CHAPTER - 6

A DISCRETE INVENTORY MODEL WHEN SALE PRICE VARIES AT RANDOM TIME

6.1 INTRODUCTION

The model presented in this chapter is an improvement of the model presented in Chapter 5. In this model we assume that the price of an item is reduced in response to some event that takes place randomly in time. It is generally observed in practice that when a new/ alternative product, with new technology, is launched in the market, the price of the current product is reduced. The time of the launch of new product is almost always random. In this chapter, we present a model for this type of inventory system. The products involved in such inventory systems can be categorized as value deteriorating items for the reasons explained in previous chapters.

Generally in a market it is observed that a price reduction results in an increase in demand. In order to enhance the reduced demand of a product, reduced due to the introduction of new models or new machines, a discounted pricing policy is frequently adopted by the retailers. This policy is commonly used in Mobile shops, electronic items, supermarkets, malls etc. As in Chapter 5, price discounts are assume to result in increased demand and thereby compensate for the reduced demand.

In the inventory model proposed in this chapter, we assume that the items are sold at a discounted price after a random time point T_0 , the time at which some event takes place, such as introduction of an alternative product. The price discount is offered by the seller in order to maintain the demand rate.

6.2 NOTATIONS

Following notations are used in this chapter.

n : Lot size (initial inventory level)

T : Cycle length.

T_0 : The random time after which the price reduction takes place.

α : Expected time at which price reduction tack place .

C_1: Holding cost per unit per unit time before time T_0.

C_2 : Holding cost per unit per unit time after time T_0.

C : Cost of inventory per unit.

$\frac{1}{\theta}$: Number of customer arriving in unit time.

6.3 THE PROPOSED MODEL

Following are the basic assumptions of the proposed model.

1. Customer arrival process is Poisson with each customer having demand of one unit.
2. Supply is instantaneous.
3. Reorder is placed as soon as inventory level reaches zero.
4. Holding cost is incurred only for the period during which the inventory items are in the stock.

Suppose inventory is maintained for a discrete item. The stock level is raised to n units in the beginning of a cycle. Items are sold at a regular price p_1 per unit. At a random time T_0 the selling price is reduced to p_2 in response to some event, such as introduction of an alternative product. The demand is assumed to be Poisson and holding cost is incurred only for the period during which the inventory items are in the stock. Since holding cost includes the cost of tied up capital, reduction in the selling price leads to an increased holding cost. In the following development, instead of directly using sale prices p_1 and p_2, we use the resultant holding costs C_1 and C_2 $(> C_1)$.

Determination of optimum inventory level

Initially at time $t_0 = 0$, the inventory level is raised to n units. Suppose 1st customer arrives at time t_1, and demands for 1 unit. So at time t_1, inventory level reduces to $(n - 1)$ units. Similarly, 2nd customer arrives at time $t_1 + t_2$, and demands for 1 unit, at time $t_1 + t_2$, reducing inventory level to $(n - 2)$ units, and so on.

Initially the holding cost is C_1 per unit per unit time. At a random time T_0, the holding cost increases to C_2 per unit per unit time (as a consequence of reduced selling price). Reorder is placed as soon as the inventory level reaches to zero.

As the customer arrival process is Poisson, the inter-arrival times $t_1, t_2, \ldots t_n$ are identically and independently distributed (iid) exponential random variables.

Since, the arrival rate of the arrival process is $\frac{1}{\theta}$,

$t_i \sim Exp(\theta)\,, i = 1,2, \dots n.$

We further assume that T_0 is a random variable with

$T_0 \sim Exp(\alpha)$

Let x be the number of units sold up to time T_0. In practice, two cases may arise $x = n,$, or $x < n$. If $x < n$, then remaining $n - x$ units will attract the higher holding cost C_2 (due to reduced selling price). Here x should be viewed as a realized value of a r.v.X .

