SOME PROBLEMS OF TESTIMATION UNDER VARIOUS ASYMMETRIC LOSS FUNCTIONS

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By

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<u>CERTIFICATE OF GUIDE</u>

This is to certify that the thesis entitled “**SOME PROBLEMS OF TESTIMATION UNDER VARIOUS ASYMMETRIC LOSS FUNCTIONS**”, submitted by Mrs. Tejal T. Shah for the award of the degree of Doctor of Philosophy in Statistics, Dept. of statistics, Faculty of Science, The Maharaja Sayajirao University of Baroda, Vadodara, Gujarat, comprises the result of independent and original investigation carried out by her under my guidance.

The matter presented in this thesis incorporates the findings of independent research work carried out by the researcher herself. The matter contained in this thesis has not been submitted elsewhere for the award of any other degree.

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DECLARATION BY CANDIDATE

I hereby declare that the entire work embodied in this thesis has been carried out by me under the supervision and guidance of Prof. Rakesh Srivastava, Department of Statistics, M.S.U., Baroda. I also declare, to the best of my knowledge and belief, this thesis no material previously published or written by any other person except where due reference is made in the text of the thesis.

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RESEARCH PUBLICATIONS

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3. Srivastava, R. and Shah, Tejal : “Some improved Double stage shrinkage testimaors for variance of Normal distribution under Asymmetric Loss Function”.

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INTRODUCTION

1. Introduction

The main Endeavour of any scientific method is to help human being towards its betterment, to this end and statistical studies have been continuously playing an important role.

Today the science of statistics is an indispensible part of any and every sphere of human activity and is extensively applied in framing polices and formulating decision in a large number of diversified fields Covering Natural, Economic, Physical, Social sciences and Life Sciences. According to Prof. P.C Mahalnobis, “statistics is essentially an applied science. Its only justification lies in the help it can give in solving a problem.”

The formulation and growth of the theory of probability during 18^{th} and 19^{th} centuries brought about a sharp and important changes in the basic premises of scientific thinking. Scientific investigators during this period began to realize a close resemblance between the laws of uncertainties governing the outcome of games of chance and the laws of variations observed by them in apparently uncontrolled phenomena in their fields of study. This led astronomers, physicists, geneticists, engineers, agriculturists etc. to believe that a stochastic or probabilistic model (or approach) could possibly explain the variability of observations in fields of scientific inquiry, where such variations were unavoidable.

It became, however, apparent even as early as in nineteenth century that no matter how strongly one believed in the deterministic model, it was not possible to use them beyond limits. A stochastic model was clearly needed as a realistic basis for explaining natural phenomenon characterized by inherent variability. Neyman in Journal of American Statistical Association (1960) reaffirmed that 'currently in the period of dynamic indeterminism in science, there is hardly a serious piece of research, which, if treated realistically does not involve operations on stochastic processes'. It was the growing complexity of physical sciences and later in biological and social sciences that inadequacy of deterministic models was realized and led to the gradual replacement of such models by stochastic models.

1.1 STATISTICS AS A SCIENCE OF INDUCTIVE INFERENCE

As we have already remarked earlier statistics is concerned with collection of data and with their analysis and interpretation. The methods by which data are to be collected has given rise to different techniques and this itself has given a branch or area in statistics called sampling. Next comes the question as to what the data tell us. This answer depends not only on the data but also on the background knowledge of the situation or phenomenon; the latter is formalized in the assumptions under which the analysis enters. The process of inference involved in statistics is of an inductive nature – inferring from particular to the general or from sample to the population. It is here that the effectiveness of statistic lies which has evolved and is in a continuous process of evolving the scientific methodology based on the theory of probability to meet the challenging needs of such inferences. Thus the development in probability theory and statistical inference are to go hand in hand. Statistics today has become an indispensible tool in planning of experiments for any scientific inquiry and in drawing valid inferences on the basis of data that could be quantified. Instead of going into how the data are to be obtained, we would assume for our purpose that they are rather given and describe in brief some principal lines of approach of statistical analysis.

CONDITIONAL INFERENCE

One of the fundamental problems in statistics is that of specification of an appropriate model to represent the phenomenon under study and to make analysis. It is indeed obvious that the validity of statistical inference depends on the appropriateness of the model. In most applications the model is parametric and if it can be determined in advance from theoretical considerations, and statistical inferences can be drawn using classical theory. This is the view of late Prof. R. A. Fisher according to whom there is 1: 1 correspondence between the model and its analysis.

There are situations when we come across data that are collected from operational studies or a researcher feels that it is extremely unlikely that any particular specification will represent exactly the phenomenon under investigation. In the former, data are not taken from well designed experiments or surveys having a specific underlying frame work. In such cases data analysis cannot confine itself to a prescribed model and hence cannot be unique. We have to examine and discuss more or less the adequacy of any proposed framework before we build statistical theories on it. The main difficulty faced by the statistician in analyzing data collected from operational studies is that he has first to evolve a model from the data, test its adequacy on it or a similar data and then to make final inferences. Thus the inferences drawn are always conditional. The decision to use conditional or unconditional inference has to be made by the experimenter (researcher) before the experiment and may be based on his prior knowledge obtained from his own experience and / or of other workers in that field. If the decision is to use unconditional inference, then available inference procedures (Classical or Bayesian) may be used. However, if the decision is to use conditional inference;

then the research worker has to base his inferences on the specification evolved through the data and then to go for final inference.

Examples and need of such inference procedures are abound. They occur in econometrics, regression analysis, ANOVA models, outliers, and other branches of statistics. In all these cases, uncertainties exist and one has to resolve them before making final inferences; hence they were given the name testi-testing, testi-mating and testi-predicting by Bancroft (1975). For testimating a new name 'testimator' has been proposed by Sclove, Morris and Radhakrishnan (1972). In all such cases, where we use conditional inference it is important that the effect of preliminary test(s) on subsequent inference should always be taken into account. This aspect was often neglected by applied statisticians.

Suppose we are interested in the estimation of θ in f(x ; θ) when a random sample of size n say $(x_1, x_2 ...., x_n)$ is available, f is completely known say for θ and in addition either a guess of θ say θ_0 or an interval (θ_1, θ_2) both known, is given in which θ is assumed to lie. This priori information is sometimes available from past experience or similar studies and we are interested in estimators of θ which behave nicely in the neighbourhood of θ_0 (or in an interval). However, we do not assume any distribution of θ but wish to utilize the information about θ.

CLASSICAL INFERENCE

In this type of inference the data are assumed to be repeated values on random variables which, we postulated to follow a joint probability distribution p belonging to some known class P. Frequently, the distributions are indexed by a parameter θ (say) taking values in a set Ω, so that $P = \{P_\theta / \theta \in \Omega\}$. The aim of statistical analysis is to specify a plausible value of θ in terms of a statistic $t = t(x_1, ..., x_n)$ where t is supposed to be measurable. But there is no unique method for the specification of t, though various methods for choosing t have been proposed in the literature. This is the problem of point estimation of θ as enunciated by Fisher about 1920s. If instead of giving a single value of 't' as, an estimate of θ, we determine a set of values for which we can plausibly assert that it does or does not contain θ , we call this estimation by confidence sets or hypothesis testing. It was fist formulated by Jerzy Neyman in his 1937 paper and later developed by Wolfwitz, Stein, Hodges, Guttman and others. It was remarked by them that in some sense estimation by confidence sets or methods may be more meaningful.

In contrast to point estimation in which we try to find out a plausible value of the parameter on the basis of the information provided by the sample observations, in statistical hypothesis testing we are to choose between two possible actions regarding the hypothesized value(s) of the parameter. i.e., to decide that the distribution is a particular member of a family which is known except for the parameters. In the context of testing of hypothesis these two actions are called acceptance or rejection of the hypothesis.

1.2 BAYESIAN INFERENCE

In Bayesian approach the parameter is assumed to be random variable with an a priori density function, this distribution expresses the state of knowledge or ignorance about θ before the sample data are analyzed. Given the probability model, the prior distribution and the data set $(x_1, x_2 ...., x_n)$, Bayes theorem is used to calculate the posterior probability density function $P(\theta / D)$ of θ, where D denotes the prior and sample information and on the basis of posterior distribution inferences about θ are drawn. Thus the Bayesian method of reasoning seems rather deductive.

Bayesian inference is an especially important consideration in those areas of application where the sample data may be either expensive or difficult to obtain, such as reliability and life testing experiments.

PRIOR DISTRIBUTION :

The prior distribution g(θ) on the parameter space **Ω** is specified before data became available and is modified using the data to determine a posterior distribution, which is the conditional distribution of θ given the observations say $x_1,.......,x_n$. The other difficulties in Bayesian analysis are :(i) There is no convincing definition of optimality, (ii) The optimal procedure depend heavily oon the assumed nature of probability model.

Some other concepts used in Bayesian analysis stem from decision theory such as risk, Bayes risk, Minimax and Bayes rules etc. Since the Bayes risk of a decision rule depends on the choice of the prior distribution and is a real number, it is possible to order. The optimal choice then would be the one which minimizes the Bayes risk. How does one select a known density g(θ) to express uncertainty about θ, is a problem which remains open and controversial? In many practical situations the statistician will possess some subjective apriori information concerning possible values of θ. This information may often be summarized and made objective by the choice of a suitable prior distribution on the parameter space. It is perhaps the most difficult task in Bayesian analysis. Although a few guidelines have been given regarding the choice of a prior distribution; yet none seems satisfactory. Summarizing about the Bayes rules we may say that we are interested in them because of (i) they are admissible and (ii) form complete class.

In Bayesian set-up the experimenter expresses his belief about the parameter by prior distribution and his misjudgment by a loss function.

Bayesian methods are now becoming widely accepted as a way to solve applied statistical problems in industries and government. Research groups in various disciplines like econometrics, education, law, archaeology, engineering, medical and life sciences are using Bayesian inferential methods to obtain optimum solutions to their problems.

DIFFERENCE BETWEEN CLASSICAL AND BAYESIAN INFERENCE

In simple language, the main difference between Bayesians and classical statistics is that the Bayesians treat the state of nature (e.g., the value of a parameter) as a random variable, whereas the classical way of looking at is that it's a fixed but unknown value, and that putting a probability distribution on it does not make sense.

Bayesian methods provide alternatives that allow one to combine prior information about a population parameter with information contained in a sample to guide the statistical inference process.

The classical estimation method originally proposed by Hamilton involves a two step procedure in which model parameters are estimated first (usually by maximum likelihood estimation), and inference on hidden states is subsequently drawn holding these parameter estimates fixed.

Advances in computational capacity have more recently spurred a number of papers employing alternative, Bayesian estimation methods based on Monte-Carlo techniques. In contrast to classical methods these methods permit simultaneous inference on both the model parameters and hidden states.

1.3 VARIOUS TYPES OF LOSS FUNCTIONS

Any decision-making situation consists a non-empty set Θ of possible states of nature, sometimes referred to as the parameter space and a non-empty set A of actions available to decision maker. Under these two situations, nature chooses a point θ in Θ and the decision maker without being informed of the choice of nature, chooses an action d in A. As a consequence, there may incur some loss which will depend on d and θ. Thus, loss is a function of θ and d defined the product space Θ x A say L(θ,d). The function L(.,.) is known as the loss function.

In point estimation problems, the action space consists of the set of all possible values of θ. Thus, it may be the whole parameter space or a subset of it. To ease the problem a sampling experiment is often conducted to collect the data. The data is considered to be an observation of the random variable x which is assumed to have a probability distribution f(x/θ), when the true state of nature is θ. The decision maker chooses an estimate/ class of estimates $\hat{\theta}$ as the value of the function of the random variable x say T(x) for the given observed value x i.e. $\hat{\theta} = T(x)$. The function T(.) is called the estimator and its value T(x) when x is observed is the estimate for θ. Naturally, the loss L(θ,d) now reduces to L(θ,T(x)) which is a random variable and depends on the sample outcome.

The basic problem of decision theory is : Given a loss function L(θ,d), a decision d and the risk R(θ,d) which criterion should one choose for adopting d? The ideal solution would be to choose a d for which R(θ,d) is minimum for all θ. Unfortunately, this is not possible. The decision theory as formulated and developed by Wald in a series of paper beginning in 1939 was an attempt to unify the statistical theories of estimation and testing of hypothesis which become especial cases now.

For point estimation a number by loss functions are available in the literature. These can be broadly classified into two groups, viz. symmetric and asymmetric. More generally we may have the idea of General Entropy Loss functions which includes Asymmetric Loss Function(ASL). Both types of loss functions have extensively been used in estimation problems. Among various symmetric loss functions (Berger (1985), Martz & Waller (1982)), the quadratic loss function or the squared error loss function (SELF) is very popular and widely used in Bayesian analysis. The main reason behind its popularity is that, it was used in estimation problems when unbiased estimators of parameter θ were being considered. A second reason is its relationship with classical least square theory. Finally the use of 'SELF' makes the calculation relatively straight forward and simple (mean of the posterior distribution). A number of situations may arise in practice where 'SELF' may be appropriately used, especially when under estimation and over estimation are of equal importance.

Inspite of above mentioned justifications for 'SELF' there may be practical situations when the real loss function may not be symmetric i.e. overestimation and underestimation are not equally penalized. Situations may exist when the overestimation may lead to more serious consequences than the underestimation or vice-versa.

For example suppose that a producer produces some electronic device, wants to estimate the failure rate of his products. If his estimate is larger than the real value, he will have to incur additional resources to improve the technology to increase the reliability of his products. On the other hand if he underestimates the real value, he may lose customers and his market share may decrease because the real reliability of his products will now be less than the value he offers. In extreme cases, underestimating the failure rate may even cause the ruin. Hence, underestimation of the failure rate will lead to worse consequences than overestimation. Similarly, overestimation (space shuttle challenger case **Ref**: Basu and Ebrahimi (1991)) may lead to worse consequences than underestimation. Due to these reasons and others, Berger (1985) points out that justification for 'SELF' has a little merit. In order to bring the statistical model nearer to practical situations, the use of asymmetric loss functions and General Entropy Loss Functions (GELF) is suggested. Varian (1975) in his applied study to real estate assessment introduced an Asymmetric Loss Function called LINEX (Linear Exponential), which rises approximately exponentially on one side of zero and approximately linearly on the other side of zero. This loss function was extensively used by Zellner (1986) in estimation of scalar parameter and prediction of a scalar random variable in Gaussian (normal) model. Use of LINEX function has been justified by Lindley (1968), Zellner and Geisel (1968), Canfield (1970), Smith (1980), Schabe (1986), Basu and Ebrahimi (1991), Pandey and Rai (1992), Srivastava & Rao (1992), Srivastava (1996), Srivastava and Kapasi (1999), Srivastava and Tank (2001), Srivastava and Tanna (2001) and others.

1.4 ASYMMETRIC LOSS FUNCTIONS

Various loss functions have been considered under the category of Asymmetric loss functions and some of them are described as below.

LINEX LOSS FUNCTION

The Linex loss function is an Asymmetric Loss Function, which was introduced by Klebanov (1972) and used by Varian (1975) in the context of real estate assessment. Zellner (1986) used it for estimation of a scalar parameter and prediction of a scalar random variable. Both Zellner (1986) and Varian (1975) have discussed its behaviour and various applications. The linex loss function is defined as $L(\theta, a) = \exp(a(\hat{\theta} - \theta)) - a(\hat{\theta} - \theta) - 1$, a $\neq 0$

For small values of $|a|$,

$$L(\theta, a) \cong \frac{a^2}{2}(\hat{\theta} - \theta)^2$$

Thus, Linex is almost symmetric and not too different from a Squared Error Loss Function (SELF) and, therefore, Bayes estimates and predictions, based on linex loss, are quite near to those obtained from SELF.

MODIFIED LINEX LOSS FUNCTION

According to Basu and Ebrahimi (1991), when the parameter θ is a scale parameter, we may take $\Delta = (\hat{\theta}/\theta) - 1$, where $\hat{\theta}$ is an estimate of θ. They define *modified linex loss* function as

$$L(\Delta) = b[\, e^{a\Delta} - a\Delta - 1], \; b > 0, \; a \neq 0 \quad \text{Where } \Delta = \left(\frac{\hat{\theta}}{\theta} - 1\right) \qquad \text{\_\_\_\_\_\_(1.4.1)}$$

The sign and magnitude of ‘a’ represents the direction and degree of asymmetry respectively. The positive value of ‘a’ is used when overestimation is more serious than under estimation, while a negative value of ‘a’ is used in reverse situations. $L(\Delta)$ rises exponentially when $\Delta < 0$ and almost linearly when $\Delta > 0$. The loss function defined by (1.4.1) is known as the LINEX loss function. ‘b’ is the factor of proportionality.

GENERAL ENTROPY LOSS FUNCTION

Calabria and Pulcini (1996) defined *generalized entropy loss function* as

$$L(\theta,\hat{\theta}) = b\left[\left(\frac{\hat{\theta}}{\theta}\right)^{p} - p\ln\left(\frac{\hat{\theta}}{\theta}\right) - 1\right], \qquad p \neq 0, b > 0$$

______(1.4.2)

as a valid alternative to the modified linex loss.

This loss is a generalization of the entropy loss used by several authors (for example, Dey and Liu, 1992; Dey et al., 1987) where the shape parameter 'p' is equal to unity (1). The more general version of (1.4.2) allows different shapes of the loss function to be considered when p > 0, a positive error $(\hat{\theta} > \theta)$ causes more serious error than a negative error and when p < 0, a negative error $(\hat{\theta} < \theta)$ causes more serious error than the positive error).

In particular, for p = 1, we have entropy loss function given by

$$L(\theta,\hat{\theta}) = b\left(\frac{\hat{\theta}}{\theta} - \ln\frac{\hat{\theta}}{\theta} - 1\right)$$

However, if $\left|\frac{\hat{\theta}-\theta}{\theta}\right| \cong 0,$ we have $L(\theta,\hat{\theta}) \cong \frac{1}{2}\left(\frac{\hat{\theta}}{\theta} - 1\right)^{2}.$ which resembles SELF.

1.5 BAYESIAN POINT ESTIMATION

In Bayesian estimation, statistical inference is made when we are given a model, a distribution of parameters and a loss function associated with the decision, we make for the parameter under this setup and experimenter expresses his belief about the real situation via a prior distribution and the misjudgment by loss function. Before collecting the sample data, the experimenter specifies a prior distribution say g(θ) which reflects his knowledge or ignorance about the parameter on the basis of the sample data. The experimenter specifies the loss function say L(x/θ). The prior information g(θ) with sample information L(x/θ) is then combined by Bayes theorem to get the posterior distribution P(θ/x) as:

$$P(\theta/x) = \frac{L(x/\theta)g(\theta)}{\int L(x/\theta)g(\theta)d\theta} \qquad \text{\_\_\_\_\_\_(1.5.1)}$$

Where integration is taken over the whole parameter space. This posterior distribution P(θ/x) is thus, an inferential statement in the Bayesian view point. Consider that we wish to obtain a point estimate for θ under some specified loss function $L(\theta, \hat{\theta})$ where $\hat{\theta}$ is the estimate of θ. In Bayesian approach an estimate $\hat{\theta}$ is selected such that it minimizes the posterior risk, which is the average loss for the specified prior distribution P(θ/x). Under different loss functions different Bayes estimaters may be obtained for the same prior distribution.

<u>ESTIMATION UNDER SQUARED ERROR LOSS FUNCTION</u>

A loss function which is often used for point estimation problem is the Squared Error Loss Function.

$$L(\theta, \hat{\theta}) = \Delta^2 \qquad \text{\_\_\_\_\_\_(1.5.2)}$$

Where $\Delta = (\hat{\theta} - \theta)$ and may be considered as error due to estimation.

The Bayes estimator under the loss (1.5.2) is the value which minimizes.

$$E[L(\theta, \hat{\theta})/x] = \int (\hat{\theta} - \theta)^2 \, P(\theta/\underline{x}) \, d\theta \qquad \text{\_\_\_\_\_(1.5.3)}$$

Obviously, $\hat{\theta} = \hat{\theta}_s = E(\theta/\underline{x}) = \int \theta \, P(\theta/\underline{x}) \, d\theta$ _____(1.5.4)

Minimizes (1.5.4) and thus posterior mean is the Bayes estimator.

<u>ESTIMATION UNDER LINEX LOSS FUNCTION</u>

The LINEX loss function suggested by Varian (1975) is

$$L(\theta, \hat{\theta}) = b\, e^{a\Delta} - c\Delta - b \;\; ; \;\; a, c \neq 0, b > 0 \qquad \text{\_\_\_\_(1.5.5)}$$

Since, the loss function should be such that it has a minimum value viz. zero at $\theta = \hat{\theta}$ we must have ab = c

Therefore (1.5.5) reduces to

$$L(\theta, \hat{\theta}) = b\,[e^{a\Delta} - a\Delta - 1] \;\; ; \;\; a \neq 0, b > 0 \qquad \_\_\_\_\_(1.5.6)$$

LINEX loss has two constants, a and b which give the freedom to tailor the loss according to our needs by choosing them appropriately. The function for various choices has been shown graphically by Zellner (1986). Thus, LINEX loss could be used in situation where loss function is asymmetric.

While estimating θ by $\hat{\theta}$, and denoting E_{POST} as the posterior expectation we have:

$$L(\Delta) = b\left[e^{a(\hat{\theta}-\theta)} - a(\hat{\theta}-\theta) - 1\right] \;\; ; \quad where \;\; \Delta = (\hat{\theta}-\theta)$$

$$E_{POST}\; L(\Delta) = b\left[\, E_{POST}\; e^{a(\hat{\theta}-\theta)} - a\, E_{POST}(\hat{\theta}-\theta) - 1\right]$$

$$\frac{dE_{POST}}{d\theta} = b\left[\, E_{POST}\; e^{a(\hat{\theta}-\theta)} - a\, E_{POST}(1) - 0\right] = 0$$

$$\Longrightarrow E_{POST}\; e^{a(\hat{\theta}-\theta)} = 1$$

i.e. $\hat{\theta} = -\frac{1}{a}\, log\, E_{POST}\; (e^{-a\theta})$

Provided $E_{POST}\; (e^{-a\theta})$ exist and is finite.

Thus, we see that Bayes estimator which is the mean of posterior probability distribution function under 'SELF', is proportional to the Moment Generating Function of posterior probability distribution function under LINEX loss.

Basu and Ebrahimi (1991) modified the loss function for estimating a scale parameter i.e. they defined Δ as:

$\Delta = \left(\frac{\hat{\theta}}{\theta} - 1\right)$ then

$$L(\Delta) = b\left[e^{a\left(\frac{\hat{\theta}}{\theta}-1\right)} - a\left(\frac{\hat{\theta}}{\theta} - 1\right) - 1\right]$$

$$therefore \ \frac{d\,E_{POST}\,L(\Delta)}{d\theta} = 0$$

$\Rightarrow E_{POST}\left(\frac{1}{\theta}\,e^{a\left(\frac{\hat{\theta}_B}{\theta}-1\right)}\right) = e^{a}E_{POST}\left(\frac{1}{\theta}\right)$, solving this we get $\hat{\theta}_B$, the estimator under L(Δ).

ESTIMATION UNDER GENERAL ENTROPY LOSS FUNCTION

A suitable alternative to modified LINEX loss is the General Entropy Loss (GEL) proposed by Calabria and Pulcini (1996) given by:

$$L_E(\hat{\theta},\theta) \propto \left\{\left(\hat{\theta}/\theta\right)^p - p\ \ln\left(\hat{\theta}/\theta\right) - 1\right\}$$

Whose minimum occurs at $\hat{\theta} = \theta$.

If we are considering prior distributions, then the Bayes estimate of θ under GELF is in a closed form and is given by $\hat{\theta} = \left[E\left(\theta^{-p}\,|x\right)\right]^{-1/p}$ provided that $E_\theta\left(\theta^{-p}\right)$ exists and is finite.

- When p = 1, the Bayes estimate (4.1.1.2) coincides with the Bayes estimate under the weighted squared error loss function $\left(\hat{\theta}-\theta\right)^2/\theta$, used by Varde (1969) for deriving Bayes estimate of R(t).
- When p = -1, the Bayes estimate (4.1.1.2) coincides with the Bayes estimate under the squared error loss function.

The Bayes estimator of θ under entropy loss function is obtained by putting p = 1 in $\hat{\theta} = \left[E\left(\theta^{-p} \mid x\right)\right]^{-1/p}$ which is the posterior harmonic mean.

For the negative values of p, i.e., p = - u (say), the form of the generalized entropy loss function reduces to $L(\theta,\hat{\theta}) = \left(\frac{\theta}{\hat{\theta}}\right)^u - u \ln\frac{\theta}{\hat{\theta}} - 1$.

In particular for u = 1, $L(\theta,\hat{\theta}) = \frac{\theta}{\hat{\theta}} - \ln\frac{\theta}{\hat{\theta}} - 1$. In this case the Bayes estimator works out to be posterior arithmetic mean.

PROBLEM OF TESTIMATION

Any given real life situation can be modeled via some probability distribution having some known mathematical form except for the constants (parameters) involved in it. Almost every parameter has its own physical interpretation in terms of real life situation.

The efforts are to estimate these parameter(s) in the best possible manner so as to provide 'best' estimator(s). Sometimes we might have 'additional' information about the parameter of interest which could be utilized to hopefully improve the estimator. Such type of informations are common in Bio-statistics and health statistics. For example we might know due to past studies that the hemoglobin level of school going girls is θ_0 and we wish to use this information for the estimation of the hemoglobin level for some population of school going girls. We might take this information as such and use it in while proposing an estimator for θ (say the hemoglobin level of entire population under study) or this available information might be tested (verified) using a test of significance and the given information is incorporated on the basis of outcome of this test.

TESTIMATION PROCEDURE:

Suppose that we have the guess information: in the form of a point θ_0 or an interval (θ_1, θ_2) and the sample information $(x_1, x_2 ...., x_n)$ then obtain: (i) The 'best' estimator of θ using $(x_1, x_2 ...., x_n)$ by Maximum Likelihood Estimator or some other suitable method of estimation (ii) Test H_0 : $\theta = \theta_0$ against a suitable alternative (one tailed or two tailed, mostly two tailed), if H_0 is accepted utilize this information, otherwise ignore it. Thus, we combine testing procedure with the estimation procedure and in the literature such procedure has been termed as 'TESTIMATION'.

1.6 **REVIEW OF LITERATURE**

In this section a review of the literature related to the problems under study in the area of inferences based on Asymmetric Loss Function and those utilizing guess information has been made.

Bancroft (1944) was the first statistician to consider the impact of preliminary test of significance on subsequent problem of estimation.

Thompson (1968) was the first to introduce the idea of shrinkage technique using point as well as interval guess. Canfield (1970) introduced the idea of Asymmetric Loss Functions.

Several authors have proposed estimator(s), weighted estimator(s), shrinkage testimators for the scale parameter of single parameter Exponential distribution. Pandey and Srivastava (1987) among others, proposed some improved shrinkage testimators, where the arbitrariness in the choice of shrinkage factor was removed.

Earlier studies were confined to the use of symmetric loss functions (mostly 'SELF') but later on in several studies the superiority of Asymmetric Loss Functions was established, like Zellner (1986), Basu and Ebrahimi (1991), Calabria and Pulcini (1994, 1996) among others..

Srivastava and kapasi (1999) have proposed Conditional – Guess testimator(s) for the mean life in single parameter & two parameter exponential distribution. Srivastava and Tank (2003) have proposed sometimes pool estimator for Exponential distribution. Properties of these estimators have been studied under asymmetric loss function.

Pandey and Srivastava (1987), Pandey and Singh (2007) have proposed shrinkage testimator(s) for the variance of Normal distribution and have studied the properties of these using 'SELF' and asymmetric loss function (ASL).

Katti (1962), Shah(1975), Arnold and Al-Bayatti (1970), Waiker et al. (1989) have proposed double stage shrinkage testimator of the mean for an Exponential distribution and the variance of Normal distribution.

Srivastava and Tanna (2007 & 2012) have proposed double stage shrinkage testimator for the mean life of an Exponential distribution under 'General Entropy Loss Function' and under asymmetric loss function.

Pandey and Singh (1984) considered estimating shape parameter of Weibull distribution by shrinkage towards an interval. Pandey, Srivastava and Malik (1989) studied some shrinkage testimators for the shape parameter of Weibull distribution.

1.7 **AN OUTLINE OF PROBLEMS UNDER INVESTIGATION**

In the present thesis, an attempt has been made to study the properties of various parameters of Exponential distribution, Normal distribution and Weibull distribution using various asymmetric loss functions and we have proposed some improved estimator(s) for various parameter(s) for different probability distributions in terms of reduced risk(s).

CHAPTER – I

Chapter - I is introductory, and it covers the basic idea of Classical and Bayesian Inference procedures. It also provides a brief review of literature. In the same chapter Bayesian estimation procedures under various loss functions have been discussed.

CHAPTER – II

Chapter - II deals with the problems of **one sample** shrinkage testimators of Exponential Distribution and Normal Distribution under Asymmetric Loss Function. The Exponential distribution has a variety of statistical applications in life testing and reliability and other fields. Normal distribution occupies a very important place in Statistical studies. Vaious testimators for different parameters of both the distributions have been proposed and their risk properties have been studied.

Several authors have proposed estimator(s), weighted estimator(s), shrinkage testimators for the scale parameter of single parameter Exponential distribution. Pandey and Srivastava (1987) among others, proposed some improved shrinkage testimators, where the arbitrariness in the choice of shrinkage factor was removed.

Srivastava and kapasi (1999) proposed Conditional – Guess testimator(s) for the same distribution. Srivastava and Tank (2003) proposed sometimes pool estimator for Exponential distribution under asymmetric loss function.

In this chapter, we have proposed single stage shrinkage testimator(s) for the scale parameter of Exponential distribution for several choices of the shrinkage factors and the properties of these have been studied using asymmetric loss functions.

Pandey and Srivastava (1987), Pandey and Singh (2007) have proposed shrinkage testimator(s) for the variance of Normal distribution and have studied the properties of these using 'SELF' and asymmetric loss function (ASL). In this chapter, we have proposed several estimators for the variance of Normal distribution for different choices of shrinkage factors, and the properties of these have been studied using asymmetric loss function. It has been found that the proposed testimators dominate the usual estimator(s) in terms of reduced risk.

