CHAPTER 4

CHARACTERIZATIONS OF STOCHASTIC PROCESSES

4.1. Introduction :

It is very important to identify a proper stochastic model for the data at hand in order to utilize the statistical theory in an optimal manner. In classical setup, where data is assumed to be a realization of iid random variables, many characterization results are available (See Kagan et al.(1973) for an extensive treatment of this topic). Very few characterization results are available for time series models. Nevertheless, such results provide an aid in the process of model identification.

Weiss (1975), Chernick et al.(1988), Arnold and Hallet(1989) and Littlejohn (1992 a, b) have obtained some results in this direction.

Weiss (1975), Chernick et al (1988) and Littlejohn (1992 a, b) gave characterizations based on the time reversibility properties of the stochastic processes. Weiss (1975) showed that discrete time ARMA models are time reversible if and only if they are Gaussian. Here by time reversibility we mean $\{X_n\} \stackrel{d}{=} \{X_{-n}\}$. Since transformation of a time-reversible process is time-reversible, an important implication of this result is that Gaussian models or their transformations can not be suitable for ARMA processes which are not time reversible.

Two processes $\{X_n\}$ and $\{Y_n\}$ are said to be mutually time reversible if $\{X_n\} \stackrel{d}{=} \{Y_{-n}\}$. Chernick et al. (1988) established that a stationary AR process and a stationary MINAR process are mutually time reversible if and only if the common marginal distribution is exponential. This result is mainly based on the following characterization result obtained by Daley (1989).

Lemma 4.1.1 : Let U and V be independent random variables such that min(U, V) and $(U-V)_+$ are independent. Then one of the following statements holds.

(i) $P(U \le V) = 1$

(ii) $P(V = v < U) = 1$ for some constant v.

(iii) U and V both have distributions on the same lattice with geometric tails.

(iv) U and V both have distributions on the same half line with exponential tails.

Here for real x, $x_+ = \max(0, x)$.

Using the Lemma 4.1.1 Littlejohn (1992 a) proved an analogous result in the case of discrete processes. It is shown that a stationary INAR process defined by (3.3.1) and a stationary discrete minification process defined by (3.4.2) are mutually time reversible if and only if the common marginal distribution is Geometric.

Arnold and Hallet (1989) gave a characterization of Pareto MINAR process in the class of stationary MINAR processes. This Characterization is based on the Markovian property of the level crossing process $\{Z_n(t)\}$ of Pareto MINAR process (See (4.4.10) for the definition of $\{Z_n(t)\}$).

In this Chapter, we give several interesting characterization results based on the structural and distributional properties of various autoregressive processes. We show that the arguments similar to those made by Arnold and Hallet (1989), with less stringent regularity conditions, characterize the class of Semi-Pareto distributions. Semi-Pareto distribution is introduced by Pillai (1991). We will discuss more about it in Section 4.4.2. We give some more characterizations of Pareto AR process using various properties of the process. We also obtain the characterizations of EAR and geometric INAR processes on similar lines, exploiting the similarities in the structures of the three processes, namely, Pareto MINAR, exponential AR and geometric INAR. Characterizations of exponential MINAR, geometric MINAR and Poisson INAR processes are also obtained.

The Chapter is organized as follows. In Section 2 we give characterizations of AR processes with exponential marginals. Characterizations of AR processes with geometric marginals are given in Section 3, whereas in Section 4 we give characterizations of Pareto processes. A new AR process with Poisson marginals is proposed in

Section 5 and a characterization for this process is also obtained.

4.2. Characterization of AR processes with exponential marginals :

4.2.1 A Characterization of exponential MINAR process:

It was shown in Chapter 1 that finding a stationary solution for MINAR processes given by

$$X_n = \theta \min(X_{n-1}, Y_n) \qquad \ldots(4.2.1)$$

is equivalent to finding a solution of the minimum problem considered by Arnold and Isaacson (1976). This fact enable us to give a characterization of exponential MINAR process of Tavares(1980), which is an immediate consequence of the following characterization of exponential distribution given by Arnold and Isaacson (1976):

Theorem 4.2.1 (Theorem 3.1 in Arnold and Isaacson (1976))

Let X *and* Y *be nondegenerate non-negative independent random variables satisfying*

$$\min(X, Y) \stackrel{d}{=} \rho X \stackrel{d}{=} (1-\rho) Y$$

for some $\rho \in (0, 1)$*. Let* F *be the distribution function of* X *and assume that* F *has a right derivative at zero. Then* F *must be an exponential distribution.*

As a consequence of this result, we have a characterization of exponential MINAR process proposed by Tavares' (1980) minification process.

Theorem 4.2.2 [Kalamkar (1995)]

Let $\{X_n\}$ *be a stationary* MINAR *process given by* (4.2.1) *with marginal distribution* F. *Let* G *be the common distribution of innovation process. If* $F(x) = G(x/(\theta-1))$ *and* F *has a right derivative at zero then* $\{X_n\}$ *is an exponential* MINAR *process.*

Proof : Let X and Y be the independent rvs with *distributions F and G respectively. Since* $F(x) = G(\frac{x}{\theta-1})$ we have

$$P(X \le x) = P((\theta-1)Y \le x).$$

Taking $\theta = 1/\rho$, we have $\rho X \overset{d}{=} (1-\rho)Y$. Further, the stationarity of $\{X_n\}$ implies that $\min(X, Y) \overset{d}{=} \rho X$. Then from Theorem 4.2.1, it follows that X must be an exponential rv. ■

Remark 4.2.1 : Since $\min(X, Y) \overset{d}{=} X/\theta$ is equivalent to $\min(X^\alpha, Y^\alpha) \overset{d}{=} X^\alpha/\theta^\alpha$ for nonnegative random variables X, Y and every real $\alpha > 0$, it follows that whenever the distribution F of X belongs to $\mathbb{F}_\theta$ (introduced in Section 3.5), the distribution of X^α belongs to $\mathbb{F}_{\theta^\alpha}$ (See Lewis and McKenzie (1991) for a detailed treatment of transformations of MINAR processes). Since a one-to-one transformation of a stationary process is stationary, this further implies that from Theorem 4.2.2 one can get a characterization of Weibull MINAR process (See Sim (1986) for the discussion of Weibull MINAR process).

Corollary 4.2.1 : *Let* $\{X_n\}$ *be a stationary* MINAR *process with parameter a. If distributions* F *and* G *satisfy* $F(x) = G(ax/b)$; *where* $a^{\alpha}+b^{\alpha} = 1$ *for some* $\alpha > 0$, *then* $\{X_n\}$ *must be a Weibull process provided* $\lim_{x \to 0+} \frac{F(x)}{x^{\alpha}} > 0$.

4.2.2 A Characterization of EAR(1) process :

Consider an AR process given by

$$X_n = \begin{cases} \rho X_{n-1} & \text{with probability } \rho \\ \rho X_{n-1} + Y_n & \text{with probability } \bar{\rho}\ (= 1-\rho) \end{cases}, \quad n = 1, 2, \ldots \qquad \ldots(4.2.2)$$

where $\rho \in (0, 1)$ and $\{Y_n\}$ is a sequence of iid nonnegative rvs, independent of nonnegative rv X_0, with common distribution function G.

Gaver and Lewis (1980) showed that if the distribution F of X_0 is same as G, and G is exponential(λ), then the process at (4.2.2) is stationary. They called the process an EAR(1) process. In the next Theorem we show that in the class of stationary AR processes with structure (4.2.2), the condition F = G in fact characterizes the EAR process.