In the proposed model, number of units sold is same as the number of customers arrived. Here value of random variable X depends on T_0. More precisely, if there are no restrictions on the values of X, we have$(X|T_0 = t_0) \sim Poission(t_0/\theta)$

That is,

$$P(X = x|T_0 = t_0) = \frac{e^{-t_0/\theta}\ (t_0/\theta)^x}{x!} \qquad , x = 1,2, \dots.$$

$$\therefore P(X = x) = \int_0^\infty \frac{e^{-t_0/\theta}\ (t_0/\theta)^x}{x!}\, \frac{e^{-t_0/\alpha}}{\alpha}\, dt_0$$

On evaluation of the gamma integral in above expression, we get

$$P(X = x) \ = \frac{\alpha^x\, \theta}{(\alpha + \theta)^{x+1}}$$

$$= (\frac{\alpha}{\alpha + \theta})^x \left(\frac{\theta}{\alpha + \theta}\right)$$

$$= (1 - \frac{\theta}{\alpha + \theta})^x \left(\frac{\theta}{\alpha + \theta}\right)$$

$$= (1-p)^x p \text{ , where } p = \frac{\theta}{\alpha+\theta}$$

The last expression is that of the pmf of Geometric distribution.

However, since the initial stock level is n, the number of units X that are sold will never exceed n. Hence, X follows truncated Geometric distribution, truncated above at n.

Further, the expected value of Cycle length $T = \sum_{i=1}^{n} t_i$ is given by

$$E(T) = n\theta$$

At time $t_1 + t_2 + \ldots + t_n = T$, inventory level becomes zero. At this time reorder is made and 2nd cycle starts.

In order to obtain the optimal order quantity, we minimize the expected total inventory cost per unit time. The total inventory cost for one cycle is given by

Total cost = Ordering cost + Inventory cost + Holding cost

Ordering cost $= C_0$ ... (6.3.1)

Cost of inventory $= nC$ ... (6.3.2)

Let x and t_0 be the realized values of X and T_0 respectively.

Before time t_0, holding cost is C_1 per unit per unit time.

Therefore, the holding cost that incurs before time t_0 is

$$HC_1 = \begin{cases} C_1(nt_1 + (n-1)t_2 + \cdots + (n-(x-1))t_x + (n-x)(t_0 - T_x)) & ; x < n \\ C_1(nt_1 + (n-1)t_2 + \cdots + (n-(x-1))t_n & ; x = n \end{cases}$$

The holding cost that incurs after time T_0, is

HC_2

$$= \begin{cases} C_2\big((n-x)(T_{x+1}-t_0) + (n-(x+1))t_{x+2} + \cdots + t_n\big) & ; x < n \\ 0\,; & ; x = n \end{cases}$$

Here T_x and T_{x+1} are the arrival times of x^{th} and $(x+1)^{th}$ customers respectively,

i.e. $T_x = \sum_{i=1}^{x} t_i$ and $T_{x+1} = T_x + t_{x+1}$

Total holding cost = $HC_1 + HC_2$

$$= \begin{cases} C_1(nt_1 + (n-1)t_2 + \cdots + (n-(x-1))t_x + (n-x)(t_0 - T_x)\,) + \\ C_2\big((n-x)(T_{x+1}-t_0) + (n-(x+1))t_{x+2} + \cdots + t_n\big) & ; x < n \quad \ldots (6.3.3) \\ C_1(nt_1 + (n-1)t_2 + \cdots + \big(n-(x-1)\big)t_n & ; x = n \end{cases}$$

Total cost for one cycle is sum of ordering cost, inventory cost and holding cost.

i.e. Total inventory cost = (6.3.1) + (6.3.2) + (6.3.3)

Thus, for the given values x and t_0 of random variables X and T_0, the total cost per unit time is

$TC(n)$

$$= \begin{cases} \dfrac{1}{T}\left\{\begin{matrix} (C_0 + nc) + C_1(nt_1 + (n-1)t_2 + \cdots + (n-(x-1))t_x + \\ (n-x)(t_0 - T_x)) + C_2\begin{pmatrix} (n-x)(T_{x+1}-t_0) + \\ \big(n-(x+1)\big)t_{x+2} + \cdots + t_n \end{pmatrix} \end{matrix}\right\} & ; x < n \\ \dfrac{1}{T}\left\{(C_0 + nc) + C_1(nt_1 + (n-1)t_2 + \cdots + \big(n-(x-1)\big)t_n\right\} & ; x = n \end{cases}$$

$$\ldots (6.3.4)$$

Since TC(n) is a random variable, we minimize $E(TC(n))$ with respect to n for obtaining the optimum inventory level.

