CHAPTER - 4

A DISCRETE INVENTORY MODEL WHEN SALE PRICE VARIES WITH STOCK LEVEL

4.1 INTRODUCTION

An interesting and active area of research in inventory theory is that of mathematical modeling of inventory for deteriorating items. One of the basic assumptions of the traditional inventory models has been the "infinite shelf life" of products while in storage. i.e., a product once in stock remains unchanged and fully usable for satisfying future demand. However, many items are known to deteriorate over time, and hence do not have infinite shelf life. If the rate of deterioration or decay is low and negligible with respect to the cycle length, its effect can be safely ignored. However, in many situations this effect plays a significant role and its impact must be considered explicitly. For example, food stuffs, medicines, photographic film etc. Several inventory models have been proposed, by different authors, which consider the effect of deterioration on inventory management.

Deteriorating items can be classified into two categories as discussed in section 1.1. The first category refers to the items that become decayed, damaged, evaporative, or expired through time, like meat, vegetables, fruit, medicine, flowers, film and so on; the other category refers to the items that lose part or total value over time for one or other reasons. For example, due to new technology or the introduction of alternatives, like computer chips, mobile phones, fashion and seasonal goods, or loss of relevance, as in case of newspapers, magazines and so on. Both the categories have the characteristic of limited life cycle. For the first category, the items have a short natural life cycle. After a certain period, the natural attributes of the items change and then items lose their useable value and hence economic value.

The second category of deteriorating items, also called value-deteriorating items, have a short market life cycle. After a period of popularity/ utility in the market, the items lose their original economic value due to the changes in consumer preference, product upgrading and other reasons. In both the categories, an item may be either completely lost or may yield a selvage value. In each case the items have limited shelf life, the length of which is random in most real life situations.

In the present chapter, we focus on value-deteriorating items. Also we restrict to only those items for which customers typically demand only one item. There are large varieties of such items. Electronic gadgets, domestic appliances, vehicles and fashion goods are some examples of this type.

An inventory problem of deteriorating item was first studied by Whitin (1957). He studied the inventory problem for fashion items deteriorating at the end of the storage period. Most of the authors developed inventory models for the first category of deteriorating items. Ghare and Schrader (1963) were among the first who discussed the usefulness of exponential distribution for modeling deterioration rate. More specifically, exponential distribution was proposed to model the distribution of "time to deteriorate" for modeling inventory of deteriorating items. Covert and Philip (1973) proposed a model for inventory that decay with deterioration modeled by Weibull distribution. Philip (1974), generalized this model by assuming a three-parameter Weibull distribution for 'time to deteriorate". Tadikamalla (1978) considered a similar model assuming a gamma distribution for the deterioration time. Y.K.Shah (1977), generalized the models proposed in above mentioned papers by introducing a model for deteriorating items where rate of deterioration can be modeled by any well behaved probability distribution. It may be noted that all the work referenced above essentially involves modeling of inventory systems for deteriorating items of the first type.

Modeling demand has also been one of the important aspects of developing inventory models. As noted above, It is well-recognized that demand is influenced by the inventory levels for various types of items such as fashion goods, home appliances, electronic gazettes etc. As inventory levels decrease, demands of the items also decrease. In practice, higher stock level for an item induces more consumers to buy it. Similarly, low stocks of certain goods might raise the perception that they are not fresh or not good. For example retail stores with higher level of stock will be able to offer more options to the customers thus pursuing them to come to their stores. Gupta and Vat (1986) were the first, as noted by Ruxian et al. (2010), who proposed an inventory model in which demand rate depends on stock. Baker and Urban (1988) presented an EOQ model for inventory-level-dependent demand pattern. Pal *et al.* (1993) extended the model of Baker and Urban for perishable products that deteriorate at a constant rate. Datta and Pal (1990) presented an inventory model in which the demand rate is dependent on the instantaneous inventory level until a given inventory level is achieved, after which the demand rate becomes constant. Urban (1992) presented a modified variant of the model of Datta and Pal. Padmanabham and Vat (1995) also proposed EOQ models for deteriorating items with stock dependent demand, with a purpose of maximizing profit.