Theorem 4.2.3 [Sreehari and Kalamkar(1997)]

Let $\{X_n\}$ *be an* AR *process given by* (4.2.2). *If* F = G *and the process is stationary for every fixed* $\rho \in (0, 1)$ *then it must be an exponential process.*

Proof : Let $L(s) = E(\exp(-sX))$ be the Laplace transform of F. From (4.2.2) it is clear that the Laplace transform of X_n is

$$L_{X_n}(s) = \rho\, L_{X_{n-1}}(\rho s) + \bar{\rho}\, L_{X_{n-1}}(\rho s)\, L_{Y_n}(s).$$

Now since $\{X_n\}$ is stationary, and $F = G$, $L(s) = L_{X_n}(s)$ satisfies for every $\rho \in (0, 1)$,

$$\begin{aligned} L(s) &= \rho L(\rho s) + \bar{\rho}\, L(\rho s)\, L(s) \\ &= L(\rho s)\, \{ \rho + \bar{\rho}\, L(s) \}. \end{aligned}$$

Define $H(s) = \dfrac{1}{L(s)} - 1$. Then $H(s)$ satisfies

$$H(\rho s) = \rho\, H(s),\ \forall\ \rho \in (0, 1],\ s > 0. \qquad \ldots(4.2.3)$$

Putting $s = t/\rho$ in above, we get

$$H(\frac{1}{\rho}\, t) = \frac{1}{\rho}\, H(t)\ \forall\ \rho \in (0, 1],\ t > 0.$$

Thus the equation (4.2.3) holds for all ρ, $s > 0$. Putting $s = 1$ in (4.2.3), we get $H(\rho) = \rho H(1)\ \forall\ \rho > 0$, or $H(s) = cs$, $s > 0$.

Here $c = H(1) > 0$, since $H(\cdot)$ must be monotonically increasing.

That is, $L(s) = (1 + cs)^{-1}$, $s > 0$,

which is the Laplace transform of an exponential distribution with mean c ∎

4.3. Characterizations of AR processes with geometric marginals:

4.3.1 A Characterization of geometric INAR process :

Consider an INAR process given by

$$X_n = \begin{cases} \rho * X_{n-1} & \text{with probability } \rho \\ \rho * X_{n-1} + Y_n & \text{with probability } \bar{\rho} \end{cases}, \quad n = 1, 2, \ldots, \qquad \ldots(4.3.1)$$

where $\rho \in (0, 1)$ and $\{Y_n\}$ is a sequence of iid rvs, independent of X_0, with common distribution function G.

McKenzie (1986) showed that if the distribution F of X_0 is same as G, and G is Geom(θ), then the process at (4.3.1) is stationary. We show that in the class of stationary INAR processes with structure (4.3.1), the condition F = G in fact characterizes the geometric process.

Theorem 4.3.1 [Sreehari and Kalamkar (1997)]

Let $\{X_n\}$ *be an* INAR *process given by* (4.3.1). *If* $F = G$ *and the process is stationary for every fixed* $\rho \in$ (0, 1) *then it must be a geometric* INAR *process.*

Proof : Let P(s) be the alternate pgf of F. Then, following the arguments of Theorem 4.2.3 with L(s) replaced by P(s), it is easily seen that P(s) is of the form

$$P(s) = (1 + cs)^{-1}, \quad s \in [0, 2],$$

where $c > 0$ is a constant. Thus P is the alternate pgf of a geometric distribution with mean c ∎

Remark 4.3.1 : From Theorem 4.2.3 and Theorem 4.3.1 it can be concluded that for an AR / INAR process $\{X_n\}$ at (4.2.2)/(4.3.1) any two of the following three statements imply the third.

(i) F is an exponential / geometric distribution.

(ii) $\{X_n\}$ is stationary.

(iii) F = G.

That (i), (ii) imply (iii) and (i), (iii) imply (ii) were proved by Gaver and Lewis (1980)/ McKenzie (1986). A similar result will be proved for a Min-geometrically stable processes in the Chapter 5 after we introduce Min-geometrically stable laws ∎

4.3.2 Characterizations of geometric MINAR process :

MINAR processes with discrete marginals were introduced in Chapter 3, where we discussed a geometric MINAR process in detail. Recall the peculiar nature of the common distribution of innovation rvs for a geometric MINAR process, that these rvs do not take the values which are multiples of K, the parameter of the process. We now show that in the class of stationary MINAR processes with discrete marginals, above mentioned peculiarity is a characteristic of geometric MINAR process. Consider a stationary MINAR process given by

$$X_n = \min(KX_{n-1}, Y_n),\ n = 1, 2, \ldots$$

for some integer $K \geq 2$. Recall that $\mathbb{F}_K$ is the class of all stationary distributions F of the MINAR process defined above, $\mathbb{F}_K^I$ is the class of all distribution in $\mathbb{F}_K$ whose support is a subset of $\mathbb{N}_0$, and $\mathbb{F}^{I*} = \bigcap_{K\geq 2} \mathbb{F}_K^I$.

THEOREM 4.3.2 [Kalamkar (1995)]

Let $\{X_n\}$ *be a stationary* MINAR *process defined above with marginal* $F \in \mathbb{F}^{I*}$. *Let* G_K *be the common distribution of the innovation rvs when the process has parameter* K ($K \geq 2$). *If* G_K *satisfies*

$$G_K(rK) = G_K(rK-1)\ ,\ r = 1,2,\ \dots \qquad \dots(4.3.2)$$

for every K *then* F *is a* Geom^+ *distribution.*

(The condition (4.3.2) means that for every $K \geq 2$ the corresponding innovation rvs do not take values which are multiples of K)

PROOF : Since $F \in \mathbb{F}_K^I$ for every integer $K \geq 2$, using the expression (3.5.6) of pmf of G_K, our hypothesis implies that

$$\frac{\bar{F}(r)}{\bar{F}(r-1)} = \frac{\bar{F}(rK)}{\bar{F}(rK-1)} \qquad \forall\ K \geq 2,\ r \geq 1.$$

In particular, we have

$$\frac{\bar{F}(1)}{\bar{F}(0)} = \frac{\bar{F}(K)}{\bar{F}(K-1)} \qquad \forall\ K \geq 2.$$

Subtracting both the sides from 1, we get

$$\lambda(1) = \lambda(K) = c, \qquad \forall\ K \geq 2 \qquad \dots(4.3.3)$$

for some $c \in (0, 1]$, where $\lambda(\cdot)$ is the discrete hazard function of F given by $\lambda(x) = \frac{P(X=x)}{P(X\geq x)}$, X having distribution F.

Let X be a rv having distribution F. We prove the result by induction.

Since $\lambda(1) = c$, we have

$$P(X = 1) = c.$$

Now suppose

$$P(X = m) = c\ (1-c)^{m-1} \text{ for } m = 1,2,\ldots x.$$

Then from (4.3.3),

$$\begin{aligned} P(X = x+1) &= c\ P(\ X > x) \\ &= c\ (1 - c\sum_{m=1}^{x} (1-c)^{m-1}\ ) \\ &= c\ (1-c)^{x}. \end{aligned}$$

Thus by mathematical induction, it is proved that F is a Geom^{+} distribution.

Remark 4.3.2 : In the statement of Theorem 4.3.2, it is necessary to assume that the condition at (4.3.2) is satisfied for every K. As a counter example of a non geometric distribution, where the condition is satisfied for K = 2, but not for every K, one may consider the example given in Remark 3.5.5 ■

The following Theorem also provides a characterization of a geometric MINAR process based on a structural relationship between the common distribution of innovations and the stationary distribution of the MINAR process. Recall that for a geometric MINAR process, an innovation rv Y and X_0 satisfy the relationship at

(3.6.3), namely

$$Y \stackrel{d}{=} X_0 + \left[\frac{X-1}{K-1}\right].$$

THEOREM 4.3.3 [Kalamkar (1995)]

Let $\{X_n\}$ *be a stationary* MINAR *process with the marginal distribution* $F \in \mathbb{F}_K^{I}$ *for some* $K \geq 2$*. If* Y *satisfies relation at* (3.6.3) *and* $P(X = 1) = p > 0$ *then* F *is* $\text{Geom}^{+}(p)$.

