CHAPTER 2

A UNIFIED APPROACH FOR MODELLING STATIONARY AR PROCESSES

2.1. INTRODUCTION

As we have mentioned in Chapter 1, first order autoregressive processes are very useful in modelling stationary time series possessing the Markovian property. Various autoregressive models have been proposed in the literature by different authors. All these models, some of which we have mentioned in Chapter 1, are seemingly of different structures.

In this Chapter we look at the problem of modelling stationary Markov processes globally and give a unified approach to the problem. We define thinning and thickening operations on arbitrary random variables and propose to obtain the models either through thickening of thinned variables or thinning of thickened variables. This approach is discussed in Section 2 and it is shown that almost all autoregressive models proposed so far by various authors can be obtained using this approach. Some new models, obtained using this approach, are proposed in Section 3. These models are studied in later chapters.

2.2. THE UNIFIED APPROACH

Definition 2.2.1 When a rv X is subjected to an operation which produces a rv not greater than X, the operation is called *thinning* of X ■

Similarly when an operation on X produces a variable not smaller than X, the operation will be called *thickening* of X. An operator T will be called deterministic if the conditional distribution of T(X) given X is degenerate, otherwise the operator will be called probabilistic.

An example of deterministic thinning of a nonnegative rv is the multiplication by a number ρ, $\rho \in (0, 1)$, whereas an example of probabilistic thinning of a nonnegative rv is the multiplication by a random variable U, which has support in (0,1). Examples of deterministic and probabilistic thickening of a nonnegative random variables are analogous to the above two operations. That is, multiplying by a number θ, $\theta > 1$, and multiplying by a rv V with support in $(1,\infty)$ respectively.

The operations of thinning and thickening are very useful for developing autoregressive type models for stationary processes. In this context it is important to answer the following two questions:

(i) When does the thickening of a thinned variable reproduce the original variable (in distribution) ?

(ii) When does the thinning of a thickened variable reproduce the original variable (in distribution) ?

Let $T^{-}(X)$ denote a thinning operation on X and $T^{+}(X)$ denote a thickening operation on X. Suppose that there exists a distribution F (of X) and operators T^{-} and T^{+}

which satisfy

$$T^{+}(T^{-}(X)) \overset{d}{=} X, \qquad \ldots(2.2.1)$$

and/or

$$T^{-}(\ T^{+}(X)) \overset{d}{=} X. \qquad \ldots(2.2.2)$$

If (2.2.1) holds then a process $\{X_n\}$ defined by

$$X_n = T^{+}(\ T^{-}(X_{n-1})), \quad n = 1,\ 2,\ \ldots \qquad \ldots(2.2.3)$$

is strictly stationary with marginal distribution F whenever X_0 is distributed as F.

Similarly, if (2.2.2) holds then a process $\{X_n\}$ defined by

$$X_n = T^{-}(\ T^{+}(X_{n-1})), \quad n = 1,\ 2,\ \ldots \qquad \ldots(2.2.4)$$

is strictly stationary with marginal distribution F whenever X_0 is distributed as F.

The processes defined by (2.2.3) and (2.2.4) are autoregressive processes in a more general sense (The term "autoregressive" is used to indicate that X_n is a function of X_{n-1}). If both T^{-} and T^{+} are deterministic operators satisfying either (1) or (2) then the process will be recognized as one where all X_n's are identical and hence trivially it is stationary. That is, X_n will be x_0 for all n if X_0 is realized as x_0. Therefore any process of practical interest should have at least one of the two operators viz., T^{-} and T^{+}, probabilistic.

Some examples of thinning and thickening operators are given below.

Thinning operators :

a. $T_1^-(X) = \rho X$, Where $\rho \in (0, 1)$ is a scalar and X is a nonnegative rv.

b. $T_2^-(X) = UX$, Where X is a nonnegative rv, and U is a rv independent of X_0, having support in (0, 1).

c. $T_3^-(X) = \min(X, Y)$, Where X and Y are independent rvs.

d. $T_4^-(X) = \rho * X$, Where X is a rv with support in $\mathbb{N}_0$, and $\rho *$ is as defined in Section 1.3.

e. $T_5^-(X) = U*X$, Where U and X are independent with U having support in [0, 1] and X having support in $\mathbb{N}_0$.

f. $T_6^-(X) = X - Y$, Where X and Y are independent rvs with Y being nonnegative.

