CHAPTER-3

A SINGLE PERIOD INVENTORY MODEL WITH POISSON DEMAND

3.1 INTRODUCTION

In classical EOQ model, developed by Harris (1913) and extensively used in the literature of inventory theory, the demand is assumed to be uniform over the period. This assumption results in a linearly decreasing inventory level until the reorder is placed (See figure 3.1(a)). However, in most real world inventory systems, the demand is rarely uniform. In fact modeling different demand patterns has been an extensive area of research as described in chapter 1. If demand is assumed to be a stochastic process, the inventory level would instead decrease, for example, as shown in figure 3.1(b) If we wish to adequately model the demand patterns that are seen in real life systems, it is necessary that demand be modeled using a stochastic component instead of a purely mathematical component. Such models are referred to as stochastic models in the literature. In stochastic inventory models such as those described in Hillier & Lieberman (1974), the cycle length is assumed to be fixed and the total demand for a given cycle is taken as a random variable.

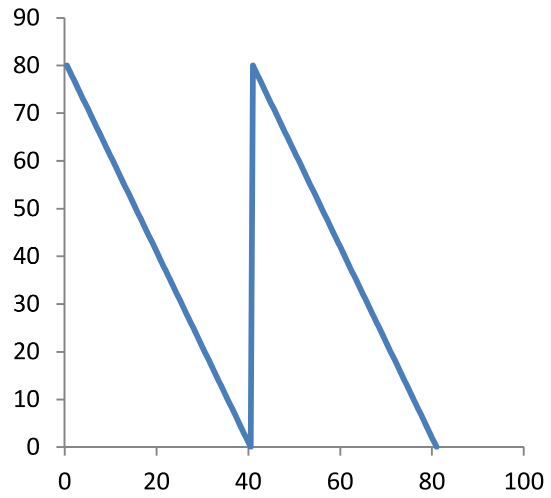

Figure 3.1(a)

Inventory Level over Time for

Uniform Demand

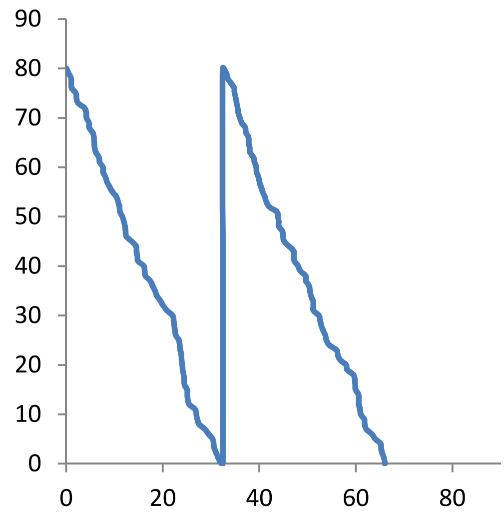

Figure 3.1(b)[2]

Inventory Level over Time for

Stochastic Demand

In other models, which are multi-period models, the inventory is reviewed periodically and depending on the inventory level at the time of review, the decision for ordering is taken as to whether to order or not, and if yes, how much to order? These models lead to optimal policies such as (s, S) policy.

In a real market scenario, demand of an item derives the customer arrival process, and the buying pattern of customer results into the consumption from inventory. Since, it is the consumption of items from inventory that ultimately matters for an inventory system, it is necessary that the demand be modeled in terms of customer arrival process and their buying pattern.

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[2] *This figure shows inventory level over time when demand is Poisson with mean 0.05 and initial inventory level is 80.*

It is common to use a Poisson process for modeling the stochastic demand. Several researchers made the assumption of Poisson demand in their model. For example, Scarf (1958), Karlin and Scarf (1958) and Galliher et al. (1959 have considered Poisson demand.

Federgruen & Schechner (1983) have studied continuous review models with a fixed delivery lag, and the stochastic demand is characterized by the renewal reward process.

Katy,Julia & Udayabhanu (2012) assumed that demand is a compound Poisson process and in other case they assumed that demand is a combination of a constant deterministic component (as in classical EOQ model) and a random component which follows a compound Poisson process, which may be referred to as the mixture of demand. Presman & Sehi (2006), Sobel & Zhang (2001) also assumed mixture of demand.

From the literature surveyed by us and referenced above, little work appears to have been done on stochastic models that are specifically designed for discrete items. In this chapter we present a stochastic model for discrete items by generalizing the model presented in Chapter 2 so as to make it more suitable for practical applications.