PROOF : Let X be a rv having distribution F. Then for every positive integer x, X satisfies

$$\begin{aligned} P(X = x) &= P(\min(KX, Y) = x) \\ &= P(X = x/K)\ P(Y > x) + P(X > x/K)\ P(Y = x) \\ &\quad + P(X = x/K)\ P(Y = x). \end{aligned}$$

The last term is zero, because Y does not take values of the form rK and X takes only integer values. Thus we have,

$$P(X = x) = P(X = x/K)\ P(Y > x) + P(X > x/K)\ P(Y = x).$$

Suppose x is not a multiple of K, then we have for $n = 0,1,\ldots$ and $r = 1,2,\ldots,K-1$,

$$\begin{aligned} P(X = nK+r) &= P(X > n + (r/K))\ P(Y = nK+r) \\ &= P(X > n)\ P(X = n(K-1) + r). \qquad \ldots(4.3.4) \end{aligned}$$

The last equality follows from (3.6.3), and the fact that X is integer valued. Next, when x is a multiple of K, we have

$$\begin{aligned} P(X = nK) &= P(X = n)\ P(Y > nK) \\ &= P(X = n)\ P(X > n(K-1)). \qquad \ldots(4.3.5) \end{aligned}$$

Using (4.3.4) and (4.3.5) we now prove that

$$P(X = r) = p(1-p)^{r-1}, \quad r = 1,2,... \tag{4.3.6}$$

by mathematical induction.

Note first that (4.3.6) holds for $r = 1$. Suppose (4.3.6) holds for $r = 0, 1, ..., m$.

Depending on whether $m+1$ is a multiple of K or not, we consider two cases.

Case (i) : $m = sK+t$; for some integer $s \geq 0$ and $0 \leq t < K-1$.

In this case $m+1$ is not a multiple of K. Therefore from (4.3.4),

$$\begin{aligned} P(X = m+1) &= P(X = sK+t+1) \\ &= P(X > s)\ P(X = s(K-1) + t+1) \\ &= p\ (1-p)^m. \end{aligned}$$

Thus (4.3.6) is true for $r = m+1$.

Case (ii) : $m = sK - 1$; for some integer $s > 0$.

In this case $m+1$ will be a multiple of K, then from (4.3.5),

$$\begin{aligned} P(X = m+1) &= P(X = sK) \\ &= P(X = s)\ P(X > s(K-1)) \\ &= p\ (1-p)^m. \end{aligned}$$

Thus (4.3.6) is true for $r = m+1$ in this case also ■

4.4 A Pareto MINAR process and related characterizations

4.4.1 A Pareto process

Yeh et al. (1988) proposed a MINAR process called ARP(1) with Pareto marginals. The process is defined by

$$X_n = \begin{cases} \beta^{-\gamma} X_{n-1} & \text{with probability } \beta \\ \min(\beta^{-\gamma} X_{n-1}, \xi_n) & \text{with probability } \bar{\beta}(=1-\beta) \end{cases}, \quad \ldots(4.4.1)$$

for $\beta \in (0, 1)$, where $\{\xi_n\}$ is a sequence of iid random variables, independent of X_0, with common distribution G, X_0 distributed as F and $\gamma > 0$. Yeh et al. (1988) showed that if G is a Pareto distribution $P(\sigma, \gamma)$ (a special case of Pareto type III distribution) and F = G then $\{X_n\}$ at (4.4.1) is stationary with $P(\sigma, \gamma)$ marginals. Here $P(\sigma, \gamma)$ is a Pareto distribution whose survival function is given by

$$\bar{G}(x) = [1+(x/\sigma)^{1/\gamma}]^{-1}, \; x \geq 0.$$

Yeh et al.(1988) investigated distributional properties of this process and also some inferential problems associated with the process. The process is useful in socio-economic studies due to the importance of Pareto distribution in that area.

It may be noted that the structure at (4.4.1) is a particular form of MINAR process. This form is obtained when the common distribution of innovations in a MINAR process assigns a positive mass at infinity. In our terminology we refer to the Pareto process defined above by MINARP(1). The important properties of MINARP process are mentioned below.

(1) The process is autoregressive.

(2) If X_0 is distributed as $P(\sigma, \gamma)$ then $\{X_n\}$ is a stationary process with $P(\sigma, \gamma)$ marginals.

(3) If X_0 is arbitrary and nonnegative then X_n converges in distribution to a $P(\sigma, \gamma)$ random variable.

(4) For every $p \in (0, 1]$,

$$\left(\frac{p}{1-q\beta}\right)^{\gamma} \max_{0\le k\le N(p)} X_k \overset{d}{=} X_0 \overset{d}{=} \left(\frac{p}{1-q\beta}\right)^{-\gamma} \min_{0\le k\le N(p)} X_k ,$$

where $N(p)$ is a geometric random variable, independent of X_k's, with $P(N(p) = 0) = p$, $q = 1-p$.

4.4.2 Semi-pareto processes

Pillai (1991) defined a Semi-Pareto distribution and proposed a MINAR process with Semi-Pareto marginal. Pillai(1991) also extended some of the results in Yeh et al.(1988) to the Semi-Pareto processes. The $P(\sigma, \gamma)$ distribution can be obtained as a particular form of Semi-Pareto distribution.

Pillai (1991) defined Semi Pareto distribution as follows :

Definition 4.4.1 : A rv X has a Semi-Pareto distribution $SP(\gamma, \beta)$, $0 < \beta < 1$, $\gamma > 0$, if its survival function $\bar{F}(\cdot)$ is of the form

$$\bar{F}(x) = P(X > x) = (1 + \phi(x))^{-1}, \qquad \ldots(4.4.2)$$

where ϕ satisfies the functional equation

$$\phi(\beta^{\gamma}x) = \beta\, \phi(x) \quad \forall\, x \ge 0 \; \blacksquare \qquad \ldots(4.4.3)$$

It is readily seen that when $\phi(x) = (x/\sigma)^{1/\gamma}$, we get the Pareto distribution $P(\sigma, \gamma)$. Pillai (1991) then proposed a Semi-Pareto process defined by the structure (4.4.1), where $\{\xi_n\}$ is a sequence of iid random variables, independent of X_0, with common distribution $SP(\gamma, \beta)$. Pillai showed that if X_0 is distributed as $SP(\gamma, \beta)$ then $\{X_n\}$ at (4.4.1) is stationary with $SP(\gamma, \beta)$ marginals, and the process possesses the properties similar to those of MINARP process. These properties are

(1) The process is autoregressive.

(2) If X_0 is distributed as $SP(\gamma, \beta)$ then $\{X_n\}$ is a stationary process with $SP(\gamma, \beta)$ marginals.

(3) If X_0 is arbitrary and nonnegative then X_n converges in distribution to an $SP(\gamma, \beta)$ random variable.

(4) For $p = \beta^2/(1+\beta+\beta^2)$,

$$\left(\frac{p}{1-q\beta}\right)^{\gamma} \max_{0\le k\le N} X_k \stackrel{d}{=} X_0 \stackrel{d}{=} \left(\frac{p}{1-q\beta}\right)^{-\gamma} \min_{0\le k\le N} X_k ,$$

where N is a geometric random variable, independent of X_k's, with $P(N(p) = 0) = p$, $q = 1-p$.