Thickening operators :

a. $T_1^+(X) = \theta X$, Where $\theta \geq 1$ is a scalar and X is a nonnegative random variable.

b. $T_2^+(X) = VX$, Where X and Y are independent nonnegative random variables with V having support in $[1, \infty)$.

c. $T_3^+(X) = \max(X, Y)$, where X and Y are independent random variables.

d $T_4^+(X) = X + Y$, Where X and Y are independent random variables with Y being nonnegative.

e. $T_5^+(X) = \sum_{i=1}^{X} Y_i$, Where Y_i's are iid random variables with support in $\mathbb{N} = \{1, 2, \ldots\}$

Remark 2.2.1 : In the probabilistic thinning/ thickening operators defined above, generally it is more convenient to take the involved rvs to be independent, although it is not a necessity ∎

We now show that most of the models proposed in the literature can be obtained as a particular form of the process defined either by (2.2.3) or by (2.2.4).

Autoregressive process (AR(1)):

If equation (2.2.1) holds with the operators T_1^- and T_4^+ then the resultant stationary process at (2.2.3) takes the form

$$X_n = \rho X_{n-1} + Y_n \ , \ n = 1, 2, \ldots;$$

which is a first order autoregressive process AR(1).

An AR process with an exponential marginal was proposed by Gaver and Lewis (1980), who also proposed the processes GAR(1) and MEAR(1) with Gamma and a mixed exponential marginals respectively.

It may be noted that the well-known AR process with Gaussian marginals can not be obtained using the approach taken in this Chapter, since neither ρX is a thinning of X nor X + Y is a thickening of X when the distributions of X and Y have $\mathbb{R}$ as a support.

AR process in a random environment

If equation (2.2.1) is satisfied by the operators T_2^- and T_4^+ then the resultant stationary process at (2.2.3) takes the form

$$X_n = U_n X_{n-1} + Y_n \ , \ n = 1, 2, \ldots;$$

which is an AR process in random environment (autoregressive coefficient being random).

Such a process with exponential marginals (NEAR(1)) was proposed by Lawrance and Lewis (1981). A process with Gamma marginals discussed in Section 1.2.4 is also of this type.

Minification process (MINAR)

If the operators T_3^- and T_1^+ satisfy the equation (2.2.1) then the resultant stationary process at (2.2.3) takes the form

$$X_n = \theta \min (X_{n-1}, Y_n) \ , \ n = 1, 2, \ldots;$$

which is a minification process (an AR process in the scheme of minima).

MINAR process was introduced by Tavares (1980), who proposed a process with exponential marginals. This type of processes were discussed in Section 1.2.2.

Extremal process ((MAXAR)

If the equation (2.2.2) is satisfied by the thinning and thickening operators T_1^- and T_3^+ then the resultant stationary process at (2.2.4) takes the form

$$X_n = \rho \max (X_{n-1}, Y_n) \ , \ n = 1, 2, \ldots$$

which is an extremal process proposed by Alpuim (1989).

Later Alpuim and Athayde (1990) proposed an extremal process under random environment, which can be obtained by replacing T_1^- in the above with T_2^-.

Integer valued autoregressive process (INAR)

If the thinning operator T_4^- and thickening operator T_4^+ satisfy the equation (2.2.1) then the resultant stationary process at (2.2.3) takes the form

$$X_n = \rho * X_{n-1} + Y_n \ , \ n = 1, 2, \ldots;$$

which is an integer valued AR process INAR(1) introduced by McKenzie (1985). This process was also proposed by Al-Osh and Alzaid (1987) independently of McKenzie. The notation INAR was coined by Al-Osh and Alzaid.

The model of Helland and Nilsen (1976)

Helland and Nilsen (1976) proposed the model

$$X_n = \mathrm{Max}(X_{n-1} - Z_n, Y_n), \ n = 1, 2, \ldots$$

to describe the water density in a still fjord as well as other phenomena such as the utility of certain industrial equipment. This model is a particular case of the process (2.2.3) with operators T_6^- and T_3^+ as thinning and thickening operators respectively.

2.3. SOME NEW MODELS

As we have noted in Section 2, many of the AR(1) type models suggested in the literature can be obtained by considering various combinations of thinning and thickening operators. We propose some new combinations of T^- and T^+ that have not been considered before. The resultant new models are as follows.

$$X_n = \rho * \max(X_{n-1}, Y_n) \ , \ n = 1, 2, \ldots$$

$$Z_n = \max \ (\rho * Z_{n-1}, W_n) \ , \ n = 1, 2, \ldots$$

$$U_n = \min(U_{n-1}, V_n) + \xi_n \ , \ n = 1, 2, \ldots$$

$$X_n = \max \ (X_n, Y_n) - Z_n, \ n = 1, 2, \ldots$$

First two of the models mentioned above are studied in Section 3.7.2 and Section 3.7.3 respectively. Necessary and sufficient conditions for stationarity of these processes are obtained. It is shown that most of the well-known discrete distributions satisfy these conditions.