Further the use of asymmetric loss function facilitates to provide better control over the 'risk' of the proposed testimators by choosing the degree of asymmetry and level of significance carefully. Recommendations regarding these two have been attempted.

CHAPTER - III

Chapter - III deals with the problems of **double stage** shrinkage testimators of Exponential Distribution and Normal Distribution under Asymmetric Loss Function.

The first stage sample is used to test $H_0 : \theta = \theta_0$ and if H_0 is not rejected, it is suggested to use the prior knowledge being supported by a test, in estimating θ.

However, if H_0 is rejected, then take $n_2 = (n - n_1)$ additional observations $x_{21}, x_{22}, \ldots . x_{2n_2}$ and use the pooled estimator i.e. we do not use the prior knowledge and obtain a second sample to make up for the loss of the prior knowledge and estimate θ using both the samples.

Such techniques were presented by Katti (1962), Shah (1975), Arnold and Al-Bayatti (1970), Waiker et al. (1989). We have proposed ‘Double stage shrinkage testimators’ for the scale parameter of an Exponential distribution and the variance of Normal distribution. Properties of these proposed testimator(s) have been studied under asymmetric loss function and attempts have been made regarding the use of such procedures.

It has been observed that General Entopy Loss Function has appeared as a valid alternative to Modified LINEX loss function, so it is of interest to study the risk properties of various testimators using General Entropy Loss Function (GELF).

In particular not many attempts have been made to study shrinkage testimators under GELF with this motivation the next chapters of the present work have been devoted to such study.

CHAPTER - IV

Chapter – IV has been devoted to the study of risk properties of single stage shrinkage testimators for various parameters of interest in Exponential and Normal distribution under ‘General Entropy Loss Function’. The risk properties of these have been studied and recommendations regarding the degrees of asymmetry and level of significance have been made.

CHAPTER - V

In the Chapter – V, we have extended the work done by Srivastava and Tanna (2007 & 2012) they have proposed double stage shrinkage testimator for the mean life of an Exponential distribution under 'General Entropy Loss Function'. Some new estimators have been proposed by removing the arbitrariness in the choice of shrinkage factors and 'Double stage shrinkage testimators' have been proposed for Exponential and Normal distributions for their mean life and variance respectively. Properties of these testimator(s) have been studied using 'General Entropy Loss Function' and recommendations for sample sizes, level(s) of significance and degrees of asymmetry have been made.

CHAPTER – VI

Pandey and Singh (1984) considered estimating shape parameter of Weibull distribution by shrinkage towards an interval. Pandey, Srivastava and Malik (1989) studied some shrinkage testimators for the shape parameter of Weibull distribution.

In this chapter, we have proposed some improved shrinkage estimators for the shape parameter of the Weibull distribution when it is known apriori that β (shape parameter) lies in the interval (β_1, β_2). We have studied the properties of this estimator using asymmetric loss function and it has been found that it is preferable to the other estimators, in terms of having smaller risk.

Chapter – 2

ONE SAMPLE SHRINKAGE TESTIMATORS UNDER ASYMMETRIC LOSS FUNCTION

2.1 Introduction

The present chapter deals with one sample shrinkage testimators under Asymmetric Loss Function (ASL) for single parameter Exponential distribution and Normal distribution.

2.1.1 Exponential Distribution

Exponential distribution plays an important part in life testing problems. For a situation where the failure rate appears to be more or less constant, the Exponential distribution would be an adequate choice.

Exponential distribution also occurs in several other contexts, such as the waiting time problems. Maguire, Pearson and Wynn (1952) studied mine accidents and showed that time intervals between accidents follow Exponential distribution.

Exponential is a very interesting continuous type distribution due to its being endowed with the Markovian character of having ‘complete lack of memory’. Its importance is stressed by Epstein (1961) by saying that the Exponential distribution occupies as commanding a position in life-testing, fatigue testing and other types of destructive test situations as does the Normal distribution in other areas of statistics. It may be defined as a special case of Gamma or Weibull distribution. Situations such as sampling from the Income-distribution, waiting time for telephonic conversation or waiting time for scooter services etc. can also be modeled by Exponential distribution.

In the estimation of reliability function use of symmetric loss function may be in appropriate as has been recognized by Canfiled (1970). Overestimate of reliability function or average failure time is usually much more serious than underestimate of reliability function or mean failure time. Also, an underestimate of the failure rate results in more serious consequences than an overestimate of the failure rate. For example, in the recent disaster of the space shuttle (Ref: Basu and Ebrahimi (1991)) the management underestimated the failure rate and therefore overestimated the reliability of solid-fuel rocket booster.

2.1.2 <u>Normal Distribution</u>

The Normal (or Gaussian) distribution is often used as a first approximation to describe real-valued random variables that tend to cluster around a single mean value. Normal distribution is commonly encountered in practice, and is used throughout statistics, natural sciences as a simple model for complex phenomenon.

The Normal distribution plays an important role in both the application and inferential statistics. In modeling applications, the normal curve is an excellent approximation to the frequency distributions of observations taken on a variety of variables and as a limiting form of various other distributions. Many psychological measurements and physical phenomena can be approximated well by the Normal distribution. In addition, there are many applications of the Normal distribution in engineering. One application deals with analysis of items which exhibit failure due to wear, such as mechanical devices. Other applications are, the analysis of the variation of component dimensions in manufacturing, modeling global irradiation data, and the intensity of laser light, and so on. Indeed the wide application and occurrence of the Normal distribution in life testing and reliability problems are a wonder. In the context of reliability problems and life testing, a number of failure time data have been examined and it was shown that the Normal distribution give quite a good fit for the most cases.

In the estimation of a parameter sometimes there exists in certain situations some prior information about the parameter which one would like to utilize in order to get a better estimator (say in the sense of efficiency). This prior information could be either in the form of an initial guessed value or an interval in which the parameter lies (Thompson 1968 a, b) or a relation between the parameter e.g. Coefficient of Variation, Kurtosis (Khan 1968, Searles 1964). In all these cases no apriori distribution of the parameter is assumed.

According to Thomson sometimes there is a natural origin say θ_0 of the parameter θ and one would like the MVUE $\hat{\theta}$ of the parameter θ to move it close to θ_0. This leads to shrinkage estimator of θ which performs better (in the sense of smaller mean square error) than $\hat{\theta}$ in the neighbourhood of θ_0. Searles (1974), Pandey and Singh (1977) and others have proposed such estimators utilizing guess value(s) of the parameter coupled with sample observations. In proposing shrinkage estimators the available prior information is always used along with the sample observations. However, if we do not want to use it, indiscriminately, we may decide to use it or not on the evidence of a test of significance. This gives us what is known a preliminary test estimator, pre-test estimator or a testimator. The pre-test estimator or a testimator has two components viz. : (i) when the outcome of the test of significance results in acceptance of the hypothesis $H_0: \theta = \theta_0$, then we use θ_0 along with sample observations which leads to a shrinkage testimator and (ii) the minimum variance unbiased estimator or the minimum mean square error estimator, when the hypothesis is rejected.

Mathematically, a Testimator of the parameter θ is defined as follows

$$\hat{\theta}_{ST} = \begin{cases} k\theta_0 + (1-k)\hat{\theta} \quad or \quad \hat{\theta}_s , & if\ H_0\ is\ accepted \\ \hat{\theta} , & if\ H_0\ is\ rejected \end{cases} \qquad \text{\_\_\_\_\_(2.1.1)}$$

$\hat{\theta}_{ST}$ is the shrinkage estimator of θ with shrinkage factor k ($0 \leq k \leq 1$) and $\hat{\theta}$ is the best estimator of θ.

In the present chapter we have considered shrinkage testimators for (i) Scale parameter of an Exponential distribution, (ii) variance of a Normal distribution, and studied their risk properties. In all these cases it has been assumed that we are given an initial estimate (or guess) of the parameter and a single random sample of size n from the underlying populations. The salient feature of the proposed testimators is that the arbitrariness in the choice of the shrinkage factors has been removed by making it dependent on the test statistics.

In section 2.2 we have proposed four different testimators for the parameter θ (mean life time) of the Exponential distribution and we have studied the risk properties of all the four shrinkage testimators under Asymmetric Loss Function. Section 2.3 deals with the derivation of the risk(s) of these four estimators. Section 2.4 deals with the relative risk(s) of these four estimators. Section 2.5 concludes with the comparison of UMVUE and the proposed shrinkage testimators in terms of their relative risks. Suggestions for the choice of shrinkage factor, level of significance and degrees of asymmetry have been made.

In section 2.6 we have proposed the two different shrinkage testimators for the variance of a Normal distribution and we have studied the risk properties of these two shrinkage testimators under Asymmetric Loss Function. Section 2.7 deals with the derivation of the risk(s) of these two estimators. Section 2.8 deals with the relative risk(s) of these two estimators. Section 2.9 concludes with the comparison of UMVUE and the proposed shrinkage testimatiors in terms of their relative risks. Further in the same section a suggestion for the choice of shrinkage factor, level of significance, degrees of asymmetry have been made.

ASYMMETRIC LOSS FUNCTIONS

The loss function L($\hat{\theta}, \theta$) provides a measure of financial consequences arising from a wrong estimate of the unknown quantity θ. As in many real life situations, particularly in insurance claims, estimating any health statistics parameter the over-estimation and under-estimation are having different impacts. So giving 'equal' importance to these as the squared error loss function (SELF) does, may not be appropriate. Several authors such as canfield (1970), zellner (1986), Basu and Ebrahimi (1991), Srivastava (1996), Srivastava and Tanna (2001), Srivstava and Shah (2010) and others have demonstrated the superiority of the Asymmetric Loss Functions, over squared error loss functions in several contexts.

A useful Asymmetric Loss Function known as LINEX loss function was introduced by Varian (1975), extended by Zellner (1986) is given by

$$L(\Delta) = b[e^{a\Delta} - a\Delta - 1] \;, a \neq 0, \; b > 0 \;\text{ where } \Delta = \left(\frac{\hat{\theta}}{\theta} - 1\right) \qquad \text{\_\_\_(2.1.2)}$$

The sign and magnitude of 'a' represents the direction and degree of asymmetry respectively. Positive values of 'a' are suggested for situations where overestimation is more serious than the under estimation, while negative values of 'a' are recommended in reverse situations. 'b' is constant of proportionality. L(Δ) rises exponentially when $\Delta < 0$ and almost linearly when $\Delta > 0$. Hence, the loss function defined by (2.1.2) is known as LINEAR EXPONENTIAL (LINEX) loss function. 'b' is the factor of proportionality.

2.2 Shrinkage Testimator(s) for Scale Parameter of an Exponential Distribution.

Let X: ($x_1, x_2 ...., x_n$) have the distribution

$$f(x;\theta) = \begin{cases} \frac{1}{\theta}\exp(-x/\theta), & x \geq 0,\ \theta > 0 \\ 0 & , \ otherwise \end{cases}, \qquad \text{\_\_\_\_\_(2.2.1)}$$

It is assumed that the prior knowledge about θ is available in the form of an initial estimate θ_0. We are interested in considering an estimator of θ possibly using the information about θ and the sample observations $x_1, x_2 ...., x_n$ from (2.2.1). We then propose a testimator of θ which can be described as follows:

1. Compute the sample mean $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, which is the best estimator of θ in the absence of any information about θ. Actually it is UMVUE.
2. Test the hypothesis $H_0 : \theta = \theta_0$ against the two sided alternative $H_1 : \theta \neq \theta_0$ at level α using the test statistic $\frac{2n\bar{x}}{\theta_0}$ which follows χ^2 – distribution with 2n degrees of freedom.
3. If H_0 is accepted, i.e., $\chi_1^2 < \frac{2n\bar{x}}{\theta_0} < \chi_2^2$, where χ_1^2 and χ_2^2 are the lower and upper points of χ^2 – distribution with 2n degrees of freedom at a given level of significance, use the conventional shrinkage estimator $\hat{\theta}_{ST}$ with shrinkage factor k ; otherwise, ignore θ_0 and use $\bar{x}$, when the hypothesis H_0 is rejected.

The shrinkage testimator $\hat{\theta}_{ST1}$ of θ is defined as:

$$\hat{\theta}_{ST1} = \begin{cases} k\bar{x} + (1-k)\theta_0 & , \ if\ H_0\ is\ accepted \\ \bar{x} & , \ if\ H_0\ is\ rejected \end{cases} \qquad \text{\_\_\_\_\_(2.2.2)}$$

Estimators of this type with 'k' arbitrary ($0 \le k \le 1$) have been defined and studied in different contexts by Bhattacharya and Srivastava (1974), Hogg (1974), Panday and Shah (1983).

We observe that 'k' defined in (2.2.2) can take any value between '0' and '1'. We know that the test statistic for testing $H_0: \theta = \theta_0$ against the two sided alternative $H_1: \theta \neq \theta_0$ at level α is given by $\frac{2n\bar{x}}{\theta_0}$ which follows χ^2 – distribution with 2n degrees of freedom. Pandey and Srivastava (1987) and others have proposed shrinkage testimator where the arbitrariness in the choice of shrinkage factor has been removed by making it dependent on the test statistics. Waiker (1984) at el. have proposed and studied the properties of shrinkage testimator of the parameter of Exponential distribution.

Now we propose a shrinkage testimator in which the shrinkage factor depends on the test statistics.

The shrinkage testimator $\hat{\theta}_{ST2}$ of θ is defined as:

$$\hat{\theta}_{ST2} = \begin{cases} \left(\frac{2n\bar{x}}{\theta_0 x^2}\right)\bar{x} + \left(1 - \frac{2n\bar{x}}{\theta_0 x^2}\right)\theta_0 & , \ if\ H_0\ is\ accepted \\ \bar{x} & , \ if\ H_0\ is\ rejected \end{cases} \qquad \text{\_\_\_\_\_(2.2.3)}$$

where $k = \frac{2n\bar{x}}{\theta_0 \chi^2}$, $x^2 = (x_2^2 - x_1^2)$. Properties of these estimators $\hat{\theta}_{ST1}$ & $\hat{\theta}_{ST2}$ have been studied by Srivastava and Shah (2010) using Asymmetric Loss Function.

In all these studies it has been shown that shrinkage testimators perform better than the conventional estimator, if k is near zero, n is small, $\boldsymbol{\theta_0}$ (the guess) is in the vicinity of θ. This motivated workers to select a shrinkage factor which approaches to zero rapidly and an obvious choice was to take the **square** of the shrinkage factor.

Thus the shrinkage testimator $\hat{\theta}_{ST3}$ of θ is defined as:

$$\hat{\theta}_{ST3} = \begin{cases} \left(\frac{2n\bar{x}}{\theta_0 x^2}\right)^2 \bar{x} + \left(1 - \left(\frac{2n\bar{x}}{\theta_0 x^2}\right)^2\right)\theta_0 & , \ if\ H_0\ is\ accepted \\ \bar{x} & , \ if\ H_0\ is\ rejected \end{cases} \qquad \_\_\_\_(2.2.4)$$

Where $k = \left(\frac{2n\bar{x}}{\theta_0 x^2}\right)^2$, $x^2 = (x_2^2 - x_1^2)$

It may be noted that different choices of 'k' have been taken by several authors keeping in mind that it should lie between '0' and '1'. But these limits are not attained unless $\chi_1^2 = 0$ or $\chi_2^2 = \infty$. So, we propose another estimator of θ as $\hat{\theta}_{ST4}$ given by

$$\hat{\theta}_{ST4} = \begin{cases} \left(\frac{2n\bar{x}}{\theta_0 x^2} - \frac{x_1^2}{x^2}\right)\bar{x} + \left(1 + \frac{x_1^2}{x^2} - \frac{2n\bar{x}}{\theta_0 x^2}\right)\theta_0 & , \ if\ H_0\ is\ accepted \\ \bar{x} & , \ if\ H_0\ is\ rejected \end{cases}$$

_____(2.2.5)

where $k = \frac{2n\bar{x}}{\theta_0 x^2} - \frac{x_1^2}{x^2}$, $x^2 = (x_2^2 - x_1^2)$ with this choice of 'k' the limits '0' and '1' can actually be attained.

Pandey, Srivastava and Malik (1989) considered another choice of shrinkage factor which lies exactly between 0 and 1.

We have considered all the four different choices of the shrinkage factor(s) and proposed four different estimators.

2.3 Risk of Testimators

In this section we derive the risk of all the four testimators which are defined in the previous section.

2.3.1 Risk of $\widehat{\theta}_{ST1}$

The risk of $\hat{\theta}_{ST_1}$ under L(Δ) is defined by

$$R(\hat{\theta}_{ST_1}) = E[\hat{\theta}_{ST_1} \,|\, L(\Delta)]$$

$$= E\left[k\bar{x} + (1-k)\theta_0 \Big/ \chi_1^2 < \frac{2n\bar{x}}{\theta_0} < \chi_2^2\right] \cdot p\left[\chi_1^2 < \frac{2n\bar{x}}{\theta_0} < \chi_2^2\right]$$

$$+ E\left[\bar{x} \,\Big|\, \frac{2n\bar{x}}{\theta_0} < \chi_1^2 \cup \frac{2n\bar{x}}{\theta_0} > \chi_2^2\right] \cdot p\left[\frac{2n\bar{x}}{\theta_0} < \chi_1^2 \cup \frac{2n\bar{x}}{\theta_0} > \chi_2^2\right]$$

______(2.3.1.1)

$$= e^{-a} \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} e^{a\left[\frac{k(\bar{x}-\theta_0)+\theta_0}{\theta}\right]} f(\bar{x})\,d\bar{x} - a \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} \left[\frac{k(\bar{x}-\theta_0)+\theta_0}{\theta} - 1\right] f(\bar{x})\,d\bar{x} - \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} f(\bar{x})\,d\bar{x}$$

$$+ e^{-a} \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} e^{a\left(\bar{x}/\theta\right)} f(\bar{x})\,d\bar{x} - a \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} \left(\frac{\bar{x}}{\theta} - 1\right) f(\bar{x})\,d\bar{x} - \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} f(\bar{x})\,d\bar{x}$$

______(2.3.1.2)

Where $f(\overline{x}) = \dfrac{1}{\Gamma n}\left(\dfrac{n}{\theta}\right)^n (\bar{x})^{n-1} e^{\frac{-n\bar{x}}{\theta}}\, d\overline{x}$

Straight forward integration of (2.3.1.2) gives

$$R(\hat{\theta}_{ST_1}) = \left\{ I\left(\frac{\chi_1^2\phi}{2}, n\right) - I\left(\frac{\chi_2^2\phi}{2}, n\right) + 1 \right\} \left\{ \frac{e^{-a}}{\left(1 - a/n\right)^n} - 1 \right\} +$$

$$\left[a\left\{ I\left(\frac{\chi_2^2\phi}{2}, n+1\right) - I\left(\frac{\chi_1^2\phi}{2}, n+1\right) \right\}\right] (1-k) +$$

$$\left\{ I\left(\frac{\chi_2^2\phi}{2}, n\right) - I\left(\frac{\chi_1^2\phi}{2}, n\right)\right\} \left\{ \frac{e^{-a}e^{a\phi(1-k)}}{\left(1 - ak/n\right)^n} - ak\phi - a\phi - 1 \right\}$$

______(2.3.1.3)

Where $I(x;p) = \left(1/\Gamma p\right) \int_0^x e^{-x} x^{p-1}\, dx$ refers to the standard incomplete gamma function and $\phi = \frac{\theta_0}{\theta}$

2.3.2 <u>Risk of</u> $\boldsymbol{\hat{\theta}_{ST2}}$

The risk of $\hat{\theta}_{ST_2}$ under L(Δ) given by

$$R(\hat{\theta}_{ST_2}) = E[\,\hat{\theta}_{ST_2} \mid L(\Delta)]$$

$$= E\left[\frac{2n\bar{x}}{\theta_0\chi^2}\left(\bar{x} - \theta_0\right) + \theta_0 \Big/ \chi_1^2 < \frac{2n\bar{x}}{\theta_0} < \chi_2^2 \right] \cdot p\left[\chi_1^2 < \frac{2n\bar{x}}{\theta_0} < \chi_2^2 \right]$$

$$+ E\left[\bar{x} \,\middle|\, \frac{2n\bar{x}}{\theta_0} < \chi_1^2 \cup \frac{2n\bar{x}}{\theta_0} > \chi_2^2 \right] \cdot p\left[\frac{2n\bar{x}}{\theta_0} < \chi_1^2 \cup \frac{2n\bar{x}}{\theta_0} > \chi_2^2 \right]$$

_______(2.3.2.1)

$$= e^{-a} \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} e^{a\left[\frac{\left(\frac{2n\bar{x}}{\theta_0\chi^2}\right)^2\left(\bar{x}-\theta_0\right)+\theta_0}{\theta}\right]} f(\bar{x})\, d\bar{x} \; - \; a \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} \left[\frac{\left(\frac{2n\bar{x}}{\theta_0\chi^2}\right)^2\left(\bar{x}-\theta_0\right)+\theta_0}{\theta} - 1 \right] f(\bar{x})\, d\bar{x}$$

$$- \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} f(\bar{x})\, d\bar{x} + e^{-a} \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} e^{a\left(\bar{x}/\theta\right)} f(\bar{x})\, d\bar{x} - a \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} \left(\frac{\bar{x}}{\theta} - 1\right) f(\bar{x})\, d\bar{x} - \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} f(\bar{x})\, d\bar{x}$$

_______(2.3.2.2)

Where $f(\overline{x}) = \frac{1}{\Gamma n}\left(\frac{n}{\theta}\right)^n (\bar{x})^{n-1} e^{\frac{-n\bar{x}}{\theta}}\, d\overline{x}$

A straight forward integration of (2.3.2.2) gives:

$$R(\hat{\theta}_{ST_2}) = I^* - \frac{2a(n+1)}{\phi(\chi^2)^2} \left\{ I\left(\frac{\chi_2^2\phi}{2}, n+2\right) - I\left(\frac{\chi_1^2\phi}{2}, n+2\right) \right\} + \left(\frac{2n}{x^2} + 1\right)$$

$$\left[a\left\{ I\left(\frac{\chi_2^2\phi}{2}, n+1\right) - I\left(\frac{\chi_1^2\phi}{2}, n+1\right) \right\} \right] + \left\{ \frac{e^{-a}}{\left(1 - a/n\right)^n} - 1 \right\}$$

$$\left\{ I\left(\frac{\chi_1^2\phi}{2}, n\right) - I\left(\frac{\chi_2^2\phi}{2}, n\right) + 1 \right\} - \left\{ I\left(\frac{\chi_2^2\phi}{2}, n\right) - I\left(\frac{\chi_1^2\phi}{2}, n\right) \right\} (a\phi + 1)$$

_______(2.3.2.3)

Where $I^* = e^{a(\phi-1)} \int_{\frac{\chi_1^2\phi}{2}}^{\frac{\chi_2^2\phi}{2}} e^{a\left[\frac{2t^2}{n\phi\chi^2} - \frac{2t}{\chi^2}\right]} \frac{1}{\Gamma n} e^{-t} t^{n-1}\, dt$

and I(x; p) as defined previously.

2.3.3 <u>Risk of</u> $\hat{\theta}_{ST3}$

The risk of $\hat{\theta}_{ST3}$ under L(Δ) defined by

$$R(\hat{\theta}_{ST_3}) = E[\hat{\theta}_{ST_3} \mid L(\Delta)]$$

$$= E\left[\left(\frac{2n\bar{x}}{\theta_0\chi^2}\right)^2\bar{x} + \left[1-\left(\frac{2n\bar{x}}{\theta_0\chi^2}\right)^2\right]\theta_0 \Big/ \chi_1^2 < \frac{2n\bar{x}}{\theta_0} < \chi_2^2\right]\cdot p\left[\chi_1^2 < \frac{2n\bar{x}}{\theta_0} < \chi_2^2\right]$$

$$+ E\left[\bar{x}\,\middle|\, \frac{2n\bar{x}}{\theta_0} < \chi_1^2 \;\cup\; \frac{2n\bar{x}}{\theta_0} > \chi_2^2\right]\cdot p\left[\frac{2n\bar{x}}{\theta_0} < \chi_1^2 \cup \frac{2n\bar{x}}{\theta_0} > \chi_2^2\right]$$

______(2.3.3.1)

$$= \quad e^{-a}\int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} e^{a\left[\frac{\left(\frac{2n\bar{x}}{\theta_0\chi^2}\right)^2(\bar{x}-\theta_0)+\theta_0}{\theta}\right]} f(\bar{x})\,d\bar{x} - \; a\int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}}\left[\frac{\left(\frac{2n\bar{x}}{\theta_0\chi^2}\right)^2(\bar{x}-\theta_0)+\theta_0}{\theta} - 1\right] f(\bar{x})\,d\bar{x}$$

$$-\int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} f(\bar{x})\,d\bar{x} + e^{-a}\int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} e^{a\left(\bar{x}/\theta\right)} f(\bar{x})\,d\bar{x} \;-\; a\int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}}\left(\frac{\bar{x}}{\theta}-1\right) f(\bar{x})\,d\bar{x} - \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} f(\bar{x})\,d\bar{x}$$

_____(2.3.3.2)

Where $\quad f(\bar{x}) = \frac{1}{\Gamma n}\left(\frac{n}{\theta}\right)^n (\bar{x})^{n-1} e^{\frac{-n\bar{x}}{\theta}}$

Straight forward integration of (2.3.3.2) gives

$$R(\hat{\theta}_{ST3}) = I_1^* - I_2^* + \left\{I\left(\frac{\chi_1^2\phi}{2}, n\right) - I\left(\frac{\chi_2^2\phi}{2}, n\right) + 1\right\}\left\{\frac{e^{-a}}{\left(1-a/n\right)^n} - 1\right\}$$

$$- a\left\{I\left(\frac{\chi_1^2\phi}{2}, n+1\right) - I\left(\frac{\chi_2^2\phi}{2}, n+1\right)\right\} - \left\{I\left(\frac{\chi_2^2\phi}{2}, n\right) - I\left(\frac{\chi_1^2\phi}{2}, n\right)\right\}(a+1)$$

_____(2.3.3.3)

Where $$I_1^* = e^{a\phi - a}\int_{\frac{\chi_1^2\phi}{2}}^{\frac{\chi_2^2\phi}{2}} e^{a\left[\frac{4t^3}{n\phi^2(\chi^2)^2} - \frac{4t^2}{\phi(\chi^2)^2}\right]}\frac{1}{\Gamma n}e^{-t}\,t^{n-1}\,dt$$

$$I_2^* = a\int_{\frac{\chi_1^2\phi}{2}}^{\frac{\chi_2^2\phi}{2}}\left\{\left[\left(\frac{2t}{\phi\chi^2}\right)^2\left(\frac{t}{n}-\phi\right)\right]+\phi-1\right\}\frac{1}{\Gamma n}e^{-t}\,t^{n-1}\,dt$$

and I(x;p) refers to incomplete gamma function defined previously.

2.3.4 Risk of $\hat{\theta}_{ST4}$

The risk of $\hat{\theta}_{ST4}$ under L(Δ) given by

$$R(\hat{\theta}_{ST_4}) = E[\hat{\theta}_{ST_4} \mid L(\Delta)]$$

$$= \quad E\left[\left(\frac{2n\bar{x}}{\theta_0\chi^2}-\frac{\chi_1^2}{\chi^2}\right)\bar{x}+\left[1+\frac{\chi_1^2}{\chi^2}-\frac{2n\bar{x}}{\theta_0\chi^2}\right]\theta_0 \Big/ \chi_1^2<\frac{2n\bar{x}}{\theta_0}<\chi_2^2\right]\cdot p\left[\chi_1^2<\frac{2n\bar{x}}{\theta_0}<\chi_2^2\right]$$
$$+E\left[\bar{x}\,\middle|\,\frac{2n\bar{x}}{\theta_0}<\chi_1^2 \;\cup\; \frac{2n\bar{x}}{\theta_0}>\chi_2^2\right]\cdot p\left[\frac{2n\bar{x}}{\theta_0}<\chi_1^2\cup\frac{2n\bar{x}}{\theta_0}>\chi_2^2\right]$$

______(2.3.4.1)

$$= \; e^{-a}\int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} e^{a\left[\frac{\left(\frac{2n\bar{x}}{\theta_0\chi^2}-\frac{\chi_1^2}{\chi^2}\right)(\bar{x}-\theta_0)+\theta_0}{\theta}\right]} f(\bar{x})\,d\bar{x} \; - a\int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}}\left[\frac{\left(\frac{2n\bar{x}}{\theta_0\chi^2}-\frac{\chi_1^2}{\chi^2}\right)(\bar{x}-\theta_0)+\theta_0}{\theta}-1\right]f(\bar{x})\,d\bar{x}$$
$$-\int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} f(\bar{x})\,d\bar{x} \; + e^{-a}\int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} e^{a\left(\bar{x}/\theta\right)} f(\bar{x})\,d\bar{x} \; - a\int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}}\left(\frac{\bar{x}}{\theta}-1\right)f(\bar{x})\,d\bar{x} - \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} f(\bar{x})\,d\bar{x}$$

______(2.3.4.2)

Where $f(\bar{x}) = \frac{1}{\Gamma n}\left(\frac{n}{\theta}\right)^{n}(\bar{x})^{n-1} e^{\frac{-n\bar{x}}{\theta}}$

A straight forward integration of (2.3.4.2) gives:

$$R(\hat{\theta}_{ST_2}) = I_1^* - I_2^* + \left\{ I\left(\frac{\chi_1^2\phi}{2}, n\right) - I\left(\frac{\chi_2^2\phi}{2}, n\right) + 1 \right\} \left\{ \frac{e^{-a}}{\left(1 - a/n\right)^n} - 1 \right\} -$$

$$a\left\{ I\left(\frac{\chi_1^2\phi}{2}, n+1\right) - I\left(\frac{\chi_2^2\phi}{2}, n+1\right) \right\} - \left\{ I\left(\frac{\chi_2^2\phi}{2}, n\right) - I\left(\frac{\chi_1^2\phi}{2}, n\right) \right\} (a+1)$$

______(2.3.4.3)

Where $I_1^* = e^{a(\phi-1)} \int_{\frac{\chi_1^2\phi}{2}}^{\frac{\chi_2^2\phi}{2}} e^{a\left[\frac{2t^2}{n\phi\chi^2} - \frac{2t}{\chi^2} - \frac{\chi_1^2 t}{\chi^2 n} + \frac{\chi_1^2\phi}{\chi^2}\right]} \frac{1}{\Gamma n} e^{-t} t^{n-1}\, dt$

$$I_2^* = a(\phi-1) \int_{\frac{\chi_1^2\phi}{2}}^{\frac{\chi_2^2\phi}{2}} a\left[\frac{2t^2}{n\phi\chi^2} - \frac{2t}{\chi^2} - \frac{\chi_1^2 t}{\chi^2 n} + \frac{\chi_1^2\phi}{\chi^2}\right] \frac{1}{\Gamma n} e^{-t} t^{n-1}\, dt$$

and I(x;p) refers to incomplete gamma function defined previously.

