Chapter 3

Testing of hypothesis for inliers in Exponential distribution

3.1. Introduction

In this chapter, we review and revisit the problem of inliers in samples and how to test the statistical significance of them. Inliers, as described in the previous chapters, are natural occurrences of a life test, and therefore, its detection is a major issue before the estimation of the model parameters. Note that, there are hardly any formal tests for inliers available. Therefore, an attempt is made to review existing test procedures for lower outliers (also called inliers). We propose tests of hypothesis consists of single and multiple inliers coming from an exponential distribution. The masking effect on Dixon type tests and Cochran type test for the case of single inliers are studied. Some of the existing tests are revisited and studied in detail. The critical values are theoretically studied and numerically computed. The power of the tests and the error probabilities for the effects of masking and swamping under outward tests are also tabulated for the number of inliers k =2 and 3.

The notion of an outliers is well known in mathematical statistics and the applied statistics. An outlier is an outstanding observation that occurs in the statistical sample (either in the positive or negative direction for one-dimensional variables or in any direction for multi-dimensional ones). An outlier is an indication of the existence

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of rare phenomena that could be a reason for further investigation. Sometimes, the mechanism generating outliers (as well as the outliers themselves) are of particular interest and can be subjects of special discussion and analysis (Barnett and Levis, 1994). Mixture models are generally a source of outlier observations, and hence the test of homogeneity is warranted. Before one tries to fit the mixture model, it would be of value to know whether the data arise from homogeneous or heterogeneous population. The rejection of the homogeneous model may also have scientific implications.

In certain situations, one may treat lower outlier as inliers for one-dimensional variables. Usually, outliers do not present any interest to the researcher and is discarded from further analysis. In inliers studies, the observations are retained completely for future analysis and inferences. Usually, the inliers situations are handled by modifying commonly used parametric models suitably incorporating inconsistent observations. The modified model with underlying distribution exponential has the form

$$g(x;p,\theta) = \begin{cases} 0, & x < d \\ 1 - pe^{-\frac{d}{\theta}}, & x = d \\ \frac{p}{\theta} e^{-\frac{x}{\theta}}, & x > d \end{cases} \qquad (3.1.1)$$

In Section 3.2, the testing of hypothesis for inlier model parameters (p, θ) is proposed. In Section 3.3 general problem of testing of hypothesis about inliers is discussed and some of the existing tests are revisited and studied in detail, which includes review of the block test constructed through identified inlier model, labeled slippage model and the outward procedure for detecting inliers. A discussion of data descriptions with inlier proness are also included in the Section 3.4. In Section 3.5, we present the masking effect in Dixon type test and Cochran type test for inliers. Simulation study is carried out to study performance of the test on the power and masking effects. Also offers an extensive Monte Carlo study to investigate the powers and the error probabilities for the effects of masking and swamping in the outward test when the number of inliers is more than one.

3.2. Testing of hypothesis for parameters $(\boldsymbol{p}, \boldsymbol{\theta})$

The most powerful (MP) test and locally most powerful (LMP) test for parameter p and θ of the density function (3.1.1) is derived in this section by referring to the test procedure given in Section 1.6.

Most Powerful test for *p*: Using Neyman-Pearson lemma, the MP test for testing $H_0: p = 1$ against $H_1: p < 1$ of size α is obtained as

$$\Phi_1(r) = \begin{cases} 1, & r < n \\ \alpha, & r = n \\ 0, & r > n \end{cases} \tag{3.2.1}$$

The above test is also uniformly most powerful similar test of size α' with power function $\beta(p) = 1 - (1 - \alpha)p^n$, $\beta(p)$ can be computed numerically for any combination of n, p and α.

Locally Most Powerful test for *p*: The LMP test of size α for testing $H_0: p = 1$ against $H_1: p < 1$ for θ known based on n iid observations from the density $g(x; p, \theta)$ according to Ferguson (1967) is given by

$$\Phi_2(x) = \begin{cases} 1, & re^{\frac{d}{\theta}} - np < \; c_\alpha\left(pe^{\frac{d}{\theta}} - p^2\right) \\ \gamma, & re^{\frac{d}{\theta}} - np = \; c_\alpha\left(pe^{\frac{d}{\theta}} - p^2\right) \\ 0, & re^{\frac{d}{\theta}} - np > \; c_\alpha\left(pe^{\frac{d}{\theta}} - p^2\right) \end{cases} \tag{3.2.2}$$

where c_α and γ are such that $E_{H_0}[\Phi_2(x)] = \alpha$.

Most Powerful test for $\boldsymbol{\theta}$: The MP test for $H_0: \theta = \theta_0$ against $H_1: \theta = \theta_1$, $(\theta_1 > \theta_0)$ for p known is given by

$$\Phi_3(x) = \begin{cases} 1, & \sum_{x_i > d} x_i > \dfrac{\left\{c_\alpha + (n-r)\left[log\left(1 - pe^{\frac{d}{\theta_0}}\right) - log\left(1 - pe^{\frac{d}{\theta_1}}\right)\right] - r(log\,\theta_0 - log\,\theta_1)\right\}}{\left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right)} \\ \gamma, & \sum_{x_i > d} x_i = \dfrac{\left\{c_\alpha + (n-r)\left[log\left(1 - pe^{\frac{d}{\theta_0}}\right) - log\left(1 - pe^{\frac{d}{\theta_1}}\right)\right] - r(log\,\theta_0 - log\,\theta_1)\right\}}{\left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right)} \\ 0, & \sum_{x_i > d} x_i < \dfrac{\left\{c_\alpha + (n-r)\left[log\left(1 - pe^{\frac{d}{\theta_0}}\right) - log\left(1 - pe^{\frac{d}{\theta_1}}\right)\right] - r(log\,\theta_0 - log\,\theta_1)\right\}}{\left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right)} \end{cases} \tag{3.2.3}$$

where c_α and γ are such that $E_{H_0}[\Phi_3(x)] = \alpha$.

As the evaluation of c_α and γ depends on the value of θ under H_1, the same test is not uniformly most powerful (UMP) for the hypothesis $\theta > \theta_0$. But it is possible to obtain locally most powerful (LMP) test for θ.

Locally Most Powerful test for $\boldsymbol{\theta}$: The LMP test of size α for testing $H_0: \theta \leq \theta_0$ against $H_1: \theta > \theta_0$ for p known based on n iid observations from the density $g(x; p, \theta)$ according to Ferguson(1967) is given by

$$\Phi_4(x) = \begin{cases} 1, & \sum_{x_i > d} x_i > \theta_0^2 c_\alpha + \dfrac{(n-r)d\,p}{\left(e^{\frac{d}{\theta_0}} - p\right)} + r\theta_0 \\ \gamma, & \sum_{x_i > d} x_i = \theta_0^2 c_\alpha + \dfrac{(n-r)d\,p}{\left(e^{\frac{d}{\theta_0}} - p\right)} + r\theta_0 \\ 0, & \sum_{x_i > d} x_i < \theta_0^2 c_\alpha + \dfrac{(n-r)d\,p}{\left(e^{\frac{d}{\theta_0}} - p\right)} + r\theta_0 \end{cases} \tag{3.2.4}$$

where c_α and γ are such that $E_{H_0}[\Phi_4(x)] = \alpha$.

3.3. Test of hypothesis about multiple inliers

In this section first we describe the problem of testing of hypothesis about inliers. Some of the existing tests are revisited and studied in detail. The critical values are theoretically studied and numerically computed. For single inlier, the masking effect is computed through a simulation study for the Dixon type test and the Cochran type test. We carry out a Monte Carlo study for evaluating the powers as well as the effect of masking and swamping in the outward test when k = 2 and 3.

3.3.1. The models and problem of testing of hypothesis

In the literature related to the detection of inliers/contaminants, several traditional tests have been proposed for testing the null hypothesis that all observations in the sample have come from a single distribution against the alternative that a small subset of observations in the sample have come from a distribution which is different from the one under the null hypothesis. See Kale and Muralidharan (2007) and the details therein. For multiple inliers, one suspects the lower k order statistics $X_{(1)}$, $X_{(2)}, \dots, X_{(k)}$ as inliers where k is often assumed known. The inliers are regarded as arising from a different population $\mathbb{Q}$ with a DF $G(x) \in \mathcal{g}$ and pdf $g(x)$, and since smaller observations are suspected it is assumed that $\mathbb{Q}$ is stochastically smaller than the population $\wp$ or $G < F$. Thus the sample $(X_1, X_2, \dots, X_n)$ is such that $(n-k)$ of these are distributed as F while k are distributed as G. Some of the existing inlier generating models can be viewed as follows:

> The *identified inliers model* of Veale (1975) is described in the following way: Suppose it is assumed that the given data have samples $(X_{i_1}, X_{i_2}, \dots, X_{i_k})$ coming from a DF G and $(X_{i_{k+1}}, X_{i_{k+2}}, \dots, X_{i_n})$ are from DF F, where $(i_1, i_2, \dots, i_n)$ is a permutation of $(1,2, \dots, n)$, and the indexing set $I_k =$

$(i_1, i_2, \dots, i_k)$ is assumed known. Then the model has k inliers. Ferguson (1961) also used this model except that I_k is not assumed known but could be any of the $\binom{n}{k}$ such sets and call it as *Ferguson type model*. This model is further used by Kale (1975), where all $\binom{n}{k}$ indexing sets are apriori assumed equally likely. Hence the model is also called *Exchangeable Model*. In case one assumes that k is not fixed and is, in fact, a random variable distributed as $b(n, p)$, then from the exchangeable model one can get the classical *Mixture Model* (Tukey, 1960, Guttman, 1973), where $(X_1, X_2, \dots, X_n)$ is regarded as a random sample from a population with DF $H = (1-p)G + pF, 0 \le p \le 1$.

Barnett and Lewis (1994) have pointed out that none of the models given above take into consideration the fact that usually $(X_{(1)}, X_{(2)}, \dots, X_{(k)})$ the smallest k observations are suspected as inliers and tested for discordancy. However, for the exchangeable model mentioned above, Kale (1975) has proved that if $\frac{dG}{dF} = \psi(x)$ is strictly decreasing, then $(X_{(1)}, X_2, \dots, X_{(k)})$ have maximum probability of being the inlier observations i.e. the set of observations distributed as G. It may be also noted that if $\frac{dG}{dF} = \psi(x)$ is strictly decreasing then the population $\mathbb{Q}$ is stochastically smaller than the population $\wp$. To emphasize the special role that lower k order statistics play in these discordancy tests, Barnett and Lewis (1994), have introduced the *labeled slippage model*. The testing for discordancy of inlier observations is now formed as the test of hypotheses problem H_0 against H_1 where H_0 and H_1 can be roughly be formulated as

$$H_0: X_{(1)}, X_{(2)}, \dots, X_{(n)} \text{ from } F \in \mathcal{F}$$

against

$$H_1: X_{(1)}, X_{(2)}, \dots, X_{(k)} \text{ from } G \in \mathcal{G} \text{ and } X_{(k+1)}, X_{(k+2)}, \dots, X_{(n)} \text{ from } F \in \mathcal{F}. \quad (3.3.1)$$

For statistical inliers test, one needs to have not only high power and computational tractability but also to estimate the number of inliers k as well. Masking and swamping effects are impediments to this task. We will be investigating these problems for various tests in subsequent sections.

In literature, two types of tests have been proposed for the lower outlier testing problem. One is the "Block test" (Chikkagoudar and Kunchur, 1983, Kimber and Stevens, 1981, Lewis and Feller, 1979), which is often used to test for discordancy of k lower outliers in the data, and then either k or 0 lower outliers are declared lower outliers in a single hypothesis test. This procedure suffers from masking and swamping effect when too many or too fewer k inliers present in the sample. The other one is the sequential procedure: Sequential identifiers can be further classified to *inward* and *outward* procedures. In the inward procedure, one start with the full sample and single-inlier test applied repeatedly, by starting with most-smallest observation, deleting discordant value at each stage and then applying the test again to the reduced sample. The process continued until a non-significant result is obtained. The estimated number of inliers is $\hat{k}$, is the number of rejected (marginal) tests. In addition to certain theoretical weakness, this inward sequential procedure is not recommended by Kimber (1982) and Chikkagoudar and Kunchur (1983) due to the fact that the presence of two or more lower outliers may well lead to a non-significant result at the very first stage. Due to the limitations in the Block test and the inward sequential procedure suffering from swamping and/or masking effects, Rosner (1975) suggested to apply an outward sequential procedure also called "inside-out" sequential procedure to the reduced sample. Here, one specifies a maximum number of inliers k, then the k^{th} smallest inlier is being tested first; if this gives a significant result, then k inliers are declared to be discordant; if a non-significant result is obtained, then test the $(k-1)^{th}$ smallest inliers and so on. This process is continued until either a significant result obtained or no inliers can be declared as discordant. This procedure minimizes the probability and magnitude of both masking and swamping effects. As such, the outward procedure has been claimed superior over the inward (see Kimber, 1982, Chikkagoudar and Kunchur, 1983, Balasooriya and Gadag, 1994). However, control of type-I error (the probability of a false alarm) is difficult in the outward procedure. Larger the value of k, more the power it loses. This outward sequential procedure for testing up to k inliers (lower outliers) in an exponential sample is proposed for $k = 2, 3$ and 4 by Kimber (1982) and a Monte-Carlo implementation was recommended by Lin and Balakrishnan (2009). For the inward sequential procedure, the type-I error level is equal to the marginal level ($\alpha = \beta$). This is because a rejection of the null only happens when the first marginal test is rejected.