We propose a model for discrete items with Poisson demand arising due to a Poisson customer arrival process and assume that the inventory holding cost is incurred only for the period during which the inventory item is in the stock. Due to our assumptions, the actual holding cost is random and depends on the time points at which actual demands occur. Also, the replenishment is assumed to be instantaneous and hence the new order is placed only when (and as soon as) the inventory level reaches zero. This also results in a random cycle length.

3.2 NOTATIONS

Notations used in this Chapter are as given below.

n : Lot size.(initial inventory level)

T : Cycle length.

C_0 : Ordering cost per order.

C_h: holding cost per unit per unit time.

C : cost of inventory per unit.

θ : Mean customer inter arrival time.(Equivalently, Customer arrival rate = $\frac{1}{\theta}$)

3.3 THE PROPOSED MODEL

Following are the basic assumptions of the proposed model.

1. Customer arrival process is Poisson with each customer having a demand of one unit. For example a retail customer of electronic goods buys only one unit.
2. Supply/ replenishment is instantaneous.
3. Reorder is placed as soon as inventory level reaches zero.
4. Holding cost is incurred only for the period during which the items are in stock.

Once an item is sold, it doesn't incur any holding cost and therefore holding cost is dependent on the time for which the item is held in stock.

Determination of Optimum Inventory Level

Suppose that initially, at time $t = 0$, the inventory level is raised to n units. Suppose 1st customer arrives at time t_1, and demand for 1 unit. So, at time t_1, inventory level reduces to $(n - 1)$. Similarly, 2nd customer arrives at time $t_1 + t_2$, and demand for 1 unit. Thus, at time $t_1 + t_2$, inventory level reduces to $(n - 2)$ units, and so on. Here t_1, t_2, … are customer inter arrival times

At time $t_1 + t_2 + t_3 + \ldots. + t_n = T$, inventory level becomes zero. At this point of time, the reorder is made and the second cycle starts.

Clearly, in this case, the cycle length T is a random variable.

As the customer arrival process is a Poisson process, the inter-arrival times $t_1, t_2, t_3, \ldots., t_n$ are identically and independently distributed (iid) exponential random variables.

Let the arrival rate of the arrival process be $\frac{1}{\theta}$ per unit time. Then

$$t_i \sim Exp(\theta), \qquad i = 1, 2, 3 \ldots \ldots, n$$

Here Exp(θ) denotes the exponential distribution with mean ¯.

The total inventory cost for one cycle is given by

Total cost = Ordering cost + Inventory cost + Holding cost

Ordering cost $= C_0$ ... (3.3.1)

Cost of inventory $= nC$ ... (3.3.2)

Total holding cost $= C_h(n\,t_1 + (n-1)t_2 + (n-2)t_3 + \ldots \ldots + t_n)$ ... (3.3.3)

Equations (3.3.1) – (3.3.3) imply that the total cost per unit time, with initial inventory level n, is

$$TC(n) = \frac{C_0 + nC}{T} + \frac{C_h(n\,t_1 + (n-1)\,t_2 + (n-2)\,t_3 + \dots\dots + t_n)}{T}$$

Note that the Total inventory cost TC(n) is a random variable. In order to obtain the optimal order quantity, we, therefore, minimize the expected total inventory cost per unit time.

The expected total cost per unit time is

$$E(TC(n)) = (C_0 + nC)E\left(\frac{1}{T}\right)$$
$$+ C_h\left(nE\left(\frac{t_1}{T}\right) + (n-1)E\left(\frac{t_2}{T}\right) + (n-2)E\left(\frac{t_3}{T}\right) + \cdots + E\left(\frac{t_n}{T}\right)\right)$$
$$= (C_0 + nC)E\left(\frac{1}{T}\right)$$
$$+ C_hE\left(\frac{t_i}{T}\right)(n + (n-1) + (n-2) + \cdots + 1) \qquad \dots(3.3.4)$$

The last equality follows from the fact that t_i's are identically distributed random variables.

The case, n=1 requires a special attention.

For $n = 1, T = t_1 \sim Exp(\theta)$. Hence

$$E\left(\frac{1}{T}\right) = \frac{1}{\theta}\int_0^{\infty} e^{-t/\theta}t^{-1}dt$$

This is a Gamma integral with shape parameter α = 0. However, it is known that Gamma integral converges if and only if α > 0.

Thus, the integral on the right hand side of $E\left(\frac{1}{T}\right)$ above diverges to ∞, which in turn leads to E(TC(1)) = ∞.

Thus n =1 can never be an optimal solution. We, therefore, assume that n is greater than 1.