2.4 Relative Risks of $\hat{\theta}_{ST_i}$

A natural way of comparing the risk of the proposed testimators, is to study its performance with respect to the best available estimator $\bar{x}$ in this case, which is also the UMVUE. For this purpose, we obtain the risk of $\bar{x}$ under L(Δ) as:

$R_E(\bar{x}) = E[\bar{x} \,|\, L(\Delta)]$

$$= e^{-a} \int_0^\infty e^{a\left(\bar{x}/\theta\right)} f(\bar{x})\, d\bar{x} - a\int_0^\infty \left(\frac{\bar{x}}{\theta} - 1\right) f(\bar{x})\, d\bar{x} - \int_0^\infty f(\bar{x})\, d\bar{x}$$

_______(2.4.1)

A straight forward integration of (2.4.1) gives

$$R_E(\bar{x}) = \frac{e^{-a}}{(1-a/n)^n} - 1 \qquad \text{\_\_\_\_\_\_\_(2.4.2)}$$

Now, we define the Relative Risk of $\hat{\theta}_{ST_i}$ $i=1...4$ with respect to $\bar{x}$ under L(Δ) as follows

$$RR_1 = \frac{R_E(\bar{x})}{R(\hat{\theta}_{ST_1})} \qquad \text{\_\_\_\_\_\_(2.4.3)}$$

Using (2.4.2) and (2.3.1.3) the expression for RR_1 is given by (2.4.3). It is observed that RR_1 is a function of ϕ, n, α, k and 'a'.

Again, we define the Relative Risk of $\hat{\theta}_{ST_2}$ by

$$RR_2 = \frac{R_E(\bar{x})}{R(\hat{\theta}_{ST_2})} \qquad \text{\_\_\_\_\_\_(2.4.4)}$$

The expression for RR_2 is given by (2.4.4) which can be obtained by using equations (2.4.2) and (2.3.2.3).

Now, we define the Relative Risk of $\hat{\theta}_{ST_3}$ as follows

$$RR_3 = \frac{R_E(\bar{x})}{R(\hat{\theta}_{ST_3})} \qquad \text{\_\_\_\_\_\_(2.4.5)}$$

Using (2.4.2) and (2.3.3.3) the expression for RR_3 is given by (2.4.5).

Finally, we define the Relative Risk of $\hat{\theta}_{ST_4}$ as follows

$$RR_4 = \frac{R_E(\bar{x})}{R(\hat{\theta}_{ST_4})} \qquad \text{\_\_\_\_\_\_(2.4.6)}$$

Same way the RR_4 is given by (2.4.6) which can be obtained by using equations (2.4.2) and (2.3.4.3)

It is observed that RR_2, RR_3 and RR_4 are functions of ϕ, n, α, and 'a'.

2.5 Recommendations for $\hat{\theta}_{ST_i}$

In this section we provide the comparison of UMVUE and the proposed shrinkage testimators in terms of their relative risks. Recommendations regarding the applications of proposed testimators are provided in terms of the range of 'k' and '∅'. The objective of present investigation is also to make recommendations for the degrees of asymmetry and level of significance. The following sections provide these separately for all the proposed testimators.

2.5.1 Recommendations for $\hat{\theta}_{ST_1}$

We observe that the expression for RR_1 is a function of 'k', '∅', 'a', 'n' and 'α'. To study the behaviour of RR_1, we have taken these values as k = 0.2 (0.2)…0.8 , ∅ = 0.2 (0.2)…1.6, α = 1%, 5%, 10%, n = 5, 8, 10 and a = ±1, ±2, ±3 ,'a' is the prime important factor and decides about the seriousness of over/under estimation in the real life situation. It is observed that $\hat{\theta}_{ST_1}$ performs better than the conventional estimator for almost the whole range of k. The performance is best at k = 0.2, n = 8, for a = -1, however as 'k' increases to k = 0.4, there is a sudden change and the performance improves at a = 1 (positive) and the same trend remains for a = 2 and 3 but the range of ∅ changes. It may be stated that for smaller weights a negative value of 'a' is suggested however for higher weights positive value particularly a = 3 should be used. We have taken α = 5% and α = 10% also, it is observed that the $\hat{\theta}_{ST_1}$ still performs better for these values of α^s**,** but the magnitude of relative risk is maximum at α = 1% out of the three values of α, so α = 1% is the **recommended** level of significance. As regards the choice of degree of asymmetry ‘a’ no fixed pattern is observed for various values of ‘k’ i.e. for some values of ‘k’, positive ‘a’ and for some values of ‘k’ (particularly lower), negative values of ‘a’ are recommended (say a = -1 for k = 0.2). Looking at the different values of ‘a’ for different choice of ‘k’ it seems more logical to remove the arbitrariness in the choice of ‘k’. $\hat{\theta}_{ST_2}$ removes this arbitrariness and our conclusions for $\hat{\theta}_{ST_2}$ are as follows:

2.5.2 Recommendations for $\hat{\theta}_{ST_2}$

There will be too many tables for varying ‘∅’, ‘α’, and ‘a’ all the tables are not presented here. However our recommendations based on all these computations are summarized as follows:

- For small n = 5 and for different levels of significance considered here $\hat{\theta}_{ST_2}$ performs better than the usual estimator in the whole range of ∅. However, its performance is best for a = ±3, (still better for a = 3) and α = 1%. Hence it is **recommended** to use the proposed estimator for the positive values of ‘a’ and small values of ‘n’. Similar results hold for n = 8 and 10 however the magnitude of RR_2 is maximum for **n = 8**.
- For α = 5% and for n = 5, 8, 10 and for 0.2 ≤ ∅ ≤ 1.6, the magnitude of relative risk is still higher, i.e. usual estimator has more risk under L(Δ) compared to $\hat{\theta}_{ST_1}$. Again, $\hat{\theta}_{ST_2}$ performs better for positive values of “a”, The higher magnitude of relative risk values implies better risk control in this situation, for the proposed testimator $\hat{\theta}_{ST_2}$ compared to $\bar{x}$.
- For α = 10%, rest of the findings are same, i.e., values of n considered here, range of ∅ (0.2 ≤ ∅ ≤ 1.6) and a = ±1, ±2, ±3. But comparing the values of

relative risks for varying α [s] (the level of significance) ; It is observed that the magnitude of these values is maximum for α = 1% and a = 1 for all the values as "n" considered here and for $0.2 \leq \emptyset \leq 1.6$

So, it is recommended to use $\hat{\theta}_{ST_2}$ for n = 8, a = 3, $0.2 \leq \emptyset \leq 1.6$ and α = 1% However, it performs well for other values of 'n' and 'a' also, considered here, but for the above values its performance is at its best.

2.5.3 **Recommendations for** $\hat{\theta}_{ST_3}$

For various values of n = 5, 8, 10 by fixing α = 1% and also varying the degree of asymmetry 'a' = ±1, ±2, ±3, it is observed that the magnitude of relative risk of $\hat{\theta}_{ST_3}$ is higher for all these values of n^s and a^s for the whole range of ∅. However, it is still higher for the positive values of 'a' in particular a=3. It is suggested therefore, to use this estimator for a=3, α = 1%.

Next we change the level of significance to α = 5% for the same set of values of other parameters, again $\hat{\theta}_{ST_3}$ performs better than the conventional estimator in the whole range of '∅' and for different values of 'n' and 'a'. However the magnitude of relative risk is higher in case of α = 1% compared to α = 5%.

While taking α = 10% and observing the behavior of relative risk, it is found that $\hat{\theta}_{ST_3}$ performs better for positive values of 'a' in particular for a = 2.

In all the above situations it is observed that the magnitude of relative risk decreases as 'α' increases and shows higher values of it for 'positive' values of a.

So, we recommend to use $\hat{\theta}_{ST_3}$ for all values of 'n' and α^s considered here. In particular its performance is at its best for α = 1%, a = 3 and n = 8, 10.

2.5.4 Recommendations for $\hat{\theta}_{ST_4}$

The testimator $\hat{\theta}_{ST_4}$ behaves nicely compared to the conventional estimator in the sense of having 'smaller' risk for different values of 'n', 'α', and 'a'. In fact $\hat{\theta}_{ST_4}$ has lower risk for almost whole range of ∅ = 0.2(0.2) 1.6. As 'n' increases the magnitude of relative risk decreases and it is lowest for n = 10, a = -1, for α = 1%. However, for smaller values of n i.e. n = 5 and n = 8, $\hat{\theta}_{ST_4}$ has better control over risk values and in particular for n = 5 and a = 3 the magnitude of relative risk is highest.

For the other value of α = 5% and different values of n = 5, 8, and 10, $\hat{\theta}_{ST_4}$ performs better for higher positive values of 'a' compared to the negative values of 'a'. Particularly for a = -3, -2 there is not much difference in the performances however, the trend starts changing from a = 1 and the highest magnitude of it is observed at a =3 for the values of n, in particular for n = 5, the gain is maximum, which remains true for n = 8 and to some extent for n = 10 for the whole range of ∅.

Finally taking α = 10% for various values of 'n' and 'a' again the performance of $\hat{\theta}_{ST_4}$ is better compared to the conventional estimator, in particular for n = 5 and a = 2, a = 3, still the magnitude of relative risk is higher for a = 3. For n = 10 and for the negative values of 'a' the performance of its relative risk is not so good as compared to conventional estimator.

CONCLUSIONS:

1. It is concluded that both $\hat{\theta}_{ST_3}$ and $\hat{\theta}_{ST_4}$ perform better than the UMVUE for almost the whole range of ∅ = 0.2 (0.2) 1.6, various values of n = 5, 8, 10 and different 'positive' values of 'a'. The performance is not so good for the negative values of 'a'.
2. Comparing the values of relative risk(s) of $\hat{\theta}_{ST_3}$ and $\hat{\theta}_{ST_4}$, it is observed that the magnitude of relative risk is higher for $\hat{\theta}_{ST_3}$, so the choice of weights (Shrinkage factor) suggested is to take the '**square**' of the shrinkage factor making it 'dependent' on test statistics.
3. It is observed that using the Asymmetric Loss Function the effective range of ∅ for which $\hat{\theta}_{ST_3}$ or $\hat{\theta}_{ST_4}$ perform better than the usual estimator **increases** considerably as compared to the **same in case of squared error loss function** .
4. In particular both the testimators $\hat{\theta}_{ST_3}$ and $\hat{\theta}_{ST_4}$ perform better for a = 3, α = 1% and n = 5. Positive value of 'a' indicate that it should be used in those situations where over-estimation is more serious than underestimation, which remains true in case of insurance and re-insurance problems.

Tables showing relative risk(s) of proposed testimator(s) with respect to the best available estimator.

Table : 2.5.1.1 **Relative Risk of** $\hat{\theta}_{ST_1}$ α = 1% , n = 5, a = 3

∅	k = 0.2	k = 0.4	k = 0.6	k = 0.8
0.20	1.14	1.139	1.132	1.104
0.40	2.35	2.363	2.231	1.796
0.60	4.994	4.847	4.182	2.747
0.80	5.484	5.384	5.558	2.985
1.00	7.01	6.884	6.813	3.743
1.20	5.007	5.805	5.08	2.414
1.40	3.547	4.792	3.578	2.097
1.60	1.872	2.516	2.499	1.813

Table : 2.5.1.2 **Relative Risk of** $\hat{\theta}_{ST_1}$ α = 1% , n = 8, a = 3

∅	k = 0.2	k = 0.4	k = 0.6	k = 0.8
0.20	1.035	1.135	1.032	1.023
0.40	2.726	2.752	2.67	1.416
0.60	4.516	4.742	3.672	2.093
0.80	5.648	6.379	4.874	2.191
1.00	7.952	7.332	5.137	4.971
1.20	5.452	5.507	2.877	2.708
1.40	2.473	2.887	1.857	1.453
1.60	1.593	1.924	1.206	1.224

Table : 2.5.1.3 **Relative Risk of** $\hat{\theta}_{ST_1}$ α = 1%, n = 8, k = 0.2

∅	a = -1	a = -2	a = -3	a = 1	a = 2	a = 3
0.20	0.777	0.824	0.826	1.163	1.059	1.035
0.40	0.31	0.393	0.417	2.525	2.409	1.726
0.60	0.41	1.548	1.611	3.185	3.17	3.516
0.80	1.327	2.836	2.147	5.678	4.517	4.648
1.00	7.861	6.177	5.585	6.878	6.385	5.952
1.20	5.257	4.641	3.796	3.946	3.958	3.452
1.40	1.796	1.663	1.831	1.928	1.139	1.473
1.60	0.833	0.835	0.96	0.434	0.493	0.593

Table : 2.5.2.1 **Relative Risk of** $\hat{\theta}_{ST_2}$ α = 1%, n = 5

∅	a = -1	a = -2	a = -3	a = 1	a = 2	a = 3
0.20	0.511	0.596	0.61	1.03	1.263	1.153
0.40	1.337	1.47	1.535	2.529	2.397	2.689
0.60	1.568	2.813	2.967	3.227	3.697	4.451
0.80	2.383	3.917	4.357	4.635	4.988	5.131
1.00	3.418	4.376	5.560	5.088	6.05	6.953
1.20	2.863	4.073	3.837	3.298	3.687	5.94
1.40	1.987	2.031	2.384	2.218	2.537	2.421
1.60	1.031	1.102	1.331	1.654	1.824	1.361

Table : 2.5.2.2 **Relative Risk of** $\hat{\theta}_{ST_2}$ α = 1%, n = 8

∅	a = -1	a = -2	a = -3	a = 1	a = 2	a = 3
0.20	0.74	0.796	0.799	1.217	1.075	1.042
0.40	1.292	1.405	1.457	2.289	2.49	2.173
0.60	2.394	2.585	2.698	3.074	4.374	4.346
0.80	3.051	3.489	4.791	4.246	5.618	6.676
1.00	4.398	5.162	6.000	5.284	6.945	7.321
1.20	2.362	4.888	4.246	3.853	3.166	4.534
1.40	1.903	2.694	2.854	2.829	2.997	2.216
1.60	0.918	1.875	1.994	1.414	1.476	1.566

Table : 2.5.2.3 **Relative Risk of** $\hat{\theta}_{ST_2}$ α = 5%, n = 5

∅	a = -1	a = -2	a = -3	a = 1	a = 2	a = 3
0.20	0.64	0.707	0.712	1.387	1.125	1.074
0.40	1.346	1.468	1.526	2.568	2.848	2.814
0.60	1.426	1.639	2.783	3.349	3.178	3.858
0.80	2.858	2.33	3.713	3.842	4.059	4.918
1.00	3.468	4.358	5.224	4.634	5.705	5.714
1.20	2.648	2.185	3.285	2.718	3.42	3.977
1.40	1.996	2.058	2.006	1.887	2.302	2.193
1.60	1.66	1.643	1.721	0.532	1.757	1.269

Table : 2.5.2.4 **Relative Risk of** $\hat{\theta}_{ST_2}$ α = 5%, n = 8

∅	a = -1	a = -2	a = -3	a = 1	a = 2	a = 3
0.20	0.865	0.895	0.894	1.082	1.03	1.017
0.40	0.346	1.457	1.501	2.889	2.213	1.523
0.60	1.317	1.48	2.583	3.834	3.592	3.959
0.80	1.633	2.006	3.286	4.481	4.075	4.202
1.00	2.505	3.319	4.006	5.561	6.554	6.761
1.20	2.44	2.757	3.861	3.184	3.692	4.207
1.40	1.856	2.484	2.739	2.567	2.806	2.052
1.60	1.256	1.607	1.456	1.334	1.445	1.555

Table : 2.5.4.1 **Relative Risk of** $\hat{\theta}_{ST_4}$ α = 1%, n = 5

∅	a = -1	a = -2	a = -3	a = 1	a = 2	a = 3
0.20	0.348	0.455	0.502	1.145	1.487	1.185
0.40	0.165	0.246	0.303	1.584	1.805	1.547
0.60	1.233	1.346	1.435	1.921	2.051	2.491
0.80	1.644	1.918	2.155	2.002	2.679	3.985
1.00	2.908	3.724	4.625	3.111	4.019	5.748
1.20	1.483	1.096	3.966	2.455	3.018	2.151
1.40	0.305	0.77	2.581	1.251	1.562	1.256
1.60	0.235	0.687	1.741	0.159	0.334	0.74

Table : 2.5.4.2 **Relative Risk of** $\hat{\theta}_{ST_4}$ α = 1%, n = 8

∅	a = -1	a = -2	a = -3	a = 1	a = 2	a = 3
0.20	0.603	0.701	0.732	1.579	1.15	1.065
0.40	1.15	1.225	1.273	2.467	1.875	2.166
0.60	1.161	1.244	1.304	2.486	3.193	3.056
0.80	1.426	2.625	2.779	3.201	4.255	3.827
1.00	2.404	3.893	4.775	3.874	4.511	5.151
1.20	1.331	1.749	1.328	2.294	3.602	3.998
1.40	1.203	1.52	1.073	1.152	2.297	1.488
1.60	0.155	0.454	1.15	0.095	1.17	0.258

2.6 Shrinkage Testimator for the Variance of a Normal Distribution

Shrinkage testimators for the mean μ of a Normal distribution N(μ, σ^2) when σ^2 is known or unknown, have been proposed by Waiker, Schuurman and Raghunandan (1984). In this section we have proposed single sample shrinkage testimator(s) for the variance of a Normal distribution. Let X be normally distributed with mean μ and variance σ^2, both being unknown. It is assumed that the prior knowledge about σ^2 is available in the form of an initial estimate σ_0^2. Using the sample observations $x_1, x_2 ......, x_n$ and possibly the given information we wish to construct a shrinkage testimator for the variance. The procedure described as follows:

1. First test with a sample of size n, the null hypothesis $H_0 : \sigma^2 = \sigma_0^2$ against the alternative $H_1 : \sigma^2 \neq \sigma_0^2$ using the test statistics $\frac{\nu s^2}{\sigma_0^2}$, where ν = (n -1) and $s^2 = \frac{1}{n-1}\sum(x_i - \bar{x})^2$. The test statistics is distributed as χ^2 with ν degrees of freedom.
2. If H_0 is accepted at α level of significance i.e. $x_1^2 < \frac{\nu s^2}{\sigma_0^2} < x_2^2$ where x_1^2 and x_2^2 are the lower and upper points of the uniformly most powerful unbiased (UMPU) test of H_0, use the conventional shrinkage estimator with shrinkage factor $k = \frac{\nu s^2}{\sigma_0^2 x^2}$, which is inversely proportional to χ^2 and it depends on the test statistic, so the arbitrariness in the choice of shrinkage factor has been removed by making it dependent on the test statistic.
3. If H_0 is rejected, use s^2, the Uniformly Minimum Variance Unbiased Estimator (UMVUE) as the estimator of σ^2.

Now, the proposed shrinkage testimator $\hat{\sigma}^2_{ST1}$ of σ^2 is

$$\hat{\sigma}^2_{ST1} = \begin{cases} k_1 s^2 + (1-k_1)\sigma_0^2 & , \ if\ H_0\ is\ accepted \\ s^2 & , \ otherwise \end{cases}$$

Where $k_1 = \frac{\nu s^2}{\sigma_0{}^2\, x^2}$

Estimators of this type with an arbitrary k ($0 \le k \le 1$) have been proposed by Pandey and Singh (1976,77), Srivastava (1976) and others. In these studies it has been shown that the shrinkage testimators work well if k is near zero, n is small and $|\sigma^2 - \sigma_0^2\ |$ is also small. Hence, we should select the shrinkage factor which approaches to zero rapidly. We have, therefore, define another shrinkage Testimator $\hat{\sigma}^2_{ST2}$ of σ^2 by taking square of the shrinkage factor $k_2 = k_1^2$.

$$\hat{\sigma}^2{}_{ST_2} = \begin{cases} \left(\frac{\nu s^2}{\sigma_0{}^2\, x^2}\right)^2 s^2 + \left[1 - \left(\frac{\nu s^2}{\sigma_0{}^2\, x^2}\right)^2\right]\sigma_0{}^2 & ;\ if\ \ H_0\ \ is\ accepted \\ s^2 & ; \quad otherwise \end{cases}$$

2.7 <u>Risk of Testimators</u>

In this section we derive the risk of these two testimators which are defined in the previous section.

2.7.1 <u>Risk of</u> $\hat{\boldsymbol{\sigma}}^2_{\boldsymbol{ST1}}$

The risk of $\hat{\sigma}^2{}_{ST1}$ under L(Δ) is defined by

$$\begin{aligned} R(\hat{\sigma}^2{}_{ST_1}) &= E[\,\hat{\sigma}^2{}_{ST_1} | \, L(\Delta)] \\ &= E\left[k_1 s^2 + (1-k_1)\sigma_0^2 \Big/ \chi_1^2 < \frac{\upsilon s^2}{\sigma_0^2} < \chi_2^2\right]\cdot p\left[\chi_1^2 < \frac{\upsilon s^2}{\sigma_0^2} < \chi_2^2\right] \\ &\quad + E\left[s^2 \;\middle|\; \frac{\upsilon s^2}{\sigma_0^2} < \chi_1^2 \ \cup\ \frac{\upsilon s^2}{\sigma_0^2} > \chi_2^2\right]\cdot p\left[\frac{\upsilon s^2}{\sigma_0^2} < \chi_1^2 \cup \frac{\upsilon s^2}{\sigma_0^2} > \chi_2^2\right] \end{aligned}$$

_____ (2.7.1.1)

$$= e^{-a} \int_{\frac{\chi_1^2 \sigma_0^2}{\upsilon}}^{\frac{\chi_2^2 \sigma_0^2}{\upsilon}} e^{a\left[\frac{\frac{\upsilon s^2}{\sigma_0^2 \chi^2}\left(s^2 - \sigma_0^2\right) + \sigma_0^2}{\sigma^2}\right]} f(s^2)\, d s^2 - a \int_{\frac{\chi_1^2 \sigma_0^2}{\upsilon}}^{\frac{\chi_2^2 \sigma_0^2}{\upsilon}} \left[\frac{\frac{\upsilon s^2}{\sigma_0^2 \chi^2}\left(s^2 - \sigma_0^2\right) + \sigma_0^2}{\sigma^2} - 1\right] f(s^2)\, d s^2$$

$$- \int_{\frac{\chi_1^2 \sigma_0^2}{\upsilon}}^{\frac{\chi_2^2 \sigma_0^2}{\upsilon}} f(s^2)\, ds^2 + e^{-a} \int_{0}^{\frac{\chi_1^2 \sigma_0^2}{\upsilon}} e^{a\left(s^2 / \sigma^2\right)} f(s^2)\, ds^2 + e^{-a} \int_{\frac{\chi_2^2 \sigma_0^2}{\upsilon}}^{\infty} e^{a\left(s^2 / \sigma^2\right)} f(s^2)\, ds^2$$

$$- a \int_{0}^{\frac{\chi_1^2 \sigma_0^2}{\upsilon}} \left(\frac{s^2}{\sigma^2} - 1\right) f(s^2)\, ds^2 - a \int_{\frac{\chi_2^2 \sigma_0^2}{\upsilon}}^{\infty} \left(\frac{s^2}{\sigma^2} - 1\right) f(s^2)\, ds^2 - \int_{0}^{\frac{\chi_1^2 \sigma_0^2}{\upsilon}} f(s^2)\, ds^2 - \int_{\frac{\chi_2^2 \sigma_0^2}{\upsilon}}^{\infty} f(s^2)\, ds^2$$

______(2.7.1.2)

Where $f(s^2) = \frac{1}{2^{\upsilon/2}\, \Gamma\left(\upsilon/2\right)} \left(s^2\right)^{\frac{\upsilon}{2} - 1} e^{\left(-\frac{1}{2} \frac{\upsilon s^2}{\sigma^2}\right)} ds^2$

Straight forward integration of (2.7.1.2) gives

$$R(\hat{\sigma}_{ST1}^2) = \left(\frac{\sigma^2}{\nu}\right)^{\nu/2} \left[\begin{array}{c} I^* - \frac{2a}{\lambda \chi^2} \left(\frac{\nu}{2} + 1\right) \left\{ I\left(\chi_2^2 \lambda, \frac{\nu}{2} + 2\right) - I\left(\chi_1^2 \lambda, \frac{\nu}{2} + 2\right) \right\} \\ + \; a \left\{ I\left(\chi_2^2 \lambda, \frac{\nu}{2} + 1\right) - I\left(\chi_1^2 \lambda, \frac{\nu}{2} + 1\right) \right\} \left(\frac{\nu}{x^2} + 1\right) \\ - a\lambda \left\{ I\left(\chi_2^2 \lambda, \frac{\nu}{2}\right) - I\left(\chi_1^2 \lambda, \frac{\nu}{2}\right) \right\} \\ + \frac{e^{-a}}{2^{\nu/2} \left(\frac{1}{2} - \frac{a}{\nu}\right)^{\frac{\nu}{2}}} \left[1 - I\left(\chi_2^2 \lambda, \frac{\nu}{2}\right) - I\left(\chi_1^2 \lambda, \frac{\nu}{2}\right) \right] + 1 \end{array}\right]$$

_____(2.7.1.3)

Where $I(x;p) = (1/\Gamma p) \int_0^x e^{-x} x^{p-1} dx$ refers to the standard incomplete gamma function, $\lambda = \frac{\sigma_0^2}{\sigma^2}$, and

$$I^* = \frac{e^{a(\lambda-1)}}{2^{v/2} \Gamma\left(\frac{v}{2}\right)} \int_{x_1^2\lambda}^{x_2^2\lambda} e^{\left[\frac{at^2}{\lambda v x^2} - \frac{at}{x^2}\right]} e^{-\frac{1}{2}t} t^{\frac{v}{2}-1} \text{dt}$$

2.7.2 <u>Risk of</u> $\hat{\sigma}^2_{ST2}$

Again, we obtain the risk of $\hat{\sigma}^2{}_{ST_2}$ under L(Δ) with respect to s^2, given by

$$R(\hat{\sigma}^2{}_{ST_2}) = E[\hat{\sigma}^2{}_{ST_2} \mid L(\Delta)]$$

$$= E\left[\left(\frac{\upsilon s^2}{\sigma_0^2 \chi^2}\right)^2 s^2 + \left(1 - \left(\frac{\upsilon s^2}{\sigma_0^2 \chi^2}\right)^2\right)\sigma_0^2 \Big/ \chi_1^2 < \frac{\upsilon s^2}{\sigma_0^2} < \chi_2^2\right] \cdot p\left[\chi_1^2 < \frac{\upsilon s^2}{\sigma_0^2} < \chi_2^2\right]$$
$$+ E\left[s^2 \,\middle|\, \frac{\upsilon s^2}{\sigma_0^2} < \chi_1^2 \cup \frac{\upsilon s^2}{\sigma_0^2} > \chi_2^2\right] \cdot p\left[\frac{\upsilon s^2}{\sigma_0^2} < \chi_1^2 \cup \frac{\upsilon s^2}{\sigma_0^2} > \chi_2^2\right]$$

______(2.7.2.1)

$$= e^{-a} \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} e^{a\left[\frac{\left(\frac{\upsilon s^2}{\sigma_0^2 \chi^2}\right)^2 (s^2 - \sigma_0^2) + \sigma_0^2}{\sigma^2}\right]} f(s^2)\, ds^2 - a \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} \left[\frac{\left(\frac{\upsilon s^2}{\sigma_0^2 \chi^2}\right)^2 (s^2 - \sigma_0^2) + \sigma_0^2}{\sigma^2} - 1\right] f(s^2) ds^2$$

$$- \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} f(s^2)\, ds^2 + e^{-a} \int_0^{\frac{\chi_1^2\sigma_0^2}{\upsilon}} e^{a\left(s^2/\sigma^2\right)} f(s^2)\, ds^2 + e^{-a} \int_{\frac{\chi_2^2\sigma_0^2}{\upsilon}}^{\infty} e^{a\left(s^2/\sigma^2\right)} f(s^2)\, ds^2$$

$$- \; a \int_{0}^{\frac{\chi_1^2\sigma_0^2}{\upsilon}} \left(\tfrac{s^2}{\sigma^2}-1\right) f(s^2)\, ds^2 - a \int_{\frac{\chi_2^2\sigma_0^2}{\upsilon}}^{\infty} \left(\tfrac{s^2}{\sigma^2}-1\right) f(s^2)\, ds^2 - \int_{0}^{\frac{\chi_1^2\sigma_0^2}{\upsilon}} f(s^2)\, ds^2 - \int_{\frac{\chi_2^2\sigma_0^2}{\upsilon}}^{\infty} f(s^2)$$

______(2.7.2.2)

Where $f(s^2) = \dfrac{1}{2^{\upsilon/2}\,\Gamma\left(\upsilon/2\right)} \left(s^2\right)^{\frac{\upsilon}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon s^2}{\sigma^2}\right)} ds^2$

Straight forward integration of (2.7.2.2) gives

$$R\left(\hat{\sigma}^2_{ST2}\right) = \left(\frac{\sigma^2}{\nu}\right)^{\nu/2} \left[\begin{array}{c} I^* - \frac{4a}{\lambda^2 \, (x^2)^2} \left(\frac{\nu}{2}+1\right)\left(\frac{\nu}{2}+2\right) \\ \left\{I\left(\chi_2^2\lambda, \frac{\nu}{2}+3\right) - I\left(\chi_1^2\lambda, \frac{\nu}{2}+3\right)\right\} \\ + \frac{4a}{\lambda(x^2)^2}\left(\frac{\nu}{2}\right)\left(\frac{\nu}{2}+1\right) \\ \left\{I\left(\chi_2^2\lambda, \frac{\nu}{2}+2\right) - I\left(\chi_1^2\lambda, \frac{\nu}{2}+2\right)\right\} \\ - \; a\left\{I\left(\chi_1^2\lambda, \frac{\nu}{2}+1\right) - I\left(\chi_2^2\lambda, \frac{\nu}{2}+1\right)\right\} \\ - a\lambda \left\{I\left(\chi_2^2\lambda, \frac{\nu}{2}\right) - I\left(\chi_1^2\lambda, \frac{\nu}{2}\right)\right\} \\ + \frac{e^{-a}}{2^{\nu/2}\left(\frac{1}{2}-\frac{a}{\nu}\right)^{\frac{\nu}{2}}} \left[1 - I\left(\chi_2^2\lambda, \frac{\nu}{2}\right) - I\left(\chi_1^2\lambda, \frac{\nu}{2}\right)\right] + 1 \end{array}\right]$$

_____(2.7.2.3)

Where $I^* = \dfrac{e^{a(\lambda-1)}}{2^{\nu/2}\;\Gamma\left(\frac{\nu}{2}\right)} \int_{x_1^2\lambda}^{x_2^2\lambda} e^{\left[\frac{at^3}{\lambda^2\, \nu(x^2)^2} - \frac{at^2}{\lambda(x^2)^2}\right]} e^{-\frac{1}{2}t}\, t^{\frac{\nu}{2}-1}\, dt$

Where $I(x;p) = \left(1/\Gamma p\right) \int_0^x e^{-x} x^{p-1} dx$ refers to the standard incomplete gamma function and λ is same as defined earlier.