3.3.2. Block test for testing k inliers using M_k model

We consider two different inlier prone(s) models to detect inliers in Block test. One is Identified inlier (M_k) model and another is Labeled slippage inlier (L_k) model. Here we describe M_k model as follows: Consider a random sample of size n, let n_0 units fails instantaneously and $(n - n_0)$ failure time is available. Suppose $X_1, X_2, \dots, X_{n-n_0}$ are the available lifetimes and $X_{(1)}, X_{(2)}, \dots, X_{(n-n_0)}$ be the corresponding order statistics. Out of these $(n - n_0)$ positive observations we have to determine which are inliers or early failures. For the null hypothesis H_0, let all the X_i are independent and identically distributed exponential with density function $f(x;\theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}$ whereas the alternative hypothesis H_1, is a labeled scale-slippage alternative such that the j observations have arisen from exponential density with pdf $g(x;\phi) = \frac{1}{\phi} e^{-\frac{x}{\phi}}; \phi < \theta$. Suppose we want to test the hypothesis (3.3.1) for single inlier, then the test is equivalent to testing $H_0: k = 0$ versus $H_1: k = 1$ then the likelihood ratio test for one inlier using M_k model is obtained as follows:

Let $M_{10}(n_0, \underline{x}|\theta)$ and $M_{11}(n_0, \underline{x}|\phi, \theta)$ denote the likelihood function under H_0 and H_1 respectively then the likelihood ratio is

$$\Lambda_1(\underline{x}) = \frac{\underset{\theta}{Max}\, M_{10}(n_0, \underline{x}|\theta)}{\underset{\theta,\phi}{Max}\, M_{11}(n_0, \underline{x}|\phi, \theta)},$$

where,

$$M_{10}(n_0, \underline{x}|\theta) = A\frac{(n-n_0)!}{\theta^{(n-n_0)}} e^{-\frac{1}{\theta}\sum_{i=1}^{n-n_0} x_{(i)}},$$

$$M_{11}(n_0, \underline{x}|\phi, \theta) = A\frac{1}{\phi}\frac{(n-n_0-1)!}{\theta^{(n-n_0-1)}} e^{-\left(\frac{x_{(1)}}{\phi}+\frac{1}{\theta}\sum_{i=2}^{n-n_0} x_{(i)}\right)},$$

and

$$A = \binom{n}{n_0}(1-p)^{n_0}p^{n-n_0}, p = P(X > 0). \qquad (3.3.2)$$

Then the likelihood ratio test reject H_0 if

$$\Lambda_1(\underline{x}) = \frac{1}{\left(\frac{1}{n-n_0}\right)\left(1-\frac{1}{n-n_0}\right)^{n-n_0-1}}\left(1-\frac{x_{(1)}}{\sum_{i=1}^{n-n_0} x_{(i)}}\right)^{n-n_0-1}\left(\frac{x_{(1)}}{\sum_{i=1}^{n-n_0} x_{(i)}}\right) < constant.$$

This is equivalent to reject H_0 if $\frac{x_{(1)}}{\sum_{i=1}^{n-n_0} x_{(i)}} < C_1(n-n_0, \alpha)$, where the value of $C_1(n-n_0, \alpha)$ is such that $P_{H_0}\left(\frac{x_{(1)}}{\sum_{i=1}^{n-n_0} x_{(i)}} < C_1(n-n_0, \alpha)\right) = \alpha$. The value of $C_1(n-n_0, \alpha)$ is obtained as

$$C_1(n-n_0, \alpha) = \frac{1-(1-\alpha)^{\frac{1}{n-n_0-1}}}{(n-n_0)}. \qquad (3.3.3)$$

Kale and Muralidharan (2007) call this test as Cochran type test as it is analogous to the test based on $\frac{x_{(1)}}{\sum_{i=1}^{n-n_0} x_{(i)}}$ derived by Cochran (1941) to test the largest of a set of variances as a fraction of that total in analysis of variance problems. The power of the test for one inlier is

$$P_1(\lambda) = 1 - \left(\frac{1-(n-n_0)\, C_1(n-n_0,\alpha)}{1-(1-\lambda)C_1(n-n_0,\alpha)}\right)^{n-n_0-1}, \lambda = \frac{\theta}{\phi}. \quad (3.3.4)$$

In general, for testing the hypothesis given in (3.3.1), the likelihood ratio reject H_0 if

$$\Lambda_k(\underline{x}) = \frac{1}{\left(\frac{k}{n-n_0}\right)^k \left(1-\frac{k}{n-n_0}\right)^{n-n_0-k}} \left(1 - \frac{\sum_{i=1}^{k} x_{(i)}}{\sum_{i=1}^{n-n_0} x_{(i)}}\right)^{n-n_0-k} \left(\frac{\sum_{i=1}^{k} x_{(i)}}{\sum_{i=1}^{n-n_0} x_{(i)}}\right)^k < constant.$$

This is equivalent to reject H_0 if $\left(\frac{\sum_{i=1}^{k} x_{(i)}}{\sum_{i=1}^{n-n_0} x_{(i)}}\right)^k < C_k$, where C_k is such that

$$P_{H_0}\left[T_k < C_k^{1/k}\right] = \alpha \text{ and } T_k = \frac{\sum_{i=1}^{k} x_{(i)}}{\sum_{i=1}^{n-n_0} x_{(i)}}.$$

Further, the null distribution of T_k was given by Lewis and Feller (1979), for $k = 1, 2$. Also, Zhang (1998) obtained the null distribution function for T_k for all possible values of k, and is given by

$$F_k(t) = 1 - \sum_{i=0}^{A_2} (-1)^i \binom{n-n_0}{i} \binom{n-n_0-i-1}{k-i-1} \left(1 - \frac{n-n_0-i}{k-i} t\right)^{n-n_0-1}$$

$$\left(\frac{k-i}{n-n_0-k}\right)^{k-1}, 0 \le t < \frac{k}{n-n_0} \text{ and } A_2 = n - n_0 - 1 - \left[\frac{n-n_0-k}{1-t}\right].$$

The critical values are tabulated for $k \le 5$ inliers for several positive observations $m \le 200$ in Table 3.1. As T_k is a Block-test statistics for k inliers, k must be specified in advance. Here, making the wrong choice of k may result in masking or swamping effects. According to, Zhang (1998) the value of k is determined this way: Let t_k be the observed value of T_k, and the p-value value of T_k is $F_k(t_k)$, then choose that k corresponds to the minimum p-value. We illustrate the way to determine k in Section 3.4.

Table 3.1. *Values of $C^{1/k}$ for k inliers in the Block test for 1% and 5% level of significance*

m	k =1		k =2		k =3		k =4		k =5	
$= n - n_0$	1%	5%	1%	5%	1%	5%	1%	5%	1%	5%
5	5.018860e-04	0.002548291								
6	3.346747e-04	0.001701036	0.008451536	0.019796390						
7	2.390934e-04	0.001216064	0.005850077	0.013760330	0.025999870	0.047160950				
8	1.793415e-04	0.000912604	0.004292071	0.010130740	0.018591871	0.033959730	0.049529630	0.078795900		
9	1.395004e-04	0.000710128	0.003287001	0.007763885	0.013985021	0.025631480	0.036410040	0.058429360		
10	1.116081e-04	0.000568304	0.002596096	0.006145675	0.010919380	0.020042980	0.027926380	0.045076610	0.057320540	0.084785090
11	9.132079e-05	0.000465109	0.002100122	0.004986026	0.008720226	0.016106910	0.022115370	0.035866840	0.044758260	0.066700960
12	7.610414e-05	0.000387681	0.001734729	0.004121043	0.007142351	0.013228010	0.017951580	0.029227340	0.035934500	0.053861175
13	6.439826e-05	0.000328101	0.001457940	0.003467500	0.005959873	0.011059097	0.014870620	0.024283130	0.029555860	0.044443990
14	5.520029e-05	0.000281276	0.001242765	0.002957074	0.005062475	0.009384657	0.012523558	0.020498500	0.024695860	0.037308390
15	4.784157e-05	0.000243807	0.001071292	0.002553099	0.004335010	0.008063012	0.010689996	0.017537160	0.020973590	0.031775950
16	4.186237e-05	2.133571e-04	0.000933129	0.002223413	0.003757558	0.007002812	0.009230559	0.015175570	0.018028870	0.027387978
17	3.693816e-05	1.882763e-04	0.000820537	0.001955950	0.003290900	0.006139070	0.008054026	0.013262160	0.015676900	0.023856470
18	3.283453e-05	1.673725e-04	0.000727382	0.001733389	0.002905530	0.005426017	0.007089150	0.011689480	0.013745853	0.020968024
19	2.937874e-05	1.497669e-04	0.000650052	0.001547034	0.002584609	0.004830412	0.006289140	0.010382060	0.012159950	0.018580398
20	2.644126e-05	1.348003e-04	0.000581827	0.001388739	0.002313983	0.004327811	0.005614851	0.009281460	0.010829331	0.016572040
21	2.392336e-05	1.219704e-04	0.000525410	0.001254838	0.002084609	0.003900911	0.005052013	0.008348919	0.009709380	0.014877570
22	2.174877e-05	1.108890e-04	0.000476860	0.001138104	0.001888079	0.003532416	0.004561006	0.007548520	0.008753380	0.013429570
23	1.985779e-05	1.012521e-04	0.000434495	0.001037313	0.001715917	0.003214820	0.004139006	0.006859643	0.007936430	0.012184310
24	1.820315e-05	9.281910e-05	0.000397344	0.000949210	0.001568214	0.002938020	0.003774913	0.006259572	0.007229543	0.011104910
25	1.674705e-05	8.539753e-05	0.000364817	0.000871935	0.001437187	0.002695329	0.003455392	0.005735666	0.006603667	0.010161578
26	1.545895e-05	7.883186e-05	0.000336439	0.000803930	0.001323587	0.002481703	0.003176011	0.005274965	0.006069596	0.009335029
27	1.431395e-05	7.299534e-05	0.000311142	0.000743575	0.001221821	0.002292595	0.002929996	0.004868161	0.005589540	0.008605209
28	1.329162e-05	6.778387e-05	0.000288329	0.000660174	0.001132613	0.002124155	0.002711166	0.004505863	0.005162540	0.007958023
29	1.237504e-05	6.311126e-05	0.000268110	0.000641435	0.001051153	0.001973807	0.002521104	0.004182970	0.004784037	0.007383906
30	1.155011e-05	5.890570e-05	0.000249959	0.000597997	0.000979049	0.001838673	0.002341645	0.003893605	0.004447037	0.006864456
35	8.444412e-06	4.307111e-05	0.000182002	0.000435504	0.000710440	0.001333990	0.001691068	0.002815284	0.003197037	0.004948000
40	6.441693e-06	3.285870e-05	0.000138276	0.000331134	0.000537383	0.001011818	0.001275565	0.002129950	0.002407617	0.003733817
45	5.075348e-06	2.589061e-05	0.000108794	0.000260330	0.000421384	0.000793759	0.000998294	0.001667699	0.001879186	0.002918090
50	4.101757e-06	2.092508e-05	0.876260e-04	0.000209973	0.000339058	0.000639218	0.000801564	0.001341018	0.001507146	0.002343145
60	2.838836e-06	1.448333e-05	6.055116e-05	0.000144932	0.000233422	0.000440119	0.000549979	0.000921268	0.001031553	0.001606352
70	2.080663e-06	1.061578e-05	4.420840e-05	1.060109e-04	0.000170226	0.000321386	0.000400630	0.000671690	0.000750113	0.001169427
80	1.590142e-06	8.113393e-06	3.374516e-05	8.092020e-05	0.000129790	0.000244950	0.000304788	0.000511350	0.000569997	0.000889469
90	1.254653e-06	6.401812e-06	2.657839e-05	6.375963e-05	0.000102068	0.000192857	0.000239635	0.000402239	0.000447659	0.000698954
100	1.015134e-06	5.179799e-06	6.58352e-06	5.154101e-05	8.241998e-5	0.000155775	0.000193343	0.000324668	0.000360899	0.000563787

