CHAPTER 1

INTRODUCTION

1.1 The need for non-Gaussian models :

In many practical problems, stationary time series possessing the Markovian property arise in a natural way. To be able to analyze such time series, we need to model them by suitable stochastic processes which have the Markovian property. Most of the traditional stochastic models make the assumption of Gaussian marginals. However many time series in practice exhibit non-Gaussian marginals. Time series of rain fall data, runoff data of rivers, wind velocity data, wind power data, economic data etc. are some of the examples of this type. The standard technique in the modelling of such processes is to remove the skewness of data by making a suitable transformation, and fit a Gaussian model to the transformed data. This technique depends heavily on the assumption that a suitable transformation is available such that the transformed series is Gaussian. The most popular transformations are those introduced by Box and Cox (1964). However, based on the analysis of an economic data, Nelson (1976) found that the Box-Cox transformed series are nowhere near Gaussian. Furthermore, Weiss (1975) showed that if $\{X_t\}$ is a stochastic process and $f(\cdot)$ is a one-to-one function, then $Y_t = f(X_t)$ is time-reversible if and only if $\{X_t\}$ is time reversible. Thus a process which is not time-reversible can not be transformed to a time-reversible Gaussian process.

As a consequence of such practical difficulties, in recent years there has been a growing interest for developing models for stationary processes with non-Gaussian marginals. In the next two Sections We mention few models that have been developed for modelling such time series. Models for continuous random phenomena are described in Section 1.2, whereas in Section 1.3, we describe models that have been developed for discrete phenomena.

1.2 Models for continuous random phenomenon

1.2.1 Autoregressive Processes :

A stochastic process $\{X_n, n = 0, \pm 1, \pm 2, \ldots\}$ is said to be an autoregressive process of order p, AR(p), if $\{\varepsilon_n\}$ is a sequence of independently and identically distributed (iid) random variables (rvs), where

$$\varepsilon_n = \sum_{k=0}^{p} b_k X_{n-k} , \qquad \ldots(1.2.1)$$

and real constants b_k's are such that $b_0, b_p \neq 0$.

Autoregressive (AR) processes are found to be very useful for modelling many time series in practice. Amongst AR processes, the first order processes, AR(1), deserve a special attention due to its simplicity in the first place. AR(1) processes are commonly represented in the form

$$X_n = \rho X_{n-1} + \varepsilon_n . \qquad \ldots(1.2.2)$$

The sequence of iid rvs $\{\varepsilon_n\}$ is frequently referred to as an *innovation process.*

By the very definition, AR(1) processes possess the stationary Markov property. For a process $\{X_n\}$ with Stationary Markov property, if the distribution of X_n does not depend on n, then the process can be shown to be stationary in strict sense. Thus, every stationary AR(1) process is a strictly stationary Markov process.

It is evident that the stationary AR(1) processes can be useful for modelling the stationary Markovian time series. The important question in this context is "Can the process at (1.2.2) be stationary?". To answer this question, rewrite (1.2.2) as

$$X_n = \sum_{k=0}^{m} \rho^k \varepsilon_{n-k} + \rho^{m+1} X_{n-m-1}.$$

In the above form, the last term represents the effect of the time before time point n-m. As m tends to infinity, this time becomes an "infinitely remote past" and in most of the situations it will have no influence. If we assume this to be the case, which perhaps always holds, then $\{X_n\}$ must be a process satisfying the limiting relation of the form

$$X_n = \sum_{k=0}^{\infty} \rho^k \varepsilon_{n-k} \qquad \ldots(1.2.3)$$

It is well-known that a stationary solution to (1.2.3) exists only if $|\rho| < 1$. Since X_{n-1} is a function of only ε_{n-1}, ε_{n-2}, ..., it is independent of ε_n.

Therefore the characteristic function (cf) ϕ_{X_n} of X_n at (1.2.2) satisfies

$$\phi_{X_n}(s) = \phi_{X_{n-1}}(\rho s)\ \phi_{\varepsilon_n}(s).$$

Clearly, the stationary solution, in which the common cf of X_n's is ϕ_X and the common cf of ε_n's is ϕ_ε, must satisfy

$$\phi_X(s) = \phi_X(\rho s)\ \phi_\varepsilon(s).$$

Thus, a necessary and sufficient condition for a distribution F to be a stationary solution of (1.2.2) is that the cf ϕ_F of F is representable as

$$\phi_F(s) = \phi_F(\rho s)\ \phi_{(\rho)}(s), \qquad \ldots(1.2.4)$$

for some cf $\phi_{(\rho)}$.