2.8 Relative Risk of $\hat{\sigma}^2_{STi}$

A natural way of comparing the risk of the proposed testimators, is to study its performance with respect to the best available estimator s^2 in this case. For this purpose, we obtain the risk of s^2 under $L_E\left(\hat{\sigma}^2, \sigma^2\right)$ as:

$$R_E(s^2) = E[\, s^2 \,|\, L(\hat{\sigma}^2, \sigma^2)\,]$$

$$= e^{-a} \int_0^\infty e^{a\left[\frac{s^2}{\sigma^2}\right]} f(s^2)\, ds^2 \; - a \int_0^\infty \left[\frac{s^2}{\sigma^2} - 1\right] f(s^2)\, ds^2 \; - \int_0^\infty f(s^2)\, ds^2$$

______(2.8.1)

Where $f(s^2) = \dfrac{1}{2^{\upsilon/2}\, \Gamma\left(\upsilon/2\right)} \left(s^2\right)^{\frac{\upsilon}{2}-1} e^{\left(-\frac{1}{2} \frac{\upsilon s^2}{\sigma^2}\right)}$

A straightforward integration of (2.8.1) gives

$$R_E(s^2) = \left(\frac{\sigma^2}{\nu}\right)^{\nu/2} \left[\frac{e^{-a}}{2^{\nu/2}\left(\frac{1}{2} - \frac{a}{\nu}\right)^{\nu/2}} - 1 \right]$$ _______(2.8.2)

Now, we define the Relative Risk of $\hat{\sigma}^2_{ST_i}$, $i = 1,2$ with respect to s^2 under $L\left(\hat{\sigma}^2, \sigma^2\right)$ as follows:

$$RR_1 = \frac{R_E(\mathrm{s}^2)}{R(\hat{\sigma}^2_{ST1})}$$ _______(2.8.3)

Using (2.8.2) and (2.7.1.3) the expression for RR_1 given in (2.8.3) can be obtained; it is observed that RR_1 is a function of 'λ', ' ν ', 'α' and 'a'.

Finally, we define the Relative Risk of $\hat{\sigma}^2_{ST_2}$ by

$$RR_2 = \frac{R_E(\mathrm{s}^2)}{R(\hat{\sigma}^2_{ST2})}$$ _______(2.8.4)

The expression for RR_2 is given by (2.8.4) can be obtained by using (2.8.2) and (2.7.2.3). Again we observed that RR_2 is a function of 'λ', 'ν', 'α' and 'a'.

2.9 Recommendations for $\hat{\sigma}^2_{STi}$

In this section we wish to compare the performance of $\hat{\sigma}^2_{ST_1}$ and $\hat{\sigma}^2_{ST_2}$ with respect to the best available (unbiased) estimator of σ^2.

2.9.1 Recommendations for $\hat{\sigma}^2_{ST1}$

It is observed that the expressions of RR_1 and RR_2 are the functions of ν, α, λ and the degrees of asymmetry " a ". For the comparison of the proposed testimators with the best available estimator we have considered $\alpha = 1\%,\ 5\%\ and\ 10\%$, $\nu = 5,\ 8, 10\ and\ 12\ \ and\ a = -2.0,\ -1.0,\ 1.0,\ 1.5,\ and\ 1.75$ and $\lambda = 0.2\ (0.2)\ 2.0$. There will be several tables and graphs for RR values for both the testimators. We have assembled some of graphs at the end of the chapter. However our recommendations based on all these computations are as follows:

(i) $\hat{\sigma}^2_{ST_1}$ Performs better than $\hat{\sigma}^2$ for a considerably large range of λ for different degrees of asymmetry. For a= -2 the range is $0.6 \le \lambda \le 1.8$, which changes slightly for a = -1 and becomes $0.6 \le \lambda \le 1.6$. For the positive values of 'a' we have observed a different pattern for RR_1 as when different values $0.8 \le \lambda \le 1.4$, the performance of $\hat{\sigma}^2_{ST_1}$ is better than $\hat{\sigma}^2$. Similar pattern is observed for the other two positive values of 'a' i.e. a = 1.5 and a = 1.75. However the values of RR_1 are smaller in magnitude. Further the magnitude of RR_1 is higher when 'a' is negative as compared to those values of when 'a' is positive.

(ii) For higher values of α i.e. 5% and 10% a similar kind of behaviour of RR values is observed but the range of 'λ' changes, it is $0.6 \leq \lambda \leq 2.0$ for a = -2 and $\alpha = 5\%$ and this becomes $0.8 \leq \lambda \leq 2.0$ for a = -2 and $\alpha = 10\%$. Similarly for other values of negative 'a' when 'a' is positive the range of 'λ' is $0.8 \leq \lambda \leq 1.8$ for a = 1.75.

(iii) It is observed that for some negative values of 'a' as well as for some positive values of 'a' the magnitude of RR_1 is greater than unity which indicates that $\hat{\sigma}^2{}_{ST_1}$ performs better than usual estimator.

(iv) As the value of 'ν' increases there is a decrease in the RR_1 values for different values of levels of significance and degrees of asymmetries. However the best performance of $\hat{\sigma}^2{}_{ST_1}$ is observed at $\alpha = 1\%$ for a = -2 and $\alpha = 1\%$ for a = 1.75

(v) It is recommended therefore to consider a smaller level of significance (preferably $\alpha = 1\%$) and smaller sample size ν = '5' or '8' for positive / negative values of 'a' in particular a = 1.75 and a = -2.0.

2.9.2 Recommendations for $\hat{\sigma}^2_{ST2}$

Next we have considered another testimator $\hat{\sigma}^2{}_{ST_2}$ which is obtained by squaring the shrinkage factor, we have evaluated the expression RR_2 for the same set of values as considered for RR_1 and our recommendations are as follows:

(i) $\hat{\sigma}^2{}_{ST_2}$ performs better than the usual estimator $\hat{\sigma}^2$ for different range of λ i.e. for a = -2, it is $0.6 \leq \lambda \leq 1.8$, however for a = -1 it becomes $0.6 \leq$

$\lambda \leq 1.6$ i.e. almost the same whole range as observed for $\hat{\sigma}^2_{ST_1}$ but the magnitude of RR_2 values are **HIGHER** than the magnitude of RR_1 values indicating a 'better' control over the risk of $\hat{\sigma}^2_{ST_2}$ as compared to $\hat{\sigma}^2_{ST_1}$. This is observed when α = 1%, ν = 5 and a = -1.0 also when a = +1.75.

(ii) A Similar kind of pattern for the performance of $\hat{\sigma}^2_{ST_2}$ is observed for $\alpha = 5\% \; and \; \alpha = 10\%$ for the range of 'λ' as mentioned above.

(iii) It is observed that the values of RR_2 are more than unity for some positive and negative values of 'a'. So, it is conclude that in both the situations i.e. over/under estimation the proposed testimators behaves nicely.

(iv) The maximum values of RR_2 are observed for α = 1%, a = -2.0 and ν = 5. Similarly for a = +1.75, α = 1% and ν = 5.

(v) The general behaviour observed is that of 'decreasing' values of RR for higher values of 'ν' and 'α' .

(vi) So, it is recommended to consider smaller level of significance along with a smaller sample size with proper choice of 'a'.

CONCLUSION:

We have proposed shrinkage testimator(s) for the variance of Normal distribution and we recommend that: A shrinkage testimator $\hat{\sigma}^2_{ST_2}$ (i.e. 'square' of shrinkage factor) should be considered for $\alpha = 1\%,\; \nu = 5 \; or \; 8 \; and \; a = 1.75$ (for situations where overestimation is more serious) and $a = -2.0$ (for situations where under estimation is more serious).

Tables showing relative risk(s) of proposed testimator(s) with respect to the best available estimator.

Table : 2.9.1.1 **Relative Risk of $\hat{\sigma}^2_{ST1}$** $\alpha = 1\%$, ν = 5

λ	a = -2	a = -1	a = 1	a = 1.5	a = 1.75
0.20	0.699	0.597	0.497	0.176	0.718
0.40	0.76	0.794	0.642	1.071	1.229
0.60	1.311	1.116	1.196	1.934	2.047
0.80	2.606	1.939	2.693	2.92	3.307
1.00	4.647	3.256	2.817	3.448	4.375
1.20	4.64	3.092	1.725	2.436	3.071
1.40	2.935	2.288	1.187	1.763	2.246
1.60	1.802	1.459	0.801	1.233	1.609
1.80	1.206	0.974	0.541	0.84	1.121
2.00	0.878	0.695	0.373	0.569	0.768

Table : 2.9.1.2 **Relative Risk of $\hat{\sigma}^2_{ST1}$** $\alpha = 1\%$, ν = 8

λ	a = -2	a = -1	a = 1	a = 1.5	a = 1.75
0.20	0.863	0.455	0.244	0.685	0.891
0.40	1.642	0.983	0.651	1.599	1.165
0.60	2.516	1.455	1.301	1.794	2.005
0.80	3.579	1.878	1.806	2.133	2.44
1.00	4.156	2.446	2.562	3.726	4.164
1.20	3.693	2.01	1.757	2.755	3.333
1.40	2.777	1.503	1.476	1.751	2.887
1.60	1.738	1.062	1.008	1.482	1.574
1.80	1.191	0.782	0.208	0.315	0.371
2.00	0.895	0.611	0.146	0.211	0.245

Table : 2.9.1.3 **Relative Risk of $\hat{\sigma}^2_{ST1}$** $\alpha = 5\%$, ν = 5

λ	a = -2	a = -1	a = 1	a = 1.5	a = 1.75
0.20	0.802	0.617	0.663	0.362	0.672
0.40	1.018	0.876	0.981	1.162	1.278
0.60	1.714	1.167	1.923	2.117	2.03
0.80	2.833	1.598	2.442	2.548	3.336
1.00	3.641	1.909	2.665	3.117	3.632
1.20	3.279	1.886	1.879	2.254	2.977
1.40	2.465	1.627	1.188	1.762	2.116
1.60	1.829	1.334	0.828	1.049	1.771
1.80	1.416	1.094	0.523	0.568	1.031
2.00	0.92	0.666	0.249	0.429	0.583

Table : 2.9.1.4 **Relative Risk of $\hat{\sigma}^2_{ST1}$** α = 5% , ν = 8

λ	a = -2	a = -1	a = 1	a = 1.5	a = 1.75
0.20	0.768	0.781	0.315	0.475	0.603
0.40	1.003	1.076	0.971	0.916	0.713
0.60	1.801	1.583	1.387	1.59	1.374
0.80	2.621	1.963	1.638	2.464	2.563
1.00	3.265	2.377	2.219	2.764	3.103
1.20	2.221	1.844	1.843	2.105	2.34
1.40	1.909	1.631	1.474	1.611	1.744
1.60	1.601	1.193	0.937	1.231	1.51
1.80	1.374	0.766	0.711	0.831	0.974
2.00	1.009	0.684	0.393	0.438	0.582

Table : 2.9.2.1 **Relative Risk of $\hat{\sigma}^2_{ST2}$** α = 1% , ν = 5

λ	a = -2	a = -1	a = 1	a = 1.5	a = 1.75
0.20	0.426	0.393	0.343	0.789	0.996
0.40	0.499	1.193	0.458	1.225	1.689
0.60	1.059	1.907	1.494	1.874	2.768
0.80	2.782	2.614	2.389	2.751	3.961
1.00	6.727	4.124	3.826	4.747	6.296
1.20	5.728	3.306	2.129	4.055	4.127
1.40	2.833	2.432	1.28	3.227	2.709
1.60	1.592	1.309	0.781	2.05	1.72
1.80	1.04	0.836	0.495	1.265	1.081
2.00	0.754	0.589	0.328	0.785	0.686

Table : 2.9.2.2 **Relative Risk of $\hat{\sigma}^2_{ST2}$** α = 5% , ν = 5

λ	a = -2	a = -1	a = 1	a = 1.5	a = 1.75
0.20	0.509	0.552	0.441	0.992	0.688
0.40	0.989	0.855	1.326	1.69	1.913
0.60	1.405	1.514	1.921	2.054	2.874
0.80	2.87	2.081	2.784	2.371	4.183
1.00	3.975	2.833	2.973	3.354	5.017
1.20	3.193	2.302	1.917	2.649	4.216
1.40	2.464	1.825	1.581	1.806	3.446
1.60	1.732	1.574	1.105	1.232	2.005
1.80	1.168	1.195	0.295	0.52	1.715
2.00	1.048	0.936	0.221	0.375	0.514

Chapter – 3

DOUBLE STAGE SHRINKAGE TESTIMATORS UNDER ASYMMETRIC LOSS FUNCTION

3.1 Introduction

In this chapter we have extended our studies of chapter 2 in the sense that now instead of drawing only one sample form the population, the experimenter may possibly drawn one or two samples. Estimation of the mean from double sample in the presence of a priori information was first considered by Katti (1962) and later by many others. Katti's method consisted in constructing a region R using the a priori information available in the form of a guess value say θ_0 of the parameter θ and the observations $x_1, x_2 ...., x_n$ from the first sample. If the estimator constructed or proposed belonged to R; there was no need to draw a second sample of size n_2. However, if it did not lie in R; a second sample of size n_2 was drawn and the proposed estimator used observations from both samples. Shah (1964) used this method in estimating variance of a Normal distribution when a guess of the population variance is given. He also proposed a pre-test estimator of the variance. The procedure adopted by Shah has something in common with the two stage procedure due to Stein (1945). Arnold and Al-Bayyati (1970) modified the estimator proposed by Katti using the shrinkage technique and studied the properties of the estimator. Waiker and Katti (1971) have also studied two stage estimation of the mean. Pandey (1979) considered estimation of variance of a normal population using a priori information.

Waiker et al (1984) have suggested and studied a two stage shrinkage testimator of the mean of a normal population when the variance of the population may be known or unknown. Their approach is different from that of Katti and others in the sense that (i) no region R is constructed in the sample space (ii) the shrinkage factor $\boldsymbol{k}$ is no longer arbitrary but is a function of the test statistic used in testing the hypothesis regarding the given a priori information. In both techniques $\boldsymbol{k}$ being arbitrary or not, no assumption is made regarding the distribution of the parameter θ on (the parameter space). At the most one may take it a singular distribution with entire mass concentrated at a single point $\theta = \theta_0$.

Similar studies for estimating the scale parameter θ in one parameter Exponential distribution with p.d.f.

$$f(x;\theta) = \begin{cases} \frac{1}{\theta}\exp(-x/\theta), & x \geq 0, \theta > 0 \\ 0 & , otherwise \end{cases} \qquad \text{\_\_\_\_\_\_ (3.1.1)}$$

have been made. Using the priori information available in the form of an initial estimate say θ_0 of the parameter θ. Shah (1975) considered estimation of θ in censored sampling. Ojha and Srivastava (1980) have studied a pre-test double stage shrunken estimators of θ using complete samples. The object of the present chapter is to propose and study shrinkage testimators for scale parameters of (3.1.1).

Recently Srivastava and Tanna (2007) have studied the risk properties of a Double stage shrinkage testimator under General Entropy Loss Function. Further Srivastava and Tanna (2012) have studied the risk properties of such estimators under Asymmetric Loss Function.

DOUBLE STAGE ESTIMATION:

The first stage sample is used to test H_0 and if H_0 is not rejected, it is suggested to use the prior knowledge being supported by a test, in estimating θ. However, if H_0 is rejected, we do not use the prior knowledge and obtain a second sample size $n_2 = (n - n_1)$ to make up for the loss of the prior knowledge and estimate θ using both the samples.

In section 3.2 we have proposed the three different shrinkage testimators for scale parameter of an Exponential Distribution and we have studied the risk properties of these three shrinkage testimators under Asymmetric Loss Function. Section 3.3 deals with the derivation of the risk(s) of these three estimators. Section 3.4 deals with the relative risk(s) of these three estimators. Section 3.5 concludes with the comparison of UMVUE and the proposed shrinkage testimators in terms of their relative risks. Further in the same section a suggestion for the shrinkage factor is made.

In section 3.6 we have proposed the two different shrinkage testimators for the variance of a Normal Distribution and we have studied the risk properties of these two shrinkage testimators under Asymmetric Loss Function. Section 3.7 deals with the derivation of the risk(s) of these two estimators. Section 3.8 deals with the relative risk(s) of these two estimators. Section 3.9 concludes with the comparison of UMVUE and the proposed shrinkage testimators regarding their choice in terms of their relative efficiency. Further in the same section a suggestion for the choice of shrinkage factor is made.

3.2 Shrinkage Testimator(s) for Scale Parameter of an Exponential Distribution.

Let x_{11}, x_{12}, _______, x_{1n1} be the first stage sample of size n_1 from and exponential population is given by (3.1.1). Let θ_0 be the guess estimate of the mean θ. Compute the sample mean $\bar{x}_1 = \frac{1}{n}\sum_{i=1}^{n1} x_{1i}$ and test the preliminary hypothesis H_0 : $\theta = \theta_0$ vs. H_1 : $\theta \neq \theta_0$, using the test statistic T = $\frac{2n_1\bar{x}_1}{\theta_0}$ which follows $x_{2n_1}^2$. It is to be noted that H_0 is accepted if $x_1^2 \leq \frac{2n_1\bar{x}_1}{\theta_0} \leq x_2^2$ and H_0 is rejected, otherwise. Then take $n_2 = n - n_1$ additional observations x_{21},x_{22},_______,x_{2n2} and use the pooled estimator $\bar{x}_p$ as the estimator of the mean where $\bar{x}_p = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1+n_2}$

x_1^2 and x_2^2 being given by

$$P\left[x_{2n_1}^2 \geq x_2^2\right] + P\left[x_{2n_1}^2 \leq x_1^2\right] = \alpha \qquad \text{\_\_\_\_\_\_\_\_(3.2.1)}$$

where α is the pre-assigned level of significance.

When $\theta = \theta_0$, the probability of avoiding the second sample is (1- α) and the expected sample size is given by

$$n^* = E[n|\,\theta = \theta_0]$$

$$= n_1\, P\left[x_1^2 < \frac{2n_1\bar{x}_1}{\theta_0} < x_2^2\right] + (n_1 + n_2)\, P\left[\frac{2n_1\bar{x}_1}{\theta_0} < x_1^2 \text{ U } \frac{2n_1\bar{x}_1}{\theta_0} > x_2^2\right]$$

or, $n^* = n_1\,(1 + u\alpha)$ where $u = {}^{n_2}/_{n_1}$.

When $\theta \neq \theta_0$, the probability of avoiding the second sample is

$$P = \frac{1}{2^{n_1}\,\Gamma n_1}\left(\frac{2n_1}{\theta_0}\right)^{n_1} (\bar{x}_1)^{n_1-1}\, e^{\left(\frac{-n_1\bar{x}_1}{\theta_0}\right)} d\bar{x}_1$$ and the expected sample size is

$$n^{**} = n_1 + n_2 \left[1 - P\left\{\lambda x_1^2 < \frac{2n_1\bar{x}_1}{\theta_0} < \lambda x_2^2\right\}\right]$$

Now we propose a shrinkage testimator $\hat{\theta}_{DST1}$ of θ defined as:

$$\hat{\theta}_{DST1} = \begin{cases} k_1\bar{x}_1 + (1-k_1)\theta_0 & ;\ if \quad x_1^2 \leq \dfrac{2n_1\bar{x}_1}{\theta_0} \leq x_2^2 \\ \bar{x}_p & ; \quad otherwise \end{cases}$$

_____(3.2.2)

Where $\bar{x}_p = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1+n_2}$ and $\bar{x}_i = \frac{1}{n_i}\sum_{j=1}^{ni} x_{ij}$; $i = 1,2$

and k_1 being dependent on test statistic is given by $k_1 = \frac{2n_1\bar{x}_1}{\theta_0\ x^2}$

where $x^2 = (x_2^2 - x_1^2)$

Now, taking the 'square' of k_1 (i.e. $k_2 = k_1^2$), another testimator is defined as

$$\hat{\theta}_{DST2} = \begin{cases} \left(\frac{2n_1\overline{x_1}}{\theta_0 x^2}\right)^2 \bar{x}_1 + \left[1 - \left(\frac{2n_1\overline{x_1}}{\theta_0 x^2}\right)^2\right]\theta_0 & ;if\ H_0\ \ is\ accepted \\ \bar{x}_p & ;\ \ otherwise \end{cases}$$

_______(3.2.3)

Finally, taking k_3 , the third testimator can be proposed as

$$\hat{\theta}_{DST3} = \begin{cases} k_3\ \bar{x}_1 + (1-k_3)\theta_0 & ;\ \ if \quad H_0\ \ is\ accepted \\ \bar{x}_p & ; \quad otherwise \end{cases}$$

_______(3.2.4)

Where $k_3 = \frac{2n_1\bar{x}_1}{\theta_0 x^2} - \frac{x_1^2}{x^2}$ and $x^2 = (x_2^2 - x_1^2)$

In this case 'k_3' exactly lies between '0' and '1'.

3.3 <u>Risk of Testimators</u>

In this section we derive the risk of all the three testimators which are defined in the previous section.

3.3.1 **Risk of** $\hat{\boldsymbol{\theta}}_{DST1}$

The risk of $\hat{\theta}_{DST_1}$ under L(Δ) is given by

$$R(\hat{\theta}_{DST_1}) = E[\hat{\theta}_{DST_1} \,|\, L(\Delta)]$$

$$= E\left[k_1\bar{x}_1 + (1-k_1)\theta_0 \Big/ \chi_1^2 < \frac{2n_1\bar{x}_1}{\theta_0} < \chi_2^2\right]\cdot p\left[\chi_1^2 < \frac{2n_1\bar{x}_1}{\theta_0} < \chi_2^2\right]$$
$$+ E\left[\bar{x}_p \;\middle|\; \frac{2n_1\bar{x}_1}{\theta_0} < \chi_1^2 \,\cup\, \frac{2n_1\bar{x}_1}{\theta_0} > \chi_2^2\right]\cdot p\left[\frac{2n_1\bar{x}_1}{\theta_0} < \chi_1^2 \cup \frac{2n_1\bar{x}_1}{\theta_0} > \chi_2^2\right]$$

_______(3.3.1.1)

$$= e^{-a}\int_{\frac{\chi_1^2\theta_0}{2n_1}}^{\frac{\chi_2^2\theta_0}{2n_1}} e^{a\left[\frac{\frac{2n_1\bar{x}_1}{\theta_0}(\bar{x}_1-\theta_0)+\theta_0}{\theta}\right]} f(\bar{x}_1)\,d\bar{x}_1$$
$$- a\int_{\frac{\chi_1^2\theta_0}{2n_1}}^{\frac{\chi_2^2\theta_0}{2n_1}} \left[\frac{\frac{2n_1\bar{x}_1}{\theta_0}(\bar{x}_1-\theta_0)+\theta_0}{\theta} - 1\right] f(\bar{x}_1)\,d\bar{x}_1$$
$$- \int_{\frac{\chi_1^2\theta_0}{2n_1}}^{\frac{\chi_2^2\theta_0}{2n_1}} f(\bar{x}_1)\,d\bar{x}_1 + e^{-a}\int_0^{\frac{\chi_1^2\theta_0}{2n_1}}\int_0^{\infty} e^{a\left(\bar{x}_p/\theta\right)} f(\bar{x}_1)f(\bar{x}_2)\,d\bar{x}_1\,d\bar{x}_2$$
$$- a\int_0^{\frac{\chi_1^2\theta_0}{2n_1}}\int_0^{\infty}\left(\frac{\bar{x}_p}{\theta}-1\right) f(\bar{x}_1)f(\bar{x}_2)\,d\bar{x}_1\,d\bar{x}_2 - \int_0^{\frac{\chi_1^2\theta_0}{2n_1}}\int_0^{\infty} f(\bar{x}_1)f(\bar{x}_2)\,d\bar{x}_1\,d\bar{x}_2$$
$$+ e^{-a}\int_{\frac{\chi_2^2\theta_0}{2n_1}}^{\infty}\int_0^{\infty} e^{a\left(\bar{x}_p/\theta\right)} f(\bar{x}_1)f(\bar{x}_2)\,d\bar{x}_1\,d\bar{x}_2$$
$$- a\int_{\frac{\chi_2^2\theta_0}{2n_1}}^{\infty}\int_0^{\infty}\left(\frac{\bar{x}_p}{\theta}-1\right) f(\bar{x}_1)f(\bar{x}_2)\,d\bar{x}_1\,d\bar{x}_2 - \int_{\frac{\chi_2^2\theta_0}{2n_1}}^{\infty}\int_0^{\infty} f(\bar{x}_1)f(\bar{x}_2)\,d\bar{x}_1\,d\bar{x}_2$$

_______(3.3.1.2)

Where $f(\bar{x}_1) = \dfrac{1}{\Gamma n_1}\left(\dfrac{n_1}{\theta}\right)^{n_1} (\bar{x}_1)^{n_1 - 1} e^{\frac{-n_1 \bar{x}_1}{\theta}}$

and $f(\bar{x}_2) = \dfrac{1}{\Gamma n_2}\left(\dfrac{n_2}{\theta}\right)^{n_2} (\bar{x}_2)^{n_2 - 1} e^{\frac{-n_2 \bar{x}_2}{\theta}}$

Straight forward integration of (3.3.1.2) gives

$$R(\hat{\theta}_{DST_1}) = I* - \frac{2a(n_1+1)}{\phi x^2}\left\{I\left(\frac{x_2^2\phi}{2}, n_1+2\right) - I\left(\frac{x_1^2\phi}{2}, n_1+2\right)\right\} +$$
$$\left\{\frac{2an_1}{x^2} + \frac{an_1}{n_1+n_2}\right\}\left\{I\left(\frac{x_2^2\phi}{2}, n_1+1\right) - I\left(\frac{x_1^2\phi}{2}, n_1+1\right)\right\} -$$
$$\left\{I\left(\frac{x_2^2\phi}{2}, n_1\right) - I\left(\frac{x_1^2\phi}{2}, n_1\right)\right\}(a\phi - a + 1) + \left\{I\left(\frac{x_2^2\phi}{2}, n_1\right) - I\left(\frac{x_1^2\phi}{2}, n_1\right)\right\}$$
$$\left\{1 - \frac{an_1}{n_1+n_2} - \frac{e^{-a}}{\left(1 - \frac{a}{n_1+n_2}\right)^{n_1+n_2}}\right\} + \left(\frac{e^{-a}}{\left(1 - \frac{a}{n_1+n_2}\right)^{n_1+n_2}} - 1\right)$$

_______(3.3.1.3)

Where $I* = e^{a\phi - a} \int_{\frac{x_1^2\phi}{2}}^{\frac{x_2^2\phi}{2}} e^{\left[\frac{2at^2}{n_1\phi x^2} - \frac{2at}{x^2}\right]} \dfrac{1}{\Gamma n_1} e^{-t}\, t^{n_1 - 1}\, dt$; $\emptyset = \frac{\theta_0}{\theta}$ and

$I(x; p) = (1/\Gamma p) \int_0^x e^{-x} x^{p-1}\, dx$ refers to the standard incomplete gamma function

3.3.2 <u>Risk of</u> $\hat{\theta}_{DST2}$

Again, we obtain the risk of $\hat{\theta}_{DST_2}$ under L(Δ) with respect to $\bar{x}_1$, given by

$R(\hat{\theta}_{DST_2}) = E[\hat{\theta}_{DST_2} \mid L(\Delta)]$

$$= E\left[\left(\frac{2n_1\bar{x}_1}{\theta_0\,\chi^2}\right)^2 (\bar{x}_1-\theta_0)+\theta_0 \Big/ \chi_1^2 < \frac{2n_1\bar{x}_1}{\theta_0} < \chi_2^2 \right] \cdot p\left[\chi_1^2 < \frac{2n_1\bar{x}_1}{\theta_0} < \chi_2^2 \right]$$

$$+ E\left[\bar{x}_p \;\middle|\; \frac{2n_1\bar{x}_1}{\theta_0} < \chi_1^2 \;\cup\; \frac{2n_1\bar{x}_1}{\theta_0} > \chi_2^2 \right] \cdot p\left[\frac{2n_1\bar{x}_1}{\theta_0} < \chi_1^2 \cup \frac{2n_1\bar{x}_1}{\theta_0} > \chi_2^2 \right]$$