3.3.3. Test of hypothesis for single inlier using L_k model

We use the L_k model as described by Kale and Muralidharan (2007). As usual, suppose the sample have $(n-n_0)$ failure times. Before the start of the experiment we do not know which units will fail instantaneously or will produce inliers. These experimental conditions using same setup given in M_k model are to be modeled in $L_k$inlier model for given k. The joint pdf of $(X_1, X_2, \dots, X_{n-n_0})$ under L_k model can be written as

$$L_k(\underline{x}|n_0, F, G) = A\frac{(n-n_0-k)!\,k!}{\varphi(1,2,\dots,k)}\prod_{i=1}^{k} g(x_{(i)})\prod_{i=k+1}^{n-n_0} f(x_{(i)}),\ \ F\in\mathcal{F}, G\in\mathcal{G} \tag{3.3.5}$$

where A is as given in (3.3.2) and $\varphi(1,2,\dots,k)$ is the normalizing constant and can be interpreted as the probability that the k smallest observations correspond to the order statistics of k inlier observations coming from G and the remaining observations coming from F under the exchangeable model. If the underlying density is exponential then it is shown that

$$\varphi(1,2,\dots,k) = \frac{(n-n_0-k)\theta}{\phi}\beta\left(k+1, \frac{(n-n_0-k)\theta}{\phi}\right) \tag{3.3.6}$$

and in order to test the hypothesis shown in (3.3.1), with $k=1$, the joint pdf under H_0 say $L_{10}(n_0, \underline{x}|\theta)$ and under H_1 say $L_{11}(n_0, \underline{x}|\phi, \theta)$ are respectively given by

$$L_{10}(n_0, \underline{x}|\theta) = A\,(n-n_0)!\ \ \theta^{n-n_0}\ \ e^{-\frac{1}{\theta}\sum_{i=1}^{n-n_0} x_{(i)}}$$

and

$$L_{11}(n_0, \underline{x}|\phi, \theta) = \ A\frac{(n-n_0-1)!}{\varphi(1)}\frac{1}{\phi(\theta^{n-n_0-1})}\ e^{-\frac{x_{(1)}}{\phi}}\ e^{-\frac{1}{\theta}\sum_{i=2}^{n-n_0} x_{(i)}}$$

where $\varphi(1) = \frac{\theta}{[(n-n_0-1)\phi+\theta]}$, $p = P(X > 0)$ and n_0 in both cases is $b(n, 1-p)$.

Note that the MLE of parameters ϕ and θ under H_1 is

$$\hat{\phi} = \left(\frac{1}{x_{(1)}} - \frac{n-n_0-1}{\hat{\theta}}\right)^{-1} \text{ and } \hat{\theta} = \frac{\sum_{i=2}^{n-n_0}\left(x_{(i)}-x_{(1)}\right)}{n-n_0-1}$$

respectively.

The likelihood ratio test is to reject H_0 if

$$\Lambda_1'(\underline{x}) = \frac{\underset{\theta}{Max} L_{10}(n_0, \underline{x}|\theta)}{\underset{\theta,\phi}{Max} L_{11}(n_0, \underline{x}|\phi, \theta)}$$

$$= \frac{(n-n_0)}{\left(1-\frac{1}{n-n_0}\right)^{n-n_0-1}}\left(1 - \frac{(n-n_0)\, x_{(1)}}{\sum_{i=1}^{n-n_0} x_{(i)}}\right)^{n-n_0-1}\left(\frac{(n-n_0)\, x_{(1)}}{\sum_{i=1}^{n-n_0} x_{(i)}}\right) < constant$$

This is equivalent to reject H_0 if

$$\frac{(n-n_0)x_{(1)}}{\sum_{i=1}^{n-n_0} x_{(i)}} < C_1'(n-n_0, \alpha),$$

where $C_1'(n-n_0, \alpha)$ is such that

$$P_{H_0}\left[\frac{x_{(1)}}{\sum_{i=1}^{n-n_0} x_{(i)}} < C_1(n-n_0, \alpha)\right] = \alpha \text{ and } C_1(n-n_0, \alpha) = \frac{C_1'(n-n_0,\alpha)}{(n-n_0)}.$$

This test is analogous to the test based on $\frac{x_{(1)}}{\sum_{i=1}^{n-n_0} x_{(i)}}$ derived by Cochran (1941) to that the largest of a set of variances as a fraction of the total in analysis of variance problem.

The value of $C_1'(n-n_0,\alpha)$ is obtained as

$$C_1'(n-n_0,\alpha) = 1-(1-\alpha)^{\frac{1}{n-n_0-1}}. \tag{3.3.7}$$

The power of the test corresponds to one inlier is obtained as

$$P_1(\lambda) = 1-\left(\frac{(n-n_0)\left(1-C_1'(n-n_0,\alpha)\right)}{(n-n_0)-(1-\lambda)C_1'(n-n_0,\alpha)}\right)^{n-n_0-1}, \lambda = \frac{\theta}{\phi}. \tag{3.3.8}$$

Note that Likelihood ratio test under L_k model and M_k model for the case of single inlier results are same, but differs for the case of more than one inliers. The likelihood ratio test under L_k model for the case of more than one inliers is in progress.

3.3.4. Outward sequential procedure for testing hypothesis for k inliers

As usual let n_0 units fails instantaneously and $(n-n_0)$ failure time is available. For the null hypothesis H_0, let all X_i are independent and identically distributed with an exponential density $f(x;\theta)$. The alternative hypothesis is H_1, is a labeled scale-slippage alternative such that the j smallest observations have arisen from exponential density with pdf $g(x;\frac{\theta}{\lambda})$ with $\lambda > 1$. Then the test statistics

$$S_j = \frac{x_{(j+1)}}{\sum_{i=1}^{j+1} x_{(i)}}, j = 1,2,\ldots,(n-n_0-1)$$

is an appropriate statistic for testing j lower outliers (inliers). If only the first j order statistics are available, then S_j is the likelihood ratio test statistics.

The size α test for up to k inliers is detected as follows:

(i) If $S_k > s_{k,(n-n_0)}$, declare the k smallest observations as inliers;

(ii) If $S_i < s_{i,(n-n_0)}$ $(i = k, k-1, \dots, l+1)$ and $S_l > s_{l,(n-n_0)}$, declare the l smallest observations as inliers $(l = k-1, k-2, \dots 1)$;

(iii) If $S_j < s_{j,(n-n_0)}$ $(j = 1, 2, \dots, k)$ declare no observations as inliers in the data.

Here $s_{j,(n-n_0)} = s_{j,(n-n_0)}(\alpha), j = 1,2,\dots,k$ are obtained such that all marginal tests to have equal level β (say), that is

$$P_{H_0}\left(S_k > s_{k,(n-n_0)}\right) = P_{H_0}\left(S_{k-1} > s_{k-1,(n-n_0)}\right) = \cdots = P_{H_0}\left(S_1 > s_{1,(n-n_0)}\right) = \beta.$$

and the level β is determined such that

$$P_{H_0}\left\{\cap_1^k\left(S_j < s_{j,(n-n_0)}\right)\right\} = 1 - \alpha, \text{ where } \alpha \text{ is the level of significance.}$$

The null joint density of $S_1, S_2, \dots, S_k$ is

$$f_k(s_1, s_2, \dots, s_k) = \frac{n!k!}{(n-n_0-k-1)!}\left(1 + (n - n_0 - k - 1)s_k\right)^{-(k-1)} \prod_{j=2}^{k}\left(1 - s_j\right)^{j-1},$$

over the region $W_k = \left\{\frac{1}{2} < s_1 < 1;\ \frac{s_j}{1+s_j} < s_{j+1} < 1,\ j = 1,2,\dots,k-1\right\}$.

The outward sequential procedure test S_j has the advantages over the usual Block test, $T_k = \frac{\sum_{i=1}^{k} x_{(i)}}{\sum_{i=1}^{n-n_0} x_{(i)}}$ of being little influenced by behavior of the smallest observations. The critical values are tabulated for k inliers for several positive observations $m \leq 200$ in Table 3.2, 3.3, 3.4 and 3.5 respectively. Note that for k =1, the critical value at α level of significance is

$$s_{1,(n-n_0)} = \frac{(n-n_0)-\alpha}{\alpha(n-n_0-2)+(n-n_0)}.$$

Table 3.2. *Critical values for the outward sequential procedure for up to k =2 inliers and associated values of β*

$m = n - n_0$	Level of significance								
	1%			5%			10%		
	$s_{2,m}$	$s_{1,m}$	β	$s_{2,m}$	$s_{1,m}$	β	$s_{2,m}$	$s_{1,m}$	β
4	0.943782	0.996245	0.005019	0.877639	0.981130	0.025481	0.829705	0.962006	0.051975
5	0.935666	0.995996	0.005020	0.862372	0.979897	0.025513	0.810766	0.959579	0.052106
6	0.930625	0.995830	0.005021	0.853143	0.979078	0.025534	0.799525	0.957968	0.052194
7	0.927186	0.995711	0.005022	0.846951	0.978493	0.025549	0.792070	0.956820	0.052256
8	0.924689	0.995622	0.005022	0.842507	0.978054	0.025561	0.786761	0.955962	0.052303
9	0.922792	0.995553	0.005022	0.839162	0.977714	0.025570	0.782785	0.955296	0.052340
10	0.921302	0.995498	0.005023	0.836552	0.977442	0.025577	0.779697	0.954763	0.052369
11	0.920101	0.995452	0.005023	0.834458	0.977220	0.025583	0.777228	0.954328	0.052393
12	0.919112	0.995415	0.005023	0.832741	0.977034	0.025588	0.775209	0.953966	0.052413
13	0.918283	0.995383	0.005023	0.831307	0.976878	0.025592	0.773527	0.953659	0.052430
14	0.917579	0.995355	0.005023	0.830093	0.976743	0.025595	0.772104	0.953397	0.052444
15	0.916973	0.995333	0.005023	0.829050	0.976627	0.025598	0.770884	0.953170	0.052457
16	0.916446	0.995311	0.005023	0.828145	0.976524	0.025601	0.769828	0.952971	0.052468
17	0.915983	0.995293	0.005024	0.827352	0.976435	0.025603	0.768903	0.952796	0.052477
18	0.915574	0.995277	0.005024	0.826652	0.976355	0.025605	0.768087	0.952640	0.052486
19	0.915209	0.995262	0.005024	0.826030	0.976283	0.025607	0.767362	0.952500	0.052493
20	0.914882	0.995249	0.005024	0.825472	0.976219	0.025609	0.766713	0.952375	0.052500
21	0.914588	0.995237	0.005024	0.824970	0.976161	0.025610	0.766130	0.952262	0.052507
22	0.914320	0.995226	0.005024	0.976109	0.824515	0.025612	0.765601	0.952159	0.052512
23	0.914077	0.995216	0.005024	0.824101	0.976060	0.025613	0.765121	0.952064	0.052517
24	0.913855	0.995207	0.005024	0.823723	0.976016	0.025614	0.764683	0.951978	0.052522
25	0.913651	0.995199	0.005024	0.823377	0.975975	0.025615	0.764281	0.951899	0.052527
26	0.913463	0.995191	0.005024	0.823058	0.975938	0.025616	0.763911	0.951826	0.052531
27	0.913289	0.995184	0.005024	0.822763	0.975903	0.025617	0.763570	0.951758	0.052534
28	0.913128	0.995178	0.005024	0.822490	0.975871	0.025618	0.763254	0.951695	0.052538
29	0.912978	0.995172	0.005024	0.822236	0.975841	0.025619	0.762960	0.951637	0.052541
30	0.912839	0.995166	0.005024	0.822001	0.975813	0.025620	0.762687	0.951581	0.052544
35	0.912263	0.995142	0.005024	0.821027	0.975697	0.025623	0.761562	0.951355	0.052557
40	0.911834	0.995124	0.005024	0.820303	0.975609	0.025625	0.760725	0.951186	0.052566
45	0.911501	0.995111	0.005024	0.819743	0.975542	0.025627	0.760079	0.951054	0.052573
50	0.911236	0.995100	0.005025	0.819297	0.975487	0.025628	0.759564	0.950948	0.052579
60	0.910840	0.995083	0.005025	0.818631	0.975406	0.025630	0.758797	0.950790	0.052588
70	0.910558	0.995071	0.005025	0.818158	0.975348	0.025632	0.758252	0.950677	0.052594
80	0.910347	0.995062	0.005025	0.817804	0.975305	0.025633	0.757844	0.950592	0.052599
90	0.910183	0.995055	0.005025	0.817530	0.975271	0.025634	0.757529	0.950527	0.052602
100	0.910052	0.995050	0.005025	0.817311	0.975244	0.025635	0.757277	0.950474	0.052605
120	0.909856	0.995041	0.005025	0.816983	0.975203	0.025636	0.756900	0.950395	0.052610
140	0.909717	0.995035	0.005025	0.816749	0.975174	0.025636	0.756632	0.950339	0.052613
160	0.909612	0.995031	0.005025	0.816574	0.975152	0.025637	0.950296	0.756431	0.052615
180	0.909531	0.995028	0.005025	0.816438	0.975135	0.025637	0.756275	0.950263	0.052617
186	0.909510	0.995027	0.005025	0.816403	0.97513	0.025637	0.756235	0.950255	0.052617
200	0.909466	0.995025	0.005025	0.816330	0.975122	0.025638	0.756150	0.950237	0.052618