Now, $E(\frac{t_i}{T})$ can be computed as

$$E\left(\frac{t_i}{T}\right) = E\left(E\left(\frac{t_i}{T} \mid T = t\right)\right)$$

It can be shown that the conditional distribution of $\frac{t_i}{T} | T = t$ is beta of Type-1 with parameters 1 and (n-1) respectively. (See Appendix A)
Thus, we have

$$E\left(\frac{t_i}{T} \mid T = t\right) = \frac{1}{n}$$

This further, implies that

$$E\left(\frac{t_i}{T}\right) = \frac{1}{n} \qquad \text{... (3.3.5)}$$

Also, for the Gamma random variable T, $E\left(\frac{1}{T}\right)$ can be computed as shown below.

$$E\left(\frac{1}{T}\right) = \frac{1}{\Gamma_n} \int_0^{\infty} \frac{\frac{1}{t} e^{-t/\theta}}{\theta^n} t^{n-1}\, dt$$

$$= \frac{1}{\Gamma_n}\frac{1}{\theta}\int_0^\infty e^{-t/\theta}(\frac{1}{\theta})^{n-1}\, t^{(n-1)-1}\, dt$$

$$= \frac{1}{\Gamma_n}\frac{1}{\theta}\int_0^\infty \frac{e^{-t/\theta}t^{(n-1)-1}}{\vartheta^{n-1}}\, dt$$

$$= \frac{1}{\Gamma_n}\frac{1}{\theta}\;\Gamma_{n-1} \qquad (\because \int_0^\infty \frac{e^{-t/\theta}t^{(n-1)-1}}{\vartheta^{n-1}}\, dt \;=\; 1\,, p.d.f\ of\ \Gamma_{n-1})$$

$$= \frac{1}{\theta}\;\frac{\Gamma_{n-1}}{(n-1)\Gamma_{n-1}}$$

$$= \frac{1}{\theta(n-1)} \qquad \ldots (3.3.6)$$

Substituting (3.3.5) and (3.3.6) in (3.3.4), we get

$$E\big((TC(n)\big) = (C_0 + nC)E\left(\frac{1}{T}\right) + \; C_h E\left(\frac{t_i}{T}\right)\left(n\left(\frac{n+1}{2}\right)\right)$$

$$= (C_0 + nC)\frac{1}{\theta(n-1)} + C_h\;\frac{1}{n}\frac{n(n+1)}{2}$$

Thus,

$$E\big(TC(n)\big) = \frac{(C_0 + \; nC)}{\theta(n - \; 1)} + \; C_h\;\frac{(n+1)}{2} \qquad \ldots (3.3.7)$$

$$E(TC(n+1)) - E(TC(n)) = \frac{C_h}{2} - \frac{C_0 + C}{\theta n(n-1)}$$

is an increasing function for all n. Thus, the expected total cost E(TC(n)) is a convex function of n. This further implies that the cost function has unique minima.

This unique optimal solution is the integer value of n that satisfies

$E(TC(n)) \leq E(TC(n-1))$ as well as $E(TC(n)) \leq E(TC(n+1))$

An optimal value of n is the one that minimizes E(TC(n)) given by above expression. This is the value of n that satisfies

$E\big(TC(n)\big) \leq E(TC(n-1))$ as well as $E\big(TC(n)\big) \leq E(TC(n+1))$

Or equivalently, the smallest value of n, that satisfies

$E\big(TC(n)\big) \leq E(TC(n+1))$. From (3.3.7),

$$E\big(TC(n)\big) \leq E(TC(n+1)$$

$$\Rightarrow \frac{(C_0 + nC)}{\theta(n-1)} + C_h \frac{(n+1)}{2} \leq \frac{(C_0 + (n+1)C)}{\theta n} + C_h \frac{(n+2)}{2}$$

$$\Rightarrow \frac{(C_0 + nC)}{\theta(n-1)} - \frac{(C_0 + (n+1)C)}{\theta n} \leq C_h \frac{(n+2)}{2} - C_h \frac{(n+1)}{2}$$

$$\Rightarrow \frac{n(C_0 + nC) - (n-1)(C_0 + nC + C)}{\theta n(n-1)} \leq \frac{C_h(n+2-n-1)}{2}$$

$$\Rightarrow \frac{2(C_0 + C)}{\theta C_h} \leq n(n-1)$$

Therefore, optimal value of n is the smallest integer value of n for which

$$n(n-1) \geq \frac{2(C_0 + C)}{\theta C_h}$$

The equality holds when,

$$n^2 - n - \frac{2(C_0 + C)}{\theta C_h} = 0$$

Comparing above equation with quadratic equation $ax^2 + bx + c = 0$,

$$a = 1, b = -1, c = -\frac{2(C_0 + C)}{\theta C_h}$$

$$\Delta = b^2 - 4ac = 1 + 8\,\frac{(C_0 + C)}{\theta C_h}$$

$$\therefore\ n = -b \pm \frac{\sqrt{\Delta}}{2a} = 1 \pm \frac{\sqrt{1 + 8\,\frac{(C_0 + C)}{\theta C_h}}}{2}$$

Note that,

$$n = 1 - \frac{\sqrt{1 + 8\,\frac{(C_0 + C)}{\theta C_h}}}{2}$$

is negative, hence not a feasible solution.

