$p(X=x) = p\, q^{x-1} \;;\; x = 1, 2, \ldots,$$

where $q = 1-p$.

Note that the operation $\rho\backslash Z$ is defined by

$$P(\rho\backslash Z = n|Z = m) = \frac{\binom{n}{m}(1-\rho)^n\, P(X = n)}{\sum_{k=m}^{\infty}\binom{k}{m}(1-\rho)^k\, P(X = k)},$$

It is clear that $P(\rho\backslash Z = 0\, |Z=m) = 0$ for all m.

Consider for n=1, 2, ...

$$P(\rho\backslash Z = n\, |Z = 0) = \frac{(\bar{\rho})^{n} p q^{n-1}}{\sum_{k=1}^{\infty}(\bar{\rho})^{k} p q^{k-1}}$$

$$= \frac{(\bar{\rho}q)^{n-1}}{\sum_{k=1}^{\infty}(\bar{\rho}q)^{k-1}}$$

$$= (\bar{\rho}q)^{n-1}\,(1-\bar{\rho}q) \qquad \ldots(1)$$

Thus conditionally on $Z = 0$, $\rho\backslash Z$ is distributed as $Geom^{+}(1-\bar{\rho}q)$.

Also for $m = 1, 2, \ldots;\; n = m, m+1, \ldots$

$$P(\rho\backslash Z = n\, |Z=m) = \binom{n}{m}(\bar{\rho}q)^{n-m}(1-\bar{\rho}q)^{m+1}. \qquad \ldots(2)$$

Thus conditionally on $Z = m$, $\rho\backslash Z$ is distributed as $m + \sum_{i=0}^{m} N_i'$, where N_i' are iid Geom($1-\bar{\rho}q$) rvs. Equations (1) and (2) suggest that $\rho\backslash Z$ is defined as

$$\rho\backslash Z = \begin{cases} N_0 & ; \text{ if } Z = 0 \\ Z + \sum_{i=0}^{Z} N_i' & ; \text{ if } Z > 0. \end{cases},$$

where N_0 is distributed as Geom$^+(1-\bar{\rho}q)$, and N_i' is distributed as Geom($1-\bar{\rho}q$) for $i = 0, 1, \ldots$

or equivalently,

$$\rho\backslash Z = \sum_{i=0}^{Z} N_i - I(Z > 0) \qquad \ldots(3)$$

where N_i are iid Geom$^+(1-\bar{\rho}q)$ variates.

Thus the Geom$^+$ discrete minification process is defined as

$$X_n = \rho\backslash\min(X_{n-1}, Y_n) \ , \ n = 1, 2, \ldots,$$

where X_0 is distributed as Geom$^+$(p), $\{Y_n\}$ is a sequence of iid Geom$\left(\frac{\bar{\rho}p}{1-\bar{\rho}q}\right)$, where $\rho\backslash Z$ is defined by (3).

Joint distribution of X_n and X_{n-1} :

First we evaluate the probability

$$P(X_n=x|X_{n-1}= y) = P(\rho\backslash\min(y,Y) = x) \quad ;Y \text{ is innovation rv}$$

$$= \sum_{k=0}^{\infty} P(\rho\backslash\min(y, k) = x)\ P(Y = k)$$

$$= \sum_{k=0}^{y} P(\rho\backslash k = x)\ P(Y = k) + P(\rho\backslash y = x) \sum_{k=y+1}^{\infty} P(Y = k)$$

Case i : $y \leq x$

In this case the above conditional probability is given by

$$(\bar{\rho}q)^{x-1}\ \bar{\rho}p + \sum_{k=1}^{y} \binom{x}{k} (\bar{\rho}q)^{x-k}\ (\bar{\rho}p)\rho^{k} + \binom{x}{y} (\bar{\rho}q)^{x-y}\ \rho^{y+1}$$