For $n = 1, T = t_1 \sim Exp(\theta)$. Hence

$$E\left(\frac{1}{T}\right) = \frac{1}{\theta}\int_0^{\infty} e^{-t/\theta} t^{-1} dt$$

As explained in earlier chapters, note that the integral on the right hand side of $E\left(\frac{1}{T}\right)$ above diverges to ∞.

Thus $n = 1$ can never be an optimal solution. We, therefore, assume that n is greater than 1.

$E(TC(n))$ is computed as $E(TC(n)) = E(E(TC(n)|X = x))$,

Where $E(TC(n)|X = x)$ is computed as $E(E(TC(n)|X = x, T_0 = t_0))$

Now,

$$E(TC(n)|X = x, T_0 = t_0)$$

$$= E\left(\begin{cases} \dfrac{1}{T}\left\{\begin{matrix}((C_0 + nc) + C_1(nt_1 + (n-1)t_2 + \cdots + (n-(x-1))t_x + \\ (n-x)(t_0 - T_x)) + C_2\left(\begin{matrix}(n-x)(T_{x+1} - t_0) + \\ (n-(x+1))t_{x+2} + \cdots + t_n\end{matrix}\right)\end{matrix}\right\} & ; x < n \\ \dfrac{1}{T}\left\{\{(C_0 + nc) + C_1(nt_1 + (n-1)t_2 + \cdots + (n-(x-1))t_n\right\} & ; x = n \end{cases}\right)$$

Thus, we get

$E(TC(n)|X = x)$

$$= \begin{cases} (C_0 + nc)E\left(\frac{1}{T}\right) + C_1\left\{n\,E\left(\frac{t_1}{T}\right) + (n-1)E\left(\frac{t_2}{T}\right) + \cdots + \left(n-(x-1)\right)E\left(\frac{t_x}{T}\right)\right\} + \\ \qquad (n-x)C_1 E\left(\frac{(T_0 - T_x)}{T}\right) + (n-x)C_2 E\left(\frac{(T_{x+1} - T_0)}{T}\right) \\ +C_2\left\{\left(n-(x+1)\right)E\left(\frac{t_{(x+1)}}{T}\right) + \cdots + 2\,E\left(\frac{t_{n-1}}{T}\right) + E\left(\frac{t_n}{T}\right)\right\} & ; x < n \\ (C_0 + nc)E\left(\frac{1}{T}\right) + C_1\,E\left(\frac{t_i}{T}\right)\left(n + (n-1) + (n-2) + \cdots + (1)\right) & ; x = n \end{cases}$$

Since, $t_i's$ are identically distributed random variables, we have

$E(TC(n)|X = x)$

$$= \begin{cases} (C_0 + nc)\,E\left(\frac{1}{T}\right) + C_1\,E\left(\frac{t_i}{T}\right)\left(n + (n-1) + (n-2) + \cdots + (n-(x-1))\right) + \\ \qquad (n-x)C_1\left\{E\left(\frac{T_0}{T}\right) - E\left(\frac{T_x}{T}\right)\right\} + (n-x)C_2\left\{E\left(\frac{T_{x+1}}{T}\right) - E\left(\frac{T_0}{T}\right)\right\} + \\ C_2\,E\left(\frac{t_i}{T}\right)((n-(x+1)) + \cdots + 2 + 1) & ; x < n \\ (C_0 + nc)E\left(\frac{1}{T}\right) + C_1\,E\left(\frac{t_i}{T}\right)\left(n + (n-1) + (n-2) + \cdots + (1)\right) & ; x = n \end{cases}$$

...(6.3.5)

As per the model assumptions, we have

$$E\left(\frac{1}{T}\right) = \frac{1}{\theta(n-1)} \;,\quad E\left(\frac{t_i}{T}\right) = \frac{1}{n}$$

$$E\left(\frac{T_x}{T}\right) = \frac{x}{n} \;, E\left(\frac{T_{x+1}}{T}\right) = \frac{x+1}{n}$$