Before we give further results, it is necessary to point out and clarify some ambiguities in the above mentioned development of Semi-Pareto distribution/ process. First note that $SP(\gamma, \beta)$ is not one distribution, as it appears from the definition, but is a family of distributions. To clarify this point, note that there exist more than one function $\phi(\cdot)$ which satisfy (4.4.3) for the same pair (γ, β) (Such an example is given in observation (c) following the Definition 4.4.2). All the distributions corresponding to these ϕ's constitute the family SP(γ, β). To make this point clearer from the definition itself, we propose to reformulate the definition as follows :

Definition 4.4.2 : A distribution F is said to belong to a Semi Pareto *family* SP(γ, β), $0 < \beta < 1$, $\gamma > 0$, if its survival function is given by (4.4.2)-(4.4.3). ■

A specific member of this family will be denoted by SP(γ, β, ϕ), where ϕ is to be specified explicitly and SP(γ, β) may be described as SP(γ, β) = { F | $\bar{F}$ is given by (4.4.2) - (4.4.3)}. We now make some observations on the SP(γ, β) family.

(a) $\phi(\beta^{-\gamma}x) = \beta^{-1}\phi(x)\ \forall\ x \geq 0$ <=> ϕ satisfies (4.4.3).

(b) For every integer m (including negative integers),

$$\phi(\beta^{m\gamma}x) = \beta^{m}\ \phi(x) \qquad \forall\ x \geq 0.$$

In particular, the inclusion SP(γ, β) $\subset$ SP(γ, β^m) can be shown to be proper for $m > 1$.

(c) If distribution of X belongs to SP(γ, β) then distribution of (1/X) also belongs to SP(γ, β). More precisely, X is distributed as SP(γ,β,ϕ) if and only if (1/X) is distributed as SP(γ, β, ψ), where $\psi(x) = 1/(\phi(1/x))$.

(d) Distribution of X belongs to SP(1, β) if and only if the distribution of X^{γ} belongs to SP(γ, β).

(e) A distribution in SP(γ, β) family need not be continuous. Consider for example, the distribution F that corresponds to $\phi(x)$ given by

$$\phi(x) = \beta^n \; ; \; \beta^n \le x \le \beta^{n-1}, \; n= 0, \pm 1, \pm 2, \ldots$$

Clearly $\phi(x)$ is a step function, and ϕ satisfies (4.4.3) with $\gamma = 1$. Thus F is discrete distribution belonging to Semi-Pareto family.

In the light of the improved definition of Semi-Pareto distribution, it should be noted that in the results obtained by Pillai (1991), it is not sufficient just to replace the phrase "SP(γ, β) distribution" by "SP(γ, β) family". Instead the phrase "SP(γ, β) distribution" should be replaced by "SP(γ, β, ϕ) distribution".

In the next Theorem we give an important characterization of Pareto distribution in the Semi-Pareto family of distributions. Let $\mathbb{C}$ be the class of all continuous distributions.

Theorem 4.4.1 :

$$SP(\gamma, \beta_1) \cap SP(\gamma, \beta_2) \cap \mathbb{C} = \{P(\sigma, \gamma) \mid \sigma > 0\},$$

provided $\dfrac{\log \beta_1}{\log \beta_2}$ *is irrational.*

Proof : Let F be expressed as $F(x) = \dfrac{1}{1+\phi(x)}$, then $\phi(\cdot)$ satisfies for i = 1, 2

$$\phi(\beta_i^{\gamma} x) = \beta_i \phi(x) \; , \; x \ge 0 \qquad \ldots(4.4.4)$$

Let $\psi(t) = \phi(e^t)e^{-t/\gamma}$, t real. Then we have for $c_i = \gamma \log \beta_i$; i = 1,2,

$$\psi(t+c_i) = \phi(e^{t+c_i})e^{-(t+c_i)/\gamma}$$
$$= \phi(e^t \beta_i^{\gamma})e^{-t/\gamma}\beta_i^{-1}$$
$$= \phi(e^t)e^{-t/\gamma} \qquad \text{using (4.4.4)}$$
$$= \psi(t).$$

Since $\phi(\cdot)$ is continuous, $\psi(\cdot)$ is also a continuous function with periods c_1 and c_2, where c_1/c_2 is irrational. This implies that $\psi(\cdot)$ is a constant function. That is

$$\psi(x) = \phi(e^x)e^{-x/\gamma} = c \text{ for all } x \text{ real.}$$

This implies $\qquad \phi(x) = cx^{1/\gamma} \quad \forall\ x$ real

This completes the proof.■

Theorem 4.4.2 *Intersection of all* SP(γ, β) *families for various values of* β *in* (0, 1) *is a Pareto family* $\mathcal{F}_1 = \{P(\sigma, \gamma) | \sigma > 0\}$.

Proof : Suppose $F \in \mathcal{F}_2 = \bigcap_{0 \le \beta \le 1} SP(\gamma,\beta)$. Define $\phi = 1/F - 1$, then ϕ satisfies for given $\gamma > 0$,

$$\phi(\beta^{\gamma}x) = \beta\ \phi(x) \qquad \forall\ x \ge 0,\ \beta \in [0, 1].$$

Property (a) mentioned earlier implies that

$$\phi(\beta^{\gamma}x) = \beta\ \phi(x) \qquad \forall\ x \ge 0,\ \beta \ge 0,$$

Taking x= 1, we get $\phi(\beta^{\gamma}) = \beta\ \phi(1)$ for all $\beta \ge 0$, or

$$\phi(x) = (x/\sigma)^{1/\gamma} \qquad \forall\ x \ge 0,$$

where $\sigma = \phi(1)^{-\gamma}$. Thus $\mathcal{F}_2 \subset \mathcal{F}_1$. The converse can be readily verified from the definition of the Pareto distribution $P(\sigma,\gamma)$. ■

Remark 4.4.1 : It may be noted that unlike in Theorem 4.4.1, the continuity of distributions is not assumed in Theorem 4.4.2. With the additional assumption of continuity, Theorem 4.4.2 would be a corollary of Theorem 4.4.1 ■

Remark 4.4.2 : The distinction between a Pareto and Semi-Pareto distributions is that $P(\sigma, \gamma)$ satisfies (4.4.3) for every $0 < \beta < 1$, whereas $SP(\gamma, \beta, \phi)$ satisfies (4.4.3) only for a specific β appearing in the parameter list (and integer powers of it) ■

Remark 4.4.3 : In the light of property (d) mentioned above, one can restrict the theoretical discussion of Pareto/ Semi-Pareto processes to the case $\gamma = 1$. All results in the case of $\gamma \neq 1$ can be derived from this special case. However, we discuss the results for general γ ■

4.4.3 A NECESSARY AND SUFFICIENT CONDITION FOR STATIONARITY

We refer to the Semi-Pareto process of Pillai (1991) by MINARSP(1) (Pillai called it ARSP(1)). It may be noted that MINARP process of Yeh et al.(1988) is a particular form of MINARSP process.

Consider a MINAR stochastic process $\{X_n\}$ given by (4.4.1), where $\{\xi_n\}$ is a sequence of iid rvs, independent of X_0, with a common distribution G, and $0 \leq \beta \leq 1$. Let F denote the distribution of X_0. The analogous MAXAR process can be defined as

$$Y_n = \begin{cases} \beta^{\gamma} Y_{n-1} & \text{with probability } \beta \\ \min(\beta^{\gamma} Y_{n-1}, \eta_n) & \text{with probability } 1-\beta \end{cases} . \qquad \ldots(4.4.5)$$

If the common distribution of η_n's is SP(γ, β, ϕ) and Y_0 follows SP(γ, β, ϕ) then it can be shown that the MAXAR process is stationary. Every result that holds for MINAR process given by (4.4.1) has an analogous result for MAXAR process given by (4.4.5), which can be proved by similar arguments. Hence hereafter we discuss only MINAR process.