The properties of the process defined at (1.2.2) will obviously depend upon the choice of ρ. The most desirable situation would be that ϕ_F can be represented in the form (1.2.4) for every $\rho \in (-1, 1)$. A cf satisfying this criterion is obviously self-decomposable (SD). The corresponding distribution is called SD distribution. Thus, a distribution F is a stationary solution to (1.2.2) for every $\rho \in (0, 1)$ if and only if F is SD. If however, we do not insist upon the stationarity for every $\rho \in (0, 1)$, then the class of stationary distributions of AR(1) processes contains the set of SD distributions as a proper subset. For example, a mixture of two exponential distributions is not an SD law but it belongs to the class of stationary distributions of AR(1) processes (See Gaver and Lewis (1980)).

Stationary AR(1) processes thus provide models for stationary Markovian time series with SD marginals, the class which includes many distributions which are non-Gaussian. It may be noted here that all SD distributions are absolutely continuous as shown by Fisz and Varadarajan (1963).

These ideas were exploited by Gaver and Lewis (1980), who appear to be the first to propose a non-Gaussian AR(1) process. The connection mentioned above between self-decomposable laws and stationary AR(1) process was also made for the first time by Gaver and Lewis (1980). This relation was pointed out by Prof. S.I. Resnick as acknowledged by the authors in their paper.

Gaver and Lewis (1980) proposed an exponential AR process EAR(1), which is defined by

$$X_n = \begin{cases} \rho X_{n-1} & \text{with probability } \rho \\ \rho X_{n-1} + Y_n & \text{with probability } (1-\rho) \end{cases} \quad \ldots(1.2.5)$$

where $\rho \in (0, 1)$, $\{Y_n\}$ is a sequence of iid rvs with common distribution exponential(λ) and Y_n is independent of X_{n-1} for every n. Here exponential(λ) stands for an exponential distribution with mean λ.

Since the sequence $\{X_n, n = 0, \pm 1, \pm 2, \ldots\}$ has its starting point at infinitely remote past, it is strictly stationary with exponential(λ) marginals. Also it can be easily verified that if X_0 is distributed as exponential(λ) then $\{X_n, n \geq 0\}$ is strictly stationary.

An important point to be noted here is that if the stationary distribution of an AR(1) process $\{X_n\}$ given by (1.2.2) is concentrated on $\mathbb{R}^+ = [0, \infty)$, then ρ must belong to [0, 1] for otherwise there can not exist an ε_n independent of X_{n-1}, such that $\rho X_{n-1} + \varepsilon_n$ is again positive.

Using the fact that Gamma distribution is self-decomposable, Gaver and Lewis (1980) also proposed a Gamma AR process GAR(1).

As pointed out earlier, if we do not insist upon the stationarity of $\{X_n\}$ for every $\rho \in (0, 1)$, then we can have stationary AR(1) processes with non-self-decomposable marginals also; ofcourse such models admit only limited values of ρ. The mixed-exponential process MEAR(1) of Gaver and Lewis(1980) is of this type, where the marginal distribution of the process is a mixture of two exponential distributions.

1.2.2 MINAR(1) process :

Tavares (1980) proposed a stationary Markovian process with exponential marginals defined by

$$X_n = \theta \min(X_{n-1}, Y_n), \quad n \geq 0, \qquad \ldots(1.2.6)$$

where $\theta > 1$ is a constant and $\{Y_n\}$ is a sequence of iid rvs, independent of X_0, with common distribution exponential($\lambda(\theta-1)$). Tavares (1980) showed that if X_0 is distributed as exponential (λ) then $\{X_n\}$ defined above is strictly stationary.

Because of its structure, the process defined by

(1.2.6) is generally referred to as a *minification* process. Also note that above structure is similar to that of an AR(1) process, with the operation of addition replaced by the minimization. thus a minification process may be called an AR(1) process in the scheme of minima. We will use the notation MINAR(1) for the processes with structure (1.2.6).

Remark 1.2.1 : MINAR process of order p, MINAR(p) can be defined by,

$$X_n = \min(\theta_1 X_{n-1},\ \theta_2 X_{n-2},\ \ldots,\ \theta_p X_{n-p},\ Z_n),$$

where $\theta_p \neq 0$ and $\{Z_n\}$ is a sequence of iid rvs, Z_n being independent of X_{n-1}, X_{n-2}, ..., X_{n-p} for every n ∎

Since a MINAR(1) process has the Markovian property, stationary MINAR(1) processes serve as another useful tool for modelling stationary Markovian time series.