Chang et al. (2010) presented models where demand rate depends not only on the on-display stock level but also on the selling price per unit as well as the amount of shelf display space. They assumed constant holding cost as well as the constant deterioration rate. Bhathavala & Rathod (2012) consider stock-level dependent demand rate and storage time dependent holding cost. They assume that holding cost is a decreasing step function of the time spent in storage.

The work presented in this chapter is an attempt to relate the stock level to the price of value deteriorating items. The general behavior of the buyer is to get attracted to those outlets where stock is well maintained. Nevertheless, from the seller's point of view, as the stock of the said items depletes he tends to reduce price in order to maintain the demand, which may otherwise reduce. A real life example is that of apparels outlets. Buyers would first enter those outlets where more apparels varieties are displayed. However, low pricing is also known to attract the buyers. Thus the effective demand is a function of the stock and price. The compromise factor plays a vital role at low stock outlets as satisfaction level is generally higher at lower prices.

As noted earlier, models developed by most of the authors are essentially for the first category of deteriorating items. In this chapter, we present an inventory model for the value-deteriorating items. We also assume that, holding cost is incurred only for the period during which the inventory items are held in the stock - an implicit assumption usually made only in EOQ models. The demand is generated according to a Poisson process with each customer demanding only one unit. Due to our assumptions, the actual holding cost is random and depends on the time points at which actual demands occur. Also the replenishment is assumed to be instantaneous and hence the new order is placed only when (and as soon as) inventory level reaches zero. This also results into the random cycle length.

4.2. NOTATIONS

Following notations are used in this chapter.

n : Lot size (initial inventory level)

T : Cycle length.

k : Critical inventory level, which when reached, the selling price is reduced.

C_0 : Ordering cost per order.

C_1: holding cost per unit per unit time in the beginning of a cycle.

C_2 : holding cost per unit per unit time after inventory level reduces to k units, $C_2 > C_1$

C : cost of inventory per unit.

θ : Mean customer inter arrival time.(Equivalently, Customer arrival rate = $\frac{1}{\theta}$)

4.3 THE PROPOSED MODEL

Following are the assumptions of the model.

1. Customer arrival process is Poisson with each customer having demand of one unit.

2. Supply is instantaneous.

3. Reorder is placed as soon as inventory level reaches zero.

4. Holding cost is incurred only for the period during which the inventory items are held in the stock.

Suppose inventory is maintained for a discrete item. The stock level is raised to n units in the beginning of a cycle. Initially, the items are sold at price p_1 per unit. However, when inventory level reduces to k units, the units are sold at a reduced price p_2 per unit. As holding cost includes the cost of tied up capital, reduction in the selling price leads to an increased holding cost. In the following development, instead of directly using sale prices p_1 and p_2, we use the resultant holding costs C_1 and C_2 ($>C_1$).

Determination of optimum inventory level

Initially at time $t_0 = 0$, the inventory level is raised to n units. Suppose 1st customer arrives at time t_1, and demand for 1 unit. So at time t_1, inventory level reduces to $(n-1)$ units. Similarly, 2nd customer arrives at time $t_1 + t_2$, and demand for 1 unit, so that at time $t_1 + t_2$, inventory level reduces to $(n-2)$ units, and so on.

At time $t_1 + t_2 + t_3 + \cdots + t_{n-k}$, inventory level reduces to *k* units. These *k* units are then sold at a reduced price, resulting in the increased holding cost C_2. We further assume that as a result of this reduced sale price, the demand rate (i.e customer arrival rate) is maintained at the same level $\frac{1}{\theta}$ as earlier.

As the customer arrival process is Poisson with arrival rate $\frac{1}{\theta}$ the inter-arrival times $t_1, t_2, t_3, \dots t_n$ are identically and independently distributed (iid) exponential random variables with mean θ.