________(3.3.2.1)

$$= \quad e^{-a} \int_{\frac{\chi_1^2\theta_0}{2n_1}}^{\frac{\chi_2^2\theta_0}{2n_1}} e^{a\left[\frac{\left(\frac{2n_1\bar{x}_1}{\theta_0\,\chi^2}\right)^2 (\bar{x}_1-\theta_0)+\theta_0}{\theta}\right]} f(\bar{x}_1)\; d\bar{x}_1$$

$$- a \int_{\frac{\chi_1^2\theta_0}{2n_1}}^{\frac{\chi_2^2\theta_0}{2n_1}} \left[\frac{\left(\frac{2n_1\bar{x}_1}{\theta_0\,\chi^2}\right)^2 (\bar{x}_1-\theta_0)+\theta_0}{\theta} - 1 \right] f(\bar{x}_1)\; d\bar{x}_1$$

$$- \int_{\frac{\chi_1^2\theta_0}{2n_1}}^{\frac{\chi_2^2\theta_0}{2n_1}} f(\bar{x}_1)\; d\bar{x}_1 \quad + e^{-a} \int_0^{\frac{\chi_1^2\theta_0}{2n_1}} \int_0^{\infty} e^{a\left(\bar{x}_p/\theta\right)} f(\bar{x}_1)\, f(\bar{x}_2)\, d\bar{x}_1\, d\bar{x}_2$$

$$- a \int_0^{\frac{\chi_1^2\theta_0}{2n_1}} \int_0^{\infty} \left(\frac{\bar{x}_p}{\theta} - 1\right) f(\bar{x}_1)\, f(\bar{x}_2)\, d\bar{x}_1\, d\bar{x}_2 - \int_0^{\frac{\chi_1^2\theta_0}{2n_1}} \int_0^{\infty} f(\bar{x}_1)\, f(\bar{x}_2)\, d\bar{x}_1\, d\bar{x}_2$$

$$+ \; e^{-a} \int_{\frac{\chi_2^2\theta_0}{2n_1}}^{\infty} \int_0^{\infty} e^{a\left(\bar{x}_p/\theta\right)} f(\bar{x}_1)\, f(\bar{x}_2)\, d\bar{x}_1\, d\bar{x}_2$$

$$- \; a \int_{\frac{\chi_2^2\theta_0}{2n_1}}^{\infty} \int_0^{\infty} \left(\frac{\bar{x}_p}{\theta} - 1\right) f(\bar{x}_1)\, f(\bar{x}_2)\, d\bar{x}_1\, d\bar{x}_2 \; - \int_{\frac{\chi_2^2\theta_0}{2n_1}}^{\infty} \int_0^{\infty} f(\bar{x}_1)\, f(\bar{x}_2)\, d\bar{x}_1\, d\bar{x}_2$$

________(3.3.2.2)

Where $f(\bar{x}_1) = \frac{1}{\Gamma n_1}\left(\frac{n_1}{\theta}\right)^{n_1}(\bar{x}_1)^{n_1-1} e^{\frac{-n_1\bar{x}_1}{\theta}}$

and $f(\bar{x}_2) = \frac{1}{\Gamma n_2}\left(\frac{n_2}{\theta}\right)^{n_2}(\bar{x}_2)^{n_2-1} e^{\frac{-n_2\bar{x}_2}{\theta}}$

A straight forward integration of (3.3.2.2) gives:

$$R(\hat{\theta}_{DST_2}) = I_1^* - \frac{4a(n_1+1)(n_1+2)}{\phi^2\,(\chi^2)^2}\left\{I\left(\frac{\chi_2^2\phi}{2}, n_1+3\right) - I\left(\frac{\chi_1^2\phi}{2}, n_1+3\right)\right\} +$$

$$\frac{4a\,n_1(n_1+1)}{\phi\,(\chi^2)^2}\left\{I\left(\frac{\chi_2^2\phi}{2}, n_1+2\right) - I\left(\frac{\chi_1^2\phi}{2}, n_1+2\right)\right\} - \left\{I\left(\frac{\chi_2^2\phi}{2}, n_1\right) - I\left(\frac{\chi_1^2\phi}{2}, n_1\right)\right\}$$

$$(a\phi - a + 1) + \frac{an_1}{n_1+n_2}\left\{I\left(\frac{\chi_2^2\phi}{2}, n_1+1\right) - I\left(\frac{\chi_1^2\phi}{2}, n_1+1\right)\right\} +$$

$$\left\{I\left(\frac{x_2^2\,\phi}{2}, n_1\right) - I\left(\frac{x_1^2\,\phi}{2}, n_1\right)\right\}\left\{1 - \frac{an_1}{n_1+n_2} - \frac{e^{-a}}{\left(1-\frac{a}{n_1+n_2}\right)^{n_1+n_2}}\right\} + \left(\frac{e^{-a}}{\left(1-\frac{a}{n_1+n_2}\right)^{n_1+n_2}} - 1\right)$$

_______(3.3.2.3)

Where $$I_1^* = e^{a(\phi-1)} \int_{\frac{\chi_1^2\phi}{2}}^{\frac{\chi_2^2\phi}{2}} e^{\left[\frac{4at^3}{n_1\phi^2(\chi^2)^2} - \frac{4at^2}{\phi(\chi^2)^2}\right]} \frac{1}{\Gamma n_1} e^{-t} t^{n_1-1}\, dt$$

3.3.3 Risk of $\hat{\theta}_{DST3}$

Finally, we obtain the risk of $\hat{\theta}_{DST_3}$ under L(Δ) with respect to $\bar{x}_1$, given by

$$R(\hat{\theta}_{DST_3}) = E[\,\hat{\theta}_{DST_3} \mid L(\Delta)]$$

$$= E\left[k_3\,\bar{x}_1 + (1-k_3)\theta_0 \Big/ \chi_1^2 < \frac{2n_1\bar{x}_1}{\theta_0} < \chi_2^2\right]\cdot p\left[\chi_1^2 < \frac{2n_1\bar{x}_1}{\theta_0} < \chi_2^2\right]$$

$$+ E\left[\bar{x}_p \;\middle|\; \frac{2n_1\bar{x}_1}{\theta_0} < \chi_1^2 \;\cup\; \frac{2n_1\bar{x}_1}{\theta_0} > \chi_2^2\right]\cdot p\left[\frac{2n_1\bar{x}_1}{\theta_0} < \chi_1^2 \cup \frac{2n_1\bar{x}_1}{\theta_0} > \chi_2^2\right]$$

____(3.3.3.1)

$$= \quad e^{-a}\int_{\frac{\chi_1^2\theta_0}{2n_1}}^{\frac{\chi_2^2\theta_0}{2n_1}} e^{a\left[\frac{\left(\frac{2n_1\bar{x}_1}{\theta_0\,x^2}-\frac{x_1^2}{x^2}\right)(\bar{x}_1-\theta_0)+\theta_0}{\theta}\right]} f(\bar{x}_1)\,d\bar{x}_1$$

$$- a\int_{\frac{\chi_1^2\theta_0}{2n_1}}^{\frac{\chi_2^2\theta_0}{2n_1}} \left[\frac{\left(\frac{2n_1\bar{x}_1}{\theta_0\,x^2}-\frac{x_1^2}{x^2}\right)(\bar{x}_1-\theta_0)+\theta_0}{\theta} - 1\right] f(\bar{x}_1)\,d\bar{x}_1$$

$$- \int_{\frac{\chi_1^2\theta_0}{2n_1}}^{\frac{\chi_2^2\theta_0}{2n_1}} f(\bar{x}_1)\,d\bar{x}_1 \;+\; e^{-a}\int_0^{\frac{\chi_1^2\theta_0}{2n_1}}\int_0^{\infty} e^{a\left(\bar{x}_p/\theta\right)} f(\bar{x}_1)\,f(\bar{x}_2)\,d\bar{x}_1\,d\bar{x}_2$$

$$- a\int_0^{\frac{\chi_1^2\theta_0}{2n_1}}\int_0^{\infty}\left(\frac{\bar{x}_p}{\theta}-1\right) f(\bar{x}_1)\,f(\bar{x}_2)\,d\bar{x}_1\,d\bar{x}_2 - \int_0^{\frac{\chi_1^2\theta_0}{2n_1}}\int_0^{\infty} f(\bar{x}_1)\,f(\bar{x}_2)\,d\bar{x}_1\,d\bar{x}_2$$

$$+ \; e^{-a}\int_{\frac{\chi_2^2\theta_0}{2n_1}}^{\infty}\int_0^{\infty} e^{a\left(\bar{x}_p/\theta\right)} f(\bar{x}_1)\,f(\bar{x}_2)\,d\bar{x}_1\,d\bar{x}_2$$

$$- \; a\int_{\frac{\chi_2^2\theta_0}{2n_1}}^{\infty}\int_0^{\infty}\left(\frac{\bar{x}_p}{\theta}-1\right) f(\bar{x}_1)\,f(\bar{x}_2)\,d\bar{x}_1\,d\bar{x}_2 \;-\; \int_{\frac{\chi_2^2\theta_0}{2n_1}}^{\infty}\int_0^{\infty} f(\bar{x}_1)\,f(\bar{x}_2)\,d\bar{x}_1\,d\bar{x}_2$$

__________(3.3.3.2)

Where $f(\bar{x}_1) = \dfrac{1}{\Gamma n_1}\left(\dfrac{n_1}{\theta}\right)^{n_1} (\bar{x}_1)^{n_1-1} e^{\frac{-n_1\bar{x}_1}{\theta}}$

and $f(\bar{x}_2) = \dfrac{1}{\Gamma n_2}\left(\dfrac{n_2}{\theta}\right)^{n_2} (\bar{x}_2)^{n_2-1} e^{\frac{-n_2\bar{x}_2}{\theta}}$

A straight forward integration of (3.3.3.2) gives:

$$
\begin{aligned}
R(\hat{\theta}_{DST_3}) = I_2^* - \frac{2a(n_1+1)}{\phi\,\chi^2}&\left\{I\left(\frac{\chi_2^2\phi}{2}, n_1+2\right) - I\left(\frac{\chi_1^2\phi}{2}, n_1+2\right)\right\} + \left\{\frac{2a\,n_1}{\chi^2} + \frac{\chi_1^2 a}{\chi^2} + \frac{an_1}{n_1+n_2}\right\} \\
&\left\{I\left(\frac{\chi_2^2\phi}{2}, n_1+1\right) - I\left(\frac{\chi_1^2\phi}{2}, n_1+1\right)\right\} - \frac{\chi_1^2\phi a}{\chi^2}\left\{I\left(\frac{\chi_2^2\phi}{2}, n_1\right) - I\left(\frac{\chi_1^2\phi}{2}, n_1\right)\right\} \\
&- \left\{I\left(\frac{\chi_2^2\phi}{2}, n_1\right) - I\left(\frac{\chi_1^2\phi}{2}, n_1\right)\right\}\left(a\phi - a - 1\right) + \left\{I\left(\frac{\chi_2^2\phi}{2}, n_1\right) - I\left(\frac{\chi_1^2\phi}{2}, n_1\right)\right\} \\
&\left\{1 - \frac{an_1}{n_1+n_2} - \frac{e^{-a}}{\left(1 - \frac{a}{n_1+n_2}\right)^{n_1+n_2}}\right\} + \left(\frac{e^{-a}}{\left(1 - \frac{a}{n_1+n_2}\right)^{n_1+n_2}} - 1\right)
\end{aligned}
$$

_____(3.3.3.3)

Where $I_2^* = e^{a\phi - a}\displaystyle\int_{\frac{x_1^2\phi}{2}}^{\frac{x_2^2\phi}{2}} e^{\left[\frac{2at^2}{n_1\phi x^2} - \frac{2at}{x^2} - \frac{x_1^2 ta}{x^2 n_1} + \frac{x_1^2 a\phi}{x^2}\right]} \frac{1}{\Gamma n_1} e^{-t}\, t^{n_1-1}\, dt$

3.4 **Relative Risks of $\hat{\theta}_{DST_i}$**

A natural way of comparing the risk of the proposed testimators, is to study its performance with respect to the best available estimator $\bar{x}_1$ in this case. For this purpose, we obtain the risk of $\bar{x}_1$ under L(Δ) as:

$$R_E(\bar{x}_1) = E[\bar{x}_1 | L(\Delta)]$$

$$= e^{-a}\int_0^{\infty}\int_0^{\infty} e^{a\left(\bar{x}_1/\theta\right)} f(\bar{x}_1) f(\bar{x}_2)\, d\bar{x}_1\, d\bar{x}_2$$

$$-a\int_0^{\infty}\int_0^{\infty}\left(\frac{\bar{x}_1}{\theta}-1\right) f(\bar{x}_1) f(\bar{x}_2)\, d\bar{x}_1\, d\bar{x}_2 - \int_0^{\infty}\int_0^{\infty} f(\bar{x}_1) f(\bar{x}_2)\, d\bar{x}_1\, d\bar{x}_2$$

_______(3.4.1)

A straightforward integration of (3.4.1) gives

$$R_E(\bar{x}_1) = \frac{e^{-a}}{(1-a/n_1)^{n_1}} - 1$$ _______(3.4.2)

Now, we define the Relative Risk of $\hat{\theta}_{DST1}$ with respect to $\bar{x}_1$ under L(Δ) as follows –

$$RR_1 = \frac{R_E(\bar{x}_1)}{R(\hat{\theta}_{DST_1})}$$ _______(3.4.3)

Using (3.4.2) and (3.3.1.3) the expression for RR_1 given in (3.4.3) can be obtained;

Similarly, we define the Relative Risk of $\hat{\theta}_{DST_2}$ with respect to $\bar{x}_1$ under L(Δ) as follows

$$RR_2 = \frac{R_E(\bar{x}_1)}{R(\hat{\theta}_{DST_2})}$$

______(3.4.4)

The expression for RR_2 given in (3.4.4) which can be obtained by using (3.4.2) and (3.3.2.3).

Finally, we define the Relative Risk of $\hat{\theta}_{DST3}$ with respect to $\bar{x}_1$ under L(Δ) as follows

$$RR_3 = \frac{R_E(\bar{x}_1)}{R(\hat{\theta}_{DST3})} \qquad \text{\_\_\_\_\_(3.4.5)}$$

Using (3.4.2) and (3.3.3.3) the expression for RR_3 given in (3.4.5) can be obtained.

Now, it is observed that RR_1 , RR_2 and RR_3 are functions of 'ϕ', 'n_1', 'n_2', 'α' and 'a'. In order to study the behaviour of Relative Risk(s), we have taken a set of values of (n_1, n_2) = (4,4), (4,6), (4,8), (4,10) and (4,12), α^s = 1%, 5% and 10% , $\emptyset$ = 0.6 (0.2) 1.8 and a = ± 1 to ±3. The recommendations regarding the applications of proposed testimators are provided as follows:

The values of n* and n** are defined in section 3.2. For some values of (n_1, n_2) these values are obtained as follows:

Table -1 shows the values of n* for ϕ = 1.0 and n_1 = 4, n_2 = 8 and table - 2 shows the values of n** for ϕ = 0.8 and n_1 = 4, n_2 = 10

Table -1 ϕ = 1.0

(n_1, n_2)	**α = 1%**	**α = 5%**
(4, 8)	4.08	4.40

Table -2 ϕ = 0.8

(n_1, n_2)	**α = 1%**	**α = 5%**
(4, 8)	4.17	4.62
(4,10)	4.21	4.78

Similarly the other values of n* and n** can be computed for other values of (n_1, n_2) considered here.

3.5 Recommendations for $\hat{\theta}_{DST_i}$

In this section we wish to compare the performance of $\hat{\theta}_{DST_1}$, $\hat{\theta}_{DST_2}$ and $\hat{\theta}_{DST_3}$ with respect to the best available (unbiased) estimator of $\bar{x}_1$.

3.5.1 Recommendations for $\hat{\boldsymbol{\theta}}_{DST_1}$

1. For various set of values of (n_1, n_2), keeping α = 1% and allowing the variations in all the values of ‘a’, it is observed that the proposed testimator $\hat{\theta}_{DST1}$ performs better than $\bar{x}_1$ for 0.6 ≤ ∅ ≤ 1.4 considered here, except for few higher values i.e. ∅ = 1.8. The magnitude of RR is higher for all the values of ‘a’ however maximum gain is achieved at a=3 and a= -3. Similar pattern is observed for other values of α^s i.e. 5% and 10% but the magnitude of Relative Risk is higher at α` = 1%. It is also observed that for a = -3 and (n_1, n_2) = (4,8), $\hat{\theta}_{DST1}$ performs better.

2. In the next comparison stage we have fixed a=3, and have allowed the variation for values of α^s such as α = 1%, 5% and 10%. Maximum gain in risk is observed at ∅ = 1.0 (though it is true for the whole range of ∅) again at α = 1%, relative risk values are higher than those at 5% and 10% so a lower level of significance i.e. α = 1% is recommended for better performance of the proposed testimator.

3. We have kept ‘a’ = 3.0 and have allowed the variation in α for n_1 = 4, n_2 = 12. It is seen that the Relative Risk values are much higher than unity, indicating superiority of the proposed testimator under Asymmetric Loss Function. A

value of α = 1% shows maximum relative risk value implying that it is the most preferred value.

4. Again, for n_1 = 4, n_2 = 10, $\emptyset$ = 1.2 for different values of α^s , the table of $\hat{\theta}_{DST\,1}$, indicates that, it dominates the usual estimator for the whole range of $\emptyset$, with best performance at α = 1% and a = 3.

5. It has also been observed that the relative risk increases as $\emptyset$ increases from 0.6 to 1.0 reaches its maximum at $\emptyset$ = 1.0 and then it decreases. The relative risk increases as n_2 increases for fixed value of n_1, and is maximum at (4, 8).

6. Thus, our recommendation for the use of $\hat{\theta}_{DST1}$ is to take n_1 = 4 and n_2 = 8 i.e. $n_2 = 2\, n_1$ and small values of α^s.

3.5.2 Recommendations for $\hat{\boldsymbol{\theta}}_{DST_2}$ and $\hat{\boldsymbol{\theta}}_{DST_3}$

We have considered two other choices of the weight functions viz. square of 'k' and making the values of 'k' to lie exactly between '0' and '1' and with these choices of shrinkage factors we have proposed $\hat{\theta}_{DST2}$ and $\hat{\theta}_{DST3}$, so it is natural to suggest which 'k' should be taken. This can be achieved by making a comparative study of the relative risks of values for all the three choices.

However a comparison of the values of relative risks for $\hat{\theta}_{DST1}$, $\hat{\theta}_{DST2}$ and $\hat{\theta}_{DST3}$ reveals that

(i) $\hat{\theta}_{DST2}$ is better than the usual estimator for $0.6 \leq \emptyset \leq 1.8$ however if n_1 is small similar pattern is observed for $\hat{\theta}_{DST3}$. However the magnitude of relative risk is smaller in case of $\hat{\theta}_{DST1}$ and $\hat{\theta}_{DST3}$ in comparison to $\hat{\theta}_{DST2}$.

So, we conclude that $\hat{\theta}_{DST2}$ is preferred in comparison to $\hat{\theta}_{DST1}$ and / or $\hat{\theta}_{DST3}$.

(ii) Our focus is also on recommending the degree(s) of asymmetry. A careful study of the table of Relative Risks, reveals following choices:

For $\hat{\theta}_{DST1}$, it is recommended that a = 3 and a = -3 for almost all the choices of n_1 and n_2

For $\hat{\theta}_{DST2}$, it is recommended to take a = -3 and a = 3 for several choices of n_1 and n_2

For $\hat{\theta}_{DST3}$, it is recommended to choose a = -3 and a = 3 and $\alpha = 1\%$. The performance of $\hat{\theta}_{DST3}$ is better than $\bar{x}_1$ in almost the whole range of $\emptyset$ ($0.6 \leq \emptyset \leq 1.4$)

CONCLUSION

To conclude it is recommended to use 'square' of the weight function (Shrinkage factor) with high positive / negative values of degrees of asymmetry along with lower level(s) of significance viz 1% and 5%. However 1% is preferable as the magnitude of relative risk is higher in this case showing better control over risk of the proposed estimator.

Tables showing relative risk(s) of proposed testimator(s) with respect to the best available estimator.

Table : 3.5.1.1 **Relative Risk of** $\hat{\theta}_{DST_1}$ α = 1%, n_1 = 4, n_2 = 4

∅	a = -1	a = -2	a = -3	a = 1	a = 2	a = 3
0.60	1.028	1.286	1.435	1.063	2.592	3.138
0.80	2.103	2.656	3.197	2.061	3.394	4.895
1.00	3.852	4.902	6.384	3.835	4.388	6.405
1.20	3.508	4.076	5.113	2.64	3.159	5.44
1.40	1.893	2.129	2.629	1.535	2.009	4.921
1.60	1.036	1.195	1.508	0.855	1.162	3.1
1.80	0.639	0.765	0.995	0.5	0.664	1.849

Table : 3.5.1.2 **Relative Risk of** $\hat{\theta}_{DST_1}$ α = 1%, n_1 = 4, n_2 = 8

∅	a = -1	a = -2	a = -3	a = 1	a = 2	a = 3
0.60	1.255	1.467	1.583	1.885	2.328	2.959
0.80	2.392	2.892	3.419	3.06	3.122	4.804
1.00	4.015	5.052	6.564	3.769	4.379	6.689
1.20	3.432	4.039	5.095	2.691	3.188	5.48
1.40	1.839	2.101	2.611	1.571	2.031	3.952
1.60	1.011	1.181	1.499	0.872	1.173	2.119
1.80	0.625	0.758	0.989	0.508	0.669	1.858

Table : 3.5.1.3 **Relative Risk of** $\hat{\theta}_{DST_1}$ α = 1%, n_1 = 4, n_2 = 10

∅	a = -1	a = -2	a = -3	a = 1	a = 2	a = 3
0.60	1.339	1.528	1.631	1.109	3.77	3.586
0.80	2.49	2.967	3.488	4.83	6.044	6.737
1.00	4.064	5.096	6.617	3.75	4.375	9.438
1.20	3.411	4.029	5.09	2.706	3.195	7.49
1.40	1.825	2.094	2.605	1.581	2.037	4.96
1.60	1.004	1.177	1.496	0.877	1.176	3.123
1.80	0.622	0.755	0.987	0.511	0.671	1.861

Table : 3.5.1.4 **Relative Risk of** $\hat{\theta}_{DST_1}$ α = 5%, n_1 = 4, n_2 = 8

∅	a = -1	a = -2	a = -3	a = 1	a = 2	a = 3
0.60	1.073	1.348	1.498	1.741	1.821	2.276
0.80	2.022	2.668	3.206	4.55	5.587	4.22
1.00	3.75	4.78	6.103	5.637	6.942	7.115
1.20	3.899	4.253	5.206	2.619	3.632	5.939
1.40	2.301	2.4	2.89	1.451	2.068	4.08
1.60	1.302	1.405	1.73	0.859	1.218	3.564
1.80	0.82	0.929	1.179	0.541	0.74	2.132

Table : 3.5.1.5 **Relative Risk of** $\hat{\theta}_{DST_1}$ α = 10%, n_1 = 4, n_2 = 8

∅	a = -1	a = -2	a = -3	a = 1	a = 2	a = 3
0.60	1.005	1.283	1.439	2.066	2.578	2.588
0.80	1.833	2.523	3.065	3.567	3.055	3.929
1.00	3.771	4.873	6.119	6.985	7.386	7.903
1.20	4.793	4.72	5.529	2.753	4.258	5.462
1.40	2.972	2.739	3.167	1.417	2.208	4.227
1.60	1.668	1.635	1.947	0.858	1.291	3.059
1.80	1.053	1.106	1.365	0.567	0.81	2.431

Table : 3.5.2.1 **Relative Risk of** $\hat{\theta}_{DST_2}$ α = 1%, n_1 = 4, n_2 = 8

∅	a = -1	a = -2	a = -3	a = 1	a = 2	a = 3
0.60	1.224	1.41	1.508	1.531	1.661	2.664
0.80	3.037	3.746	4.437	3.93	3.367	3.359
1.00	6.154	7.193	8.043	6.574	7.717	8.331
1.20	4.046	4.508	5.469	3.635	4.902	5.35
1.40	1.61	1.817	2.243	1.512	2.192	3.766
1.60	0.82	0.969	1.241	0.724	1.02	2.237
1.80	0.501	0.62	0.822	0.399	0.522	1.578

3.6 Shrinkage Testimator for the Variance of a Normal Distribution

Let X be normally distributed with mean μ and variance σ^2, both unknown. It is assumed that the prior knowledge about σ^2 is available in the form of an initial estimate σ_0^2. We are interested in constructing an estimator of σ^2 using the sample observations and possibly the guess value σ_0^2. We define a double stage shrinkage testimator of σ^2 as follows:

1. Take a random sample x_{1i} $(i = 1,2,\_\_, n_1)$ of size n_1 from N(μ, σ^2) and compute $\bar{x}_1 = \frac{1}{n_1}\sum x_{1i}$, $s_1^2 = \frac{1}{n_1-1}\sum(x_{1i} - \bar{x}_1)^2$.
2. Test the hypothesis $H_0 : \sigma^2 = \sigma_0^2$ against the alternative $H_1 : \sigma^2 \neq \sigma_0^2$ at level α using the test statistic $\frac{\nu_1 s_1^2}{\sigma_0^2}$, which is distributed as χ^2 with $\nu_1 = (n_1 - 1)$ degrees of freedom.
3. If H_0 is accepted at α level of significance i.e. $x_1^2 < \frac{\nu_1 s_1^2}{\sigma_0^2} < x_2^2$,where x_1^2 and x_2^2 refer to lower and upper critical points of the unbiased portioning of the test statistic at a given level of significance α, take $k_1 s_1^2 + (1 - k_1)\sigma_0^2$ as the shrinkage estimator of σ^2 with shrinkage factor k_1 dependent on the test statistic.
4. If H_0 is rejected, take a second sample x_{2j} $(j = 1,2,\_\_, n_2)$ of size $n_2 = (n - n_1)$ compute $\bar{x}_2 = \frac{1}{n_2}\sum x_{2j}$, $s_2^2 = \frac{1}{n_2-1}\sum(x_{2j} - \bar{x}_2)^2$ and take $(\nu_1 s_1^2 + \nu_2 s_2^2)/(\nu_1 + \nu_2)$ where $\nu_2 = (n_2 - 1)$ as the estimator of σ^2.

To summarize, we define the double- stage shrinkage Testimator $\hat{\sigma}_{DST1}^2$ of σ^2 as follows:

$$\hat{\sigma}^2_{DST1} = \begin{cases} k_1\, s_1^2 + (1-k_1)\sigma_0^2 \;, & if \;\; H_0 \;\; is\, accepted \\ s_p^2 = \dfrac{(\nu_1 s_1^2 + \nu_2 s_2^2)}{(\nu_1+\nu_2)}, & if\; H_0 \;\; is \;\; rejected \end{cases}$$

Where $k_1 = \frac{\nu_1 s_1^2}{\sigma_0{}^2\, \chi^2}$

Estimators of this type with k arbitrary and lying between 0 and 1 have been proposed by Katti (1962), Shah(1964), Arnold and Al-Bayyati (1970), Waiker and Katti (1971), Pandey (1979) and k being dependent on the test statistics by Waiker Schuurman and Raghunandan (1984).

We define another double stage shrinkage Testimator $\hat{\sigma}^2_{DST2}$ of σ^2 by taking square of the shrinkage factor as $k_2 = k_1^2 = \left(\frac{\nu_1 s_1^2}{\sigma_0{}^2\, \chi^2}\right)^2$ which tends to zero more rapidly than k_1 as follows

$$\hat{\sigma}^2_{DST2} = \begin{cases} \left(\dfrac{\nu_1 s_1^2}{\sigma_0{}^2\, \chi^2}\right)^2 s_1^2 + \left(1 - \left(\dfrac{\nu_1 s_1^2}{\sigma_0{}^2\, \chi^2}\right)^2\right)\sigma_0^2 \;, if \;\; H_0 \;\; is\, accepted \\ s_p^2 \qquad\qquad\qquad\qquad\qquad , \quad if\; H_0 \;\; is \;\; rejected \end{cases}$$

3.7 **<u>Risk of Testimators</u>**

In this section we derive the risk of two proposed testimators which are defined in the previous section.