Sim (1986) showed that the structure of MINAR(1) process also accommodates Weibull marginals. Yeh et al. (1988) proposed a stationary MINAR(1) process with Pareto marginals, which they called an ARP(1) process. ARP(1) process is defined by

$$X_n = \begin{cases} \beta^{-\gamma} X_{n-1} & \text{with probability } \beta \\ \min(\beta^{-\gamma} X_{n-1},\ Y_n) & \text{with probability } (1-\beta) \end{cases} \qquad \ldots(1.2.7)$$

where $\beta \in (0,\ 1)$ and $\gamma \geq 1$ are constants and ξ_n's, independent of X_0, are iid rvs with Pareto distribution $P(\sigma,\ \gamma)$, whose survival function is given by $\bar{F}(y)$ =

$P(\xi_n > y) = \{ 1 + (y/\sigma)^{1/\gamma} \}^{-1}$, $y \geq 0$.

It is easy to verify that if X_0 is distributed as $P(\sigma, \gamma)$ then $\{X_n\}$ defined above is a strictly stationary process. It may be noted that the structure (1.2.7) is a particular form of the structure (1.2.6), when the common distribution of Y_n's assigns a positive mass at infinity.

An obvious question that arises is : "What is the class of stationary distributions for MINAR(1) processes with structure given at (1.2.6) ?"

Under the assumption of stationarity, let the common distribution of X_n's be same as that of a rv X and the common distribution of Y_n's be same as that of a rv Y. Then X and Y must satisfy the stochastic equation

$$X \stackrel{d}{=} \theta \min(X, Y). \qquad \ldots(1.2.8)$$

or equivalently,

$$\min(X, Y) \stackrel{d}{=} \rho X \quad ; \rho = 1/\theta.$$

Conversely, suppose X and Y be rvs satisfying equation (1.2.8). If $X_0 \stackrel{d}{=} X$, and the common distribution of Y_n's is same as that of Y, then it can be proved by mathematical induction that the MINAR(1) process defined by structure (1.2.6) is strictly stationary. Thus the problem of finding stationary distributions for MINAR(1) processes is same as that of solving the stochastic equation (1.2.8).

Arnold and Issacson (1976) considered the problem of solving equation (1.2.8) for nonnegative rvs X and Y. Let F and G denote the distributions of X and Y respectively.

It was shown that for $0 < \theta < 1$, no nondegenerate solution exists. For $\theta=1$, the support of F should lie to the left of the support of G (except for this F and G are arbitrary). For $\theta > 1$, G must satisfy

$$\bar{F}(x)\ \bar{G}(x) = \bar{F}(\theta x), \qquad \ldots(1.2.9)$$

where $\bar{F}$ and $\bar{G}$ are survival functions of rvs X and Y respectively. Iterating above equation we find that

$$\bar{F}(x) = \bar{F}(x/\theta^{n}) \prod_{i=1}^{n} \bar{G}(x/\theta^{i}) \quad \forall n \text{ and } \forall x \geq 0$$

Letting $n \to \infty$, we get

$$\bar{F}(x) = \bar{F}(0) \prod_{i=1}^{\infty} \bar{G}(x/\theta^{i}) \quad \forall x \geq 0$$

This will be satisfied in the degenerate case where $\bar{F}(x)\equiv 0$, $x \geq 0$. Nontrivial solutions will arise if for every $x>0$, the indicated infinite product converges. Or alternatively if

$$\sum_{i=1}^{\infty} G(\theta^{-i}x) < \infty \text{ for some } x > 0, \qquad \ldots(1.2.10)$$

and the corresponding F is given by

$$F(x) = 1 - \alpha \prod_{i=1}^{\infty} (1 - G(\theta^{-i}x)), \qquad \ldots(1.2.11)$$

where $\alpha \in (0, 1]$ is an arbitrary constant. The case $\alpha = 1$ refers to a distribution which is continuous at 0.

Remark 1.2.2 : If $\alpha < 1$, then we have $P(X = 0) > 0$. Now for a MINAR(1) process $\{X_n\}$, the event $X_0 = 0$ requires special attention. Observe from (1.2.9) that $Y > 0$ with probability 1. Therefore if X_0 takes the value 0 then all subsequent X_n's will also be 0, whereas if X_0 takes a positive value then all subsequent X_n's will also take positive values. This peculiarity suggests that a MINAR process is more naturally defined with the marginals having support $(0, \infty)$ rather than $[0, \infty)$ ■

The connection observed above between the Arnold and Issacson's work and the stationarity of MINAR(1) processes was pointed out in Kalamkar (1995).This relationship can be utilized in two ways (i) If we want to know whether F can be the marginal distribution of a stationary MINAR(1) process, we need to check whether for some $\theta > 1$, G defined by (1.2.9) is a proper survival function or not. If it is, then it is taken as the common survival function of the innovation process. (ii) If we know that G is the common distribution of the innovation process, then check whether it satisfies (1.2.10) for some $\theta > 0$. If it does and X_0 follows the F in (1.2.11), then F is the stationary distribution of the MINAR(1) process with scale parameter θ. If instead, X_0 is an arbitrary random variable then F is the limiting distribution of X_n as $n \rightarrow \infty$.