Table 3.3. *Critical values for the outward sequential procedure for up to k =3 inliers and associated values of β*

$m = n - n_0$	Level of significance											
	1%				5%				10%			
	$s_{3,m}$	$s_{2,m}$	$s_{1,m}$	β	$s_{3,m}$	$s_{2,m}$	$s_{1,m}$	β	$s_{3,m}$	$s_{2,m}$	$s_{1,m}$	β
6	0.850980	0.942745	0.997210	0.003355	0.760280	0.877025	0.985825	0.017205	0.707572	0.830184	0.971099	0.035502
7	0.839013	0.939829	0.997130	0.003357	0.744618	0.871478	0.985405	0.017238	0.690906	0.823187	0.970221	0.035626
8	0.830791	0.937706	0.997069	0.003358	0.734092	0.867476	0.985089	0.017262	0.679837	0.818171	0.969561	0.035719
9	0.824787	0.936091	0.997022	0.003359	0.726521	0.864451	0.984843	0.017281	0.671939	0.814397	0.969047	0.035791
10	0.820208	0.934821	0.996984	0.003359	0.720810	0.862085	0.984646	0.017296	0.666015	0.811454	0.968636	0.035849
11	0.816598	0.933796	0.996954	0.003360	0.716346	0.860183	0.984484	0.017309	0.661405	0.809094	0.968299	0.035895
12	0.813679	0.932951	0.996928	0.003360	0.712760	0.858620	0.984350	0.017319	0.657714	0.807160	0.968018	0.035934
13	0.811268	0.932243	0.996907	0.003361	0.709815	0.857314	0.984240	0.017328	0.654692	0.805546	0.967780	0.035967
14	0.809245	0.931640	0.996888	0.003361	0.707353	0.856205	0.984138	0.017336	0.652171	0.804178	0.967576	0.035995
15	0.807521	0.931122	0.996872	0.003361	0.705265	0.855252	0.984054	0.017342	0.650037	0.803005	0.967399	0.036020
16	0.806035	0.930670	0.996858	0.003362	0.703471	0.854425	0.983980	0.017348	0.648206	0.801986	0.967245	0.036041
17	0.804742	0.930274	0.996845	0.006662	0.701912	0.853699	0.983914	0.017353	0.646618	0.801094	0.967108	0.036060
18	0.803605	0.929923	0.996834	0.003362	0.700546	0.853058	0.983856	0.017357	0.645227	0.800307	0.966986	0.036077
19	0.802599	0.929611	0.996824	0.003362	0.699339	0.852487	0.983804	0.017361	0.644000	0.799606	0.966878	0.036091
20	0.801701	0.929331	0.996815	0.003362	0.698264	0.851976	0.983757	0.017365	0.642908	0.798979	0.966780	0.036105
21	0.800895	0.929078	0.996807	0.003362	0.697301	0.851515	0.983715	0.017368	0.641930	0.798414	0.966691	0.036117
22	0.800167	0.928849	0.996800	0.003363	0.696433	0.851098	0.983676	0.017371	0.641050	0.797903	0.966611	0.036128
23	0.799508	0.928640	0.996793	0.003363	0.695647	0.850718	0.983641	0.017374	0.640253	0.797437	0.966537	0.036138
24	0.798906	0.928450	0.996787	0.003363	0.694932	0.850371	0.983609	0.017376	0.639529	0.797012	0.966470	0.036147
25	0.798356	0.928274	0.996781	0.003363	0.694278	0.850052	0.983579	0.017379	0.638867	0.796623	0.966408	0.036156
26	0.797852	0.928113	0.996776	0.003363	0.693679	0.849759	0.983552	0.017381	0.638260	0.796264	0.966351	0.036164
27	0.797386	0.927964	0.996771	0.003363	0.693126	0.849488	0.983526	0.017383	0.637701	0.795933	0.966298	0.036171
28	0.796956	0.927826	0.996767	0.003363	0.692616	0.849237	0.983503	0.017384	0.637186	0.795626	0.966248	0.036178
29	0.796557	0.927697	0.996763	0.003363	0.692143	0.849004	0.983481	0.017386	0.636708	0.795341	0.966203	0.036184
30	0.796185	0.927578	0.996759	0.003364	0.691704	0.848787	0.983460	0.017388	0.636264	0.795076	0.966160	0.036190
35	0.794662	0.927083	0.996742	0.003364	0.689904	0.847891	0.983375	0.017394	0.634447	0.793983	0.965983	0.036214
40	0.793534	0.926710	0.996730	0.003364	0.688575	0.847224	0.983312	0.017399	0.633107	0.793169	0.965850	0.036232
45	0.792665	0.926428	0.996721	0.003364	0.687553	0.846707	0.983262	0.017403	0.632078	0.792539	0.965746	0.036246
50	0.791975	0.926200	0.996713	0.003364	0.686743	0.846296	0.983222	0.017406	0.631262	0.792038	0.965663	0.036257
60	0.790949	0.925859	0.996702	0.003365	0.685539	0.845681	0.983163	0.017410	0.630052	0.791290	0.965539	0.036274
70	0.790222	0.925617	0.996694	0.003365	0.684689	0.845244	0.983120	0.017414	0.629198	0.790758	0.965450	0.036286
80	0.789680	0.925435	0.996688	0.003365	0.684056	0.844917	0.983089	0.017416	0.628561	0.790361	0.965384	0.036295
90	0.789261	0.925294	0.996683	0.003365	0.683566	0.844664	0.983064	0.017418	0.628070	0.790052	0.965332	0.036302
100	0.788927	0.925182	0.996680	0.003365	0.683175	0.844461	0.983044	0.017420	0.627678	0.789806	0.965291	0.036308
120	0.788427	0.925013	0.996674	0.003365	0.682593	0.844158	0.983014	0.017422	0.627094	0.789438	0.965229	0.036316
140	0.788072	0.924893	0.996670	0.003365	0.682179	0.843942	0.982993	0.017424	0.626679	0.789176	0.965184	0.036322
160	0.787806	0.924803	0.996667	0.003365	0.681870	0.843781	0.982977	0.017425	0.626369	0.788980	0.965151	0.036326
180	0.787600	0.924733	0.996664	0.003365	0.681630	0.843655	0.982965	0.017426	0.626128	0.788827	0.965125	0.036330
186	0.787547	0.924715	0.996664	0.003365	0.681568	0.843622	0.982961	0.017426	0.626066	0.788788	0.965118	0.036331
200	0.787436	0.924677	0.996662	0.003365	0.681439	0.843555	0.982955	0.017427	0.625937	0.788705	0.965104	0.036333

Table 3.4. *Critical values for the outward sequential procedure for up to k =4 inliers*

$m = n - n_0$	Level of significance											
	1%				5%				10%			
	$s_{4,m}$	$s_{3,m}$	$s_{2,m}$	$s_{1,m}$	$s_{4,m}$	$s_{3,m}$	$s_{2,m}$	$s_{1,m}$	$s_{4,m}$	$s_{3,m}$	$s_{2,m}$	$s_{1,m}$
11	0.720072	0.830718	0.942114	0.997706	0.623386	0.735434	0.876163	0.988174	0.574863	0.682146	0.829262	0.975540
12	0.714613	0.827931	0.941357	0.997686	0.617679	0.731870	0.874691	0.988062	0.569322	0.678372	0.827366	0.975294
13	0.710196	0.825628	0.940722	0.997669	0.613093	0.728938	0.873458	0.987968	0.564883	0.675277	0.825781	0.975084
14	0.706547	0.823693	0.940182	0.997655	0.609326	0.726485	0.872411	0.987886	0.561245	0.672691	0.824436	0.974903
15	0.703482	0.822044	0.939717	0.997642	0.606175	0.724401	0.871510	0.987816	0.558209	0.670500	0.823281	0.974749
16	0.700871	0.820621	0.939311	0.997631	0.603500	0.722609	0.870728	0.987754	0.555635	0.668618	0.822278	0.974492
17	0.698619	0.819382	0.938956	0.997621	0.601201	0.721051	0.870041	0.987700	0.553427	0.666984	0.821398	0.974492
18	0.696656	0.818292	0.938641	0.997613	0.599204	0.719685	0.869433	0.987651	0.551510	0.665552	0.820621	0.974385
19	0.694931	0.817326	0.938360	0.997605	0.599204	0.719685	0.869433	0.987651	0.549831	0.664288	0.819930	0.974289
20	0.693403	0.816465	0.938108	0.997598	0.595903	0.717400	0.868408	0.987569	0.548348	0.663162	0.819310	0.974202
21	0.692039	0.815691	0.937881	0.997592	0.594524	0.716436	0.867970	0.987533	0.547028	0.662154	0.818752	0.974124
22	0.690815	0.814993	0.937675	0.997586	0.593288	0.715566	0.867574	0.987501	0.545846	0.661246	0.818246	0.974053
23	0.689710	0.814360	0.937488	0.997581	0.592174	0.714778	0.867214	0.987472	0.544781	0.660424	0.817786	0.973988
24	0.688707	0.813781	0.937316	0.997576	0.591164	0.714060	0.866884	0.987445	0.543817	0.659676	0.817366	0.973928
25	0.687793	0.813254	0.937159	0.997572	0.590245	0.713404	0.866582	0.987420	0.542940	0.658992	0.816980	0.973874
26	0.686956	0.812769	0.937013	0.997568	0.589405	0.712803	0.866303	0.987397	0.542138	0.658365	0.816625	0.973823
27	0.686188	0.812322	0.936879	0.997564	0.588634	0.712248	0.866046	0.987376	0.541403	0.657788	0.816297	0.973776
28	0.685480	0.811908	0.936755	0.997561	0.587923	0.711736	0.865807	0.987356	0.540726	0.657255	0.815993	0.973733
29	0.684825	0.811524	0.936639	0.997557	0.587267	0.711261	0.865585	0.987338	0.540101	0.656761	0.815711	0.973692
30	0.684217	0.811167	0.936531	0.997554	0.586659	0.710820	0.865379	0.987320	0.539522	0.656302	0.815448	0.973654
35	0.681739	0.809702	0.936086	0.997542	0.584183	0.709011	0.864526	0.987249	0.537165	0.654422	0.814363	0.973497
40	0.679919	0.808616	0.935754	0.997532	0.582371	0.707674	0.863891	0.987196	0.535442	0.653034	0.813556	0.973379
45	0.678527	0.807780	0.935496	0.997525	0.580986	0.706645	0.863399	0.987155	0.534127	0.651968	0.812931	0.973288
50	0.677426	0.807116	0.935291	0.997519	0.579894	0.705830	0.863007	0.987121	0.533090	0.651122	0.812433	0.973214
60	0.675799	0.806127	0.934984	0.997510	0.578281	0.704618	0.862421	0.987072	0.531560	0.649867	0.811689	0.973104
70	0.674653	0.805427	0.934765	0.997504	0.577148	0.703760	0.862004	0.987036	0.530486	0.648980	0.811160	0.973026
80	0.673802	0.804904	0.934601	0.997499	0.576307	0.703122	0.861692	0.987009	0.529689	0.648320	0.810765	0.972966
90	0.673145	0.804500	0.934474	0.997496	0.575658	0.702628	0.861450	0.986988	0.529075	0.647810	0.810458	0.972920
100	0.672623	0.804178	0.934373	0.997493	0.575143	0.702235	0.861257	0.986972	0.528587	0.647403	0.810213	0.972884
120	0.671845	0.803696	0.934221	0.997488	0.574376	0.701647	0.860967	0.986947	0.527861	0.646796	0.809847	0.972829
140	0.671293	0.803354	0.934112	0.997485	0.573833	0.701230	0.860761	0.986929	0.527347	0.646365	0.809586	0.972789
160	0.670881	0.803098	0.934031	0.997483	0.573427	0.700918	0.860606	0.986916	0.526963	0.646043	0.809390	0.972760
180	0.670562	0.802899	0.933968	0.997481	0.573113	0.700675	0.860486	0.986905	0.526667	0.645793	0.809238	0.972737
186	0.670480	0.802848	0.933952	0.997480	0.573032	0.700610	0.860455	0.986902	0.526590	0.645729	0.809199	0.972731
200	0.670307	0.802740	0.933918	0.997479	0.572863	0.700482	0.860390	0.986897	0.526430	0.645593	0.809117	0.972718