Recall that,

$$n + (n-1) + (n-2) + \cdots + (n - (x-1)) = \frac{x}{2}\,(n + n - (x-1))$$

and,

$$(n - (x+1)) + \cdots + 2 + 1) = \frac{n - (x+1)}{2}\,(n - (x+1) + 1)$$

Therefore, $E(TC(n)|X = x)$

$$= \begin{cases} \frac{C_0 + nC}{\theta(n-1)} + C_1 \frac{x(2n-x+1)}{2n} + (n-x)C_1\left(\frac{\alpha}{\theta(n-1)} - \frac{x}{n}\right) + \\ (n-x)C_2\left(\frac{x+1}{n} - \frac{\alpha}{\theta(n-1)}\right) + C_2 \frac{(n-x-1)(n-x)}{2n} & ; x < n \\ \frac{C_0 + nC}{\theta(n-1)} + C_1 \frac{n+1}{2} & ; x = n \end{cases}$$

$$= \begin{cases} \frac{C_0 + nC}{\theta(n-1)} - \frac{n}{(n-1)}\frac{\alpha}{\theta}(C_2 - C_1) + \frac{x}{(n-1)}\frac{\alpha}{\theta}(C_2 - C_1) + \frac{n+1}{2}\,C_2 - \\ \frac{x}{2n}\,(C_2 - C_1) - \frac{(C_2 - C_1)}{2n}x^2 & ;\ x < n \\ \frac{C_0 + nc}{\theta(n-1)} + C_1 \frac{n+1}{2} & ; x = n \end{cases}$$

$$E\big(TC(n)\big) = \sum_{x=0}^{n-1} \left(\frac{C_0 + nC}{\theta(n-1)} - \frac{n}{n-1}\frac{\alpha}{\theta}\,(C_2 - C_1) + \frac{x}{n-1}\,\frac{\alpha}{\theta}\,(C_2 - C_1) \right.$$

$$\left. + \frac{n+1}{2}\,C_2 - \frac{x}{2n}\,(C_2 - C_1) - \frac{(C_2 - C_1)}{2n}x^2\right)\frac{p(1-p)^x}{1-(1-p)^{n+1}}$$

$$+ \left(\frac{C_0 + nC}{\theta(n-1)} + C_1 \frac{n+1}{2}\right)\frac{p(1-p)^n}{1-(1-p)^{n+1}}$$

$$= \sum_{x=0}^{n-1} \left(\frac{C_0 + nC}{\theta(n-1)}\right) \frac{p(1-p)^x}{1-(1-p)^{n+1}} + \left(\frac{C_0 + nC}{\theta(n-1)}\right) \frac{p(1-p)^n}{1-(1-p)^{n+1}}$$

$$- \sum_{x=0}^{n-1} \frac{n}{n-1} \frac{\alpha}{\theta} (C_2 - C_1) \frac{p(1-p)^x}{1-(1-p)^{n+1}}$$

$$- \frac{n}{n-1} \frac{\alpha}{\theta} (C_2 - C_1) \frac{p(1-p)^n}{1-(1-p)^{n+1}}$$

$$+ \frac{n}{n-1} \frac{\alpha}{\theta} (C_2 - C_1) \frac{p(1-p)^n}{1-(1-p)^{n+1}}$$

$$+ \sum_{x=0}^{n-1} \frac{x}{n-1} \frac{\alpha}{\theta} (C_2 - C_1) \frac{p(1-p)^x}{1-(1-p)^{n+1}}$$

$$+ \sum_{x=0}^{n-1} \frac{n+1}{2} C_2 \frac{p(1-p)^x}{1-(1-p)^{n+1}} + \frac{n+1}{2} C_2 \frac{p(1-p)^n}{1-(1-p)^{n+1}}$$

$$+ \frac{n+1}{2} C_1 \frac{p(1-p)^n}{1-(1-p)^{n+1}} - \frac{n+1}{2} C_2 \frac{p(1-p)^n}{1-(1-p)^{n+1}}$$