Remark 1.2.3 : The condition (1.2.10) is a necessary and sufficient condition for the convergence of infinite product in (1.2.11). For some distributions it may be easier to establish the convergence of infinite product directly. For example, if G is an exponential distribution with mean $1/\theta$, then $\bar{G}(x) = \exp(-\theta x)$. This implies $\prod_{i=1}^{\infty} \bar{G}(\rho^i x) = \exp(-\theta x \ \Sigma\rho^i) = \exp(-\theta x\rho/(1-\rho))$ for all x > 0. Thus the infinite product converges for all x > 0 ■

Note that (1.2.9) can be rewritten as

$$\bar{F}(x) = \bar{F}(\rho x)\ \bar{G}_{(\rho)}(x) , \qquad \ldots(1.2.12)$$

where $\bar{G}_{(\rho)}(x) = \bar{G}(\rho x)$ with $\rho \in (0,\ 1)$. It may be noted that the equation (1.2.12) is similar to equation (1.2.4), Which defines self-decomposable distributions, except that cf's are replaced by survival functions. We define the self-decomposable distributions in the scheme of minima (min-self-decomposable) as follows.

Definition 1.2.1 : A distribution F is called min-self-decomposable (min-SD) if for every $\rho \in (0,\ 1)$, its survival function $\bar{F}$ can be represented in the form (1.2.12), for some survival function $\bar{G}_{(\rho)}$ ■

Then it follows that F is the stationary distribution of a MINAR(1) process for every $\rho \in (0,\ 1)$ if and only if F is min-SD. This observation is comparable with the observation made by Gaver and Lewis (1980) regarding SD laws and AR(1) processes. As will be shown in Chapter 3, the min-SD distributions are necessarily continuous. Thus only continuous distributions can be marginals of a stationary MINAR(1) process which admits all values of the autoregressive parameter ρ in interval (0, 1). If stationarity of the process is not insisted upon for every $\rho \in (0,\ 1)$, then the class of stationary distributions for MINAR(1) processes contains the class of min-SD laws as a proper subset.

It may be noted that this connection between minification processes and min-SD laws has not been pointed out earlier in the literature.

Remark 1.2.4 : Min-SD laws defined above are analogous to the Max-self-decomposable (max-SD) laws defined by Pancheva (1990)■

1.2.3 MAXAR(1) processes :

AR processes in the scheme of maxima can be defined in an analogous manner. The first order processes in this scheme will be denoted by MAXAR(1).

MAXAR(1) processes are introduced by Alpuim (1989), which are called extremal processes by the author. These processes take the form

$$X_n = \rho \max (X_{n-1}, Y_n), \qquad \ldots(1.2.13)$$

where $\rho \in (0, 1)$ and $\{Y_n\}$ is a sequence of iid rvs, independent of X_0.

A necessary and sufficient condition for a distribution F to be the marginal distribution of a stationary MAXAR(1) process is that F satisfies

$$F(x) = F(x/\rho)\, G(x/\rho),$$

or

$$F(x) = F(x/\rho)\, G_{(\rho)}(x) \text{ say,} \qquad \ldots(1.2.14)$$

where $G_{(\rho)}$ is a distribution function.

Note that the equation (1.2.14) is also similar to (1.2.4), the defining equation for SD laws, with cf's now replaced by distribution functions. Max-self-decomposable (max-SD) distributions are defined as follows.

Definition 1.2.2 : A distribution F is said to be max-SD if for every $\rho \in (0, 1)$, F can be expressed in the form (1.2.14) for some distribution function $G_{(\rho)}$ ∎

It is clear that F is the marginal distribution of a stationary MAXAR(1) process for every $\rho \in (0, 1)$ if and only if F is max-SD. However, if stationarity of a process is not insisted upon for every $\rho \in (0, 1)$ then the class of stationary distributions for MAXAR(1) processes contains the class of max-SD laws as a proper subset.