Table 3.5. *Critical values for the outward sequential procedure for up to k =5 inliers*

m	Level of significance														
	1%					5%					10%				
	$s_{5,m}$	$s_{4,m}$	$s_{3,m}$	$s_{2,m}$	$s_{1,m}$	$s_{5,m}$	$s_{4,m}$	$s_{3,m}$	$s_{2,m}$	$s_{1,m}$	$s_{5,m}$	$s_{4,m}$	$s_{3,m}$	$s_{2,m}$	$s_{1,m}$
12	0.633996	0.726152	0.838283	0.947139	0.998140	0.545980	0.630813	0.745793	0.885812	0.990288	0.503882	0.582636	0.693440	0.841429	0.979683
13	0.627247	0.721776	0.836056	0.946556	0.998125	0.539540	0.626145	0.742853	0.884614	0.990201	0.497829	0.578039	0.690262	0.839836	0.979483
14	0.621789	0.718159	0.834182	0.946051	0.998113	0.622305	0.622305	0.740389	0.883596	0.990127	0.492977	0.574267	0.687604	0.838483	0.979311
15	0.617281	0.715119	0.832585	0.945620	0.998102	0.530110	0.619090	0.738295	0.882718	0.990062	0.488999	0.571116	0.685349	0.837320	0.979162
16	0.613495	0.712525	0.831207	0.945245	0.998093	0.526552	0.616360	0.736492	0.881955	0.990006	0.485676	0.568444	0.683412	0.836310	0.979031
17	0.610269	0.710287	0.830004	0.944915	0.998085	0.523531	0.614011	0.734924	0.881286	0.989956	0.482860	0.566149	0.681729	0.835424	0.978916
18	0.607486	0.708337	0.828947	0.944622	0.998078	0.520933	0.611969	0.733548	0.880693	0.989911	0.480441	0.564156	0.680253	0.834640	0.978814
19	0.605062	0.706621	0.828010	0.944362	0.998071	0.518676	0.610177	0.732331	0.880165	0.989871	0.478341	0.562410	0.678949	0.833943	0.978722
20	0.602930	0.705100	0.827174	0.944128	0.998065	0.516695	0.608592	0.731246	0.879691	0.989835	0.476500	0.560867	0.677788	0.833318	0.978639
21	0.601041	0.703743	0.826423	0.943917	0.998060	0.514943	0.607180	0.730273	0.879264	0.989803	0.474873	0.559493	0.676748	0.832754	0.978565
22	0.599356	0.702524	0.825744	0.943726	0.998055	0.513383	0.605914	0.729395	0.878877	0.989773	0.473425	0.558262	0.675810	0.832244	0.978497
23	0.597843	0.701424	0.825129	0.943551	0.998051	0.511984	0.604772	0.728600	0.878525	0.989747	0.472128	0.557153	0.674961	0.831780	0.978434
24	0.596476	0.700425	0.824568	0.943392	0.998047	0.510722	0.603738	0.727875	0.878202	0.989722	0.470959	0.556148	0.674188	0.831355	0.978378
25	0.595237	0.699514	0.824054	0.943246	0.998043	0.509579	0.602795	0.727213	0.877906	0.989699	0.469900	0.555234	0.673482	0.830966	0.978325
26	0.594106	0.698681	0.823582	0.943111	0.998040	0.508539	0.601934	0.726605	0.877634	0.989678	0.468936	0.554399	0.672834	0.830607	0.978277
27	0.593072	0.697915	0.823147	0.942986	0.998036	0.507587	0.601143	0.726044	0.877382	0.989658	0.468055	0.553632	0.672238	0.830276	0.978232
28	0.592122	0.697209	0.822745	0.942870	0.998033	0.506713	0.600415	0.725527	0.877148	0.989640	0.467246	0.552927	0.671687	0.978190	0.978190
29	0.591245	0.696556	0.822372	0.942763	0.998030	0.505908	0.599742	0.725047	0.876931	0.989623	0.466502	0.552275	0.671176	0.829683	0.978151
30	0.590435	0.695950	0.822024	0.942662	0.998028	0.505165	0.599118	0.724601	0.876729	0.989608	0.465814	0.551670	0.670701	0.829418	0.978115
35	0.587152	0.693478	0.820598	0.942248	0.998017	0.502157	0.596576	0.722771	0.875894	0.989542	0.463036	0.549212	0.668757	0.828322	0.977965
40	0.584764	0.691662	0.819541	0.941939	0.998009	0.499976	0.594714	0.721418	0.875272	0.989493	0.461022	0.547414	0.667321	0.827505	0.977853
45	0.582950	0.690271	0.818726	0.941699	0.998003	0.498321	0.593292	0.720377	0.874789	0.989455	0.459496	0.546041	0.666217	0.826873	0.977765
50	0.581525	0.689173	0.818079	0.941508	0.997998	0.497023	0.592170	0.719551	0.874404	0.989425	0.458300	0.544958	0.665341	0.826369	0.977695
60	0.579428	0.687547	0.817115	0.941222	0.997991	0.497023	0.590511	0.718323	0.873830	0.989379	0.456544	0.543360	0.664041	0.825617	0.977590
70	0.577960	0.686401	0.816432	0.941018	0.997985	0.495117	0.589345	0.717454	0.873420	0.989346	0.455317	0.542237	0.663122	0.825081	0.977515
80	0.576875	0.685551	0.815923	0.941865	0.997981	0.492801	0.588480	0.716806	0.873114	0.989322	0.454412	0.541404	0.662438	0.824681	0.977458
90	0.576040	0.684894	0.815529	0.940747	0.997978	0.492045	0.587812	0.716306	0.872877	0.989303	0.453716	0.540762	0.661909	0.824371	0.977415
100	0.575378	0.684372	0.815214	0.940652	0.997976	0.491445	0.587282	0.715906	0.872687	0.989287	0.453165	0.540252	0.661488	0.824123	0.977380
120	0.574394	0.683594	0.814744	0.940510	0.997972	0.490555	0.586493	0.715310	0.872403	0.989264	0.452347	0.539493	0.660859	0.823752	0.977327
140	0.573698	0.683041	0.814410	0.940409	0.997969	0.489926	0.585933	0.714887	0.872200	0.989248	0.451768	0.538956	0.660412	0.823488	0.977290
160	0.573180	0.682630	0.814160	0.940333	0.997967	0.489458	0.585516	0.714570	0.872048	0.989236	0.451338	0.538554	0.660078	0.823290	0.977261
180	0.572779	0.682310	0.813966	0.940275	0.997966	0.489095	0.585192	0.714324	0.871930	0.989226	0.451005	0.538244	0.659818	0.823136	0.977240
186	0.572676	0.682228	0.813916	0.940259	0.997965	0.489002	0.585109	0.714261	0.871900	0.989224	0.450919	0.538164	0.659752	0.823096	0.977234
200	0.572459	0.682055	0.813811	0.940228	0.997964	0.488807	0.584934	0.714128	0.871836	0.989219	0.450740	0.537996	0.659612	0.823013	0.977221

3.4. Illustrative examples

There are several data sets which can be modeled using inliers. The Vannman (1991) data on drying of woods under different experiments and schedules was used by Muralidharan and Lathika (2006), Muralidharan and Arti (2008), Muralidharan (2010), Muralidharan and Bavagosai (2016a, b, c). Various inferences based on this data were given by the above authors. Another application of inliers is the identification of rainfall pattern and probability of occurrence of the rainfall events, usually observed in hydrology and economic evaluation of water resources projects. Muralidharan and Lathika (2005) have analyzed various aspects of modeling of rainfall data using a modified version of the Weibull distribution. They proposed a modified Weibull distribution as a singular model at zero and a two-parameter Weibull distribution. The method is illustrated with actual measured data on rainfall from two stations in India namely, Jalgaon and Coimbatore from 1961 to 1970. Another application is visualized in Demographic studies, where age on death of infants was observed in the form of inliers and positive observations. The NFHS-3/India data on the child's age on death is considered in the analysis of lifetime model with a discrete mass at zero and one by Muralidharan and Bavagosai (2017). The inliers prone data is based on tumor size in invasive ductal carcinoma of female patients considered in Weibull model with inliers at zero and one based on type II censored samples (Muralidharan and Bavagosai, 2018). Recently, Louzada et al. (2018) used two real data sets, where one is on rainfall and another on aircraft data with occurrence of instantaneous failures, and modeled using Exponential-Poisson distribution. See Appendix A for detailed discussion on data and their applications.

For the illustration of the tests of hypothesis on inliers problems proposed in this chapter, we discuss two data sets in detail in the following subsections below.

3.4.1. NEFT outward amount in banks of India

RBI as an apex body of banks, evaluate the performance of all banks in terms of NEFT transactions, the volume of transactions and e-banking related activities every year to classify banks as performing and non-performing types. From appendix, we consider the dataset A.4, on average outward debits of National Electronic Fund Transfer (NEFT) made in all the banks of India for the month January-2018. There are 187 banks in the data set and out of this one bank (n_0=1) has average outward debits of NEFT amount zero.

According to the sequential procedure, the observed test statistics as S_1=0.7777132, S_2=0.477540, S_3=0.6904526, S_4=0.4283232, and S_5=0.3020174. This test statistic values are compared with the critical values $s_{i,(n-n_0)}$, i = 1,2,3,4 and 5 with k =1,2,3,4, and 5. From Table 3.5 we see that at 10% level of significance both S_5 and S_4 is not significant but S_3 is significant, indicates that as per the procedure is given above, the first three observations are declared as inliers. From Table 3.4 we see that at 10% level of significance S_4 is not significant but S_3 is significant, indicates that the first three observations are inliers, whereas from Table 3.3 we see that at 5% level of significance S_3 is significant, again indicates that first three observations are inliers. Hence, the average amount of outward debits of National Australia bank (1.476 NEFT amount), North East small finance bank limited **(**5.164071 NEFT amount**)** and Fino payments bank limited (6.06917 NEFT amount) are declared as inliers (non-performing banks) and in all, the total number of non-performing banks in the data set is 4 including the Export-Import Bank of India, whose average amount of outward debits is zero. Note that from Table 3.2, both S_2 and S_1 are not significant at 10% level of significance, indicates that there is no inliers in the sample. Also, at 10% level of significance, the value S_1 is not greater than the critical value $s_{1,186}$=0.9094912, indicates that there are no inliers in the sample. Thus, it is concluded that the outward sequential procedure may have been affected by the masking effect.

For likelihood ratio test under M_k model, the observed value of T_k and its p-value, are given in Table 3.6. Observe that, the p-value is minimum when k=3. Hence, observations 1.476, 5.164071 and 6.069173 are declared as inlier observations as concluded above in outward sequential procedure.

Table 3.6. *Observed value t_k and the p-value of* average outward debits of NEFT data

k	1	2	**3**	4	5
t_k	1.021e-05	4.594e-05	8.792e-05	2.840e-04	4.968e-04
p-value	0.296502	0.297665	**0.213333**	0.560432	0.672202

It is seen that determining the value of k is crucial in this case, as the wrong choice of k=1, 2, may result in masking effect and for k=4, 5 may results in swamping effects. However, inward sequential tests are invalid when applied to this example, because $\frac{x_{(2)}}{\sum_{i=2}^{n-n_0} x_{(i)}}$ (=1.021088e-05), $\frac{x_{(3)}}{\sum_{i=3}^{n-n_0} x_{(i)}}$ (=3.572510e-05) and $\frac{x_{(4)}}{\sum_{i=4}^{n-n_0} x_{(i)}}$ (=4.198810e-05), which are respectively not smaller than the corresponding critical values 1.49044e-06, 1.50664e-06 and 1.52311e-06 (from Table 3.1).

3.4.2. Rainfall data

Consider the dataset A.5.from appendix, on monthly rainfall (in mm) of Rayagada district (Odisha) – 2016. Here n = 12, n_0 = 1. According to outward sequential procedure, the observed test statistics are S_1=0.500000., S_2=0.949367, S_3=0.732203, S_4=0.469424, S_5=0.540875. They are compared with the critical values $S_{i,(n-n_0)}$, i = 1,2,3,4 and 5 with k =1,2,3,4, and 5. From Table 3.5 we see that at 1% level of significance S_5, S_4 and S_3 is not significant but S_2 is significant, confirming that two observations are inliers. From Table 3.4 we see that at 1% level of significance S_4 and S_3 is not significant but S_2 is significant, indicating that two observations are inliers.

From Table 3.3 we see that at 1% level of significance S_3 is not significant but S_2 is significant and from Table 3.2 we see that at 1% level of significance S_2 is significant, indicating that two observations are inliers. Hence, two observations with magnitude 0.2 are declared as inliers in this data. Note that for 10% level of significance the value S_1 is not greater than the critical value $s_{1,11}$=0.9159664, indicates that there is no inlier in the sample; this may be due to the masking effect. For M_k test, the observed value of T_k and its p-value, are given in Table 3.7. Here p-value is minimum when k=2, which again confirms that two observations with magnitude 0.2 are declared as inliers.