3.7.1 **<u>Risk of</u>** $\hat{\boldsymbol{\sigma}}^2_{DST1}$

The risk of $\hat{\sigma}^2{}_{DST1}$ under L(Δ) is defined by

$$R(\hat{\sigma}^2{}_{DST_1}) = E\,[\,\hat{\sigma}^2{}_{DST_1} \,|\, L(\Delta)]$$

$$= E\left[k_1\, s_1{}^2 + (1-k_1)\sigma_0^2 \Big/ \chi_1^2 < \frac{\upsilon_1\, s_1{}^2}{\sigma_0^2} < \chi_2^2\right] \cdot p\left[\chi_1^2 < \frac{\upsilon_1\, s_1{}^2}{\sigma_0^2} < \chi_2^2\right]$$
$$+ E\left[s_p{}^2 \,\left|\, \frac{\upsilon_1\, s_1{}^2}{\sigma_0^2} < \chi_1^2 \;\cup\; \frac{\upsilon_1\, s_1{}^2}{\sigma_0^2} > \chi_2^2\right.\right] \cdot p\left[\frac{\upsilon_1\, s_1{}^2}{\sigma_0^2} < \chi_1^2 \cup \frac{\upsilon_1\, s_1{}^2}{\sigma_0^2} > \chi_2^2\right]$$

_____ (3.7.1.1)

$$= \quad e^{-a} \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}} e^{a\left[\frac{\frac{\upsilon_1 s_1^{\,2}}{\sigma_0^2\,\chi^2}\left(s_1^{\,2}-\sigma_0^2\right)+\sigma_0^2}{\sigma^2}\right]} f(s_1^{\,2})\; d\,s_1^{\,2}$$

$$-\; a \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}} \left[\frac{\frac{\upsilon_1 s_1^{\,2}}{\sigma_0^2\,\chi^2}\left(s_1^{\,2}-\sigma_0^2\right)+\sigma_0^2}{\sigma^2} - 1\right] f(s_1^{\,2})\; d\,s_1^{\,2}$$

$$-\int_{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}} f(s_1^{\,2})\; ds_1^{\,2} \;+\; e^{-a} \int_0^{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}} \int_0^{\infty} e^{a\left(s_p^{\,2}/\sigma^2\right)} f(s_1^{\,2})\, f(s_2^{\,2}) ds_1^{\,2} ds_2^{\,2}$$

$$+\; e^{-a} \int_{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}}^{\infty} \int_0^{\infty} e^{a\left(s_p^{\,2}/\sigma^2\right)} f(s_1^{\,2})\, f(s_2^{\,2}) ds_1^{\,2} ds_2^{\,2}$$

$$-\; a \int_0^{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}} \int_0^{\infty} \left(\frac{s_p^{\,2}}{\sigma^2}-1\right) f(s_1^{\,2})\, f(s_2^{\,2}) ds_1^{\,2} ds_2^{\,2}$$

$$-\; a \int_{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}}^{\infty} \int_0^{\infty} \left(\frac{s_p^{\,2}}{\sigma^2}-1\right) f(s_1^{\,2})\, f(s_2^{\,2}) ds_1^{\,2} ds_2^{\,2}$$

$$-\int_0^{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}} \int_0^{\infty} f(s_1^{\,2})\, f(s_2^{\,2}) ds_1^{\,2} ds_2^{\,2} - \int_{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}}^{\infty} \int_0^{\infty} f(s_1^{\,2})\, f(s_2^{\,2}) ds_1^{\,2} ds_2^{\,2}$$

______(3.7.1.2)

Where $$f(s_1^2) = \frac{1}{2^{\upsilon_1/2}\,\Gamma\left(\upsilon_1/2\right)} \left(s_1^2\right)^{\frac{\upsilon_1}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon_1 s_1^2}{\sigma^2}\right)} ds_1^2$$

$$f(s_2^2) = \frac{1}{2^{\upsilon_2/2}\,\Gamma\left(\upsilon_2/2\right)} \left(s_2^2\right)^{\frac{\upsilon_2}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon_2 s_2^2}{\sigma^2}\right)} ds_2^2$$

Straight forward integration of (3.7.1.2) gives

$$R(\hat{\sigma}^2_{DST1}) = \left(\frac{\sigma^2}{\mathcal{V}_1}\right)^{\mathcal{V}_1/2} \left(\frac{\sigma^2}{\mathcal{V}_2}\right)^{\mathcal{V}_2/2}$$

$$R(\hat{\sigma}^2_{DST1}) = \left(\frac{\sigma^2}{\mathcal{V}_1}\right)^{\mathcal{V}_1/2} \left(\frac{\sigma^2}{\mathcal{V}_2}\right)^{\mathcal{V}_2/2} \left[\begin{array}{c} I_1{}^* - \frac{2a}{\lambda\,\chi^2}\left(\frac{\mathcal{V}_1}{2}+1\right) \\ \left\{I\left(\chi_2^2\lambda\,,\ \frac{\mathcal{V}_1}{2}+2\right) - I\left(\chi_1^2\lambda\,,\ \frac{\mathcal{V}_1}{2}+2\right)\right\} \\ + \quad \frac{a\mathcal{V}_1}{\chi^2}\left\{I\left(\chi_2^2\lambda\,,\ \frac{\mathcal{V}_1}{2}+1\right) - I\left(\chi_1^2\lambda\,,\ \frac{\mathcal{V}_1}{2}+1\right)\right\} \\ -(a\lambda - a + 1)\left\{I\left(\chi_2^2\lambda\,,\ \frac{\mathcal{V}_1}{2}\right) - I\left(\chi_1^2\lambda\,,\ \frac{\mathcal{V}_1}{2}\right)\right\} \\ -\frac{a\mathcal{V}_1}{\mathcal{V}_1+\mathcal{V}_2}\left\{I\left(\chi_1^2\lambda\,,\ \frac{\mathcal{V}_1}{2}+1\right) - I\left(\chi_2^2\lambda\,,\ \frac{\mathcal{V}_1}{2}+1\right)\right\} \\ + \frac{a\mathcal{V}_1}{\mathcal{V}_1+\mathcal{V}_2}\left\{I\left(\chi_1^2\lambda\,,\ \frac{\mathcal{V}_1}{2}\right) - I\left(\chi_2^2\lambda\,,\ \frac{\mathcal{V}_1}{2}\right)\right\} \\ -\left\{I\left(\chi_1^2\lambda\,,\ \frac{\mathcal{V}_1}{2}\right) - I\left(\chi_2^2\lambda\,,\ \frac{\mathcal{V}_1}{2}\right) + 1\right\} + I_2{}^* \end{array}\right]$$

_____ (3.7.1.3)

Where $I(x;p) = \left(1/\Gamma p\right)\int_0^x e^{-x}\,x^{p-1}\,dx$ refers to the standard incomplete gamma function, $\lambda = \frac{\sigma_0^2}{\sigma^2}$ and

$$I_1{}^* = \frac{e^{a(\lambda-1)}}{2^{\mathcal{V}_1/2}\ \Gamma\left(\frac{\mathcal{V}_1}{2}\right)} \int_{x_1^2\lambda}^{x_2^2\lambda} e^{\left[\frac{a t_1{}^2}{\lambda \mathcal{V}_1 x^2} - \frac{a t_1}{x^2}\right]}\, e^{-\frac{1}{2}t_1}\,(t_1)^{\frac{\mathcal{V}_1}{2}-1}\,dt_1$$

$$I_2^{\,*} = \frac{e^{-a}}{2^{\left(\frac{v1}{2}+\frac{v2}{2}\right)}\left(\frac{1}{2}-\frac{a}{v1+v2}\right)^{\left(\frac{v1}{2}+\frac{v2}{2}\right)}} \left[I\left(\chi_1^2\lambda,\ \frac{v_1}{2}\right) - I\left(\chi_2^2\lambda,\ \frac{v_1}{2}\right) + 1 \right]$$

3.7.2 <u>Risk of</u> $\hat{\sigma}^2_{DST2}$

Again, we obtain the risk of $\hat{\sigma}^2{}_{DST_2}$ under L(Δ) with respect to ${s_p}^2$, given by

$$R(\hat{\sigma}^2{}_{DST_2}) = E[\,\hat{\sigma}^2{}_{DST_2} \,|\, L(\Delta)]$$

$$= E\left[\left(\frac{\upsilon_1 {s_1}^2}{\sigma_0^2\,\chi^2}\right)^2 {s_1}^2 + \left(1-\left(\frac{\upsilon_1 {s_1}^2}{\sigma_0^2\,\chi^2}\right)^2\right)\sigma_0^2 \Big/ \chi_1^2 < \frac{\upsilon_1 {s_1}^2}{\sigma_0^2} < \chi_2^2\right] \cdot p\left[\chi_1^2 < \frac{\upsilon_1 {s_1}^2}{\sigma_0^2} < \chi_2^2\right]$$

$$+ E\left[{s_p}^2 \;\middle|\; \frac{\upsilon_1 {s_1}^2}{\sigma_0^2} < \chi_1^2 \cup \frac{\upsilon_1 {s_1}^2}{\sigma_0^2} > \chi_2^2\right] \cdot p\left[\frac{\upsilon_1 {s_1}^2}{\sigma_0^2} < \chi_1^2 \cup \frac{\upsilon_1 {s_1}^2}{\sigma_0^2} > \chi_2^2\right]$$

______(3.7.2.1)

$$= \quad e^{-a} \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}} e^{a\left[\frac{\left(\frac{\upsilon_1 {s_1}^2}{\sigma_0^2\,\chi^2}\right)^2\left({s_1}^2 - \sigma_0^2\right)+\sigma_0^2}{\sigma^2}\right]} f({s_1}^2)\, d{s_1}^2$$

$$- \quad a \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}} \left[\frac{\left(\frac{\upsilon_1 {s_1}^2}{\sigma_0^2\,\chi^2}\right)^2\left({s_1}^2 - \sigma_0^2\right)+\sigma_0^2}{\sigma^2} - 1\right] f({s_1}^2)\, d{s_1}^2$$

$$- \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}} f({s_1}^2)\, d{s_1}^2 + e^{-a} \int_0^{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}} \int_0^{\infty} e^{a\left({s_p}^2 \big/ \sigma^2\right)} f({s_1}^2)\, f({s_2}^2)\, d{s_1}^2\, d{s_2}^2$$

$$+ \; e^{-a} \int_{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}}^{\infty} \int_0^{\infty} e^{a\left(s_p^2/\sigma^2\right)} f(s_1^2)\, f(s_2^2) ds_1^2 ds_2^2 - a \int_0^{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}} \int_0^{\infty} \left(\frac{s_p^2}{\sigma^2} - 1\right) f(s_1^2)\, f(s_2^2) ds_1^2 ds_2^2$$

$$- a \int_{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}}^{\infty} \int_0^{\infty} \left(\frac{s_p^2}{\sigma^2} - 1\right) f(s_1^2)\, f(s_2^2) ds_1^2 ds_2^2 - \int_0^{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}} \int_0^{\infty} f(s_1^2)\, f(s_2^2) ds_1^2 ds_2^2$$

$$- \int_{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}}^{\infty} \int_0^{\infty} f(s_1^2)\, f(s_2^2) ds_1^2 ds_2^2$$

______(3.7.2.2)

Where $f(s_1^2) = \dfrac{1}{2^{\upsilon_1/2}\, \Gamma\left(\upsilon_1/2\right)} \left(s_1^2\right)^{\frac{\upsilon_1}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon_1 s_1^2}{\sigma^2}\right)} ds_1^2$

$$f(s_2^2) = \frac{1}{2^{\upsilon_2/2}\, \Gamma\left(\upsilon_2/2\right)} \left(s_2^2\right)^{\frac{\upsilon_2}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon_2 s_2^2}{\sigma^2}\right)} ds_2^2$$

Straight forward integration of (3.7.2.2) gives

$$\left(\hat{\sigma}^2_{DST2}\right) = \left(\frac{\sigma^2}{\mathcal{V}_1}\right)^{\mathcal{V}_1/2} \left(\frac{\sigma^2}{\mathcal{V}_2}\right)^{\mathcal{V}_2/2} \left[\begin{array}{c} I_1^* - \frac{4a}{\lambda^2\,(x^2)^2}\left(\frac{\mathcal{V}_1}{2}+1\right)\left(\frac{\mathcal{V}_1}{2}+2\right) \\ \left\{I\left(\chi_2^2\lambda,\, \frac{\mathcal{V}_1}{2}+3\right) - I\left(\chi_1^2\lambda,\, \frac{\mathcal{V}_1}{2}+3\right)\right\} \\ + \frac{4a}{\lambda(x^2)^2}\left(\frac{\mathcal{V}_1}{2}\right)\left(\frac{\mathcal{V}_1}{2}+1\right) \\ \left\{I\left(\chi_2^2\lambda,\, \frac{\mathcal{V}_1}{2}+2\right) - I\left(\chi_1^2\lambda,\, \frac{\mathcal{V}_1}{2}+2\right)\right\} \\ -(a\lambda - a + 1)\left\{I\left(\chi_2^2\lambda,\, \frac{\mathcal{V}_1}{2}\right) - I\left(\chi_1^2\lambda,\, \frac{\mathcal{V}_1}{2}\right)\right\} \\ -\frac{a\mathcal{V}_1}{\mathcal{V}_1+\mathcal{V}_2}\left\{I\left(\chi_1^2\lambda\,,\, \frac{\mathcal{V}_1}{2}+1\right) - I\left(\chi_2^2\lambda\,,\, \frac{\mathcal{V}_1}{2}+1\right)\right\} \\ +\frac{a\mathcal{V}_1}{\mathcal{V}_1+\mathcal{V}_2}\left\{I\left(\chi_1^2\lambda\,,\, \frac{\mathcal{V}_1}{2}\right) - I\left(\chi_2^2\lambda\,,\, \frac{\mathcal{V}_1}{2}\right)\right\} \\ -\left\{I\left(\chi_1^2\lambda\,,\, \frac{\mathcal{V}_1}{2}\right) - I\left(\chi_2^2\lambda\,,\, \frac{\mathcal{V}_1}{2}\right) + 1\right\} + I_2^* \end{array}\right]$$

_____(3.7.2.3)

Where $I(x;p) = \left(1/\Gamma p\right) \int\limits_0^x e^{-x} x^{p-1} dx$ refers to the standard incomplete gamma function and

$$I_1^* = \frac{e^{a(\lambda-1)}}{2^{\nu_1/2}\ \Gamma\left(\frac{\nu_1}{2}\right)} \int_{\chi_1^2\lambda}^{\chi_2^2\lambda} e^{\left[\frac{a t_1^3}{\lambda^2 \nu_1 (x^2)^2} - \frac{a t_1^2}{\lambda (x^2)^2}\right]} e^{-\frac{1}{2}t_1} (t_1)^{\frac{\nu_1}{2}-1} dt_1$$

$$I_2^* = \frac{e^{-a}}{2^{\left(\frac{\nu_1}{2}+\frac{\nu_2}{2}\right)}\left(\frac{1}{2}-\frac{a}{\nu_1+\nu_2}\right)^{\left(\frac{\nu_1}{2}+\frac{\nu_2}{2}\right)}} \left[I\left(\chi_1^2\lambda,\ \frac{\nu_1}{2}\right) - I\left(\chi_2^2\lambda,\ \frac{\nu_1}{2}\right) + 1\right]$$

3.8 Relative Risk of $\hat{\sigma}^2_{DSTi}$

A natural way of comparing the risk of the proposed testimators, is to study its performance with respect to the best available estimator $s_p{}^2$ in this case. For this purpose, we obtain the risk of $s_p{}^2$ under $L_E\left(\hat{\sigma}^2, \sigma^2\right)$ as:

$$R_E(s_p^2) = E[\, s_p^2 \,|\, L(\hat{\sigma}^2, \sigma^2)]$$

$$= e^{-a} \int\limits_0^\infty \int\limits_0^\infty e^{a\left[\frac{s_p{}^2}{\sigma^2}\right]} f(s_1{}^2) f(s_2{}^2)\, ds_1{}^2 ds_2{}^2$$

$$-a \int\limits_0^\infty \int\limits_0^\infty \left[\frac{s_p^2}{\sigma^2} - 1\right] f(s_1{}^2) f(s_2{}^2)\, ds_1{}^2 ds_2{}^2 - \int\limits_0^\infty \int\limits_0^\infty f(s_1{}^2) f(s_2{}^2)\, ds_1{}^2 ds_2{}^2$$

______(3.8.1)

Where $f(s_1{}^2) = \dfrac{1}{2^{\upsilon_1/2}\ \Gamma\left(\upsilon_1/2\right)} \left(s_1{}^2\right)^{\frac{\upsilon_1}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon_1 s_1{}^2}{\sigma^2}\right)} ds_1{}^2$

$$f(s_2^{\;2}) = \frac{1}{2^{\upsilon_2/2}\,\Gamma\left(\upsilon_2/2\right)}\left(s_2^{\;2}\right)^{\frac{\upsilon_2}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon_2 s_2^{\;2}}{\sigma^2}\right)} ds_2^{\;2}$$

A straightforward integration of (3.8.1) gives

$$R_E(s_p^2) = \left(\frac{\sigma^2}{\nu_1}\right)^{\nu_1/2}\left(\frac{\sigma^2}{\nu_2}\right)^{\nu_2/2}\left[\frac{e^{-a}}{2^{\left(\nu_1/2+\nu_2/2\right)}\left(\frac{1}{2}-\frac{a}{\nu_1+\nu_2}\right)^{\nu_1/2+\nu_2/2}} - 1\right]$$

_______(3.8.2)

Now, we define the Relative Risk of $\hat{\sigma}^2_{DST_i}$, $i = 1,2$ with respect to s_p^2 under $L\left(\hat{\sigma}^2, \sigma^2\right)$ as follows:

$$RR_1 = \frac{R_E(\mathrm{s_p}^2)}{R(\hat{\sigma}^2_{DST1})} \qquad \text{\_\_\_\_\_\_\_(3.8.3)}$$

Using (3.8.2) and (3.7.1.3) the expression for RR_1 given in (3.8.3) can be obtained; it is observed that RR_1 is a function of 'ν_1', 'ν_2', 'λ', 'α' and 'a'.

Finally, we define the Relative Risk of $\hat{\sigma}^2_{ST_2}$ by

$$RR_2 = \frac{R_E(\mathrm{s_p}^2)}{R(\hat{\sigma}^2_{DST2})} \qquad \text{\_\_\_\_\_\_\_(3.8.4)}$$

The expression for RR_2 is given by (3.8.4) can be obtained by using (3.8.2) and (3.7.2.3). Again we observed that RR_2 is a function of 'ν_1', 'ν_2', 'λ', 'α' and 'a'.

3.9 Recommendations for $\hat{\boldsymbol{\sigma}}^2_{\boldsymbol{DSTi}}$

In this section we wish to compare the performance of $\hat{\sigma}^2_{DST_1}$ and $\hat{\sigma}^2_{DST_2}$ with respect to the best available (unbiased) estimator of σ^2 $i.e.$ $\hat{\sigma}^2$.

3.9.1 Recommendations for $\hat{\sigma}^2_{DST1}$

It is observed that the above expressions (3.8.3) and (3.8.4) are functions of $\alpha, \lambda, \nu_1, \nu_2$ and the degrees of asymmetry " a ". For the comparison purpose we have considered several values for these viz. (ν_1, ν_2) = (6,6), (6,9), (6,12), (6,15), (6,18); (8,8), (8,12), (8,16), (8,20), (8,24) and (10,10), (10,15), (10,20), (10,25), (10,30) ; α = 1%, 5% and 10%, and a = -3, -2, -1, 1, 1.25, 1.50 and λ = 0.2 (0.2) 2.0 .

In all there will be several tables for these data sets of Relative Risk (RR_1). We have presented some of the tables at the end of the chapter. However, our recommendations based on all these findings are as follows:

(i) The proposed testimator $\hat{\sigma}^2_{DST_1}$ performs better than the pooled estimator ${s_p}^2$ for almost all the values considered as above. However some of the best performances are outlined specifically.

(ii) $\hat{\sigma}^2_{DST_1}$ dominates the usual estimator when (ν_1, ν_2) = (6,6) ; α = 1% ; a = -1 for $0.2 \leq \lambda \leq 2.0$ and for a = +1 the range of λ is $0.2 \leq \lambda \leq 2.0$.

(iii) As 'ν_2' increases the RR_1 values are still greater than unity, but decrease in magnitude also the range of 'λ' changes slightly now it becomes $0.6 \leq \lambda \leq 1.8$ for negative values of 'a'. A similar pattern is observed when 'a' is positive for almost $0.6 \leq \lambda \leq 1.8$.

(iv) The performance of $\hat{\sigma}^2_{DST_1}$ is the best when a = +1 or a = -1 in terms of the range of λ, the magnitude of RR_1 values for the first data set i.e. (6,6). The same remains true when ν_2 increases i.e. (6,9) etc. Here we have considered these values for α = 1%.

(v) As the other quantity of interest i.e. the level of significance in addition to the degrees of asymmetry. We change ‘α’ to 5% and 10% it is observed that still the proposed testimator performs better for the ‘ranges’ mentioned as above. i.e. when ‘a’ is negative $0.2 \leq \lambda \leq 2.0$ and when ‘a’ is positive it becomes $0.2 \leq \lambda \leq 1.6$ indicating that range shrinks for overestimation case. Still the values of RR_1 are more than unity but their magnitude decreases slightly.

(vi) Now, we have considered the other values of (ν_1, ν_2) as mentioned above and it is observed that RR_1 values are still higher than unity for these different data sets, with almost the same ranges of ‘λ’ as above for positive as well as negative values of ‘a’. Again as ν_2 increases the magnitude of RR_1 values decreases but not falling below 1.

(vii) Overall recommendations are: ν_1 should be small i.e. $\nu_1 \ngtr 10$ and $\nu_2 \leq 3\nu_1$, α = 1% i.e. a smaller level of significance and for various degrees of asymmetry i.e. ‘a’ could be extreme negative as a = -3 or it could be considerably positive i.e. a = 1.5. The best suggested values are a = -1 or a = +1.

(viii) When these RR_1 values are compared with the Mean Square values of $\hat{\sigma}^2_{DST_1}$ proposed by Pandey and Srivastava (1987) it is observed that the magnitude of RR_1 values are **HIGHER**, the range of ‘λ’ increases considerably as it was (0.5 – 1.5) and now it becomes almost (0.2 – 2.0) earlier it was recommended that $\nu_2 \leq 2\nu_1$ now it becomes $\nu_2 \leq 3\nu_1$ a considerable increase in the choice of ν_2. Implying that the use of ASL not only allows to take account for various degrees of asymmetry (i.e. choose ‘a’ accordingly when over / under estimation is more serious) but also increases the range of ‘λ’, ν_2 etc.

3.9.2 Recommendations for $\hat{\sigma}^2_{DST2}$

We have also proposed $\hat{\sigma}^2{}_{DST_2}$ which is obtained by squaring the shrinkage factor. The performance of it, is compared with respect to $s_p{}^2$ for the **same** data as considered for $\hat{\sigma}^2{}_{DST_1}$. Again, similar tables of RR_2 will be generated for these data sets. Our recommendations based on all these computations are as follows:

(i) It is observed that the magnitude of RR_2 values is higher than RR_1 values. The proposed testimator performs better than the best available estimator for almost all the values considered here. The best performing data sets are mentioned briefly.

(ii) $\hat{\sigma}^2{}_{DST_2}$ dominates $s_p{}^2$ when $(\nu_1, \nu_2) =$ (6,6), α = 1% for a = -1, $0.2 \leq \lambda \leq 2.0$ and for a = +1, $0.2 \leq \lambda \leq 2.0$ as obtained earlier.

(iii) As 'ν_2' increases the RR_2 values decrease in their magnitude (but still above unity). Here the range of 'λ' change shortens slightly as it is now $0.6 \leq \lambda \leq 1.8$ for negative values of 'a' however when 'a' is positive it remains unchanged i.e. $0.2 \leq \lambda \leq 2.0$.

(iv) The performance of $\hat{\sigma}^2{}_{DST_2}$ is at its best when a = ± 1. As 'ν_2' increases i.e. for the other data set (6,9), (6,12), (6,15) or (6,18) the magnitude of RR_2 decreases slightly but not below unity. Again, if we increase ν_1 i.e. (8,8), (8,12) etc. Similar behaviour of RR_2 values is observed but their magnitude change.

(v) Now taking $\alpha = 5\%$ and $\alpha = 10\%$ when the values of RR_2 are obtained again these values are 'good' in the sense of being more than unity. But there is a decrease in the magnitude of RR_2 values as 'α' increase.

(vi) We therefore recommend as, ν_1 should be small i.e. $\nu_1 \ngtr 10$ and $\nu_2 \leq 3\nu_1$, and choose α = 1% . However the degree of asymmetry could chosen for a fairly large range i.e. from a = -3 to a = 1.5. The best performing values are observed for a = ±1.

(vii) Comparing these RR_2 values with those obtained by Pandey and Srivastava (1987) under the MSE criterion (or the use of 'SELF') indicate that these values are 'better' than those values showing that the application of Asymmetric Loss Function yields better result also providing a choice to tailor the risk by choosing 'a' appropriately. Further the range of 'λ' increases.

CONCLUSIONS:

Two shrinkage testimators viz. $\hat{\sigma}^2_{DST_1}$ and $\hat{\sigma}^2_{DST_2}$ have been proposed for the variance of a Normal distribution. It is concluded that (i) use asymmetric loss function to study the risk properties. (ii) ν_1 should be small preferably should not exceed 10 for both the cases. (iii) $\nu_2 \leq 3\nu_1$ (iv) take α = 1% and take $0.2 \leq \lambda \leq 2.0$ for negative values of 'a' and take $0.2 \leq \lambda \leq 1.8$ for positive values of 'a'. (v) take 'SQUARE' of the shrinkage factor.

Tables showing relative risk(s) of proposed testimator(s) with respect to the best available estimator.

Table : 3.9.1.1 **Relative Risk of** $\hat{\sigma}^2_{DST_1}$ α = 1% , (ν_1, ν_2) = (6, 6)

λ	a = -3	a = -2	a = -1	a = 1	a = 1.25	a = 1.50
0.20	1.059	1.531	1.229	1.835	1.884	1.890
0.40	1.257	1.649	2.081	2.06	1.984	1.975
0.60	1.658	2.618	3.714	3.514	3.762	3.509
0.80	3.484	4.013	5.103	5.913	4.623	3.974
1.00	4.433	5.486	6.834	7.02	5.153	4.851
1.20	4.086	5.332	6.08	6.884	4.774	3.368
1.40	3.753	4.414	5.827	4.499	3.213	2.336
1.60	2.357	3.417	4.518	2.909	2.087	1.541
1.80	1.637	2.339	3.117	1.911	1.354	0.999
2.00	1.239	1.735	2.236	1.295	0.899	0.654

Table : 3.9.1.2 **Relative Risk of** $\hat{\sigma}^2_{DST_1}$ α = 5% , (ν_1, ν_2) = (6, 6)

λ	a = -3	a = -2	a = -1	a = 1	a = 1.25	a = 1.50
0.20	1.379	1.831	1.813	1.49	1.692	1.057
0.40	1.972	1.912	2.436	2.592	2.54	1.536
0.60	1.339	2.021	3.939	2.855	2.711	2.676
0.80	2.271	3.074	4.42	3.909	3.568	3.334
1.00	3.593	4.462	5.081	5.001	4.111	4.67
1.20	4.153	4.172	5.051	3.563	2.842	2.267
1.40	3.549	3.736.	4.476	2.299	1.837	1.472
1.60	2.754	2.888	3.762	1.634	1.294	1.034
1.80	2.182	2.166	3.133	1.212	0.945	0.748
2.00	1.815	1.658	2.649	0.926	0.707	0.551

Table : 3.9.1.1 **Relative Risk of** $\hat{\sigma}^2_{DST_1}$ α = 1% , (ν_1, ν_2) = (8, 8)

λ	a = -3	a = -2	a = -1	a = 1	a = 1.25	a = 1.50
0.20	1.486	1.74	1.481	1.775	1.792	1.851
0.40	1.989	1.839	1.861	2.693	2.606	2.56
0.60	1.617	2.722	2.195	3.211	3.123	3.249
0.80	3.301	3.041	3.476	4.793	4.437	4.311
1.00	4.105	5.296	6.005	6.403	5.446	5.405
1.20	4.077	4.315	5.212	5.968	4.572	3.507
1.40	3.886	3681	4.75	3.679	2.869	2.256
1.60	2.5	2.75	3.089	2.326	1.797	1.413
1.80	1.782	2.675	2.395	1.534	1.159	0.897
2.00	1.392	2.069	2.338	1.059	0.779	0.589

Table : 3.9.2.1 **Relative Risk of** $\hat{\sigma}^2_{DST_2}$ α = 1% , (ν_1, ν_2) = (6, 6)

λ	a = -3	a = -2	a = -1	a = 1	a = 1.25	a = 1.50
0.20	0.647	1.117	1.897	1.214	1.164	1.156
0.40	0.652	1.997	2.725	1.452	1.347	1.299
0.60	1.381	2.054	3.202	2.629	2.496	3.603
0.80	3.826	4.273	4.73	4.669	3.67	4.078
1.00	5.225	5.732	5.933	5.684	5.396	6.077
1.20	4.882	4.758	4.747	4.077	4.857	4.248
1.40	3.444	3.075	3.385	3.59	3.369	2.531
1.60	2.035	2.952	3.022	2.702	1.951	1.464
1.80	1.397	1.976	2.755	1.68	1.181	0.871
2.00	1.06	1.463	2.735	1.103	0.752	0.541

Table : 3.9.2.2 **Relative Risk of** $\hat{\sigma}^2_{DST_2}$ α = 1% , (ν_1, ν_2) = (8, 8)

λ	a = -3	a = -2	a = -1	a = 1	a = 1.25	a = 1.50
0.20	0.74	1.306	1.328	1.143	1.074	1.036
0.40	0.61	1.962	2.865	1.177	1.173	1.909
0.60	1.285	2.009	4.475	3.825	3.595	3.514
0.80	3.539	4.15	5.735	4.962	4.863	3.643
1.00	6.627	7.176	7.917	6.658	5.38	4.939
1.20	4.728	5.556	5.968	4.261	4.945	3.906
1.40	3.439	3.151	4.721	3.488	2.737	2.179
1.60	2.076	2.192	3.519	2.069	1.577	1.233
1.80	1.466	1.692	3.76	1.313	0.968	0.736
2.00	1.151	1.406	2.141	0.885	0.631	0.465

Table : 3.9.2.3 **Relative Risk of** $\hat{\sigma}^2_{DST_2}$ α = 5% , (ν_1, ν_2) = (6, 6)

λ	a = -3	a = -2	a = -1	a = 1	a = 1.25	a = 1.50
0.20	0.848	1.521	1.017	1.577	1.549	1.582
0.40	0.957	1.87	2.343	1.947	1.967	1.927
0.60	1.987	1.302	2.97	2.942	2.777	2.698
0.80	3.798	2.522	4.836	4.478	3.225	3.658
1.00	4.434	4.561	5.914	5.711	4.446	5.626
1.20	3.403	3.652	4.607	3.579	2.887	2.326
1.40	2.456	3.585	3.919	2.183	1.751	1.413
1.60	1.887	2.773	2.884	1.487	1.171	0.936
1.80	1.555	2.272	2.015	1.065	0.818	0.641
2.00	1.357	1.964	1.388	0.79	0.589	0.45

Chapter – 4

SINGLE SAMPLE SHRINKAGE TESTIMATORS UNDER GENERAL ENTROPY LOSS FUNCTION

4.1 Introduction

The present chapter deals with one sample shrinkage testimators under General Entropy Loss Function (GELF) for single parameter Exponential distribution and Normal distribution.

The aim of systems reliability is to forecast of various system performance measures such as mean life time, guarantee period and reliability etc. In general, the type of failure distribution depends on the failure mechanism of components. If the failure rate is constant, which is mostly true for electronic components during the major part of their useful life, the failure time follows an exponential distribution with the p.d.f.

$$f(x;\theta) = \begin{cases} \frac{1}{\theta}\exp(-x/\theta), & x \geq 0, \theta > 0 \\ 0 & , otherwise \end{cases} \qquad \text{\_\_\_\_\_\_(4.1.1)}$$

In the context of life testing and reliability estimation, numerous data have been examined and it has been found that exponential distribution fits well for most of the cases. Several authors have proposed estimators, testimators with different choices of shrinkage factors (S.F.) under different loss functions. The choice of an appropriate loss function is guided by financial consideration apart from other considerations such as over estimation being more serious than under-estimation or vice-versa.

Shrinkage testimators for the mean μ of a Normal distribution N(μ, σ^2) when variance σ^2 is known or unknown, have been proposed by Waiker, Schuurman and Raghunandan (1984). Recently Pandey et. al. (1987) considered some shrinkage testimators for the variance estimator under Mean Square Error criterion (MSE). Parisan and Farsipour (1999), Misra and Meuten (2003), Pandey et. al. (2004), Ahmadi et. al. (2005), Xiao et. al. (2005), Prakash and Singh (2006), Prakash and Pandey (2007) and others have considered the estimation procedures under the LINEX loss function in various contexts. Pandey et. al. (2007) have proposed shrinkage testimator(s) variance and have studied the properties of these under the Asymmetric loss function (ASL). The present work is an attempt to study the risk properties of shrinkage testimator(s) for the variance of Normal distribution under a more general loss function viz. (GELF). Pandey et. al. (2007) have studied the risk properties of the same for positive degree of asymmetry only, under ASL. Where as this study attempts to find the range for positive as well as negative degrees of asymmetry under GEL where the shrinkage testimator of variance performs better than the UMVUE.