Alpuim (1989) showed that the max-SD distributions are necessarily continuous. The proof given by Alpuim (1989) is different from our proof given in Chapter 3 for min-SD laws. However our arguments are relatively elementary. Generalized Pareto distribution specified by

$$F(x) = 1 - (1-\alpha x)^{1/\alpha},\ 0 < x < 1/\alpha,\ \alpha > 0;$$

Frechet-type extreme value distribution

$$F(x) = \exp(-(\delta x)^{-\alpha}),\ x > 0,\ \alpha,\delta > 0;$$

and Pareto distribution $P(\sigma, \gamma)$ are examples of max-SD distributions.

1.2.4 AR(1) processes with random coefficient :

AR processes with random autoregressive coefficient are also introduced in the literature for all the three schemes viz. addition, minimization and maximization. Such processes are frequently referred to as stochastic processes in random environment.

Lewis(1982) presented a gamma AR(1) process defined by

$$X_n = U_n X_{n-1} + Y_n ,$$

where $\{U_n\}$ and $\{Y_n\}$ are sequences of iid rvs, which are independent of each other and also independent of X_0. The common distribution of Y_n's is Ga($\beta-\alpha$, λ) and the common distribution of U_n's is Be(α, $\beta-\alpha$). If X_0 is distributed as Ga(β, λ) then $\{X_n\}$ defined above is a stationary process.

Remark 1.2.5 : Note that the Gamma AR(1) process described above is different from GAR(1) process of Gaver and Lewis (1980), which also is a stationary process having Gamma marginals ■

An AR(1) process in random environment with exponential marginals was proposed by Lawrence and Lewis(1981), which was called NEAR(1) process by the authors.

AR(1) processes with random coefficient in the scheme of maxima are studied by Alpuim and Athayde(1990). These processes are defined by

$$X_n = Z_n \max(X_{n-1}, Y_n),$$

where $\{Z_n\}$ and $\{Y_n\}$ are sequences of iid rvs, independent of each other and also independent of X_0.

Alpuim and Athayde (1990) gave a criterion for a distribution to be the stationary distribution of such a process. It is shown that Be(α, β), Ga(β, 1) and P(1, 1) distributions are accommodated as stationary distributions for the process defined above.

1.2.5 Some more models :

Haslett (1979) used the following model to describe a solar thermal energy storage system.

$$X_n = \max(\beta X_{n-1}, \alpha\beta X_{n-1} + Y_n),$$

where $0 < \beta < 1$, and $0 \le \alpha \le 1$. This model was developed by Daley and Haslett (1982), Hooghiemstra and Keane (1985), and Hooghiemstra and Scheffer (1986).

Helland and Nilsen (1976) used the model

$$X_n = \max(X_{n-1} - Z_n, Y_n)$$

to describe the water density in a still fjord as well as the other phenomena such as the utility of an industrial equipment.

1.3 MODELS FOR DISCRETE RANDOM PHENOMENON

The models described in Section 1.2 serve the purpose of modelling time series of continuous phenomena. Many time series in practice assume discrete values. Mainly these are counting processes. To model stationary time series of this type, we need stationary stochastic processes which can accommodate discrete distributions as marginals. Very few models of autoregressive nature have been proposed so far for this purpose as compared to the continuous case. Some models that have been proposed in the literature are described below.

1.3.1 Integer valued autoregressive processes : INAR(1)

We have noted earlier that only self-decomposable distributions are marginal distributions of stationary AR(1) processes. It is Well-known that all self-decomposable distributions are absolutely continuous. Therefore a different approach is required for finding the models for integer valued time series.

Steutel and Van Harn (1979) introduced the discrete analogue of self-decomposability.

Definition 1.3.1 : A distribution F concentrated on $\mathbb{N}_0$ with probability generating function (pgf) Q_F is called discrete self-decomposable (DSD) if for every $\rho \in (0, 1)$, Q can be expressed as

$$Q_F(s) = Q_F(1-\rho + \rho s)\, Q_{(\rho)}(s) \;;\; |s| \le 1, \qquad \ldots(1.3.1)$$

where $Q_{(\rho)}$ is a pgf ■

In terms of rvs, X is said to be a DSD rv if for every $\rho \in (0, 1)$, we have

$$X \stackrel{d}{=} \rho * X + X_\rho ,$$

where X_ρ is a nonnegative integer valued rv, independent of X, and the operation $\rho * X$ is defined as follows.

Definition 1.3.2 : For $\rho \in [0, 1]$,

$$\rho * X = \sum_{i=1}^{X} B_i \ , \qquad \ldots(1.3.2)$$

where X is a nonnegative integer valued rv and B_i's are iid $B(1, \rho)$ rvs, independent of X ■

The operation $\rho * X$ is called thinning of X. This thinning operation is frequently used in the study of point processes. See for example, Cox and Isham (1980).