$$- \sum_{x=0}^{n-1} \frac{x}{2n} (C_2 - C_1) \frac{p(1-p)^x}{1-(1-p)^{n+1}} - \frac{n}{2n} (C_2 - C_1) \frac{p(1-p)^n}{1-(1-p)^{n+1}}$$

$$+ \frac{n}{2n} (C_2 - C_1) \frac{p(1-p)^n}{1-(1-p)^{n+1}} - \sum_{x=0}^{n-1} \frac{x^2}{2n} (C_2 - C_1) \frac{p(1-p)^n}{1-(1-p)^{n+1}}$$

$$- \frac{n^2}{2n} (C_2 - C_1) \frac{p(1-p)^n}{1-(1-p)^{n+1}} + \frac{n^2}{2n} (C_2 - C_1) \frac{p(1-p)^n}{1-(1-p)^{n+1}}$$

$$= \sum_{x=0}^{n}\left(\frac{C_0+nC}{\theta(n-1)}\right)\frac{p(1-p)^x}{1-(1-p)^{n+1}} - \sum_{x=0}^{n}\frac{n}{n-1}\frac{\alpha}{\theta}(C_2-C_1)\frac{p(1-p)^x}{1-(1-p)^{n+1}}$$

$$+ \sum_{x=0}^{n}\frac{x}{n-1}\frac{\alpha}{\theta}(C_2-C_1)\frac{p(1-p)^x}{1-(1-p)^{n+1}}$$

$$+ \sum_{x=0}^{n}\frac{n+1}{2}C_2\frac{p(1-p)^x}{1-(1-p)^{n+1}} - \sum_{x=0}^{n}\frac{x}{2n}(C_2-C_1)\frac{p(1-p)^x}{1-(1-p)^{n+1}}$$

$$- \sum_{x=0}^{n}\frac{x^2}{2n}(C_2-C_1)\frac{p(1-p)^x}{1-(1-p)^{n+1}}$$

$$= \frac{C_0+nC}{\theta(n-1)} - \frac{n}{n-1}\frac{\alpha}{\theta}(C_2-C_1) + \frac{n+1}{2}C_2$$

$$+ \frac{1}{n-1}\frac{\alpha}{\theta}(C_2-C_1)\sum_{x=0}^{n}x\frac{p(1-p)^x}{1-(1-p)^{n+1}} - \frac{1}{2n}(C_2$$

$$- C_1)\sum_{x=0}^{n}x\frac{p(1-p)^x}{1-(1-p)^{n+1}} - \frac{1}{2n}(C_2-C_1)\sum_{x=0}^{n}x^2\frac{p(1-p)^x}{1-(1-p)^{n+1}}$$

... (6.3.6)

$$= \frac{C_0+nC}{\theta(n-1)} - \frac{n}{n-1}\frac{\alpha}{\theta}(C_2-C_1) + \frac{n+1}{2}C_2$$

$$+ \frac{1}{n-1}\frac{\alpha}{\theta}(C_2-C_1)\mu_1'(p,n) - \frac{1}{2n}(C_2-C_1)(\mu_2'(p,n)+\mu_1'(p,n))$$

where $\mu_1'(p,n)$ and $\mu_2'(p,n)$ are the first two raw moments of the truncated geometric distribution with parameter p truncated above at n.

As moments of truncated above Geometric distribution are not available in a closed form, obtaining the formula of E(TC(n)) and hence that of the optimal value of n in a closed form is difficult. Hence, we present an algorithm for the calculation of optimal value of n and the associated minimum cost. An implementation of the same in C++ is presented in the appendix. The algorithm is described in the next section.

6.4 ALGORITHM

Step 1: Enter the value of $C_1, C_2, C_0, C, \theta, \alpha$.

Step 2: Compute $= \frac{\theta}{\theta+\alpha}$.