4.1.1 General Entropy Loss Function (GELF)

A suitable alternative to modified LINEX loss is the General Entropy Loss (GEL) proposed by Calabria and Pulcini (1996) given by:

$$L_E(\hat{\theta}, \theta) \propto \left\{ \left(\hat{\theta}/\theta\right)^p - p \ln \left(\hat{\theta}/\theta\right) - 1 \right\}, \quad p \neq 0 \qquad \text{______ (4.1.1.1)}$$

Whose minimum occurs at $\hat{\theta} = \theta$.

This loss is a generalization of the entropy loss used by several authors (for example, Dey and Liu, 1992) where the shape parameter 'p' is equal to unity (1). The more general version of (4.1.1.1) allows different shapes of the loss function to be considered (say when p > 0, a positive error $(\hat{\theta} > \theta)$ causes more serious consequence than a negative error and when p < 0, then negative error is more serious). If we are considering prior distributions, then the Bayes estimate of θ under GELF is in a closed form and is given by

$$\hat{\theta}^E = \left[E_\theta\left(\theta^{-p}\right)\right]^{-1/p} \qquad \text{\_\_\_\_\_\_(4.1.1.2)}$$

provided that $E_\theta\left(\theta^{-p}\right)$ exists and is finite.

4.1.2 Incorporating a Point Guess and $\hat{\theta}_{ST}$

In many real life situations the experimenter may have some prior information regarding the parameter being estimated due to some past experience or similar kind of studies and it is thought to apply this information to inference procedures of the original model. If the prior information is available only in the form of a point (a single) value (say) θ_0 for θ. For example a medical practitioner knows that in how many days the patient may get cured (say) 7 days or 10 days due to his past experience of treatment. Here we may take θ_0 = 7 days. For such situations it is suggested to start with the current (sample) information, construct an estimator $\hat{\theta}$ (MVUE or UMVUE) and modify it by incorporating the guess θ_0 (sometimes called natural origin) so that the resulting estimator or testimator though perhaps biased, has smaller risk than that of $\hat{\theta}$ in some interval around θ_0.

In this chapter an attempt has been made to demonstrate that how shrinkage testimation procedure works under GELF.

We have proposed the shrinkage testimators for the scale parameter of an Exponential distribution in section 4.2. The risks of the proposed testimators have been derived in section 4.3. The section 4.4 deals with the relative risk(s) of these two estimators. Section 4.5 concludes with the comparison of UMVUE and the proposed shrinkage testimators in terms of their relative risks. Suggestion for the choice of shrinkage factor is made and recommendations regarding the choice of level of significance and degree of asymmetry have been made.

In section 4.6 we have proposed the two different shrinkage testimators for the variance of a Normal distribution and we have studied the risk properties of these two shrinkage testimators under General Entropy Loss Function. Section 4.7 deals with the derivation of the risk(s) of these two estimators. Section 4.8 deals with the relative risk(s) of these two estimators. Section 4.9 concludes with the comparison of UMVUE and the proposed shrinkage testimatiors in terms of their relative risks. Further in the same section a suggestion for the shrinkage factor is made, along with the choices of degrees of asymmetry and level of significance.

4.2 Shrinkage Testimator(s) for Scale Parameter of an Exponential Distribution.

Let x have the distribution defined in (4.1.1). It is assumed that the prior knowledge about θ is available in the form of an initial estimate θ_0. We are interested in constructing an estimator of θ possibly using the information about θ and the sample observations: $x_1, x_2 ...., x_n$. The proposed shrinkage testimator can be described as follows:

(i) Compute the sample mean $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ which is the 'best' estimator of θ in absence of any information about θ. (ii) Test the hypothesis H_0 : $\theta = \theta_0$ against H_1 : $\theta \neq \theta_0$ at α level using the test statistic $\frac{2n\bar{x}}{\theta_0}$ which follows χ^2 – distribution with 2n degrees of freedom.

We define the shrinkage testimator $\hat{\theta}_{ST_1}$ and $\hat{\theta}_{ST_2}$ of θ as follows:

$$\hat{\theta}_{ST_1} = \begin{cases} k\,\bar{x} + (1-k)\theta_0 & ;\ if\ \ \chi_1^2 \le \left(2n\,\bar{x}/\theta_0\right) \le \chi_2^2 \\ \bar{x} & ;\ otherwise \end{cases} \qquad \_\_\_\_\_\_\_(4.2.1)$$

where k being dependent on test statistic is given by $k = 2n\,\bar{x}/\theta_0\,\chi^2$ and $\chi^2 = \left(\chi_2^2 - \chi_1^2\right)$

Now, taking the 'square' of k (i.e. $k = k^2$), another testimator is defined as

$$\hat{\theta}_{ST_2} = \begin{cases} \left(2n\,\bar{x}/\theta_0\ \chi^2\right)^2 \bar{x} + \left[1 - \left(2n\,\bar{x}/\theta_0\ \chi^2\right)^2\right]\theta_0 & ;\ if\ \ H_0\ is\ accepted \\ \bar{x} & ;\ otherwise \end{cases} \qquad \_\_\_\_\_(4.2.2)$$

4.3 Risk of Testimators

In this section we derive the risk of these two testimators which are defined in the previous section.

4.3.1 Risk of $\hat{\theta}_{ST1}$

The risk of $\hat{\theta}_{ST_1}$ under $L_E(\hat{\theta},\theta)$ is defined by

$$R(\hat{\theta}_{ST_1}) = E[\,\hat{\theta}_{ST_1} \,|\, L_E(\hat{\theta},\theta)]$$

$$= E\left[k\,\bar{x} + (1-k)\theta_0 / \chi_1^2 < 2n\bar{x}/\theta_0 < \chi_2^2\right] \cdot p\left[\chi_1^2 < 2n\bar{x}/\theta_0 < \chi_2^2\right]$$
$$+ E\left[\bar{x} \,\middle|\, 2n\bar{x}/\theta_0 < \chi_1^2 \ \cup\ 2n\bar{x}/\theta_0 > \chi_2^2\right] \cdot p\left[2n\bar{x}/\theta_0 < \chi_1^2 \cup 2n\bar{x}/\theta_0 > \chi_2^2\right]$$

$$\_\_\_\_\_\_\_(4.3.1.1)$$

$$= \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} \left[\left(\frac{2n\bar{x}}{\theta_0\,\chi^2}\right)(\bar{x}-\theta_0) + \theta_0 \Big/ \theta\right]^p f(\bar{x})\,d\bar{x} - \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} p \ln\left[\frac{2n\bar{x}}{\theta_0\,\chi^2}(\bar{x}-\theta_0) + \theta_0 \Big/ \theta\right] f(\bar{x})\,d\bar{x}$$

$$- \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} f(\bar{x})\,d\bar{x} + \int_0^{\frac{\chi_1^2\theta_0}{2n}} \left[(\bar{x}/\theta)^p - p\ln(\bar{x}/\theta) - 1\right] f(\bar{x})\,d\bar{x} + \int_{\frac{\chi_2^2\theta_0}{2n}}^{\infty} \left[(\bar{x}/\theta)^p - p\ln(\bar{x}/\theta) - 1\right] f(\bar{x})\,d\bar{x}$$

$$\_\_\_\_\_\_\_(4.3.1.2)$$

Where $f(\bar{x}) = (1/\Gamma n)\ (n/\theta)^n (\bar{x})^{n-1} \exp(-n\bar{x}/\theta)$

A Straight forward integration of (4.3.1.2) gives

$$R(\hat{\theta}_{ST_1}) = I_1 - I_2 - \left\{ I\left(\frac{x_2^2\phi}{2}, n\right) - I\left(\frac{x_1^2\phi}{2}, n\right)\right\} +$$
$$(1/n)^p \frac{\Gamma(p+n)}{\Gamma n}\left\{ I\left(\frac{x_1^2\phi}{2}, n+p\right) - I\left(\frac{x_2^2\phi}{2}, n+p\right) + 1\right\} -$$
$$\left\{ I\left(\frac{x_1^2\phi}{2}, n\right) - I\left(\frac{x_2^2\phi}{2}, n\right) + 1\right\} - I_3 - I_4$$

_______(4.3.1.3)

Where $I(x;p) = (1/\Gamma p)\int_0^x e^{-x} x^{p-1}\,dx$ refers to the standard incomplete gamma function and

$$I_1 = \int_{\frac{x_1^2\phi}{2}}^{\frac{x_2^2\phi}{2}} \left[\left(2t^2/n\phi\chi^2\right) - \left(2t/\chi^2\right) + \phi\right]^p (1/\Gamma n)\, e^{-t}\, t^{n-1}\, dt$$

$$I_2 = \int_{\frac{x_1^2\phi}{2}}^{\frac{x_2^2\phi}{2}} p\ln\left[\left(2t^2/n\phi\chi^2\right) - \left(2t/\chi^2\right) + \phi\right](1/\Gamma n)\, e^{-t}\, t^{n-1}\, dt$$

$$I_3 = \int_0^{\frac{x_1^2\phi}{2}} p\ln\left(t/n\right)\left(1/\Gamma n\right) e^{-t}\, t^{n-1}\, dt$$

$$I_4 = \int_{\frac{x_2^2\phi}{2}}^{\infty} p\ln\left(t/n\right)\left(1/\Gamma n\right) e^{-t}\, t^{n-1}\, dt$$

4.3.2 Risk of $\hat{\theta}_{ST2}$

Similarly, we obtain the risk of $\hat{\theta}_{ST_2}$ under $L_E(\hat{\theta},\theta)$ given by

$$R(\hat{\theta}_{ST_2}) = E[\hat{\theta}_{ST_2} \mid L_E(\hat{\theta},\theta)]$$

$$= E\left[\left(2n\bar{x}/\theta_0\, \chi^2\right)^2 (\bar{x}-\theta_0)+\theta_0/\chi_1^2 < 2n\bar{x}/\theta_0 < \chi_2^2\right]\cdot p\left[\chi_1^2 < 2n\bar{x}/\theta_0 < \chi_2^2\right]$$
$$+ E\left[\bar{x} \mid 2n\bar{x}/\theta_0 < \chi_1^2 \cup 2n\bar{x}/\theta_0 > \chi_2^2\right]\cdot p\left[2n\bar{x}/\theta_0 < \chi_1^2 \cup 2n\bar{x}/\theta_0 > \chi_2^2\right]$$

________(4.3.2.1)

$$= \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} \left[\left(\frac{2n\bar{x}}{\theta_0\,\chi^2}\right)^2 (\bar{x}-\theta_0)+\theta_0 \Big/ \theta\right]^p f(\bar{x})\, d\bar{x} - \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} p \ln\left[\left(\frac{2n\bar{x}}{\theta_0\,\chi^2}\right)^2 (\bar{x}-\theta_0)+\theta_0 \Big/ \theta\right] f(\bar{x})\, d\bar{x}$$

$$- \int_{\frac{\chi_1^2\theta_0}{2n}}^{\frac{\chi_2^2\theta_0}{2n}} f(\bar{x})\, d\bar{x} + \int_0^{\frac{\chi_1^2\theta_0}{2n}} \left[(\bar{x}/\theta)^p - p\ln(\bar{x}/\theta)-1\right] f(\bar{x})\, d\bar{x} + \int_{\frac{\chi_2^2\theta_0}{2n}}^{\infty} \left[(\bar{x}/\theta)^p - p\ln(\bar{x}/\theta)-1\right] f(\bar{x})\, d\bar{x}$$

________(4.3.2.2)

Where $f(\bar{x}) = (1/\Gamma n)\,(n/\theta)^n (\bar{x})^{n-1} \exp(-n\bar{x}/\theta)$

A straight forward integration of (4.3.2.3) gives:

$$R(\hat{\theta}_{ST_2}) = I_1 - I_2 - \left\{ I\left(\frac{x_2^2\,\phi}{2}, n\right) - I\left(\frac{x_1^2\,\phi}{2}, n\right)\right\} +$$
$$(1/n)^p \frac{\Gamma(p+n)}{\Gamma n} \left\{ I\left(\frac{x_1^2\,\phi}{2}, n+p\right) - I\left(\frac{x_2^2\,\phi}{2}, n+p\right) + 1\right\} -$$
$$\left\{ I\left(\frac{x_1^2\,\phi}{2}, n\right) - I\left(\frac{x_2^2\,\phi}{2}, n\right) + 1\right\} - I_3 - I_4$$

_______(4.3.2.3)

Where $I(x;p) = (1/\Gamma p) \int_0^x e^{-x}\, x^{p-1}\, dx$ refers to the standard incomplete gamma function and

$$I_1 = \int_{\frac{x_1^2\,\phi}{2}}^{\frac{x_2^2\,\phi}{2}} \left[\left(4t^3/n\phi^2 (\chi^2)^2\right) - \left(4t^2/\phi(\chi^2)^2\right) + \phi\right]^p (1/\Gamma n)\, e^{-t}\, t^{n-1}\, dt$$

$$I_2 = \int_{\frac{x_1^2\phi}{2}}^{\frac{x_2^2\phi}{2}} p\ln\left[\left(4t^3/n\phi^2\left(\chi^2\right)^2\right) - \left(4t^2/\phi\left(\chi^2\right)^2\right) + \phi\right]\left(1/\Gamma n\right) e^{-t}\, t^{n-1}\, dt$$

$$I_3 = \int_0^{\frac{x_1^2\phi}{2}} p\ln\left(t/n\right)\left(1/\Gamma n\right) e^{-t}\, t^{n-1}\, dt$$

$$I_4 = \int_{\frac{x_2^2\phi}{2}}^{\infty} p\ln\left(t/n\right)\left(1/\Gamma n\right) e^{-t}\, t^{n-1}\, dt$$

4.4 Relative Risks of $\hat{\theta}_{ST_i}$

A natural way of comparing the risk of the proposed testimators, is to study its performance with respect to the best available estimator $\bar{x}$ in this case. For this purpose, we obtain the risk of $\bar{x}$ under $L_E(\hat{\theta},\theta)$ as:

$$R_E(\bar{x}) = E[\bar{x} \mid L_E(\hat{\theta},\theta)]$$

$$= \int_0^{\infty} \left[(\bar{x}/\theta)^p - p\ln\left(\bar{x}/\theta\right) - 1\right] f(\bar{x})\, d\bar{x} \qquad \text{\_\_\_\_\_\_\_(4.4.1)}$$

A straightforward integration of (4.4.1) gives

$$R_E(\bar{x}) = \left[\frac{\Gamma(n+P)}{\Gamma n\,(n^p)} - p\{\psi(n) - \ln(n)\}\right] - 1 \qquad \text{\_\_\_\_\_\_(4.4.2)}$$

Where $\psi(n) = (d/dn)\ln\Gamma n$ refers to the Euler's psi function.

Now, we define the Relative Risk of $\hat{\theta}_{ST1}$ with respect to $\bar{x}$ under $L_E(\hat{\theta},\theta)$ as follows:

$$RR_1 = \frac{R_E(\bar{x})}{R(\hat{\theta}_{ST_1})} \qquad \text{\_\_\_\_\_\_\_(4.4.3)}$$

Using (4.4.2) and (4.3.1.3) the expression for RR_1 given in (4.4.3) can be obtained; Similarly, we define the Relative Risk of $\hat{\theta}_{ST_2}$ by

$$RR_2 = \frac{R_E(\bar{x})}{R(\hat{\theta}_{ST_2})} \qquad \text{\_\_\_\_\_\_(4.4.4)}$$

The expression for RR_2 is given by (4.4.4) which can be obtained by using equations (4.4.2) and (4.3.2.3).

Now, it is observed that both RR_1 and RR_2 are functions of '∅', 'n', 'α' and 'p'.

4.5 Recommendations for $\hat{\theta}_{ST_i}$

In this section we provide the comparison of UMVUE and the proposed shrinkage testimators in terms of their relative risks. Recommendations regarding the applications of proposed testimators are provided.

- In order to study the behaviour of $\hat{\theta}_{ST_1}$ and $\hat{\theta}_{ST_2}$ and the effect of shrinkage factor (S.F.) on the proposed testimators we have computed the values of Relative Risk (RR_1) for the following set of values. n = 5, 8, 10, 12; α = 1%, 5%, 10% ; p = -3, -2, -1 and p = 2, 3, 4. In all there will be several tables of RR for different variations in 'p', 'α' and 'n'. We have considered ϕ = 0.2 (0.2) 1.6. Some of the tables have been assembled in the appendix by (i) keeping 'α' to be fixed and varying 'p' (ii) keeping 'p' to be fixed and varying 'α' as we wish to recommend for these two values.

- For n = 5, α = 1% and for different values of 'p' (positive as well as negative) $\hat{\theta}_{ST_1}$ performs better than the conventional estimator for all the values of 'p' with its best performance for p = -3 and p = 2 for the whole range of ϕ. Considered here i.e. $0.2 \leq \phi \leq 1.6$.

- Next we have changed to α = 5%. Similar pattern of behaviour is observed for the relative risk and p = -3 and p = 2 provide the best results. However the magnitude of RR is small compared to α = 1% values.

- We have also considered α = 10%. In order to observe the behaviour for still higher level of significance just to confirm whether under different loss function the value of 'α' gets changed or not. We found that $\hat{\theta}_{ST_1}$ performs still better than the conventional estimator but the magnitude of RR values is still small though in all the cases it is above unity.

- So, a small value of α = 1% is recommended. Also by varying 'n' it is observed that RR values are higher for n = 5 compared to its other values of 8, 10 and 12. Hence a smaller 'n' is suggested. A higher RR_1 value indicates a 'better' control over the risk. So, by choosing appropriate value of 'p' and 'α' a better gain in terms of performance of $\hat{\theta}_{ST_1}$ can be achieved.

- $\hat{\theta}_{ST_2}$, is another testimator proposed by taking the 'SQUARE' of shrinkage factor. We have again prepared the relevant tables of Relative Risk (RR_2) of $\hat{\theta}_{ST_2}$ with respect to the conventional estimator for the same set of values as we have considered to study the behaviour of $\hat{\theta}_{ST_1}$. We observe the following:

- For $\hat{\theta}_{ST_2}$ where we have considered the square of S.F. Following behaviour of RR is observed. For almost the entire range of ϕ i.e. $0.2 \le \phi \le 1.4$ the values of RR (in terms of magnitude) are higher than those for S.F.(without square).

- Almost similar pattern of RR for different values of ‘p’ and ‘α’ has been observed for the values of n considered here. The S.F. can be made small either by taking smaller values of α or by fixing α and taking higher powers of ‘k’.

- So, the proposed testimator is having smaller risk than the conventional estimator provided n is small, α is small and square of S.F. is considered.

Tables showing relative risk(s) of proposed testimator(s) with respect to the best available estimator.

Table : 4.5.1.1 **Relative Risk of** $\hat{\theta}_{ST_1}$ α = 1% , n = 5

ϕ	**p = -3**	**p = -2**	**p = -1**	**p = 2**	**p = 3**	**p = 4**
0.20	0.959	0.777	0.568	1.002	0.953	0.968
0.40	1.327	0.918	0.595	1.935	1	1
0.60	1.893	1.327	0.745	2.369	1.144	1.071
0.80	2.183	1.821	0.949	3.448	1.966	1.476
1.00	3.003	1.934	1.048	4.583	3.359	2.257
1.20	1.669	1.641	1.626	3.008	2.301	1.453
1.40	1.383	1.291	1.362	1.772	1.654	1.464
1.60	1.175	1.026	1.113	0.744	0.723	0.741

Table : 4.5.1.2 **Relative Risk of** $\hat{\theta}_{ST_1}$ α = 1% , n = 8

ϕ	**p = -3**	**p = -2**	**p = -1**	**p = 2**	**p = 3**	**p = 4**
0.20	0.995	0.995	0.984	0.957	0.976	0.968
0.40	1.046	0.998	1	1.004	0.977	0.986
0.60	1.429	1.215	1.087	1.006	1.002	1.001
0.80	2.149	1.742	1.371	2.11	1.394	1.197
1.00	2.435	2.124	1.603	4.259	3.894	2.992
1.20	1.943	1.839	1.505	3.227	2.768	1.824
1.40	1.411	1.351	1.211	1.408	1.166	1.096
1.60	1.071	1.002	0.942	0.48	0.513	0.528

Table : 4.5.1.3 **Relative Risk of** $\hat{\theta}_{ST_1}$ α = 5% , n = 5

ϕ	**p = -3**	**p = -2**	**p = -1**	**p = 2**	**p = 3**	**p = 4**
0.20	1.12	1.026	0.991	1.049	1.022	1.012
0.40	1.371	1.179	1.051	1.09	1.04	1.021
0.60	1.449	1.437	1.197	1.314	1.122	1.058
0.80	1.575	1.589	1.36	2.17	1.375	1.17
1.00	1.587	1.63	1.404	3.488	2.793	1.404
1.20	1.28	1.391	1.299	2.844	1.771	1.35
1.40	1.139	1.182	1.132	1.524	1.233	1.136
1.60	1.035	1.02	0.979	0.722	0.739	0.76

Table : 4.5.1.4 **Relative Risk of** $\hat{\theta}_{ST_1}$ α = 5% , n = 8

ϕ	**p = -3**	**p = -2**	**p = -1**	**p = 2**	**p = 3**	**p = 4**
0.20	1.033	1.017	1.007	1.008	1.004	1.002
0.40	1.151	1.088	1.047	1.096	1.053	1.035
0.60	1.341	1.203	1.097	1.11	1.047	1.023
0.80	1.545	1.403	1.216	1.303	1.089	1.007
1.00	1.555	1.487	1.304	2.055	1.396	1.18
1.20	1.324	1.326	1.232	1.998	1.319	1.111
1.40	1.099	1.094	1.063	1.062	0.992	0.951
1.60	0.947	0.918	0.907	0.556	0.606	0.627

Table : 4.5.2.1 **Relative Risk of** $\hat{\theta}_{ST_2}$ α = 1% , n = 5

ϕ	**p = -3**	**p = -2**	**p = -1**	**p = 2**	**p = 3**	**p = 4**
0.20	0.919	0.883	0.916	0.82	0.883	0.923
0.40	1.302	1.027	0.932	0.968	0.983	0.99
0.60	2.034	1.608	1.231	1.261	1.091	1.035
0.80	2.215	2.396	1.671	3.366	2.363	1.621
1.00	2.463	2.508	1.843	6.484	5.819	3.158
1.20	1.786	1.993	1.617	4.733	4.083	3.031
1.40	1.449	1.5	1.288	1.617	1.273	1.237
1.60	1.219	1.169	1.023	0.495	0.513	0.54

Table : 4.5.2.2 **Relative Risk of** $\hat{\theta}_{ST_2}$ α = 1% , n = 8

φ	p = -3	p = -2	p = -1	p = 2	p = 3	p = 4
0.20	0.979	0.943	0.952	0.874	0.881	0.886
0.40	0.978	0.994	1.001	0.906	0.948	0.969
0.60	1.405	1.183	1.057	1.008	1.004	1.003
0.80	2.361	1.878	1.431	2.305	1.385	1.144
1.00	2.628	2.306	1.697	5.258	3.935	2.094
1.20	1.883	1.774	1.467	2.069	2.689	1.952
1.40	1.29	1.205	1.09	0.768	0.759	0.764
1.60	0.963	0.871	0.816	0.291	0.322	0.331

4.6 Shrinkage Testimator for the Variance of a Normal Distribution

Shrinkage testimators for the mean μ of a Normal distribution N(μ, σ^2) when variance σ^2 is known or unknown, have been proposed by Waiker, Schuurman and Raghunandan (1984). Recently Pandey et. al. (2007) have studied the risk properties for the positive degree of asymmetry. Where as this study finds the range for positive as well as negative degrees of asymmetry where the shrinkage testimator perform better than the UMVUE.

Let X be Normally distributed with mean μ and variance σ^2. We have proposed a single sample shrinkage testimator. It is assumed that the prior knowledge about σ^2 is available in the form of an initial estimate σ_0^2. Using the sample observations $x_1, x_2 ...., x_n$ and possibly the given information we wish to construct a shrinkage testimator. The procedure is as follows:

1. First test with a sample of size n, the null hypothesis $H_0 : \sigma^2 = \sigma_0^2$ against the alternative $H_1 : \sigma^2 \neq \sigma_0^2$ using the test statistics $\frac{\nu s^2}{\sigma_0^2}$, where ν = (n -1)

and $s^2 = \frac{1}{n-1}\sum(x_i - \bar{x})^2$. The test statistics is distributed as χ^2 with ν degrees of freedom.

2. If H_0 is accepted at α level of significance i.e. $x_1^2 < \frac{\nu s^2}{\sigma_0^2} < x_2^2$ where x_1^2 and x_2^2 are the lower and upper points of the uniformly most powerful unbiased (UMPU) test of H_0, use the conventional shrinkage estimator with shrinkage factor $k = \frac{\nu s^2}{\sigma_0^2 x^2}$, which is inversely proportional to χ^2 and it depends on the test statistic, so the arbitrariness in the choice of shrinkage factor has been removed by making it dependent on the test statistic.
3. If H_0 is rejected, use s^2, the Uniformly Minimum Variance Unbiased Estimator (UMVUE) as the estimator of σ^2.

Now, the proposed shrinkage testimator $\hat{\sigma}_{ST1}^2$ of σ^2 is

$$\sigma_{ST1}^2 = \begin{cases} k\, s^2 + (1-k)\sigma_0^2 & , \; if\ H_0\ is\ accepted \\ s^2 & , \; otherwise \end{cases}$$

The next proposed shrinkage testimator $\hat{\sigma}_{ST2}^2$ of σ^2 is

$$\hat{\sigma}_{ST2}^2 = \begin{cases} k_1\, s^2 + (1-k_1)\sigma_0^2 & , \; if\ H_0\ is\ accepted \\ s^2 & , \; otherwise \end{cases}$$

Where $k_1 = \frac{\nu s^2}{{\sigma_0}^2\, x^2}$

Estimators of this type with and arbitrary k (0 ≤ k ≤ 1) have been proposed by Pandey and Srivastava (1987) and others. In all such studies it has been found that the shrinkage estimators work well if k is near zero and 'n' is small and 'α' is also small. The present work deal with the shrinkage factor dependent on the test statistic and arbitrary 'k'.

We have studied the risk properties for several choices of level of significance, sample sizes, a wide range of λ and several values of degrees of asymmetry.

4.7 Risk of Testimators

In this section we derive the risk of these two testimators which are defined in the previous section.