That is,

$t_i \sim Exp(\,\theta)\,, i = 1,2, \dots , n.$

Then the expected value of Cycle length $T = \Sigma t_i$ is given by

$E(T) = n\theta$

In order to obtain the optimal order quantity, we minimize the expected total inventory cost per unit time. The total inventory cost for one cycle is given by

Total cost = Ordering cost + Inventory cost + Holding cost

Ordering cost $= C_0$ ... (4.3.1)

Cost of inventory $= nC$ ... (4.3.2)

Total holding cost

$$= C_1(nt_1 + (n-1)t_2 + (n-2)t_3 + \cdots + (k+1)t_{n-k})$$
$$+ C_2\big(kt_{n-(k-1)} + \cdots + 2t_{n-1} + t_n\big) \quad \dots (4.3.3)$$

Total inventory cost = (4.3.1) + (4.3.2) + (4.3.3)

Thus, the total cost per unit time as a function of initial inventory level n, is

$$TC(n) = \frac{1}{T}\{C_0 + nC + C_1(nt_1 + (n-1)t_2 + (n-2)t_3 + \cdots + (k+1)t_{n-k})$$
$$+ C_2\big(kt_{n-(k-1)} + \cdots + 2t_{n-1} + t_n\big)\}$$

The expected total cost per unit time is

$$E\big(TC(n)\big) = (C_0 + nC)E\left(\frac{1}{T}\right)$$

$$+ C_1\left\{n\,E\left(\frac{t_1}{T}\right) + (n-1)E\left(\frac{t_2}{T}\right) + \cdots + (k+1)\,E\left(\frac{t_{n-k}}{T}\right)\right\}$$

$$+ C_2\left\{kE\left(\frac{t_{n-(k-1)}}{T}\right) + \ldots + 2E\left(\frac{t_{n-1}}{T}\right) + E\left(\frac{t_n}{T}\right)\right\}$$

$$= (C_0 + nC)\,E\left(\frac{1}{T}\right) + C_1\,E\left(\frac{t_i}{T}\right)\left(\frac{(n-k)(n+k+1)}{2}\right)$$

$$+ C_2E\left(\frac{t_i}{T}\right)\left(\frac{\mathrm{k}(k+1)}{2}\right)$$

Now, $E(\frac{t_i}{T})$ can be computed as

$E\left(\frac{t_i}{T}\right) = E\left(E\left(\frac{t_i}{T} \mid T = t\right)\right)$, where inner expectation is conditional on T=t.

As we have already noted in Chapter 3, Appendix 3, the conditional

distribution of $\frac{t_i}{T}|T = t$ is beta of Type-1 with parameters 1 and (n-1)

respectively.

Thus, we have

$$E\left(\frac{t_i}{T}\middle|T = t\right) = \frac{1}{n}$$

Since this conditional expectation is independent of t, this, further, implies that

$$E\left(\frac{t_i}{T}\right) = \frac{1}{n} \qquad \ldots (4.3.4)$$

Also, when n > 1, for the Gamma random variable T, we have

$$E\left(\frac{1}{T}\right) = \frac{1}{\theta(n-\ 1)} \qquad \text{... (4.3.5)}$$

For n=1, T = t_1~ Exp(θ). Hence

$$E\left(\frac{1}{T}\right) \ = \ \frac{1}{\theta}\int_0^\infty e^{-t/\theta}t^{-1}dt$$

This is a Gamma integral with α = 0. However Gamma integral converges if and only if α > 0.

Thus, the integral on the right hand side of $E\left(\frac{1}{T}\right)$ above diverges to ∞. This in turn implies that E(TC(1)) =∞.

Thus n =1 cannot be an optimal solution. We, therefore, assume that n is greater than 1 in the optimization process.

From (4.3.4) and (4.3.5),

$$E(TC(n)) = \ \frac{C_0 + nC}{\theta(n-1)} + \frac{C_1(n-k)(n+k+1)}{2n} + \frac{C_2k(k+1)}{2n} \qquad \text{... (4.3.6)}$$

$$E(TC(n+1)) - E(TC(n)) = \frac{C_1(n^2+n+k^2+k)}{2n(n+1)} - \frac{C_2k(k+1)}{2n(n+1)} - \frac{C_0+C}{\theta n(n-1)}$$

is an increasing function for all n. Thus, the expected total cost $E(TC(n))$ is a convex function of n. This further implies that the cost function has unique minima.