Remark 4.4.4 : While dealing with maximum process it is convenient to work with F instead of $\bar{F}$; In this connection it may be useful to observe that if F $\in$ SP(γ, β) then F(x) is of the form $(1 + \psi(x))^{-1}$ where $\psi(x)$ satisfies $\psi(\beta^{\gamma}x) = \beta^{-1}\psi(x)$ for every $x \geq 0$ ■

Recall that MINAR process at (4.4.1) is a particular form of the MINAR process given by

$$X_n = \min\ (\rho^{-1}X_{n-1}, Z_n) \qquad \ldots(4.4.6)$$

where $\{Z_n\}$ is a sequence of iid rvs, independent of X_0, having a common distribution H, and $\rho \in [0, 1]$.

If H has a positive mass $\beta = \rho^{1/\gamma}$ concentrated at infinity then this process takes the form (4.4.1) with $G = H/\bar{\beta}$. Recall that a necessary and sufficient condition for the process at (4.4.6) to be stationary with marginal F is that F is a min-SD distribution. This requirement can also be expressed as :

$$\bar{H}(x) = \frac{\bar{F}(x)}{\bar{F}(\rho x)} \quad \forall\ x \geq 0. \qquad \ldots(4.4.7)$$

It is known that if X_0 has an arbitrary distribution and H satisfies (4.4.7) for some F, then X_n converges in distribution to a rv X having distribution F. The condition (4.4.7) also implies that if the process at (4.4.6) is stationary, then H is uniquely determined by the stationary distribution F and vice versa. From (4.4.7), it follows that a necessary and sufficient condition for MINAR process at (4.4.1) to be stationary is that

$$\bar{\beta}G(x) = \frac{\bar{F}(\beta^{\gamma}x) - \bar{F}(x)}{\bar{F}(\beta^{\gamma}x)} = P(X_0 \leq x \mid X_0 > \beta^{\gamma}x). \qquad \ldots(4.4.8)$$

From these observations, it follows that

(i) a process at (4.4.6) is a MINARSP process with $SP(\gamma, \beta, \phi)$ marginal if and only if H is an improper distribution given by $H(x) = (1-\beta)G(x)$, where $G = SP(\gamma, \beta, \phi)$, and

(ii) For the process $\{X_n\}$ defined at (4.4.1), X_n converges in distribution to a $SP(\gamma, \beta, \phi)$ rv. if X_0 has an arbitrary distribution and $G = SP(\gamma, \beta, \phi)$.

4.4.4 SOME CHARACTERIZATION RESULTS

Having noted that in both MINARP and MINARSP processes we have F = G, that is, common distribution of innovations is same as the stationary distribution of the process, it is of interest to answer the question "If $F = G$, is $\{X_n\}$ at (4.4.1) necessarily stationary?". The question is answered by the following Theorem which shows that the condition "$F = G$" characterizes the Semi-Pareto process in the class of stationary MINAR processes given by (4.4.1).

Theorem 4.4.3 : *For the* MINAR *process given by* (4.4.1), *with* F *and* G *being the distribution of* X_0 *and the common distribution of innovation rvs respectively, any two of the following statements imply the third.*

(i) $F = G$.

(ii) $\{X_n\}$ *is stationary.*

(iii) G *belongs to* SP(γ, β) *family.*

Proof : Suppose (i) and (ii) hold. Then from (4.4.8), we get

$$\bar{\beta}G(x) = 1 - \frac{\bar{G}(x)}{\bar{G}(\beta^{\gamma}x)}$$

Writing $\bar{G}(x) = (1 + \phi(x))^{-1}$, we get

$$\frac{\bar{\beta}\ \phi(x)}{1 + \phi(x)} = \frac{\phi(x) - \phi(x\beta^{\gamma})}{1 + \phi(x)}, \qquad \forall\ x \geq 0.$$

That is, $\phi(x\beta^{\gamma}) = \beta\ \phi(x)$. Thus (iii) holds.

That (i) and (iii) imply (ii) is established in Section 4.4.3 and that (ii) and (iii) imply (i) follows from (4.4.8) ∎

Yeh et al.(1988) also studied the behaviour of geometric minima and geometric maxima for MINARP process. Let N be a nonnegative integer valued rv independent of the MINAR process $\{X_n\}$. Define

$$T_N = \min_{0 \le k \le N} X_k \qquad \text{and} \qquad M_N = \max_{0 \le k \le N} X_k .$$

Yeh et al.(1988) proved that if $\{X_n\}$ is a MINARP process then for every $p \in (0, 1)$

$$\left(\frac{p}{1-q\beta}\right)^{\gamma} M_{N(p)} \stackrel{d}{=} X_0 \stackrel{d}{=} \left(\frac{p}{1-q\beta}\right)^{-\gamma} T_{N(p)} , \qquad \ldots(4.4.9)$$

when N(p) is geometric with P(N(p) = 0) = p, q = 1-p. In proving this result the authors have used the level crossing process $\{Z_n(t)\}$ defined by

$$Z_n(t) = \begin{cases} 1, & \text{if } X_n > t, \\ 0, & \text{if } X_n \le t. \end{cases} \qquad \ldots(4.4.10)$$

The authors used the argument that $\{Z_n(t)\}$ is a Markov chain for every fixed $t > 0$. That the level crossing process $\{Z_n(t)\}$ forms a Markov chain for a fixed t is not an obvious result and its proof uses the structure of Semi-Pareto family. This result is proved by Arnold and Hallet (1989) for MINARP process, where they also give a characterization of MINARP process, in the class of stationary minification processes defined at (4.4.6), based on the Markovian property of $\{Z_n(t)\}$, under the regularity condition

$$\lim_{t \to 0} t^{-1/\gamma} H(t) = \lambda > 0,$$

where H(t) is the common distribution of innovation rvs Z_n.

Latter Pillai (1991) proved that (4.4.9) holds for a MINARSP process also for an appropriate choice of p. It is interesting to observe that Pillai does not refer to Arnold and Hallet (1989), although he uses the argument that $\{Z_n(t)\}$ possesses Markovian property for MINARSP process also. Having noted the characterization of Pareto given by Arnold and Hallet, it appears that Pillai (1991) might have committed an error. However, as we show it in the next Theorem, with a weaker regularity condition than that assumed by Arnold and Hallet, similar arguments characterize the Semi-Pareto family of distributions.

Theorem 4.4.4 : *Consider a stationary* MINAR *process defined by* (4.4.6) *with the corresponding stationary distribution* F. *Let* H *be the common distribution of rvs* Z_n. Assume that for some $\delta > 0$,

$$\lim_{t \to 0} \frac{H(t)}{F(t)} = \delta \qquad \ldots(4.4.11)$$

The level crossing processes $\{Z_n(t)\}$ *defined by* (4.4.10) *are Markovian for every* t *if and only if* F *belongs to Semi-Pareto family.*

Proof : The Markovian property of $\{Z_n(t)\}$ can be proved for MINARSP process using the arguments similar to those made by Arnold and Hallet (1989).

Conversely, suppose $\{Z_n(t)\}$ is Markovian for every $t > 0$, for a stationary MINAR process defined by (4.4.6). Then following the arguments of Arnold and Hallet, we get

$$\frac{H(t)}{F(t)} = \frac{H(\rho t)}{F(\rho t)} \ \forall \ t > 0.$$

Denote $\dfrac{H(t)}{F(t)}$ by $\psi(t)$. Then on iteration of above equation we get

$$\psi(t) = \frac{H(\rho^n t)}{F(\rho^n t)} \ \forall\ n \geq 1$$

This implies $\psi(t) = \lim\limits_{n\to\infty} \dfrac{H(\rho^n t)}{F(\rho^n t)} = \delta$ from (4.4.11)

Thus, we have $\delta F(t) = H(t)$.