Thus, the optimal value n^* is the smallest integer greater than or equal to the other solution. Hence

$$n^* = \left\lceil 1 + \frac{\sqrt{1 + \;\; 8\frac{(C_o + C)}{\theta C_h}}}{2} \right\rceil \qquad \ldots(3.3.8)$$

With expected cycle length

$$E(T) = \left\lceil 1 + \frac{\sqrt{1 + \;\; 8\frac{(C_o + C)}{\theta C_h}}}{2} \right\rceil \theta$$

3. 4 AN ILLUSTRATIVE EXAMPLE

A company stocks an item that is consumed at the rate of θ = 0.05 per unit time. The holding cost per unit per unit time is Rs.1000, ordering cost is Rs.10000 per order. Suppose shortages are not allowed and the purchasing cost per unit is Rs.15000. Determine expected total cost and optimal quantity.

This example is solved with the help of a computer.

C_o= 10000 Rs.

C_h =1000 RS.

C = 15,000 Rs.

a = $\frac{1}{\theta}$ = 1/ 0.05 = 20

Using the classical EOQ formula we get Q^* = 20 resulting in

E(TC) = 336815.8 Rs.

Using the formula 3.3.10 we get n* = 32.62 ≅ 33 and from 3.3.6

E(TC) = 332625 Rs.

From the result obtained in above example, it is clear that the use of approximate EOQ model gives the lot size that incurs higher expected total cost than that for the optimal lot size obtained for the model proposed in this chapter due to hodling cost and Poisson process for modeling the stochastic demand.

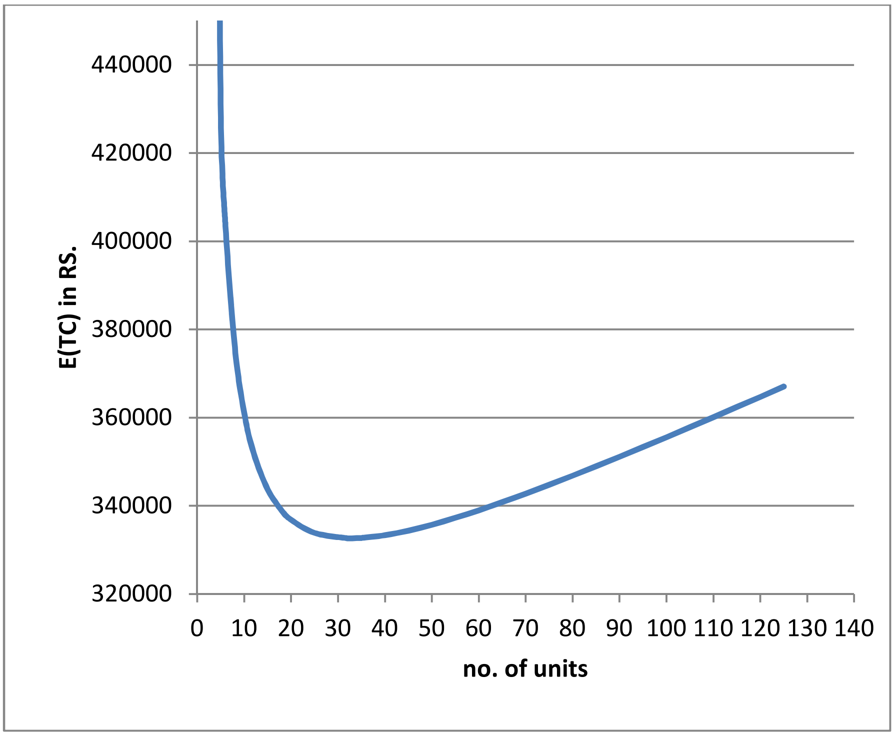

Figure 3.4 Expected total cost as a function of n for Poisson Demand

The graph describes how expected total cost varies with the initial inventory level. The expected total cost of inventory reduces initially. The minimum cost is achieved at 33 units and after that it rises again.

From the result obtained in above example, it is clear that the use of approximate EOQ model gives the lot size that incurs higher expected total cost than that for the optimal lot size obtained for the model proposed in this chapter.