Table 3.7. *Observed value t_k and the p-value of monthly rainfall of Rayagada district*

k	1	**2**	3	4	5	6
t_k	0.000176	0.000351	0.006938	0.025909	0.048832	0.106359
p-value	0.019155	**0.000301**	0.005331	0.017245	0.014455	0.040077

Note that the inward sequential tests are valid when applied to this example. Since $\frac{x_{(1)}}{\sum_{i=1}^{n-n_0} x_{(i)}}$ (=0.0001757), $\frac{x_{(2)}}{\sum_{i=2}^{n-n_0} x_{(i)}}$ (=0.0001757) which are respectively smaller than the corresponding critical values 0.0004651088, 0.0005683045 and $\frac{x_{(3)}}{\sum_{i=3}^{n-n_0} x_{(i)}}$ (=0.0065893) not smaller than the corresponding critical value 0.0007101279 at 5% level of significance (see Table 3.1). We conclude that the first two observations are inliers. Hence according to inward procedure also, two observations with magnitude 0.2 are declared as inliers observations.

3.5. Masking and swamping effect in the test for inliers

It is known and to be expected that all testing procedures will suffer from the swamping and masking effects to some extent. The masking effect has been widely discussed by various authors and some of the references are Barnett and Lewis (1994), Hawkins (1980) and David (1981), Kale and Muralidharan (2007), Bipin and Mintu (2015) among others. Bendre and Kale (1985), Chikkagoudar and Kunchur (1987), Lin and Balakrishnan (2014) have quantified the masking and swamping effect as a difference between powers of the test under the different alternatives.

Let $T(X)$ be a test statistics to detect a single discordant observation, with the critical region, say $C_{n,\alpha}$. Then, technically, the masking effect is defined as

$$M_\lambda = P_1(\lambda) - P_k(\lambda)$$

where $P_1(\lambda) = P\big(T(X) \in C_{n,\alpha} | H_1\big)$ is the power of the test.. $P_k(\lambda) = P\big(T(X) \in C_{n,\alpha} | k \geq 2\big)$ is the power of the test under the alternative is different from the one specified by H_1.

The limiting masking effect is then quantified as $M = \lim_{\lambda \to \lambda_0} M_\lambda$, where λ_0 is the limiting discordancy value. A test is said to suffer from masking effect if the measure M is positive and is said to be free from masking effect if M is zero. For a test which has the property, $\lim_{\lambda \to \lambda_0} P_1(\lambda) = 1$, is a consistent test, although usually, the consistent test has power tending to one as sample size n tends to infinity.

3.5.1. Masking effect in the test of single inlier for Dixon type test and Cochran type test

Kale and Muralidharan (2007) have studied the masking effect in Dixon type test and Cochran type tests (L_k test) for the testing number of inliers defined and quantified the loss of power due to the presence of more than anticipated discordant observations in the sample. Suppose that $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n-n_0)}$ are the $n - n_0$ ordered failures with the life greater than zero and $n_0$is the total failures with life zero. The Dixon type test for single inlier is given by

$$\phi(x) = \begin{cases} 1, & \frac{x_{(2)}-x_{(1)}}{x_{(1)}} > C(n-n_0,\alpha) \\ 0, & otherwise \end{cases}, \quad (3.5.1)$$

where, $C(n-n_0,\alpha)$ is such that $P\left(\frac{x_{(2)}-x_{(1)}}{x_{(1)}} > C(n-n_0,\alpha)|H_0\right) = \alpha$.

The value of $P_1(\lambda)$ and $P_k(\lambda), k \geq 2$ are respectively obtained as

$$P_1(\lambda) = P\left(X_{(2)} > X_{(1)}|H_1\right) = \frac{\lambda+n-n_0-1}{\lambda+(n-n_0-1)d_1}, \quad (3.5.2)$$

and

$$P_k(\lambda) = \frac{k\lambda+n-n_0-k}{\lambda[1+(k-1)d_1]+(n-n_0-k)d_1}, \quad (3.5.3)$$

where $d_1 = 1 + C(n-n_0,\alpha)$.

Clearly, for (3.5.2) and (3.5.3) both $P_1(\lambda) \to 1$ and $P_k(\lambda) \to \frac{k}{[1+(k-1)d_1]}$ as $\lambda \to \infty$. The limiting masking effect for the Dixon type test is computed as $\frac{(d_1-1)(k-1)}{1+(k-1)d_1}$, which is strictly positive.

Similarly the Cochran type test for one inlier is given as

$$\phi_1(x) = \begin{cases} 1, & \frac{x_{(1)}}{\sum_{i=1}^{n-n_0} x_{(i)}} < \frac{C_1'(n-n_0,\alpha)}{(n-n_0)} \\ 0, & otherwise \end{cases}, \quad (3.5.4)$$

where $C_1'(n-n_0,\alpha)$ is given in (3.3.7) and $P_1(\lambda)$ is as given in (3.3.8). The value of $P_2(\lambda)$ is obtained as

$$P_2(\lambda) = P_{H_2}\left(\frac{x_{(1)}}{\sum_{i=1}^{n-n_0} x_{(i)}} < \frac{C_1'(n-n_0,\alpha)}{(n-n_0)}\right)$$

$$=P_{H_2}\left[\sum_{i=3}^{n-n_0}\left(x_{(i)} - x_{(2)}\right) > (C^* - 1)x_{(1)} - (n - n_0 - 1)x_{(2)}\right] \text{ where } C^* = \frac{(n-n_0)}{C_2(n-n_0,\alpha)}$$

$$= \begin{cases} 1 & if\ (C^* - 1)x_{(1)} - (n - n_0 - 1)x_{(2)} \le 0 \\ 1 - \frac{(n-1)(2\lambda+n-n_0-2)}{\lambda(n-n_0-1)+(\lambda+n-n_0-2)(C^*-1)} & if\ (C^* - 1)x_{(1)} - (n - n_0 - 1)x_{(2)} > 0. \end{cases}$$

(3.5.5)

For single inlier, the masking effect is computed through a simulation study. We have generated about 4,00,000 samples and obtained the values of power and the masking effect for the outward sequential test, the Cochran type test and the Dixon type test for the various combination of (n, n_0), and λ at α= 1% and 5% percentile points. They are given in Table 3.8. The upper entries are for the outward sequential test, middle for the Cochran type test while the lower for the Dixon type test. Note that for $\lambda = 1$, tests attains the level of the test. We observe that the power is large for the Dixon type test and small for the outward sequential test for all combination of (n, n_0), and λ. The Cochran type test seems to be a comprise test between the above two test. The masking effect increases as the number of instantaneous failures n_0 increases for all sample sizes in each test. Also, the masking effect increases with the increase in the number of inliers k, especially for the Cochran type tests. The other two types of tests do not show any serious masking effect. Thus, the Cochran type test is highly affected by the masking effect and the outward sequential test have low masking effect in the case of single inlier. So, it is important to examine further about masking and swamping effects of the outward sequential test for more than one inliers. This is done in the next section.

Table 3.8. *The values of probability of power and probability of masking effect for the outward sequential test (upper),* the *Cochran type test (middle) and the Dixon type test (lower) at α= 1% and 5%*

n	n_0	k	$\lambda = \theta/\phi$											
			1		10		14		15		16		20	
			1%	5%	1%	5%	1%	5%	1%	5%	1%	5%	1%	5%
20	0	1	0.01	0.05	0.01113	0.05453	0.01159	0.05642	0.01136	0.05726	0.01164	0.05810	0.01250	0.06443
			0.01	0.05	0.01405	0.06889	0.01559	0.07660	0.01602	0.07947	0.01644	0.08110	0.01865	0.09057
			0.01	0.05	0.01443	0.07090	0.01639	0.07990	0.01688	0.08212	0.01737	0.08434	0.01932	0.09308
		2	0.01	0.05	0.01120	0.05534	0.01150	0.05738	0.01176	0.05929	0.01208	0.06022	0.01285	0.06346
			0.01	0.05	0.01713	0.08470	0.02070	0.09982	0.02175	0.10480	0.02260	0.10869	0.02624	0.12510
			0.01	0.05	0.01286	0.06354	0.01361	0.06706	0.01377	0.06780	0.01391	0.06849	0.01443	0.07090
		3	0.01	0.05	0.01100	0.05553	0.01153	0.05749	0.01219	0.05922	0.01188	0.05876	0.01242	0.06064
			0.01	0.05	0.02038	0.09888	0.02532	0.12074	0.02726	0.12742	0.02772	0.13249	0.03241	0.15374
			0.01	0.05	0.01204	0.05972	0.01242	0.06152	0.01250	0.06188	0.01257	0.06220	0.01280	0.06327
	4	1	0.01	0.05	0.01126	0.05634	0.01227	0.05975	0.01244	0.06055	0.01212	0.06121	0.01328	0.06526
			0.01	0.05	0.01468	0.07296	0.01721	0.08408	0.01779	0.08601	0.01808	0.08877	0.02050	0.09883
			0.01	0.05	0.01554	0.07599	0.01798	0.08709	0.01859	0.08982	0.01915	0.09254	0.02162	0.10324
		2	0.01	0.05	0.01152	0.05615	0.01238	0.06063	0.01247	0.06098	0.01243	0.06223	0.01331	0.06538
			0.01	0.05	0.01855	0.01913	0.02299	0.11170	0.02367	0.11597	0.02493	0.12033	0.02900	0.13863
			0.01	0.05	0.01324	0.06533	0.14001	0.06891	0.01416	0.06965	0.01431	0.07034	0.01482	0.07267
		3	0.01	0.05	0.01142	0.05591	0.01192	0.05893	0.01228	0.05985	0.01233	0.06058	0.01269	0.06259
			0.01	0.05	0.02188	0.10730	0.02819	0.13412	0.02948	0.14108	0.03109	0.14708	0.03680	0.17143
			0.01	0.05	0.01219	0.06041	0.01254	0.06208	0.01261	0.06240	0.01267	0.06269	0.01288	0.06364
	8	1	0.01	0.05	0.01176	0.05830	0.01276	0.06336	0.01324	0.06479	0.01373	0.06664	0.01500	0.07130
			0.01	0.05	0.16130	0.07893	0.01913	0.09230	0.01984	0.09643	0.02086	0.09959	0.02543	0.11204
			0.01	0.05	0.01737	0.08434	0.02061	0.09881	0.02142	0.10236	0.02222	0.10588	0.02543	0.11969
		2	0.01	0.05	0.01972	0.05907	0.01273	0.06452	0.01310	0.06462	0.01314	0.06585	0.01432	0.07041
			0.01	0.05	0.02095	0.10178	0.02653	0.12651	0.02744	0.13192	0.02886	0.13815	0.03415	0.16069
			0.01	0.05	0.01370	0.06748	0.01445	0.07096	0.01460	0.07166	0.01474	0.07230	0.01520	0.07442
		3	0.01	0.05	0.01202	0.05802	0.01229	0.06083	0.01246	0.06200	0.01263	0.06282	0.01300	0.06450
			0.01	0.05	0.02502	0.01206	0.03224	0.15221	0.03405	0.16047	0.03563	0.16836	0.04255	0.19751
			0.01	0.05	0.01230	0.06093	0.01260	0.06235	0.01266	0.06262	0.01271	0.06286	0.01287	0.06361

Table 3.8. Continuous…

			0.01	0.05	0.00998	0.05020	0.01018	0.05190	0.01031	0.05250	0.01044	0.05230	0.01083	0.05325
		1	0.01	0.05	0.01162	0.05710	0.01226	0.06111	0.01250	0.06299	0.01263	0.06347	0.01357	0.06770
			0.01	0.05	0.01178	0.05847	0.01257	0.06219	0.01276	0.06311	0.01296	0.06404	0.01375	0.06771
			0.01	0.05	0.01043	0.05208	0.01068	0.05279	0.01086	0.05274	0.01090	0.05365	0.01069	0.05418
	0	4	0.01	0.05	0.01593	0.07895	0.01878	0.09248	0.01946	0.09440	0.02042	0.09331	0.02286	0.11194
			0.01	0.05	0.01108	0.05515	0.01134	0.05641	0.01140	0.05668	0.01145	0.05692	0.01163	0.05777
			0.01	0.05	0.01030	0.05194	0.01061	0.05286	0.01028	0.05220	0.01063	0.05250	0.01086	0.05315
		8	0.01	0.05	0.02069	0.10109	0.02638	0.12483	0.02712	0.13187	0.02869	0.13724	0.03372	0.16003
			0.01	0.05	0.01067	0.05319	0.01077	0.05369	0.01079	0.05378	0.01081	0.05387	0.01087	0.05415
			0.01	0.05	0.01035	0.05107	0.01049	0.05262	0.01067	0.05312	0.01076	0.05309	0.01106	0.05483
		1	0.01	0.05	0.01184	0.05938	0.01265	0.06412	0.01320	0.06494	0.01345	0.06625	0.01445	0.07126
			0.01	0.05	0.01222	0.06057	0.01321	0.06519	0.01345	0.06634	0.01370	0.06748	0.01468	0.07204
			0.01	0.05	0.01045	0.05267	0.01088	0.05373	0.01064	0.05421	0.01103	0.05470	0.01109	0.05538
50	10	4	0.01	0.05	0.01724	0.08505	0.02102	0.10095	0.02178	0.10531	0.02258	0.10886	0.02600	0.12508
			0.01	0.05	0.01121	0.05579	0.01148	0.05707	0.01154	0.05733	0.01159	0.05757	0.01176	0.05839
			0.01	0.05	0.01032	0.05221	0.01023	0.05259	0.01055	0.05348	0.01082	0.05314	0.01073	0.05407
		8	0.01	0.05	0.02214	0.11026	0.02910	0.13932	0.03044	0.14694	0.03237	0.15342	0.03861	0.17980
			0.01	0.05	0.01070	0.05334	0.01079	0.05378	0.01081	0.05387	0.01082	0.05394	0.01087	0.05418
			0.01	0.05	0.01052	0.05310	0.01108	0.05523	0.01122	0.05583	0.01134	0.05635	0.01205	0.05847
		1	0.01	0.05	0.01293	0.06528	0.01464	0.07196	0.01526	0.07400	0.01533	0.07545	0.01702	0.08266
			0.01	0.05	0.01355	0.06680	0.01512	0.07407	0.01551	0.07587	0.01590	0.07767	0.01747	0.08478
			0.01	0.05	0.01077	0.05408	0.01102	0.05552	0.01138	0.05619	0.01136	0.05639	0.05797	0.01180
	25	4	0.01	0.05	0.02051	0.10046	0.02573	0.12529	0.02752	0.12944	0.02863	0.13710	0.16001	0.03396
			0.01	0.05	0.01146	0.05699	0.01171	0.05816	0.01176	0.05839	0.01180	0.05860	0.05927	0.01195
			0.01	0.05	0.01071	0.05269	0.01110	0.05305	0.01061	0.05316	0.01068	0.05321	0.05378	0.01080
		8	0.01	0.05	0.02736	0.13125	0.03595	0.16900	0.03788	0.17655	0.03972	0.18602	0.22142	0.04830
			0.01	0.05	0.01070	0.05333	0.01076	0.05364	0.01077	0.05369	0.01078	0.05374	0.05389	0.01081