This unique optimal solution is the integer value of n that satisfies

$E(TC(n)) \le E(TC(n-1))$ as well as

$E(TC(n)) \le E(TC(n+1))$

An optimal value of n is the one that minimizes $E(TC(n))$. This is the smallest value of n that satisfies

$$E\big(TC(n)\big) \le E\big(TC(n+1)\big)$$

$$\Rightarrow \frac{C_0 + nC}{\theta(n-1)} + \frac{C_1(n-k)(n+k+1)}{2n} + \frac{C_2 k(k+1)}{2n}$$

$$\le \frac{C_0 + (n+1)C}{\theta n} + \frac{C_1(n+1-k)(n+k+2)}{2(n+1)} + \frac{C_2 k(k+1)}{2(n+1)}$$

$$\Rightarrow \frac{C_0 + nC}{\theta(n-1)} - \frac{C_0 + (n+1)C}{\theta n}$$

$$\le \frac{C_1(n+1-k)(n+k+2)}{2(n+1)} + \frac{C_2 k(k+1)}{2(n+1)}$$

$$- \left(\frac{C_1(n-k)(n+k+1)}{2n} + \frac{C_2 k(k+1)}{2n}\right)$$

$$\Rightarrow \frac{n(C_0 + nC) - (n-1)(C_0 + nC + C)}{\theta n\,(n-1)}$$

$$\le \frac{1}{2n(n+1)}\{C_1(n(n+1-k)(n+k+2) - (n+1)(n-k)(n$$

$$+ k + 1)) + C_2((nk(k+1) - (n+1)(k(k+1))\}$$

$$\Rightarrow \frac{n(C_0 + nC) - n(C_0 + nC) + (C_0 + nC + C)}{\theta(n-1)}$$

$$\le \frac{C_1(n^2 + n + k^2 + k) \; + \; C_2(-k^2 - k)}{2(n+1)}$$

$$\Rightarrow \frac{2(C_0 + C)}{\theta} \leq \frac{(n-1)(C_1(n^2 + n + k^2 + k) - C_2(k^2 + k)}{n+1} \qquad \text{... (4.3.7)}$$

Above inequality implies that the optimal value of n is the smallest integer that satisfies

$$\frac{2(C_0 + C)}{\theta} \leq \frac{n-1}{n+1}(C_1(n^2 + n + k^2 + k) - C_2 k(k+1))$$

The equality holds when the following cubic equation is satisfied

$$n^3 C_1 + n\left(C_1(k^2 + k - 1) - C_2 k(k+1) + 2\frac{C_0 + C}{\theta}\right) + C_2 k(k+1) - C_1 k(k+1) - 2\frac{C_0 + C}{\theta} = 0$$

Using the solution of cubic equation as given by Francois Viete (2006), we obtain solution of above cubic equation as

$$n_i = 2 * \sqrt{\frac{-p}{3}} * cos\left(\frac{1}{3} arccos\left(\frac{3q}{2p}\sqrt{\frac{-3}{p}}\right) - i\frac{2\pi}{3}\right) \quad , i = 0,1,2 \qquad \text{...(4.3.8)}$$

$$where\ \ p = \frac{\left(C_1(k^2+k-1) - C_2 k(k+1) - 2\frac{C_0+C}{\theta}\right)}{C_1}\ , q = \frac{C_2 k(k+1) - C_1 k(k+1) - 2\frac{C_0+C}{\theta}}{C_1}$$

Since the value of arccosine is in the interval $(0, \pi)$, It follows that the argument of cosine at equation (4.3.8) is in the interval $(-\pi, 0)$ for i = 1 and 2. As a result n_1 and n_2 are always negative. Thus n_0 is the only feasible solution of the cubic equation.