Remark 1.3.1 : Whenever we write $\rho * X_1$ and $\rho * X_2$, it is implicitly understood that the thinning is performed independently on X_1 and X_2 ■

Using the Concepts discussed above, McKenzie (1985) and Al-Osh and Alzaid (1987) independently introduced the discrete analogue of an AR(1) process. This model was called INAR(1) by Al-Osh and Alzaid. INAR(1) process is defined by

$$X_n = \rho * X_{n-1} + Y_n , \qquad \ldots(1.3.3)$$

where $\rho \in (0, 1)$ and $\{Y_n\}$ is a sequence of iid nonnegative integer valued rvs, independent of a nonnegative integer valued rv X_0.

A distribution F on $\mathbb{N}_0$ is the marginal distribution of a stationary INAR(1) process if and only if its pgf Q_F satisfies (1.3.1), where $Q_{(\rho)}$ is the common pgf of rvs Y_n. Clearly, a distribution F is the stationary distribution of an INAR(1) process for every $\rho \in (0, 1)$ if and only if F is DSD. The class of stationary distributions of INAR(1) processes, in general, contains the DSD laws as a proper sub-class.

McKenzie (1986) proposed a geometric INAR(1) process, which is a discrete analogue of EAR(1) process of Gaver and Lewis (1980). The geometric INAR(1) process is defined by

$$X_n = \begin{cases} \rho * X_{n-1} & \text{with probability } \rho \\ \rho * X_{n-1} + Z_n & \text{with probability } (1-\rho) \end{cases}, \qquad \ldots(1.3.4)$$

where $\{Z_n\}$ is a sequence of iid rvs, independent of X_0, with common distribution Geom(p). If X_0 is distributed as Geom(p) then the process defined above is strictly stationary.

Analogous to the GAR(1) process of Gaver and Lewis (1980), a negative binomial INAR(1) is also proposed by McKenzie (1986). A Poisson INAR(1) process was introduced by Alzaid and Al-Osh(1988).

Remark 1.3.2 : It is interesting to note that the structures of ARP(1) (or MINARP(1) in our terminology), EAR(1) and INAR(1) processes are similar. Also for all the three processes the common distribution of innovation rvs is same as the stationary marginal of the process. The observation of this similarity leads to another interesting study discussed in Chapter 5 ∎

Remark 1.3.3 : The structure of INAR(1) process was generalized to obtain INAR(p) processes by Alzaid and Al-Osh (1990), whose structure is similar to the classical AR(p) processes with scalar multiplication replaced by the thinning operation ■

1.3.2 Discrete Minification Processes :

Littlejohn (1992) introduced discrete minification processes. For this purpose the author first defined a thickening operator $\rho\backslash$ in such a way that it serves as a left-inverse when operated on $\rho * X$, for a nonnegative integer valued rv X. Then a rv Y is to be selected in such a way that $\min(X, Y) \stackrel{d}{=} \rho * X$. Then a discrete minification process is defined by

$$X_n = \rho\backslash\min(X_{n-1}, Y_n), \qquad \ldots(1.3.5)$$

where $\{Y_n\}$ is a sequence of iid rvs, independent of X_0, whose common distribution is same as that of Y. If $X_0 \stackrel{d}{=} X$ then the process $\{X_n\}$ at (1.3.5) is strictly stationary.

Littlejohn (1992) showed that geometric, negative binomial, Poisson and binomial distributions are accommodated as stationary distributions of the above discrete minification process.

1.3.3 INAR(1) Processes with Random coefficient

In the definition of INAR(1) process, if we replace ρ by a rv with support on (0, 1), then the resultant process is an INAR(1) process with random coefficient. More specifically such a process is defined by

$$X_n = U_n * X_{n-1} + Y_n, \qquad \ldots(1.3.6)$$

where $\{U_n\}$ is a sequence of iid rvs with support on (0, 1), $\{U_n\}$ and $\{Y_n\}$ are sequences of iid rvs, independent of each other and also of X_0. Here by $U*X$ we mean that conditionally on $(U=\rho)$ $U*X$ is equal to $\rho*X$.

McKenzie (1986) proposed two INAR(1) processes in random environment with negative binomial and geometric marginals, which are discrete analogues of the gamma AR(1) process of Lewis (1982) and NEAR(1) process of Lawrence and Lewis (1981) respectively.

Remark 1.3.4 : Throughout the thesis we are mainly concerned with autoregressive processes of order 1. Henceforth by notations AR, MINAR, MAXAR, INAR etc. wherever they appear, we mean AR(1), MINAR(1), MAXAR(1), INAR(1) respectively. For the processes with higher order, the order will be mentioned explicitly ∎

1.4 PROBLEMS TAKEN UP IN THE THESIS

In the present thesis we take up mainly three types of problems.