Table 3..8. Continuous…

			0.01	0.05	0.01023	0.05099	0.01000	0.05036	0.01016	0.05078	0.01012	0.05081	0.01023	0.05124
		1	0.01	0.05	0.01117	0.05463	0.01106	0.05528	0.01122	0.05597	0.01138	0.05637	0.01175	0.05909
			0.01	0.05	0.01089	0.05426	0.01128	0.05613	0.01138	0.05660	0.01148	0.05707	0.01188	0.05894
			0.01	0.05	0.01024	0.05140	0.01031	0.05110	0.01026	0.05124	0.01024	0.05126	0.01078	0.05273
	0	10	0.01	0.05	0.01717	0.08510	0.02046	0.01053	0.02145	0.10484	0.02215	0.10994	0.02652	0.12613
			0.01	0.05	0.01044	0.05213	0.01053	0.05257	0.01055	0.05265	0.01057	0.05273	0.01063	0.05300
			0.01	0.05	0.01028	0.05084	0.01046	0.05103	0.01045	0.05095	0.01018	0.05123	0.01033	0.05066
		25	0.01	0.05	0.02516	0.12068	0.03244	015340	0.03438	0.16202	0.03586	0.17053	0.04329	0.20027
			0.01	0.05	0.01021	0.05102	0.01023	0.05112	0.01024	0.05114	0.01024	0.05116	0.01025	0.05122
			0.01	0.05	0.00999	0.05056	0.01001	0.05101	0.01012	0.05026	0.01030	0.05166	0.01048	0.05138
		1	0.01	0.05	0.01086	0.05452	0.01125	0.05617	0.01149	0.05686	0.01175	0.05765	0.01225	0.59830
			0.01	0.05	0.01099	0.05473	0.01143	0.05681	0.01154	0.05733	0.01165	0.05785	0.01209	0.05992
			0.01	0.05	0.01026	0.05161	0.01020	0.05083	0.01035	0.05168	0.01056	0.05181	0.01042	0.05180
100	10	10	0.01	0.05	0.01819	0.08822	0.02132	0.10584	0.02270	0.11082	0.02394	0.11507	0.02768	0.13260
			0.01	0.05	0.01047	0.05223	0.01056	0.05266	0.01057	0.05274	0.01059	0.05282	0.01064	0.05307
			0.01	0.05	0.01013	0.05069	0.00998	0.05110	0.01031	0.05089	0.01013	0.05117	0.01008	0.05089
		25	0.01	0.05	0.02607	0.12568	0.03347	0.16113	0.03558	0.16981	0.03794	0.17759	0.04522	0.21045
			0.01	0.05	0.01021	0.05101	0.01023	0.05111	0.01023	0.05113	0.01024	0.05114	0.01025	0.05119
			0.01	0.05	0.00991	0.05042	0.01020	0.05088	0.01040	0.05128	0.01028	0.05141	0.01010	0.05100
		1	0.01	0.05	0.01104	0.05543	0.01184	0.05875	0.12098	0.05879	0.01217	0.05974	0.01224	0.06192
			0.01	0.05	0.01127	0.05607	0.01183	0.05874	0.01198	0.05940	0.01212	0.06007	0.01268	0.06270
			0.01	0.05	0.01038	0.05166	0.01050	0.05149	0.01057	0.05132	0.01033	0.05200	0.01055	0.05289
	30	10	0.01	0.05	0.02024	0.09758	0.02469	0.11882	0.02552	0.23430	0.02701	0.12983	0.03201	0.15214
			0.01	0.05	0.01051	0.05244	0.01059	0.05284	0.01061	0.05291	0.01062	0.05298	0.01067	0.05321
			0.01	0.05	0.01001	0.05129	0.01013	0.05093	0.01004	0.05018	0.01007	0.05101	0.01017	0.05101
		25	0.01	0.05	0.02852	0.13696	0.03694	0.17563	0.03890	0.18447	0.04180	0.19479	0.05006	0.23126
			0.01	0.05	0.01020	0.05096	0.01022	0.05104	0.01220	0.05105	0.01022	0.05107	0.01023	0.05110

3.5.2. Masking and swamping effect in the outward sequential test for more than one inliers

As the choice of k used in test procedure is critical, Kimber (1982) pointed out that, unless the sample size is very large, the value of k, should be chosen in advance, is going to be small, but should also be somewhat larger than necessary. With regard to choice of the testing procedure itself, we should consider the one that demonstrates lesser effects of swamping and masking. The masking and swamping effect for the outward sequential test is quantified in this section by means of a Monte Carlo study. We carry out a Monte Carlo study with 4,00,000 simulations for evaluating the powers as well as the effect of masking and swamping in the outward test when k=2 and 3.

Let $P_{ij}^{k} = P(\text{Accept } H_i|H_j)$, i, j = 1,2,…, k. For k=2, the probabilities of the correct decisions (or the powers) be P_{11}^{2} and P_{22}^{2}, the probability of the masking effect P_{12}^{2}, and the probability of the swamping effect P_{21}^{2} where all estimated for significance level α = 1% and 5% and different combinations of (n, n_0) and λ. The result so obtained are reported in Table 3.9. Here,

$$P_{11}^{2} = P\left(S_1 > s_{1,(n-n_0)}, S_2 \leq s_{2,(n-n_0)}|H_1\right),$$

$$P_{22}^{2} = P\left(S_2 > s_{2,(n-n_0)}|H_2\right),$$

$$P_{12}^{2} = P\left(S_1 > s_{1,(n-n_0)}, S_2 \leq s_{2,(n-n_0)}|H_2\right),$$

and

$$P_{21}^{2} = P\left(S_1 > s_{1,(n-n_0)}|H_1\right),$$

wherein the required $s_{i,(n-n_0)}, i = 1,2$ were taken from Table 3.2. Similarly, the probabilities $P_{ij}^3, i,j = 1,2$ and 3, for $k = 3$ and the results are presented in Tables 3.10-3.12.

From Table 3.9 and 3.10, it is inferred that as the number of inliers k increases the power estimates decreases. That is, outward sequential test substantially weakened for the higher value of k. Also, the power increases as λ increases for each combination of (n, n_0), and k. Also, it is inferred that for each k as the number of instantaneous failures increases, the power increases and decreases as sample size increases. This is true for all value λ and each combination of (n, n_0). It is noticed that at 1% level of significance, for a small value of n = 20, and $\lambda \leq 15$ the power P_{kk}^k increases as the number of inliers k increases, whereas at 5% level of significance it increases for n = 20 and $\lambda \leq 10$.

From Table 3.9 and 3.11, it is inferred that, as k increases the probabilities of masking effects decreases but it increases as λ increases for each combination of (n, n_0). Also, it is inferred that for each k as the number of instantaneous failures increases, the probability of masking effect increases and decreases as sample size increases for all value of λ and each combination of (n, n_0).

From Table 3.9 and 3.12, we conclude that, as k increases the probabilities of swamping effects decreases but it increases as λ increases for each combination of (n, n_0). Also, it is concluded that for each k as the number of instantaneous failures increases, the probability of swamping effect increases and decreases as n increases for all value of λ and each combination of (n, n_0).

Table 3.9. *P_{11}^2, P_{22}^2, P_{12}^2 and P_{21}^2 values for the outward sequential test when k=2*

P_{ij}^3	n	n_0	$\lambda = \theta/\phi$									
			1		14		15		16		20	
			1%	5%	1%	5%	1%	5%	1%	5%	1%	5%
P_{11}^2	20	0	0.00531	0.02649	0.00569	0.02777	0.00584	0.02796	0.00553	0.02793	0.00613	0.02971
		4	0.00563	0.02707	0.00591	0.02868	0.00604	0.02945	0.00635	0.02972	0.00644	0.03150
		8	0.00583	0.02844	0.00644	0.03125	0.00645	0.03102	0.00683	0.03178	0.00724	0.03451
	50	0	0.00498	0.02485	0.00516	0.02501	0.00522	0.02512	0.00508	0.02531	0.00529	0.02612
		10	0.00513	0.02407	0.00507	0.02559	0.00533	0.02585	0.00526	0.02540	0.00542	0.02705
		25	0.00566	0.02572	0.00541	0.02690	0.00542	0.02720	0.00563	0.02790	0.00598	0.02850
	100	0	0.00507	0.02475	0.00520	0.02436	0.00503	0.02476	0.00515	0.02485	0.00509	0.02461
		10	0.00500	0.02409	0.00517	0.02499	0.00509	0.02506	0.00502	0.02445	0.00508	0.02480
		30	0.00500	0.02477	0.00492	0.02473	0.00497	0.02499	0.00507	0.02489	0.00519	0.02578
P_{22}^2	20	0	0.00706	0.03367	0.00806	0.03876	0.00832	0.03998	0.00839	0.04097	0.00994	0.04652
		4	0.00754	0.03639	0.00915	0.04370	0.00948	0.04473	0.01034	0.04645	0.01194	0.05396
		8	0.00845	0.02844	0.01151	0.05162	0.01234	0.05379	0.01246	0.05535	0.01573	0.06515
	50	0	0.00573	0.02805	0.00599	0.02943	0.00586	0.02960	0.00587	0.03020	0.00633	0.03182
		10	0.00548	0.02866	0.00617	0.03108	0.00640	0.03143	0.00632	0.03221	0.00701	0.03436
		25	0.00649	0.03128	0.00739	0.03571	0.00762	0.03679	0.00770	0.03757	0.00871	0.04185
	100	0	0.00520	0.02592	0.00521	0.02682	0.00537	0.02737	0.00548	0.02745	0.00562	0.02824
		10	0.00520	0.02667	0.00548	0.02743	0.00549	0.02791	0.00539	0.02754	0.00570	0.02829
		30	0.00548	0.02680	0.00550	0.02788	0.00537	0.02782	0.00550	0.02843	0.00585	0.02954
P_{12}^2	20	0	0.00541	0.02643	0.00574	0.02821	0.00604	0.02878	0.00603	0.02881	0.00633	0.03042
		4	0.00561	0.02745	0.00598	0.02882	0.00645	0.02989	0.00644	0.03037	0.00670	0.03140
		8	0.00608	0.02873	0.00655	0.03043	0.00664	0.03112	0.00678	0.03202	0.00710	0.03340
	50	0	0.00503	0.02490	0.00508	0.02586	0.00526	0.02583	0.00532	0.02569	0.00526	0.02616
		10	0.00504	0.02541	0.00533	0.02606	0.00536	0.02630	0.00548	0.02638	0.00570	0.02706
		25	0.00538	0.02652	0.00569	0.02765	0.00562	0.02742	0.00563	0.02790	0.00613	0.02918
	100	0	0.00502	0.02462	0.00527	0.02476	0.00548	0.02533	0.00497	0.02486	0.00545	0.02550
		10	0.00498	0.02469	0.00498	0.02484	0.00514	0.02517	0.00510	0.02481	0.05227	0.02674
		30	0.00503	0.02478	0.00529	0.02535	0.00513	0.02523	0.00506	0.02470	0.00544	0.02562
P_{21}^2	20	0	0.00644	0.03173	0.00738	0.03549	0.00775	0.03661	0.00780	0.03746	0.00881	0.04152
		4	0.00694	0.03376	0.00821	0.03922	0.00838	0.04016	0.00883	0.04163	0.01063	0.04708
		8	0.00779	0.03791	0.01005	0.04522	0.01038	0.04680	0.01092	0.04979	0.01316	0.05673
	50	0	0.00523	0.02719	0.00543	0.02790	0.00571	0.02863	0.00584	0.02849	0.00589	0.03006
		10	0.00558	0.02799	0.00573	0.02917	0.00586	0.02926	0.00596	0.02981	0.00666	0.03139
		25	0.00602	0.03005	0.00674	0.03226	0.00678	0.03305	0.00679	0.03384	0.00781	0.03746
	100	0	0.00508	0.02597	0.00513	0.02665	0.00537	0.02655	0.00518	0.02652	0.00531	0.02720
		10	0.00517	0.02674	0.00532	0.02653	0.00527	0.02707	0.00536	0.02708	0.00523	0.02674
		30	0.00523	0.02669	0.00538	0.02732	0.00559	0.02785	0.00535	0.02730	0.00566	0.02806