Step 3: Set $n = 2, E(TC(1)) = large$

Step 4: Compute

$$e_1 = \frac{C_0 + nC}{\theta(n-1)}$$

$$e_2 = \frac{n}{n-1} \frac{\alpha}{\theta} (C_2 - C_1)$$

$$e_3 = \frac{n+1}{2} C_2$$

Step 5: Compute the value of $\mu_1'(p,n)$ as

$$sum1 = \sum_{x=0}^{n} x \frac{P(1-P)^x}{1-(1-P)^{n+1}}$$

Step 6: Compute

$$e_4 = \frac{\alpha}{\theta} \frac{(C_2 - C_1)}{n-1} sum1$$

Step 7: Compute the value of $\mu_2'(p,n)$ as

$$sum2 = \sum_{x=0}^{n} x^2 \frac{P(1-P)^x}{1-(1-P)^{n+1}}$$

Step 8: Compute

$$e_5 = \frac{C_2 - C_1}{2n}(sum1 + sum2)$$

Step 9: Compute $ETC(n) = e_1 - e_2 + e_3 + e_4 - e_5$

Step 10: If $E\big(TC(n)\big) > E(TC(n-1))$ then out put the value of $n-1$ as the optimal solution and stop.

Else set $n = n+1$, and go to step 4.

6.5 AN ILLUSTRATIVE EXAMPLE

Suppose that demand for a product is 20 per month and customer arrival process is a Poisson process with each customer having demand of one unit. Holding cost per unit per unit time is Rs. 500 before time T_0. Holding cost per unit per unit time is Rs. 750 after time T_0. Ordering cost is Rs.10000 per order. Suppose shortages are not allowed and the purchasing cost per unit is Rs.12000. Here we determine expected total cost and optimal quantity. Value of α = 3 months ..

This example is solved with the help of program develop in C++. (see Appendix-F)

$C = 12000$, $C_0 = 10000$, $C_1 = 500$, $C_2 = 750$, $\alpha = 3$, $\theta = 0.05$, $1/\theta = 20$

Using a C++ program presented in Appendix-F,

We get the optimal value of n= 32 with E(TC)= 261002.1 .

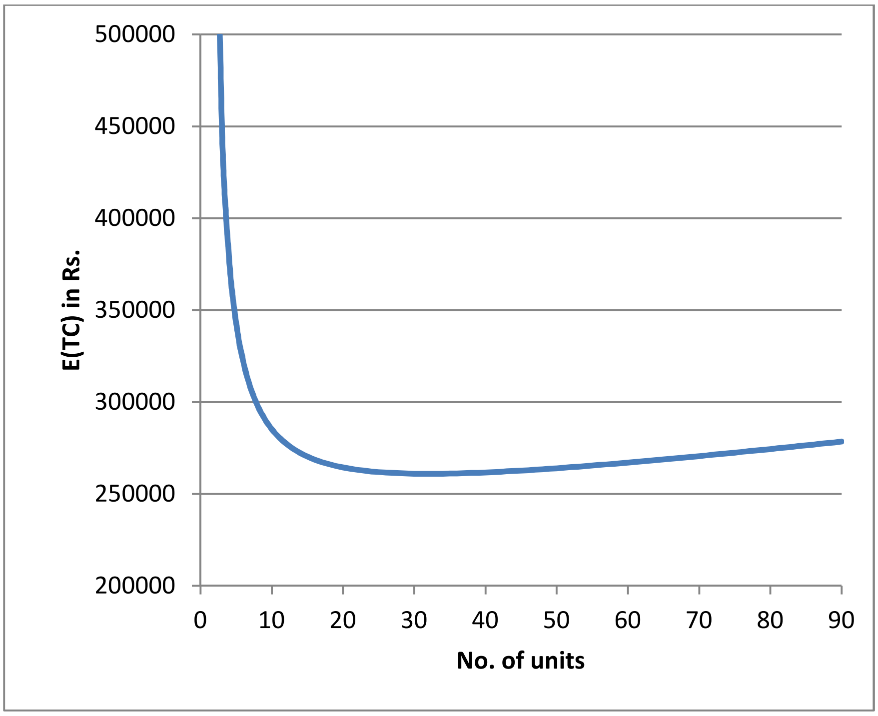

Figure 6.5

Expected total cost as a function of n when Sale Price varies at Random Time

The graph describes the relationship between expected total cost and initial inventory level. As it can be observed, the curve for E(TC(n)) is a convex function of n, with minimum cost achieved at n=32 units and after that it rises again.