First problem is that of constructing AR models for non-Gaussian time series. This includes models for discrete variate time series also.

Second problem is that of characterization of stochastic models, either by their distributional properties or structural properties. Such a study is important for the purpose of identification of appropriate models.

Lastly, we present analogues of geometrically stable laws, viz. discrete geometrically stable laws, max-geometrically stable laws and min-stable geometrically stable laws. Analogues of geometric domain of attraction are also presented. The autoregressive processes with such marginals are also proposed.

<u>Autoregressive Models</u>

The need of stochastic models for non-Gaussian time series is already discussed in Section 1.1. We mainly concentrate on AR type models $\{X_n\}$, where the n^{th} term X_n is determined by $(n-1)^{th}$ term X_{n-1} and a random factor. Various models that have already been proposed in the literature are summarized in Sections 1.2 and 1.3.

All the AR models that have been described in Sections 1.2 and 1.3 are seemingly of different structures. We look at the problem of constructing AR models for time series in a unified manner. As a result we propose a unified approach for constructing AR models. We define thinning and thickening operations on the rvs and show that all the processes described earlier can be obtained by the following general rule. "$(n+1)^{st}$ member of the sequence $\{X_n\}$ is obtained by, either thickening of a thinned version of the n^{th} member or, thinning of a thickened version of the n^{th} member". We feel that such an approach obviously widens the vision, while looking for various stochastic models, and makes the construction process very simple. The unification proposed by us is limited only to the construction of models. Once a model is formed, the further analysis of it must be taken up separately for each model. We show that most of the known models can be constructed using our approach. New models, constructed using this approach, are proposed and some of them are studied in the thesis.

As we have already noted in Section 1.2, only min-SD laws can be the marginals of stationary MINAR processes, when stationary is required for every value of ρ in (0, 1). Since only continuous laws are discussed so far in the literature of MINAR processes, a natural question that arises is whether discrete distributions are not admitted as stationary laws for MINAR processes. We investigate this question and show that min-SD laws are necessarily continuous. However, if stationarity for all values of ρ in (0,1) is not insisted upon, then we can also accommodate laws that are not min-SD. A necessary and sufficient condition is obtained for a distribution on $\mathbb{N}_0$ to be a stationary distribution of a MINAR process. A Geometric MINAR process is studied in detail.

We mentioned earlier about a discrete minification process proposed by Littlejohn (1992). This process requires an operator $\rho\backslash$, which serves as a left inverse of thinning operator $\rho*$. However, as we will see later in Chapter3, it turns out that the definition of $\rho\backslash$ can not be given independently of the marginal distribution of the process. That is, for every marginal distribution, the definition of $\rho\backslash$ is to be obtained afresh. This is rather strange and makes one uncomfortable. Generally one expects that the underlying mechanism of the process should be specified clearly. After that one would ask whether a specific distribution can be accommodated by the given process. In case of discrete minification process, it is not happening so. Instead the structure of the process is decided only after the marginal distribution is fixed.

Motivated by above considerations, and following the unified approach suggested by us, we define a thickening operator in an unambiguous manner and propose a new model based on it. A Poisson process with above structure is presented and studied.

As we noted earlier in Section 1.2.3, only max-SD laws can be the stationary laws of MAXAR processes. We investigate a similar question, as we did in case of a MINAR process, whether discrete laws can be accommodated by MAXAR processes if stationarity for all values of θ is not insisted upon.

We show that a big class of discrete distributions, which contains most of the potentially useful distributions, is not accommodated by MAXAR processes. Thus it turns out that MAXAR process can not be used as a model for discrete time series. We propose a discrete version of MAXAR process, where we replace a scalar multiplication in the MAXAR process by a thinning operator $\rho *$, and call the new model a discrete maximum process. Two different versions of discrete maximum process are proposed. The conditions that must be satisfied by a stationary distribution of such process are also obtained. We show that both the models accommodate most of the well-known discrete distributions.

Characterizations of AR processes :

Characterization results are important in the process of identification of suitable stochastic models for the phenomenon under study. We consider the problem of characterizing AR processes based on their structural and distributional properties.

Some work in this direction is done by Chernick et al.(1988), Arnold and Hallett (1989) and Littlejohn (1992). We give characterizations of various processes such as EAR(1), exponential MINAR, Pareto MINAR, geometric MINAR, Semi-Pareto MINAR, geometric INAR and Poisson INAR based on various properties of these processes.