The optimal inventory level is, therefore,

$$n^* = \left\lceil 2 * \sqrt{\frac{-p}{3}} * cos\left(\frac{1}{3}\arccos\left(\frac{3q}{2p}\sqrt{\frac{-3}{p}}\right)\right) \right\rceil$$

4.4 ALGORITHM

Step 1: Enter the value of C, C_1, C_2, k, θ

Step 2: Compute *p and q using the following formula*.

$$p = \frac{\left(C_1(k^2+k-1) - C_2k(k+1) - 2\frac{C_0 + C}{\theta}\right)}{C_1}$$

p is always negative because $C_2 > C_1$ and $k^2+k > k^2+k-1$

$$q = \frac{C_2k(k+1) - C_1k(k+1) - 2\frac{C_0 + C}{\theta}}{C_1}$$

Step 3: Compute $s_1 = \sqrt{\frac{-p}{3}}$ and $s_2 = \sqrt{\frac{-3}{p}}$

Step 4: Compute $d_1 = \frac{3q}{2p}$

Step 5: Compute $m_1 = d_1 * s_2$

Step 6: Compute $m_2 = \frac{1}{3}\arccos(m_1)$

Step 7: Compute $n_i = 2 * s_1 * cos(m_2),\ \ i = 0$

Step 8: Output $n = round(n_i + 0.5)$

4.5 AN ILLUSTRATIVE EXAMPLE

Suppose demand for a product is 20 per unit time and customer arrival process is a Poisson process with each customer having demand of one unit. Holding cost per unit per unit time is Rs. 500 before inventory level reduces to k =5 units. Holding cost increases to Rs. 750 when inventory level reduces to 5 units. Ordering cost is Rs.7500 per order. Further, suppose that the purchasing cost per unit is Rs.10000. Here we determine expected total cost and optimal quantity.

Here,

C_0= 7500 , C=10000, C_1=500, C_2=750, θ =0.05., k=5.

Different intermediate values are computed as

p= -1416

q= -1385

s_1= 21.7256

s_2= 0.0460287

d_1= 1.46716

m_1= 0.0675316

m_2= 0.501071

n_0 = 38.1096

Thus, the optimal solution is

n*= 39

with expected total cost = Rs. 219307

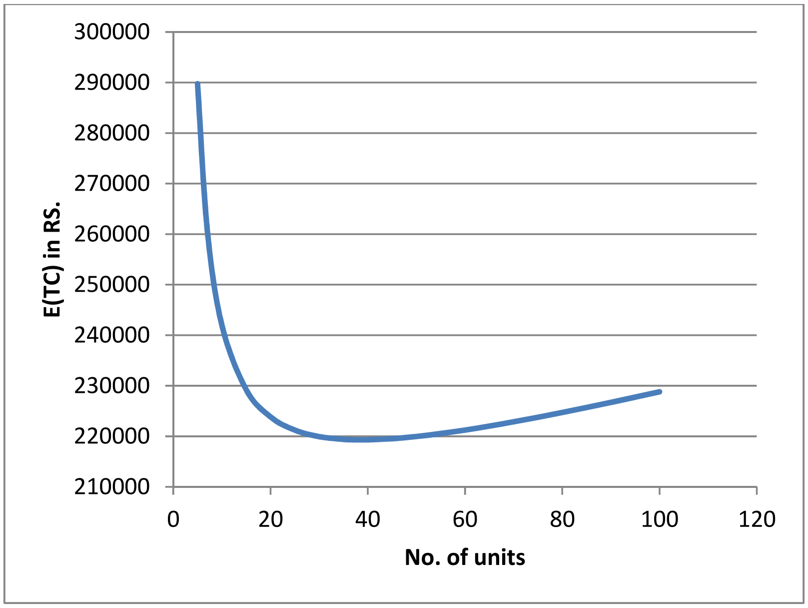

Figure 4.5 Expected total cost as a function of n when

Sale Price varies with Stock Level

The above figure 4.5 indicates the relationship between expected total cost and initial inventory level. The curve is convex, indicating unique minima occurring at n = 39.