Table 3.10. *P_{11}^3, P_{22}^3 and P_{33}^3 values for the outward sequential test when k=3*

P_{ij}^3	n	n_0	$\lambda = \theta/\phi$									
			10		14		15		16		20	
			1%	5%	1%	5%	1%	5%	1%	5%	1%	5%
P_{11}^3	20	0	0.00362	0.01785	0.00379	0.01847	0.00387	0.01861	0.00402	0.01894	0.00417	0.02019
		4	0.00365	0.01081	0.00391	0.01938	0.00389	0.01920	0.00396	0.01970	0.00421	0.02083
		8	0.00404	0.01870	0.00439	0.02050	0.00442	0.02054	0.00452	0.02117	0.00483	0.02308
	50	0	0.00336	0.01667	0.00344	0.01716	0.00347	0.01715	0.00352	0.01736	0.00351	0.01726
		10	0.00338	0.01676	0.00360	0.01726	0.00348	0.01734	0.00351	0.01736	0.00356	0.01782
		25	0.00357	0.01724	0.00364	0.01814	0.00358	0.01816	0.00363	0.01790	0.00393	0.01908
	100	0	0.00334	0.01686	0.00350	0.01680	0.00344	0.01689	0.00336	0.01627	0.00337	0.01689
		10	0.00343	0.01697	0.00346	0.01691	0.00342	0.01656	0.00339	0.01639	0.00336	0.01691
		30	0.00342	0.01688	0.00335	0.01667	0.00339	0.01684	0.00327	0.01679	0.00345	0.01682
P_{22}^3	20	0	0.00457	0.02142	0.00537	0.02413	0.00570	0.02501	0.00578	0.02606	0.00689	0.02963
		4	0.00501	0.02280	0.00609	0.02756	0.00633	0.02811	0.00670	0.02954	0.00794	0.03396
		8	0.00591	0.02669	0.00738	0.03250	0.00784	0.03357	0.00857	0.03589	0.01068	0.04146
	50	0	0.00363	0.01719	0.00400	0.01851	0.00392	0.01907	0.00409	0.01888	0.00424	0.02052
		10	0.00367	0.01800	0.00429	0.01967	0.00396	0.01940	0.00417	0.01997	0.00457	0.02092
		25	0.00417	0.01975	0.00490	0.02248	0.00486	0.02310	0.00547	0.02358	0.00605	0.02668
	100	0	0.00345	0.01676	0.00340	0.01665	0.00347	0.01680	0.00354	0.01703	0.00365	0.01746
		10	0.00348	0.01664	0.00356	0.01709	0.00342	0.01730	0.00351	0.01726	0.00365	0.01749
		30	0.00364	0.01689	0.00358	0.01756	0.00381	0.01772	0.00358	0.01814	0.00391	0.01837
P_{33}^3	20	0	0.00619	0.02142	0.00863	0.03698	0.00867	0.03900	0.00934	0.04036	0.01203	0.04902
		4	0.00745	0.03378	0.01042	0.04374	0.01105	0.04590	0.01209	0.04863	0.01532	0.05889
		8	0.00967	0.04093	0.01428	0.05614	0.01598	0.06007	0.01715	0.06313	0.02313	0.07804
	50	0	0.00392	0.02031	0.00466	0.02274	0.00479	0.02390	0.00506	0.02391	0.00539	0.02644
		10	0.00442	0.02164	0.00507	0.02469	0.00537	0.02562	0.00554	0.02661	0.00659	0.02976
		25	0.00531	0.02655	0.00701	0.03163	0.00724	0.03300	0.00782	0.03467	0.00934	0.04128
	100	0	0.00369	0.01815	0.00387	0.01935	0.00390	0.01940	0.00380	0.01999	0.00420	0.02094
		10	0.00365	0.01875	0.00390	0.01963	0.00390	0.02036	0.00397	0.02011	0.00434	0.02147
		30	0.00394	0.01968	0.00420	0.02097	0.00413	0.02104	0.00422	0.02169	0.00470	0.02290

Table 3.11. *P_{12}^3, P_{13}^3 and P_{23}^3 values for the outward sequential test when k=3*

P_{ij}^3	n	n_0	$\lambda = \theta/\phi$									
			10		14		15		16		20	
			1%	5%	1%	5%	1%	5%	1%	5%	1%	5%
P_{12}^3	20	0	0.00367	0.01777	0.00387	0.01868	0.00400	0.01907	0.00392	0.01937	0.00429	0.02043
		4	0.00372	0.01843	0.00394	0.01894	0.00415	0.01952	0.04008	0.01941	0.00425	0.02058
		8	0.00398	0.01914	0.00431	0.02021	0.00430	0.02060	0.00437	0.02090	0.00456	0.02184
	50	0	0.00335	0.01701	0.00361	0.01717	0.00348	0.01696	0.00369	0.01733	0.00358	0.01766
		10	0.00364	0.01695	0.00367	0.01748	0.00362	0.01734	0.00368	0.01787	0.00366	0.01798
		25	0.00347	0.01753	0.00373	0.01803	0.00363	0.01820	0.00393	0.01872	0.00388	0.01919
	100	0	0.00335	0.01680	0.00522	0.01675	0.00337	0.01669	0.00354	0.01712	0.00344	0.01700
		10	0.00332	0.01651	0.00360	0.01665	0.00347	0.01654	0.00346	0.01723	0.00341	0.01688
		30	0.00345	0.01705	0.00339	0.01733	0.00343	0.01683	0.00323	0.01677	0.00352	0.01737
P_{13}^3	20	0	0.00376	0.01781	0.00371	0.01808	0.00406	0.01870	0.00387	0.01900	0.00413	0.01926
		4	0.00372	0.01805	0.00411	0.01882	0.00395	0.01864	0.00408	0.01924	0.00414	0.01942
		8	0.00380	0.01858	0.00400	0.01937	0.00408	0.01919	0.00415	0.01921	0.00423	0.01967
	50	0	0.00333	0.01714	0.00350	0.01724	0.00360	0.01724	0.00356	0.01767	0.00364	0.01790
		10	0.00350	0.01720	0.00357	0.01729	0.00354	0.01758	0.00363	0.01760	0.00383	0.01826
		25	0.00366	0.01798	0.00378	0.01808	0.00363	0.00382	0.00390	0.01850	0.00403	0.01861
	100	0	0.00344	0.01660	0.00336	0.01678	0.00352	0.01705	0.00361	0.01693	0.00357	0.01741
		10	0.00328	0.01636	0.00335	0.01659	0.00347	0.01649	0.00333	0.01633	0.00363	0.01723
		30	0.00355	0.01698	0.00353	0.01676	0.00351	0.01708	0.00350	0.01725	0.00360	0.01748
P_{23}^3	20	0	0.00480	0.02170	0.00526	0.02434	0.00552	0.02516	0.00562	0.02566	0.00650	0.02844
		4	0.00483	0.02281	0.00590	0.02659	0.00638	0.02760	0.00648	0.02813	0.00737	0.03079
		8	0.00535	0.02511	0.00712	0.03000	0.00707	0.03108	0.00764	0.03229	0.00932	0.03716
	50	0	0.00376	0.01781	0.00401	0.01874	0.00406	0.01943	0.00396	0.01922	0.00433	0.02064
		10	0.00368	0.01828	0.00409	0.01950	0.00427	0.02032	0.00430	0.02069	0.00484	0.02184
		25	0.00437	0.02042	0.00493	0.02272	0.00494	0.02328	0.00495	0.02297	0.00549	0.02525
	100	0	0.00340	0.01668	0.00342	0.01663	0.00350	0.01742	0.00350	0.01753	0.00383	0.01778
		10	0.00331	0.01676	0.00355	0.01738	0.00358	0.01747	0.00376	0.01802	0.00376	0.01803
		30	0.00342	0.01706	0.00365	0.01772	0.00362	0.01832	0.00365	0.01835	0.00409	0.01953

Table 3.12. *P_{21}^3, P_{31}^3 and P_{32}^3 values for the outward sequential test when k=3*

P_{ij}^3	n	n_0	$\lambda = \theta/\phi$									
			10		14		15		16		20	
			1%	5%	1%	5%	1%	5%	1%	5%	1%	5%
P_{21}^3	20	0	0.00407	0.01988	0.00491	0.02253	0.00503	0.02334	0.00483	0.02302	0.00581	0.02587
		4	0.00452	0.02096	0.00534	0.02451	0.00552	0.02551	0.00609	0.02667	0.00684	0.03000
		8	0.00503	0.02375	0.00664	0.02925	0.00707	0.03049	0.00758	0.03186	0.00896	0.03661
	50	0	0.00351	0.01735	0.00353	0.01767	0.00372	0.01827	0.00381	0.01808	0.00406	0.01884
		10	0.00340	0.01726	0.00384	0.01827	0.00396	0.01862	0.00403	0.01891	0.00435	0.01983
		25	0.00409	0.01930	0.00439	0.02063	0.00451	0.02077	0.00451	0.02155	0.00531	0.02359
	100	0	0.00333	0.01622	0.00348	0.01671	0.00343	0.01663	0.00336	0.01655	0.00367	0.01721
		10	0.00326	0.01642	0.00348	0.01695	0.00340	0.01668	0.00334	0.01709	0.00348	0.01712
		30	0.00340	0.01655	0.00355	0.01722	0.00366	0.01753	0.00354	0.01727	0.00383	0.01790
P_{31}^3	20	0	0.00513	0.02430	0.00653	0.02871	0.00658	0.03002	0.00680	0.03095	0.00847	0.03594
		4	0.00580	0.02687	0.00739	0.03309	0.00796	0.03452	0.00814	0.03583	0.01036	0.04183
		8	0.00727	0.03194	0.00962	0.04003	0.01025	0.04203	0.01108	0.04403	0.01400	0.05209
	50	0	0.00376	0.01872	0.00387	0.02001	0.00433	0.02075	0.00426	0.02084	0.00483	0.02290
		10	0.00401	0.01985	0.00435	0.02126	0.00438	0.02182	0.00450	0.02228	0.00505	0.02496
		25	0.00456	0.02242	0.00518	0.02551	0.00548	0.02673	0.00573	0.02712	0.00696	0.03138
	100	0	0.00353	0.01809	0.00347	0.01806	0.00373	0.01884	0.00376	0.01879	0.00347	0.01868
		10	0.00351	0.01804	0.00362	0.01856	0.00363	0.01848	0.00356	0.01885	0.00386	0.01970
		30	0.00368	0.01847	0.00383	0.01945	0.00386	0.01953	0.00399	0.01967	0.00404	0.02053
P_{32}^3	20	0	0.00612	0.02801	0.00798	0.03551	0.00861	0.03754	0.00926	0.03881	0.01148	0.04654
		4	0.00711	0.03242	0.00988	0.04198	0.01042	0.04467	0.01137	0.04690	0.01472	0.05681
		8	0.00964	0.04032	0.01398	0.05393	0.01557	0.05805	0.01641	0.06112	0.02223	0.07581
	50	0	0.00369	0.02003	0.00469	0.02280	0.00474	0.02260	0.00470	0.02339	0.00523	0.02535
		10	0.00423	0.02114	0.00502	0.02411	0.00496	0.02476	0.00511	0.02519	0.00595	0.02832
		25	0.00493	0.02489	0.00641	0.03022	0.00701	0.03156	0.00722	0.03333	0.00890	0.03901
	100	0	0.00366	0.01822	0.00379	0.01931	0.00379	0.01934	0.00376	0.01933	0.00397	0.02046
		10	0.00369	0.01837	0.00376	0.01906	0.00373	0.01981	0.00403	0.02035	0.00408	0.02087
		30	0.00376	0.01904	0.00400	0.02007	0.00398	0.02020	0.00430	0.02078	0.00462	0.02288