Since stationarity of the process at (4.4.6) implies $\bar{F}(t) = \bar{F}(\rho t)\bar{H}(t)$, we have

$$\delta F(t) = 1 - \frac{\bar{F}(t)}{\bar{F}(\rho t)},$$

or equivalently,

$$\bar{F}(\rho t) = \frac{\bar{F}(t)}{1 - \delta F(t)}.$$

That is,

$$F(\rho t) = \frac{(1-\delta)F(t)}{1 - \delta F(t)}.$$

Hence

$$\frac{F(\rho t)}{\bar{F}(\rho t)} = \frac{(1-\delta)F(t)}{\bar{F}(t)}$$

That is,

$$\phi(\rho t) = (1-\delta)\ \phi(t)\ , \qquad \ldots(4.4.12)$$

where $\phi(t) = \dfrac{F(t)}{\bar{F}(t)}$.

Note that, since $\bar{F}(t) = \bar{F}(\rho t)\bar{H}(t)$, we have $\delta = \dfrac{H(t)}{F(t)}$ < 1 for some t. Therefore $\beta = (1-\delta) \in (0, 1)$. Since both ρ and β are in $(0, 1)$, there exists $\gamma > 0$ such that $\rho = \beta^{\gamma}$. Then from (4.4.12), we have

$$\phi(\beta^{\gamma} t) = \beta\phi(t).$$

Hence F belongs to Semi-Pareto family ■

The choice of p ($p = \beta^2/(1+\beta+\beta^2)$) made by Pillai (1991) for which (4.4.9) holds for Semi-Pareto process with SP(γ, β, ϕ) marginals reduces (4.4.9) to the following form

$$\beta^{2\gamma} M_N \stackrel{d}{=} X_0 \stackrel{d}{=} \beta^{-2\gamma} T_N .$$

In the light of property (b) of the Semi-Pareto family, a more general result stated below holds.

Theorem 4.4.5 : *Let* $\{X_n\}$ *be a* MINARSP *process. For every positive integer* m, *there exists a geometric rv* N(m) *such that*

$$\beta^{m\gamma} M_{N(m)} \stackrel{d}{=} X_0 \stackrel{d}{=} \beta^{-m\gamma} T_{N(m)}, \qquad \ldots(4.4.13)$$

where $P(N(m) = 0) = p$ *is the solution of equation* $\frac{p}{1-q\beta} = \beta^m$, $q = 1-p$.

(For m = 2 this result reduces to the result obtained by Pillai(1991)).

Proof : Let $c > 0$, and N be a Geom(p) rv, where p is a solution of $\frac{p}{1-q\beta} = c$. The survival function of $c^{-\gamma}T_N$ is given by

$$\bar{H}(x) = P(c^{-\gamma}T_N > x) = P(T_N > c^{\gamma}x)$$

$$= \sum_{k=0}^{\infty} pq^k \, P(X_i > c^{\gamma}x;\ 0 \le i \le k). \qquad \ldots(4.4.14)$$

Now,

$$P(X_i > c^{\gamma}x;\ 0 \le i \le k) = P(Z_0(c^{\gamma}x)=1, \ldots, Z_k(c^{\gamma}x) = 1),$$

where $Z_i(c^{\gamma}x) = I_{\{X_i > c^{\gamma}x\}}$.

$= P(A_0 \cap A_1 \cap \ldots \cap A_k)$ say.

$= P(A_0)P(A_1|A_0) \ldots P(A_k|A_0 \cap A_1 \cap \ldots \cap A_{k-1})$

$= \bar{G}(c^{\gamma}x) \{\beta + \bar{\beta}\bar{G}(c^{\gamma}x)\}^k.$

Substituting in (4.4.14), we have

$$H(x) = \frac{p\bar{G}(c^{\gamma}x)}{1-q\{\beta+\bar{\beta}G(c^{\gamma}x)\}}$$

That is,

$$\frac{1}{\bar{H}(x)} - 1 = \frac{1 - q\beta}{p} \frac{1}{\bar{G}(c^{\gamma}x)} - \frac{q\bar{\beta}}{p} - 1$$

$$= c^{-1}\left(\frac{1}{\bar{G}(c^{\gamma}x)} - 1\right)$$

$$= c^{-1}\phi(c^{\gamma}x)$$

$$= \phi(x), \quad \text{provided } c \text{ is of the form } \beta^m.$$

(See property (b))

Thus it follows that

$$c^{-\gamma}T_N \stackrel{d}{=} X_0 \quad \text{when } c \text{ is of the form } \beta^m.$$

In other words,

$$\beta^{-m\gamma}T_{N(m)} \stackrel{d}{=} X_0,$$

where $N(m)$ is a $Geom(p)$ rv, and p is a solution of $\frac{p}{1-q\beta} = \beta^m$.

The first equality can be proved in a similar fashion by incorporating property (b) in the original proof of Pillai's (1991) result ∎

Remark 4.4.5 : The result of Yeh et al.(1988) can also be stated as follows. Let $\{X_n\}$ be a MINARP process. Then for every $c \in (0, 1)$ there exists geometric rv $N(c)$ such that

$$c^{\gamma} M_{N(c)} \overset{d}{=} X_0 \overset{d}{=} c^{-\gamma} T_{N(c)}, \qquad \ldots(4.4.15)$$

where $P(N(c) = 0) = p$ is the solution of equation $\frac{p}{1-q\beta} = c$, $q = 1-p$ ■

Remark 4.4.6 : The result (4.4.15) for MINARP process is obviously more general than the result (4.4.13) for MINARSP process, as (4.4.15) holds for every $c \in (0, 1)$. This fact characterizes the Pareto distribution, as we prove it later in this Section ■

Recall that MINAR process at (4.4.1) is stationary with SP(γ, β, ϕ) marginals only when F = G is SP(γ, β, ϕ). If F = G is SP(α, c, ϕ), where $c \neq \beta$, or $\alpha \neq \gamma$, then the process at (4.4.1) is not stationary. However it is interesting to note that the second equality in (4.4.15), with γ replaced by α, still holds for this non-stationary process. This is shown in the next Theorem.

Theorem 4.4.6 : *For a* MINAR *process of the form given at* (4.4.1). *Suppose* F = G *is* SP(α, c, ϕ), $\alpha > 0$, $c \in (0, 1)$, *and let* N *be a geometric rv with* $P(N = 0) = p$, *where* p *is the solution of* $p/(1-q\,\beta) = c$. *Then*

$$c^{-\alpha} T_N \overset{d}{=} X_0. \qquad \ldots(4.4.16)$$

Proof : Recall that $q = 1-p$ and $Z_n(t) = I_{\{X_n > t\}}$. The survival function of $c^{-\alpha}T_N$ is given by

$$\bar{H}(x) = P(c^{-\alpha}T_N > x) = \sum_{K=0}^{\infty} p\, q^K\, P(X_s > c^{\alpha}x,\ 0 \le s \le K\ ).$$

Now, using arguments in the proof of Theorem 4.4.5,

$P(X_s > c^{\alpha}x,\ 0 \le s \le \kappa\ ) = \bar{G}(c^{\alpha}x)(\ \beta + \bar{\beta}\ \bar{G}(c^{\alpha}x)\ )^{\kappa}.$

Thus we have,

$$\bar{H}(x) = \frac{p\ \bar{G}(c^{\alpha}x)}{1-q\{\beta + \bar{\beta}\ \bar{G}(c^{\alpha}x)\}}$$

$$\text{or } \frac{1}{\bar{H}(x)} - 1 = \frac{1 - q\beta}{p}\ \frac{1}{\bar{G}(c^{\alpha}x)} - \frac{q\bar{\beta}}{p} - 1$$

$$= c^{-1}\ \phi(c^{\alpha}x) = \phi(x) \quad \forall\ x \ge 0.$$

Thus, $H(x) = G(x)$ ■

Remark 4.4.7 : Clearly if we take $F = G = P(\sigma,\ \gamma)$ in above Theorem, then (4.4.16) holds for every $c \in (0,\ 1)$. In The light of property (b) of Semi-Pareto family, a more general result given below also holds.

$$c^{-m\alpha}T_N \overset{d}{=} X_0,$$

where N is a Geom(p) rv and m is a positive integer ■

Next we show that (4.4.16), for a specific value of c, characterizes the Semi Pareto structure of MINAR process, whereas (4.4.16) for every $c \in (0,\ 1)$ characterizes the Pareto structure of the MINAR process, when N is geometric.