Analogues of geometric stability and related processes:

While studying the structures of various autoregressive processes, we observe a striking similarity between three autoregressive processes (See Remark 1.3.2). These are EAR(1) process of Gaver and Lewis(1980) defined at (1.2.5), ARP(1) process of Yeh et al. (1988) defined at (1.2.7) and geometric INAR process of McKenzie (1986) defined at (1.3.4).

The structures of the three processes, with ARP(1) process for $\gamma = 1$, are exactly same except that they are in three different schemes viz. addition with scalar multiplication, minimization with scalar multiplication, and addition with thinning. In all the three cases marginals of the stationary processes are same as the common marginals of corresponding innovation processes. Also the Laplace transform $L(\cdot)$, survival function $\bar{F}(\cdot)$, and alternate probability generating function[†1] $P(\cdot)$ of respectively exponential distribution, Pareto distributions, and geometric distribution are given by

$$L(s) = \bar{F}(s) = P(s) = \frac{1}{1+\theta s};\ s \geq 0$$

with $\theta > 0$.

The similarity among the three processes obviously demands further investigation. This investigation leads us to the development of analogues of geometrically stable laws. Geometrically stable laws are introduced by Klebanov et al. (1988) and further studied by Mohan et al. (1993).

In particular we define analogues of geometrically stable and geometrically Semi-stable laws for the discrete case as well as in the scheme of maxima. Analogues of geometric domain of attraction/ partial attraction are also defined. Analogous results of those

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[1†] Alternate probability generating function is defined in Section 3.2.

in the classical case are established for both the analogues. Similar concepts in the scheme of minima are completely parallel to those in the scheme of maxima.

The concepts developed above lead to the generalized versions of EAR(1), INAR(1) and ARP(1) processes.

1.5 ORGANIZATION OF THE THESIS :

In Chapter 2 we describe a unified approach for developing stationary first order autoregressive models. It will be shown that all the models described so far can be obtained through this approach by choosing appropriate pairs of thinning and thickening operators. Many new models arise due to this new approach. Some of these models are studied in Chapter 3.

Chapter 3 is concerned with the stationary autoregressive models for discrete valued phenomena. Four different models are presented and studied in this Chapter. The first model is MINAR (or minification process) described in Section 1.2.2. We show that stationary MINAR processes accommodate distributions on $\mathbb{N}_0$ as marginals. A necessary and sufficient condition is obtained for a distribution F on $\mathbb{N}_0$ to be the marginal of a stationary MINAR process. It is shown that Negative binomial and Poisson distributions satisfy this condition. A geometric MINAR process is studied in some detail. The innovation process of geometric MINAR process has an interesting structure, which in fact characterizes the geometric process (Two characterizations of geometric

MINAR process are given in Chapter 4, which is devoted to the characterizations of autoregressive processes).

The remaining three models are suggested by the unified approach described in Chapter 2. In the light of first model mentioned above, in is natural for one to expect that MAXAR processes also accommodate distributions on $\mathbb{N}_0$. However, this is not true. We propose two models (second and third respectively) for discrete maximum processes. All important distributions can be the stationary distributions for these two models. The fourth model is based on a new thickening operator combined with the thinning operator $\rho*$. It is shown that only Poisson distribution can be accommodated by this model (this characterization is given in Chapter 4).

Chapter 4 deals with the characterizations of autoregressive processes based on their distributional and structural properties. Characterizations of geometric MINAR process, exponential MINAR process, EAR(1) process, geometric INAR process and a process with Poisson marginals are given. We also show that most of the properties of ARP(1) process characterize the process. Characterizations of Semi-Pareto processes (ARSP(1)) introduced by Pillai (1991) can be easily obtained from above characterizations by relaxing certain conditions.

In Chapter 5, we develop two different analogues of geometric stability viz. discrete analogue and analogue in the scheme of maxima. We consider a more general form of the structure of EAR(1) process and show that the class of stationary distributions of this new model is essentially the class of geometrically stable laws. The similarity in the structures of EAR(1), geometric INAR, and ARP(1) processes suggests similar generalizations in the other two cases also. Also this leads to the development of new concepts, namely, discrete geometric stability and max geometric stability. The corresponding Semi-stable laws are also introduced. Results analogous to the classical theory are obtained. Following the approach of Mohan et al. (1993) we also define geometric domains of discrete attraction/ partial attraction, whose limit laws are discrete geometrically stable/ Semi-stable laws. Similar discussion, in brief, is also made for the schemes of maxima and minima.

Lastly, we return to the development of autoregressive models. Enlightened by the new concepts mentioned above, we propose some new processes whose marginals are geometrically infinite divisible, geometrically stable, discrete geometrically stable and max/min geometrically stable laws.