Case ii : $y > x$

In this case the above conditional probability is given by

$$(\bar{\rho}q)^{x-1}\ \bar{\rho}p + \sum_{k=1}^{x} \binom{x}{k} (\bar{\rho}q)^{x-k}\ (\bar{\rho}p)\rho^{k} + 0$$

$$= (\bar{\rho}q)^{x-1}\ \bar{\rho}p + \bar{\rho}p\ (\ (\rho + \bar{\rho}q)^{x} - (\bar{\rho}q)^{x}\ )$$

$$= \bar{\rho}p\ (\ (\rho + \bar{\rho}q)^{x} + (\bar{\rho}q)^{x-1}(1-\bar{\rho}q)\ )$$

The Joint pgf of X_n and X_{n-1} is given by

$$Q(s_1, s_2) = \sum_{x=1}^{\infty} \sum_{y=1}^{\infty} s_1^{x}\ s_2^{y}\ P(X_n = x,\ X_{n-1} = y)$$

$$= \sum_{x=1}^{\infty} s_1^{x} \sum_{y=1}^{\infty} s_2^{y}\ P(X_n = x \mid X_{n-1} = y) P(X_{n-1} = y) \qquad \ldots(4)$$

The inner sum in (4) is equal to

$$\sum_{y=1}^{x} s_2^{y}\ P(X_n = x \mid X_{n-1} = y) P(X_{n-1} = y)$$

$$+ \sum_{y=x+1}^{\infty} s_2^{y}\ P(X_n = x \mid X_{n-1} = y) P(X_{n-1} = y)$$

Substituting the conditional probability expressions from Case i and Case ii, after simplification above equals

$$(\rho q)^{x-1}\left\{\frac{\bar{\rho}p^2 s_2}{1-s_2 q} - \frac{(\bar{\rho})^2 p^2}{1-s_2 q} - \rho\bar{\rho}p\right\} + (\rho q+\rho s_2 q)^x\left\{\frac{\bar{\rho}p^2}{q(1-s_2 q)} + \frac{\rho p}{q}\right\}$$

Substituting in (4) we get after simplification

$$Q(s_1, s_2) = \frac{\bar{\rho}p(1-\bar{\rho}q)(s_2-1)}{(1-s_2 q)\ (1-s_1 q\bar{\rho})} + \frac{s_1 p(1-q(\bar{\rho}+\rho s_2))(\bar{\rho}+\rho s_2)}{(1-s_2 q)\ (1-s_1 q(\bar{\rho}+\rho s_2))}\ .$$

Autocorrelation :

Using this joint pgf, we get the following results.

$$E(X_n) = E(X_{n-1}) = \frac{1}{p},\quad V(X_n) = V(X_{n-1}) = \frac{q}{p}$$

and

$$E(X_n X_{n-1}) = \frac{\bar{\rho}}{1-\bar{\rho}q} + \frac{\rho + q}{p^2}$$

Hence autocorrelation of lag 1 is given by

$$\text{Corr}\ (X_n, X_{n-1}) = \frac{\dfrac{\bar{\rho}}{1-\bar{\rho}q} + \dfrac{\rho + q}{p^2} - \dfrac{1}{p^2}}{\dfrac{q}{p^2}}$$

$$= \frac{\rho^2}{p+\rho q} \qquad \text{after simplification.}$$

Making the sample paths comparable.

To make the sample paths of our $Geom^+$ MINAR process and $Geom^+$ discrete minification process comparable, we choose the parameter ρ (in the latter process) in such a way that autocorrelation of lag 1 becomes equal for both.

That is to find ρ such that

$$\frac{\rho^2}{p+\rho q} = \frac{p}{q}\,\frac{q^K}{1-q^K}\,.$$

That is such that

$$\rho^2 q(1-q^K) - \rho(pq^{K+1}) - p^2q^K = 0,$$

which is quadratic equation in ρ, having the solution

$$\rho = \frac{pq^{K+1} + \sqrt{(pq^{K+1})^2 + 4q^{K+1}(1-q^K)P^2}}{2q(1-q^K)}$$

It may be noted that the second solution of the quadratic equation is negative.