Theorem 4.4.7 : *Let* $\{X_n\}$ *be a* MINAR *process of the form given at* (4.4.1) *and let* $F = G$. *Suppose* (4.4.16) *holds for some* $c \in (0,\ 1)$ *and* $\alpha > 0$, *where* $N = N(c)$ *is the* Geom(p) *rv, and* p *is the solution of* $p/(1-q\beta) = c$. *Then* F *belongs to* $SP(\alpha,\ c)$. *Further if* (4.4.16) *holds for every* $c \in (0,\ 1)$ *then* F *must be a Pareto distribution.*

Proof : Proceeding as in Theorem 4.4.6 and using (4.4.16), we have

$$P(c^{-\alpha}T_{N(c)} > x) = \frac{p\bar{G}(c^{\alpha}x)}{1-q\{\ \beta + \bar{\beta}\ \bar{G}(c^{\alpha}x)\}} = \bar{G}(x) \quad \forall\ x \geq 0,$$

That is, G must satisfy

$$c^{-1}\left[\frac{1}{\bar{G}(c^{\alpha}x)} - 1\right] = \frac{1}{\bar{G}(x)} - 1 \quad \forall\ x \geq 0.$$

Writing $\phi(x)$ for $\{1/\bar{G}(x)\} - 1$, ϕ must satisfy,

$\phi(c^{\alpha}x) = c\ \phi(x) \quad \forall\ x \geq 0.$

Above result along with Theorem 4.4.2 completes the remaining proof ■

Remark 4.4.8 : The first equality alone in (4.4.15) for all $c \in (0, 1)$ may not characterize the Semi-Pareto (and hence Pareto) structure of MINAR process even in the class of stationary processes.

Following corollary is a consequence of Theorem 4.4.1 and Theorem 4.4.7.

Corollary 4.4.1 :*Let* $\{X_n\}$ *be a* MINAR *process of the form given at* (4.4.1) *with* F = G *which is continuous. Suppose*

$$c_1^{-\alpha}\ T_{N_1} \stackrel{d}{=} c_2^{-\alpha}\ T_{N_2} \stackrel{d}{=} X_0$$

holds for some $\alpha > 0$ *and* $c_1,\ c_2 \in (0,\ 1)$ *such that* $\dfrac{\log c_1}{\log c_2}$ *is irrational, where for* $i = 1,\ 2$, N_i *is a* Geom(p_i) *rv, and* p_i *is a solution of* $p_i/(1-q_i\beta) = c_i$. *Then F must be Pareto.*

In the next Theorem we show that (4.4.16) characterizes the geometric nature of N for a MINAR process when F = G is Semi Pareto.

Theorem 4.4.8 : *Let* $\{X_n\}$ *be a* MINAR *process of the form given at* (4.4.1) with F = G = SP(α, c, ϕ) *which is continuous. Suppose* (4.4.16) *holds for a nonnegative integer valued rv* N *independent of* $\{X_n\}$ *where* P(N=0) = p *is the solution of* $p/(1-q\beta) = c$. *Then* N *must be geometric.*

Proof : Proceeding as in Theorem 4.4.5, we obtain for all $x \geq 0$,

$$P(c^{-\alpha}T_N > x) = \sum_{k=0}^{\infty} P(N=k)\ \{\beta + \bar{\beta}\ \bar{F}(c^{\alpha}x)\}^k\ \bar{F}(c^{\alpha}x) = \bar{F}(x).$$

That is, the probability generating function (pgf) $Q_N(s)$ of N satisfies,

$$Q_N(s) = \frac{\bar{F}(x)}{\bar{F}(c^{\alpha}x)} = \frac{1+\phi(c^{\alpha}x)}{1+\phi(x)} \quad \forall\ x \geq 0,$$

where $s = \beta + \bar{\beta}(1 + \phi(c^{\alpha}x))^{-1}$. Also we note that

$$1 + \phi(c^{\alpha}x) = \frac{\bar{\beta}}{s-\beta}\ , \text{ and}$$

$$1 + \phi(x) = \frac{1 - s + cs - c\beta}{c(s - \beta)}\ .$$

Therefore,

$$Q_N(s) = \frac{\bar{\beta}c}{(1-c\beta)(1-sq)}\ . \quad \text{because } \frac{1-c}{1-c\beta} = q$$

Substituting $c = \frac{1-q}{1-q\beta}$, we get

$$Q_N(s) = (1-q)/(1-qs) \qquad \forall\ s \in (\beta,\ 1].$$

Thus N has geometric distribution with parameter q (See Rudin (1976), Theorem 8.5) ■

Next we prove that first equality in (4.4.13) also characterizes the geometric nature of N(m) for MINARSP process.

Theorem 4.4.9 : *Let* $\{X_n\}$ *be a* MINARSP *process with continuous* SP(γ, β, ϕ) *marginals. Suppose for positive integer* m, *and a nonnegative integer valued rv* N, *independent of* $\{X_n\}$,

$$\beta^{m\gamma} M_N \overset{d}{=} X_0, \qquad \ldots(4.4.17)$$

holds, where $P(N = 0) = p$ *is the solution of* $p/(1-q\beta)=\beta^m$, $q = 1-p$. *Then* N *must be geometric.*

Proof : From (4.4.17), the distribution function of $\beta^{m\gamma}M_N$ satisfies for all $x \geq 0$,

$$P(X_0 \leq x) = P(\beta^{m\gamma}M_N \leq x)$$

$$= \sum_{k=0}^{\infty} P(N = k)\left(\frac{\beta + \phi(\beta^{-m\gamma}x)}{1 + \phi(\beta^{-m\gamma}x)}\right)^k \frac{\phi(\beta^{-m\gamma}x)}{1 + \phi(\beta^{-m\gamma}x)}$$

Writing $s = \frac{\beta + \phi(\beta^{-m\gamma}x)}{1 + \phi(\beta^{-m\gamma}x)}$, the pgf $Q_N(s)$ of N must satisfy,

$$Q_N(s) = \frac{\phi(x)}{1+\phi(x)}\ \frac{1+\phi(\beta^{-m\gamma}x)}{\phi(\beta^{-m\gamma}x)}.$$

Note that

$$\phi(\beta^{-\gamma}x) = \frac{s-\beta}{1-s}, \text{ and } \phi(x) = \frac{\beta^m(s-\beta)}{1-s}.$$

Therefore,

$$Q_N(s) = \frac{\beta^m \bar{\beta}}{1-\beta^{m+1}-s(1-\beta^m)},$$

Using the relation $\frac{p}{1-q\beta} = \beta^m$, we get

$$Q_N(s) = (1-q)/(1-qs) \quad \forall\ s \in [\beta, 1],$$

where $q = \frac{1-\beta^m}{1-\beta^{m+1}}$.

Using the same argument as in the proof of Theorem 4.4.8, it follows that N is geometric with parameter q ∎

Remark 4.4.9 : Since Pareto distribution is a particular case of Semi-Pareto, Theorem 4.4.9 also applies to MINARP process ∎

Remark 4.4.10 : From Theorems 4.4.8 and 4.4.9 we are able to characterize the geometric nature of N if either $\beta^{-\gamma}T_N$ or $\beta^{\gamma}M_N$ is distributionally equivalent to X_0 for a MINARSP process. However if $\beta^{-\gamma}T_N \overset{d}{=} \beta^{\gamma}M_N$ for this process for some positive integer valued rv N, it may not follow that N is geometric ∎

4.5. Characterization of an AR process with Poisson marginals

4.5.1 The new thickening operator $\theta\circ$:

Using the unified approach of Chapter 2, we propose a new thickening operator $\theta\circ$ to be used along with the thinning operator $\rho *$ to construct a new AR model. To have motivation for the new operator that we propose to define, consider the following identity :

$$\rho * X = \sum_{i=1}^{X} B_i \quad , B_i\text{'s are iid Bernoulli}(\rho).$$

$$= \sum_{i=1}^{X} (1-C_i) = X - \sum_{i=1}^{X} C_i \quad , C_i\text{'s are iid Bernoulli}(\bar{\rho}).$$

(Recall $\bar{\rho} = 1-\rho$). Thus $\rho * X = X - \bar{\rho} * X$. In other words, by performing an operation $\rho *$ on X, we are, in effect, subtracting a thinned version of X (although not independent of X) from X. Hence it appears quite natural to add a thinned version of X to it as a thickening operator. This leads to the following definition of new thickening operator.

Definition 4.5.1 :

$$\theta \circ X = X + \theta * X_1,$$

where $\theta \in (0, 1)$ and X_1, independent of X, is distributed as X ∎

The operator $\theta\circ$ has the following properties :

(a) $\theta\circ(\rho * X) \overset{d}{=} \rho * (\theta\circ X)$ for every nonnegative integer valued rv X.

(b) $E(\theta\circ X) = (1+\theta)E(X)$

(c) If $Q_X(s)$ is the pgf of rv X, then pgf of $\theta\circ X$ is given by $Q_X(s)\, Q_X(1-\theta + \theta s)$.

Let X be a nonnegative integer valued rv. Given $\rho \in (0, 1)$, suppose that there exists $\theta \in (0, 1)$ such that $\theta \circ (\rho * X) \overset{d}{=} X$. Then using the unified approach proposed in Chapter 2, the stationary process at (2.2.3) takes the form

$$X_n = \theta \circ (\rho * X_{n-1}), \ n = 1, 2, \ldots$$

i.e. $$X_n = \rho * X_{n-1} + \theta * Y_n, \ n = 1, 2, \ldots,$$

where $\{Y_n\}$ is a sequence of independent rvs, independent of X_0, whose common distribution is same as that of $\rho * X_0$. This further implies that $\theta * Y_n$ will be distributed as $(\rho\theta) * Z_n$, where $\{Z_n\}$ is a sequence of independent rvs, independent of X_0, whose common distribution is same as that of X_0. Thus, the above mentioned stationary process can be expressed by

$$X_n = \rho * X_{n-1} + (\rho\theta) * Z_n, \ n = 1, 2, \ldots$$

If X is a rv with a finite mean, then Property (b) of operator $\theta\circ$ implies that θ must be equal to $\bar{\rho}/\rho$. In that case our thickening operator is defined only when $\rho \in [0.5, 1)$, and the above stationary process takes the form

$$X_n = \rho * X_{n-1} + \bar{\rho} * Z_n, \ n = 1, 2, \ldots \qquad \ldots(4.5.1)$$

Note that the process at (4.5.1) is defined for all $\rho \in (0, 1)$ although to visualize it as a consequence of our thickening operator $\theta\circ$, ρ is required to belong to the restricted range [0.5, 1). This thickening operator will be modified in Section 4.5.3 to permit all values of ρ in (0, 1). It may also be observed that this process is a particular form of INAR process with a specific representation for innovation rv.

4.5.2 A stationary process with poisson marginals and a characterization :

Suppose X is a Poisson(λ) rv and let $W = \rho * X + \bar{\rho} * Z$, where Z is a Poisson($\lambda$) rv and is independent of X. Then the pgf of W is given by,

$$\begin{aligned} Q_W(s) &= Q_X(\bar{\rho} + \rho s)\, Q_X(\rho + \bar{\rho} s) \\ &= \exp\{ \rho\lambda(s-1) + \bar{\rho}\lambda(s-1)\} \\ &= \exp(\lambda(s-1)) \\ &= Q_X(s). \end{aligned}$$

Thus, W is distributed as X. This implies that the process at (4.5.1) is a stationary process with Poisson marginals, when X_0 is Poisson(λ) and $\{Z_n\}$ is a sequence of independent Poisson(λ) rvs, independent of X_0. Since a thinned Poisson variable is again a Poisson variable, it follows that this process is same as the autoregressive process with Poisson marginals defined by Al-Osh and Alzaid (1987).

It, however, turns out that the Poisson is the only distribution which can be a stationary distribution of the INAR process expressed by (4.5.1). We thus have the following result.

Theorem 4.5.1 : [Sreehari and Kalamkar (1997]

Let $\{X_n\}$ *be an* INAR *process defined at* (4.5.1). *If* $\{X_n\}$ *is stationary for every* $\rho \in (0, 1)$, *then it must have Poisson marginals.*

Proof : Since $\{X_n\}$ is stationary, the rv X having the common distribution of X_n's satisfies

$$X \stackrel{d}{=} \rho * X + \bar{\rho} * Z \qquad \forall\ \rho \in (0,\ 1),$$

where X and Z are iid rvs. But this implies that the distribution of X is discrete stable (Steutel and Van Harn (1979)) with exponent 1, which is Poisson.■

In view of the discussion following equation (4.5.1), it appears that the thickening operator $\theta\circ$ will work as an inverse of $\rho *$ only when $\rho \geq 1/2$. However if we ignore this point at the stage of definition of $\theta\circ$, then we have seen that the process is well defined for every $\rho \in (0,\ 1)$. In order to overcome this difficulty, we propose the following modification in our thickening operator.

4.5.3 Modified definition of thickening and a related stationary process :

Let $\theta > 0$ be an arbitrary real number. Choose an integer k such that $(\theta/k) < 1$, Then define $\theta\circ$ as

$$\theta\circ X = X + \sum_{i=1}^{k} (\theta/k) * Y_i \ ,$$

To have uniqueness of k in the definition, k may be chosen to be smallest positive integer such that $(\theta/k) < 1$. It may be noted that the modified definition still satisfies properties (a) and (b) mentioned earlier, whereas the pgf of $\theta\circ X$ is given by

$$Q_{\theta\circ X}(s) = \begin{cases} Q_X(s)\ Q_X(1-(1-s)\theta)) & , 0 < \theta \le 1 \\ Q_X(s)\{Q_X(1-(1-s)\theta/k)\}^k & , \theta > 1 \end{cases} .$$

It can be easily verified that Poisson distribution satisfies

$$\theta\circ\ (\rho * X) \stackrel{d}{=} X$$

with the modified definition of operator $\theta\circ$ also, where $\theta = \bar{\rho}/\rho$. Consequently the stationary process at (2.2.3) takes the form

$$X_n = \rho * X_{n-1} + \sum_{i=1}^{k} (\bar{\rho}/k) * Y_{n,i} \quad , \ n=1,2,\ \ldots \qquad \ldots(4.5.2)$$

where X_0 is a Poisson(λ) rv and $Y_{n,i}$'s, independent of X_0, are iid Poisson(λ) rvs.

It may be noted again that the stationary process is still defined properly when k = 1 and for every $\rho \in (0,\ 1)$. In fact, the stationary process at (4.5.2) is defined for an arbitrary positive integer k and for every $\rho \in (0,\ 1)$. A necessary and sufficient condition for the process at (4.5.2) to be stationary is that the pgf Q_X of X_0 satisfies

$$Q_X(s) = Q_X(\bar{\rho} + \rho s)\ \{Q_X(1 - (1-s)\bar{\rho}/k)\ \}^k. \qquad \ldots(4.5.3)$$

A Poisson distribution satisfies (4.5.3) for every $\rho \in (0,\ 1)$ and every positive integer k. Obviously, Poisson is the only distribution that satisfies (4.5.3), since equation (4.5.3) with k=1 characterizes the Poisson distribution.