4.7.1 Risk of $\hat{\sigma}^2_{ST1}$

The risk of $\hat{\sigma}^2{}_{ST1}$ under $L_E(\hat{\theta},\theta)$ is defined by

$$R(\hat{\sigma}^2{}_{ST_1}) = E[\hat{\sigma}^2{}_{ST_1} \mid L_E(\hat{\theta},\theta)]$$

$$= E\left[ks^2 + (1-k)\sigma_0^2 \Big/ \chi_1^2 < \frac{\upsilon s^2}{\sigma_0^2} < \chi_2^2\right] \cdot p\left[\chi_1^2 < \frac{\upsilon s^2}{\sigma_0^2} < \chi_2^2\right]$$
$$+ E\left[s^2 \;\Big|\; \frac{\upsilon s^2}{\sigma_0^2} < \chi_1^2 \cup \frac{\upsilon s^2}{\sigma_0^2} > \chi_2^2\right] \cdot p\left[\frac{\upsilon s^2}{\sigma_0^2} < \chi_1^2 \cup \frac{\upsilon s^2}{\sigma_0^2} > \chi_2^2\right]$$

______(4.7.1.1)

$$= \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} \left[\frac{k\left(s^2-\sigma_0^2\right)+\sigma_0^2}{\sigma^2}\right]^p f(s^2)\, ds^2 - \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} p\ \ln\left[\frac{k\left(s^2-\sigma_0^2\right)+\sigma_0^2}{\sigma^2}\right] f(s^2)\, ds^2$$

$$- \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} f(s^2)\, ds^2 + \int_0^{\frac{\chi_1^2\sigma_0^2}{\upsilon}} \left[\left(\frac{s^2}{\sigma^2}\right)^p - p\ \ln\left(\frac{s^2}{\sigma^2}\right) - 1\right] f(s^2)\, ds^2 + \int_{\frac{\chi_2^2\sigma_0^2}{\upsilon}}^{\infty} \left[\left(\frac{s^2}{\sigma^2}\right)^p - p\ \ln\left(\frac{s^2}{\sigma^2}\right) - 1\right] f(s^2)\, ds^2$$

______(4.7.1.2)

Where $$f(s^2) = \frac{1}{2^{\upsilon/2}\,\Gamma\left(\upsilon/2\right)} \left(s^2\right)^{\frac{\upsilon}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon s^2}{\sigma^2}\right)} ds^2$$

Straight forward integration of (4.7.1.2) gives

$$R\left(\hat{\sigma}^2_{ST1}\right) = \left(\frac{\sigma^2}{\nu}\right)^{\nu/2} \left[\begin{array}{c} I_1 - I_2 - \left\{I\left(\chi_2^2\lambda, \frac{\nu}{2}\right) - I\left(\chi_1^2\lambda, \frac{\nu}{2}\right)\right\} \\ -\left[I\left(\chi_1^2\lambda, \frac{\nu}{2}\right) - I\left(\chi_2^2\lambda, \frac{\nu}{2}\right) + 1\right] \\ + \frac{\Gamma\left(\frac{\nu}{2}+p\right)}{\Gamma\left(\frac{\nu}{2}\right)\left(\frac{\nu}{2}\right)^p} \left[I\left(\chi_1^2\lambda, \frac{\nu}{2}+p\right) - I\left(\chi_2^2\lambda, \frac{\nu}{2}+p\right) + 1\right] \\ - I_3 - I_4 \end{array}\right]$$

_____(4.7.1.3)

Where $I(x;p) = \left(1/\Gamma p\right) \int_0^x e^{-x} x^{p-1}\, dx$ refers to the standard incomplete gamma function, $\lambda = \frac{\sigma_0^2}{\sigma^2}$, and

$$I_1 = \frac{1}{2^{\nu/2}\ \Gamma\left(\frac{\nu}{2}\right)} \int_{x_1^2\lambda}^{x_2^2\lambda} \left(k\left(\frac{t}{\nu} - \lambda\right) + \lambda\right)^p e^{-\left(\frac{1}{2}\right)t}\, t^{\frac{\nu}{2}-1}\, dt$$

$$I_2 = \frac{p}{2^{\nu/2}\ \Gamma\left(\frac{\nu}{2}\right)} \int_{x_1^2\lambda}^{x_2^2\lambda} ln\left(k\left(\frac{t}{\nu} - \lambda\right) + \lambda\right) e^{-\left(\frac{1}{2}\right)t}\, t^{\frac{\nu}{2}-1}\, dt$$

$$I_3 = \frac{p}{2^{\nu/2}\ \Gamma\left(\frac{\nu}{2}\right)} \int_0^{x_1^2\lambda} ln\left(\frac{t}{\nu}\right) e^{-\left(\frac{1}{2}\right)t}\, t^{\frac{\nu}{2}-1}\, dt$$

$$I_4 = \frac{p}{2^{\nu/2}\ \Gamma\left(\frac{\nu}{2}\right)} \int_{x_2^2\lambda}^{\infty} ln\left(\frac{t}{\nu}\right) e^{-\left(\frac{1}{2}\right)t}\, t^{\frac{\nu}{2}-1}\, dt$$

4.7.2 Risk of $\hat{\sigma}^2_{ST2}$

Again, we obtain the risk of $\hat{\sigma}^2{}_{ST_2}$ under $L_E\left(\hat{\theta},\theta\right)$ with respect to s^2, given by

$$R(\hat{\sigma}^2{}_{ST_2}) = E[\,\hat{\sigma}^2{}_{ST_2} \big| L_E\left(\hat{\theta},\theta\right)\,]$$

$$= E\left[k_1 s^2 + (1-k_1)\sigma_0^2 \Big/ \chi_1^2 < \frac{\upsilon s^2}{\sigma_0^2} < \chi_2^2\right] \cdot p\left[\chi_1^2 < \frac{\upsilon s^2}{\sigma_0^2} < \chi_2^2\right]$$
$$+ E\left[s^2 \;\Big|\; \frac{\upsilon s^2}{\sigma_0^2} < \chi_1^2 \cup \frac{\upsilon s^2}{\sigma_0^2} > \chi_2^2\right] \cdot p\left[\frac{\upsilon s^2}{\sigma_0^2} < \chi_1^2 \cup \frac{\upsilon s^2}{\sigma_0^2} > \chi_2^2\right] \qquad \text{\_\_\_(4.7.2.1)}$$

$$= \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} \left[\frac{\frac{\upsilon s^2}{\sigma_0^2\chi^2}\left(s^2-\sigma_0^2\right)+\sigma_0^2}{\sigma^2}\right]^p f(s^2)\, ds^2 - \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} p\ \ln\left[\frac{\frac{\upsilon s^2}{\sigma_0^2\chi^2}\left(s^2-\sigma_0^2\right)+\sigma_0^2}{\sigma^2}\right] f(s^2)\, ds^2$$

$$- \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} f(s^2)\, ds^2 + \int_0^{\frac{\chi_1^2\sigma_0^2}{\upsilon}} \left[\left(\frac{s^2}{\sigma^2}\right)^p - p\ \ln\left(\frac{s^2}{\sigma^2}\right)-1\right] f(s^2)\, ds^2 + \int_{\frac{\chi_2^2\sigma_0^2}{\upsilon}}^{\infty} \left[\left(\frac{s^2}{\sigma^2}\right)^p - p\ \ln\left(\frac{s^2}{\sigma^2}\right)-1\right] f(s^2)\, ds^2$$

_______(4.7.2.2)

Where $f(s^2) = \frac{1}{2^{\upsilon/2}\,\Gamma\left(\upsilon/2\right)} \left(s^2\right)^{\frac{\upsilon}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon s^2}{\sigma^2}\right)} ds^2$

Straight forward integration of (4.7.2.2) gives

$$R\left(\hat{\sigma}_{ST1}^2\right) = \left(\frac{\sigma^2}{\nu}\right)^{\nu/2} \left[\begin{array}{c} \mathrm{I}_1 - \mathrm{I}_2 - \left\{\mathrm{I}\left(\chi_2^2\lambda, \frac{\nu}{2}\right) - \mathrm{I}\left(\chi_1^2\lambda, \frac{\nu}{2}\right)\right\} \\ -\left[\mathrm{I}\left(\chi_1^2\lambda, \frac{\nu}{2}\right) - \mathrm{I}\left(\chi_2^2\lambda, \frac{\nu}{2}\right) + 1\right] \\ + \frac{\Gamma\left(\frac{\nu}{2}+\mathrm{p}\right)}{\Gamma\left(\frac{\nu}{2}\right)\left(\frac{\nu}{2}\right)^{\mathrm{p}}} \left[\mathrm{I}\left(\chi_1^2\lambda, \frac{\nu}{2}+\mathrm{p}\right) - \mathrm{I}\left(\chi_2^2\lambda, \frac{\nu}{2}+\mathrm{p}\right) + 1\right] \\ -\mathrm{I}_3 - \mathrm{I}_4 \end{array}\right]$$

_____(4.7.2.3)

Where $I(x;p) = \left(1/\Gamma p\right) \int_0^x e^{-x} x^{p-1}\, dx$ refers to the standard incomplete gamma function, $\lambda = \frac{\sigma_0^2}{\sigma^2}$, and

$$I_1 = \frac{1}{2^{\nu/2}\ \Gamma\left(\frac{\nu}{2}\right)} \int_{x_1^2\lambda}^{x_2^2\lambda} \left(\frac{t^2}{\nu\lambda\chi^2} - \frac{t}{\chi^2} + \lambda\right)^p e^{-\left(\frac{1}{2}\right)t}\, t^{\frac{\nu}{2}-1}\, dt$$

$$I_2 = \frac{p}{2^{\nu/2}\ \Gamma\left(\frac{\nu}{2}\right)} \int_{x_1^2\lambda}^{x_2^2\lambda} ln\left(\frac{t^2}{\nu\lambda\chi^2} - \frac{t}{\chi^2} + \lambda\right) e^{-\left(\frac{1}{2}\right)t}\, t^{\frac{\nu}{2}-1}\, dt$$

$$I_3 = \frac{p}{2^{\nu/2}\ \Gamma\left(\frac{\nu}{2}\right)} \int_0^{x_1^2\lambda} ln\left(\frac{t}{\nu}\right) e^{-\left(\frac{1}{2}\right)t}\, t^{\frac{\nu}{2}-1}\, dt$$

$$I_4 = \frac{p}{2^{\nu/2}\ \Gamma\left(\frac{\nu}{2}\right)} \int_{x_2^2\lambda}^{\infty} ln\left(\frac{t}{\nu}\right)\ e^{-\left(\frac{1}{2}\right)t}\ t^{\frac{\nu}{2}-1}\ dt$$

4.8 <u>Relative Risk of</u> $\hat{\sigma}^2_{STi}$

A natural way of comparing the risk of the proposed testimators, is to study its performance with respect to the best available estimator s^2 in this case. For this purpose, we obtain the risk of s^2 under $L_E\left(\hat{\sigma}^2, \sigma^2\right)$ as:

$$R_E(s^2) = E[\, s^2 \,|\, L(\hat{\sigma}^2, \sigma^2)\,]$$

$$= \int_0^{\infty} \left[\left(s^2/\sigma^2\right)^p - p\ln\left(s^2/\sigma^2\right) - 1\right] f(s^2)\, ds^2 \qquad \text{\_\_\_\_\_\_(4.8.1)}$$

Where $\quad f(s^2) = \frac{1}{2^{\upsilon/2}\,\Gamma\left(\upsilon/2\right)} \left(s^2\right)^{\frac{\upsilon}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon s^2}{\sigma^2}\right)}$

A straightforward integration of (4.8.1) gives

$$R_E(s^2) = \left[\frac{\Gamma\left(\frac{\nu}{2}+P\right)}{\Gamma\left(\frac{\nu}{2}\right)\left(\frac{\nu}{2}\right)^p} - p\left\{\psi\left(\frac{\nu}{2}\right) - \ln\left(\frac{\nu}{2}\right)\right\}\right] - 1 \qquad \text{\_\_\_\_\_\_(4.8.2)}$$

Where $\quad \psi(n) = \left(d/dn\right)\ln\Gamma n \quad$ refers to the Euler's psi function.

Now, we define the Relative Risk of $\hat{\sigma}^2_{ST_i}$, $i = 1,2$ with respect to s^2 under $L\left(\hat{\sigma}^2, \sigma^2\right)$ as follows:

$$RR_1 = \frac{R_E(\mathrm{s}^2)}{R(\hat{\sigma}^2_{ST1})} \qquad \text{\_\_\_\_\_\_\_(4.8.3)}$$

Using (4.8.2) and (4.7.1.3) the expression for RR_1 given in (4.8.3) can be obtained; it is observed that RR_1 is a function of 'λ', ' ν ', 'α', 'k' and 'p'.

Finally, we define the Relative Risk of $\hat{\sigma}^2_{ST_2}$ by

$$RR_2 = \frac{R_E(s^2)}{R(\hat{\sigma}^2{}_{ST2})} \qquad \text{\_\_\_\_\_\_\_(4.8.4)}$$

The expression for RR_2 is given by (4.8.4) which can be obtained by using (4.8.2) and (4.7.2.3). Again we observed that RR_2 is a function of 'λ', 'ν', 'α' and 'p'.

4.9 Recommendations for $\hat{\sigma}^2_{STi}$

In this section we wish to compare the performance of $\hat{\sigma}^2{}_{ST_1}$ and $\hat{\sigma}^2{}_{ST_2}$ with respect to the best available (unbiased) estimator of σ^2.

4.9.1 Recommendations for $\hat{\sigma}^2_{ST1}$

It is observed that RR_1 is a function of ν, α, λ, k and the degrees of asymmetry "p". In order to study the behaviour of $\hat{\sigma}^2{}_{ST_1}$ with respect to the best available estimator we have considered several values of above mentioned quantities viz. k = 0.2 (0.2) 1.0, λ = 0.2 (0.2) 2.0, ν = 5,8,10,12, and p = -2, -1.75, -1.5, -1.25, -1.0 , 1.0 and smaller values of α = 1% and 0.1%. As we have observed that RR_1 values start getting negative even for p = +1, so other higher values of 'p' are not considered with a view that for positive values of 'p' the usual estimator may perform better than the proposed one. Also, several studies have pointed out that smaller level of significance should be taken, this motivated us to consider smaller values of α^s considered as above. There will be several tables of RR_1 . Some of these have been assembled at the end of the chapter. However our recommendations based on all these tables are as follows.

1. $\hat{\sigma}^2{}_{ST_1}$ performs better than $\hat{\sigma}^2$ at α = 1% for the whole range of 'λ' for p = -2 i.e. the values of RR_1 are greater than unity for $0.4 \leq \lambda \leq 1.8$. In this situation the range of 'k' is $0.2 \leq k \leq 0.8$. It is observed that as p assume

other negative values upto p = -1, still the performance is better but the range of 'λ' changes and for p = -1 it is $0.8 \leq \lambda \leq 1.2$. i.e. it reduces. These values are obtained for ν = 5. However, for other values of ν i.e. 8, 10 and 12 again a similar pattern is observed but now the recommended values of p are upto -1.50.

2. The positive values of 'p' (p = +1 reported here) are indicative of better performance of σ^2, so it is suggested that the use of GEL would be beneficial for under estimation situations.
3. We have considered α = 0.1% also to observe the behaviour of $\hat{\sigma}^2_{ST_1}$, here the range of 'λ' is increased as now it is $0.4 \leq \lambda \leq 2.0$ which holds even for 'p' upto -1.25 again when p = -1 the range changes slightly and becomes $0.4 \leq \lambda \leq 1.8$. As ν is increased to '8' the range of 'λ' decreases for different negative values of 'p' and it is now $0.6 \leq \lambda \leq 1.8$ for p = -2 and $0.8 \leq \lambda \leq 1.2$ for p = -1.

4. Still increasing ν to 10 and 12 we have observed that the range of λ reduces to $0.6 \leq \lambda \leq 1.6$ and now the values are better upto p = -1.50.

5. For both the values of α^s considered here the RR_1 values are more than '1' but the magnitude of these values are higher for α = 0.1% and the range of shrinkage factor for all the above recommendations is $0.2 \leq k \leq 0.8$.

6. So, it is recommended to consider higher degrees of underestimation with a small sample size and smaller level of significance. i.e. take ν = 5, p = -2, α = 0.1% than $\hat{\sigma}^2_{ST_1}$ performs better than $\hat{\sigma}^2$ for $0.4 \leq \lambda \leq 2.0$ and $0.2 \leq k \leq 0.8$.

4.9.2 Recommendations for $\hat{\sigma}^2_{ST2}$

As the arbitrariness in the choice of ‘k’ is removed by making it dependent on test statistic, now the relative risk of $\hat{\sigma}^2{}_{ST_2}$ with respect to $\hat{\sigma}^2$ is a function of p, λ, c, and α. In order to study the behaviour of RR_2 we have considered p = -2, -1.75, -1.50, -1.25, -1.0 and 1.0, λ = 0.2 (0.2) 2.0, ν = 5,8,10 and 12, α = 1% and 0.1% . Again the reason for considering only one positive value for degree is that RR_2 values turn negative even at p = +1. Again there will be several tables of RR_2 some of these have been assembled at the end of the chapter however our recommendations for $\hat{\sigma}^2{}_{ST_2}$ are as follows:

1. For $0.2 \leq \lambda \leq 1.6$, p =-2, ν = 5 and α = 1% $\hat{\sigma}^2{}_{ST_2}$ dominates $\hat{\sigma}^2$. However the range of ‘λ’ decreases as ‘p’ becomes -1.75, now it is $0.2 \leq \lambda \leq 1.4$ and it remains true upto -1.25. But for p = -1 the range of ‘λ’ is shorter as it is now $0.8 \leq \lambda \leq 1.2$. These values of RR_2 were observed for ν = 5. For the other values of ‘ν’ almost similar pattern of RR_2 values is observed but the values become smaller as ν increase.

2. Here also for positive values of ‘p’ $\hat{\sigma}^2$ the usual estimator performs better than $\hat{\sigma}^2{}_{ST_2}$ as the RR_2 values are negative in this case.

3. For another lower level of significance i.e. α = 0.1% the values of RR_2 are higher in magnitude as compared to those at α = 1%. Also the range of ‘λ’ increases and it becomes $0.2 \leq \lambda \leq 2.0$ upto p = -1.50, it slightly decreases and becomes $0.6 \leq \lambda \leq 1.6$ for p = -1. Again for p = +1 the RR_2 values are negative for the whole range of λ.

4. Changing ν = 8, 10, 12 we observe that the range of 'λ' reduces further and it becomes $0.6 \leq \lambda \leq 1.6$. However for ν = 12 none of the RR_2 values is greater than '1'.

5. For both the values of α^s considered here the RR_2 values are more than unity but the magnitude of RR_2 values is higher for lower level of significance.

6. So, it is recommended to consider the higher values of degree of asymmetry when under estimation is more serious than over estimation and a lower values of 'ν'.

CONCLUSION:

Two shrinkage testimators for the variance of Normal distribution have been proposed viz. $\hat{\sigma}^2{}_{ST_1}$ and $\hat{\sigma}^2{}_{ST_2}$.

The values of RR_1 (i.e. $\hat{\sigma}^2{}_{ST_1}$ with respect to $\hat{\sigma}^2$) and RR_2 (i.e. $\hat{\sigma}^2{}_{ST_2}$ with respect to $\hat{\sigma}^2$) are not much different in their magnitudes. However $\hat{\sigma}^2{}_{ST_2}$ is a shrinkage testimator based on test statistic, so it could be used. It is observed that the use of GELF does not provide good result for positive values of degrees of asymmetry (i.e. overestimation being more serious). So, it is recommended for the reverse situations.

A lower value ν = 5 with p = -2, α = 0.1% provide better result for almost the whole range of 'λ'. However both the estimators perform better than the usual estimator for other values also but the reported values are indicative of the best performance.

Tables showing relative risk(s) of proposed testimator(s) with respect to the best available estimator.

Table : 4.9.1.1 **Relative Risk of** $\hat{\sigma}^2_{ST_1}$ $\alpha = 0.1\%$, $\nu_1 = 5$, k = 0.2

λ	p = -2	p = -1.75	p = -1.50	p = -1.25	p = -1	p = 1
0.20	0.833	0.671	0.558	0.476	0.411	-2.82
0.40	1.432	1.298	1.079	0.869	0.687	-2.13
0.60	1.801	1.962	1.827	1.522	1.18	-1.37
0.80	1.831	2.171	2.214	1.95	1.527	-1.101
1.00	1.727	2.058	2.16	1.966	1.575	-1.028
1.20	1.599	1.856	1.932	1.775	1.448	-1.081
1.40	1.48	1.661	1.691	1.547	1.277	-1.268
1.60	1.379	1.494	1.485	1.346	1.117	-1.719
1.80	1.293	1.358	1.319	1.182	0.983	-3.129
2.00	1.221	1.247	1.186	1.051	0.873	-3.934

Table : 4.9.1.2 **Relative Risk of** $\hat{\sigma}^2_{ST_1}$ $\alpha = 0.1\%$, $\nu_1 = 5$, a = -1.75

λ	k = 0.2	k = 0.4	k = 0.6	k = 0.8	k = 1.0
0.20	0.671	0.867	0.994	1.027	0.839
0.40	1.298	1.428	1.411	1.255	0.821
0.60	1.962	1.911	1.733	1.429	0.818
0.80	2.171	2.05	1.84	1.502	0.807
1.00	2.058	1.976	1.818	1.517	0.796
1.20	1.856	1.829	1.738	1.5	0.786
1.40	1.661	1.674	1.639	1.466	0.777
1.60	1.494	1.533	1.538	1.423	0.769
1.80	1.358	1.411	1.444	1.376	0.762
2.00	1.247	1.308	1.358	1.328	0.756

Table : 4.9.1.3 **Relative Risk of** $\hat{\sigma}^2_{ST_1}$ $\alpha = 0.1\%$, $\nu_1 = 8$, a = -1.75

λ	k = 0.2	k = 0.4	k = 0.6	k = 0.8	k = 1.0
0.20	0.346	0.486	0.623	0.745	0.825
0.40	0.562	0.703	0.782	0.79	0.709
0.60	1.237	1.288	1.197	1.013	0.738
0.80	1.914	1.741	1.466	1.131	0.723
1.00	1.913	1.756	1.506	1.16	0.695
1.20	1.565	1.532	1.406	1.136	0.666
1.40	1.24	1.28	1.255	1.081	0.639
1.60	1.003	1.072	1.106	1.014	0.615
1.80	0.837	0.914	0.977	0.945	0.594
2.00	0.719	0.795	0.871	0.881	0.577

Table : 4.9.1.4 **Relative Risk of** $\hat{\sigma}^2_{ST_1}$ α = 1% , ν_1 = 5, a = -2

λ	k = 0.2	k = 0.4	k = 0.6	k = 0.8	k = 1.0
0.20	0.859	0.991	1.058	1.047	0.863
0.40	1.127	1.172	1.157	1.073	0.816
0.60	1.277	1.263	1.21	1.099	0.81
0.80	1.273	1.247	1.196	1.093	0.805
1.00	1.21	1.193	1.156	1.072	0.801
1.20	1.138	1.132	1.111	1.048	0.798
1.40	1.074	1.078	1.07	1.025	0.797
1.60	1.02	1.031	1.033	1.003	0.797
1.80	0.976	0.991	1	0.984	0.798
2.00	0.939	0.957	0.973	0.967	0.801

Table : 4.9.2.1 **Relative Risk of** $\hat{\sigma}^2_{ST_2}$ α = 0.1% , ν_1 = 5

λ	p = -0.2	p = -1.75	p = -1.5	p = -1.25	p = -1.0
0.20	1.242	1.057	0.876	0.728	0.601
0.40	1.632	1.558	1.317	1.053	0.815
0.60	1.839	2.029	1.899	1.579	1.215
0.80	1.824	2.151	2.179	1.907	1.489
1.00	1.714	2.026	2.106	1.904	1.521
1.20	1.584	1.824	1.879	1.714	1.396
1.40	1.464	1.627	1.639	1.488	1.226
1.60	1.361	1.459	1.433	1.288	1.065
1.80	1.273	1.321	1.267	1.125	0.931
2.00	1.2	1.209	1.134	0.995	0.822

Table : 4.9.2.2 **Relative Risk of** $\hat{\sigma}^2_{ST_2}$ α = 1% , ν_1 = 5

λ	p = -0.2	p = -1.75	p = -1.5	p = -1.25	p = -1.0
0.20	1.169	1.085	0.967	0.846	0.727
0.40	1.256	1.217	1.084	0.908	0.723
0.60	1.3	1.347	1.282	1.125	0.914
0.80	1.266	1.339	1.323	1.207	1.013
1.00	1.199	1.254	1.24	1.146	0.978
1.20	1.129	1.152	1.121	1.03	0.884
1.40	1.066	1.058	1.008	0.915	0.782
1.60	1.012	0.979	0.913	0.816	0.694
1.80	0.967	0.913	0.835	0.736	0.621
2.00	0.929	0.86	0.773	0.673	0.564

Chapter – 5

DOUBLE STAGE SHRINKAGE TESTIMATORS UNDER GENERAL ENTROPY LOSS FUNCTION

5.1 Introduction

In situations when there is no a priori knowledge is available for the parameter θ (scale parameter) the sample mean $\bar{x}$ is the BLUE (Best Linear Unbiased Estimator) of θ based on complete set of observations.

However in many real life situations such as mean life time of a component / system, average number of days required to get cured from a disease, etc. A guess value of θ in terms of a point (single) or interval is available to the experimenter either due to past studies or similar studies or his familiarity with behavior of the characteristic under study. Then this guess may be utilized to improve the estimation procedure. In order to use this information for constructing an estimator for θ, the use of preliminary test of significance has been suggested by Bancroft (1944). An extensive bibliography in this area is provided by Han and Bancroft (1977) and Han, Rao and Ravichandran (1988).

Several authors have proposed estimators / testimators for the mean life (scale parameter) with different shrinkage factors and under different loss functions mostly under Squared Error Loss Function (SELF). Recently Srivastava and Tanna (2007) have proposed a double stage shrinkage testimator under General Entropy Loss Function (GELF) and they have shown the superiority of the proposed testimators, over the usual estimator.

The shrinkage testimators are proposed when the shrinkage factor can take any arbitrary value between ‘0’ and ‘1’. In the present paper this arbitrariness in the choice of shrinkage factor is removed by making it dependent on the test statistics and hence for a given level of significance and degrees of freedom, the shrinkage factor is no longer arbitrary. The choice of an appropriate loss function is often guided by economic considerations and the situation(s) under which the parameter is being estimated.

In this chapter the problem of estimation of the mean life θ of exponential population is considered when a guess θ_0 of θ is available to the experimenter. The double stage estimation for θ is to use the mean of the first sample and the guess value if $H_0 : \theta = \theta_0$ is accepted; otherwise use pooled mean $\bar{x}_p$ of the two samples if H_0 is rejected.

In section 5.2 we have proposed the two different shrinkage testimators for scale parameter of an Exponential Distribution and we have studied the risk properties of these two shrinkage testimators under General Entropy Loss Function defined in section 4.1.1. Section 5.3 deals with the derivation of the risk(s) of these two estimators. Section 5.4 deals with the relative risk(s) of these two estimators. Section 5.5 concludes with the comparison of unbiased pooled estimator and the proposed shrinkage testimators in terms of their relative risks. Further in the same section a suggestion for the shrinkage factor is made.

In section 5.6 we have proposed the two different shrinkage testimators for the variance of a Normal Distribution and we have studied the risk properties of these two shrinkage testimators under General Entropy Loss Function. Section 5.7 deals with the derivation of the risk(s) of these two estimators. Section 5.8 deals with the relative risk(s) of these two estimators. Section 5.9 concludes with the comparison of unbiased pooled estimator and the proposed shrinkage testimators in terms of their relative risks. Further in the same section a suggestion for the shrinkage factor is made.

5.2 Shrinkage Testimator(s) for Scale Parameter of an Exponential Distribution.

Let x_{11}, x_{12}, ---------, x_{1n1} be the first stage sample of size n_1 from the exponential population

$$f(x;\theta)=\begin{cases}(1/\theta)\,e^{-x/\theta} & ;\ x,\ \theta>0\\ 0 & ;\ otherwise\end{cases} \qquad \text{\_\_\_(5.2.1)}$$

Let θ_0 be the guess value of θ. Compute the sample mean $\bar{x}_1 = \frac{1}{n}\sum_{i=1}^{n} x_{1i}$ and test the preliminary hypothesis H_p : $\theta = \theta_0$, using the test statistic $\left(2n_1\bar{x}_1/\theta_0\right)$ which has $\chi^2_{2n_1}$ distribution. It is to be noted that H_P is accepted if $x_1^2 \leq \frac{2n_1\bar{x}_1}{\theta_0} \leq x_2^2$ and H_P is rejected, otherwise where x_1^2 and x_2^2 being given by $P\left[x^2_{2n_1} \geq\right.$ $x2\ 2+Px2n1\ 2\leq x1\ 2=\alpha$ where α is the pre-assigned level of significance.

Now, if H_p is accepted, take the estimator $k\left(\bar{x}_1-\theta_0\right)+\theta_0$ $(0\leq k\leq 1)$ and if it is rejected then take $n_2 = n - n_1$ additional observations x_{21}, x_{22},_ _ _ _, X_{2n2} and use the pooled estimator $\bar{x}_p = \frac{n_1\bar{x}_1+n_2\bar{x}_2}{(n_1+n_2)}$ as the estimator of θ. The properties of such estimators have been studied by Srivastava and Tanna (2007) under General Entropy Loss Function.

Now, we define the shrinkage testimator $\hat{\theta}_{DST_1}$ and $\hat{\theta}_{DST_2}$ of θ as follows:

$$\hat{\theta}_{DST_1} = \begin{cases} k\,\bar{x}_1 + (1-k)\theta_0 & ;\ if\ \ \chi_1^2 \le \left(2n_1\bar{x}_1/\theta_0\right) \le \chi_2^2 \\ \bar{x}_p & ;\ otherwise \end{cases} \qquad \text{\_\_\_\_\_(5.2.2)}$$

where k being dependent on test statistic is given by $k = 2n_1\,\bar{x}_1/\theta_0\,\chi^2$ and $\chi^2 = \left(\chi_2^2 - \chi_1^2\right)$

Finally, taking the 'square' of k (i.e. $k_1 = k^2$), another testimator is defined as

$$\hat{\theta}_{DST_2} = \begin{cases} \left(2n_1\,\bar{x}_1/\theta_0\,\chi^2\right)^2 \bar{x} + \left[1 - \left(2n_1\,\bar{x}_1/\theta_0\,\chi^2\right)^2\right]\theta_0 & ;\ if\ \ H_0\ is\ accepted \\ \bar{x}_p & ;\ otherwise \end{cases} \qquad \text{\_\_\_\_(5.2.3)}$$

5.3 Risk of Testimators

In this section we derive the risk of all the two testimators which are defined in the previous section.

5.3.1 Risk of $\hat{\boldsymbol{\theta}}_{\boldsymbol{DST1}}$

The risk of $\hat{\theta}_{DST_1}$ under $L_E\left(\hat{\theta},\theta\right)$ is defined by

$$R(\hat{\theta}_{DST_1}) = E[\,\hat{\theta}_{DST_1} \big|\, L_E\left(\hat{\theta},\theta\right)]$$

$$= \ E\left[k\,\bar{x}_1 + (1-k)\theta_0 / \chi_1^2 < 2n_1\bar{x}_1/\theta_0 < \chi_2^2\right]\cdot\, p\left[\chi_1^2 < 2n_1\bar{x}_1/\theta_0 < \chi_2^2\right]$$
$$+\ E\left[\bar{x}_p \ \big|\ 2n_1\bar{x}_1/\theta_0 < \chi_1^2\ \cup\ 2n_1\bar{x}_1/\theta_0 > \chi_2^2\right]\cdot\, p\left[2n_1\bar{x}_1/\theta_0 < \chi_1^2 \cup 2n_1\bar{x}_1/\theta_0 > \chi_2^2\right] \qquad \text{\_\_\_\_\_\_\_(5.3.1.1)}$$

$$= \int_{\frac{\chi_1^2\theta_0}{2n_1}}^{\frac{\chi_2^2\theta_0}{2n_1}} \left[\left(\frac{2n_1\bar{x}_1}{\theta_0\,\chi^2}\right)(\bar{x}_1-\theta_0)+\theta_0 \Big/ \theta\right]^p f(\bar{x}_1)\,d\bar{x}_1 - \int_{\frac{\chi_1^2\theta_0}{2n_1}}^{\frac{\chi_2^2\theta_0}{2n_1}} p\ln\left[\frac{2n_1\bar{x}_1}{\theta_0\,\chi^2}(\bar{x}_1-\theta_0)+\theta_0 \Big/ \theta\right] f(\bar{x}_1)\,d\bar{x}_1 - \int_{\frac{\chi_1^2\theta_0}{2n_1}}^{\frac{\chi_2^2\theta_0}{2n_1}} f(\bar{x}_1)\,d\bar{x}_1$$
$$+ \int_0^{\frac{\chi_1^2\theta_0}{2n_1}}\int_0^{\infty} \left[\left(\frac{\bar{x}_p}{\theta}\right)^p - p\ln\left(\frac{\bar{x}_p}{\theta}\right) - 1\right] f(\bar{x}_1)\,f(\bar{x}_2)\,d\bar{x}_1\,d\bar{x}_2 + \int_{\frac{\chi_2^2\theta_0}{2n_1}}^{\infty}\int_0^{\infty} \left[\left(\frac{\bar{x}_p}{\theta}\right)^p - p\ln\left(\frac{\bar{x}_p}{\theta}\right) - 1\right] f(\bar{x}_1)\,f(\bar{x}_2)\,d\bar{x}_1\,d\bar{x}_2 \qquad \text{\_\_\_\_\_\_\_(5.3.1.2)}$$

Where $f(\bar{x}_1) = (1/\Gamma n_1)\ (n_1/\theta)^{n_1} (\bar{x}_1)^{n_1 -1} \exp(-n_1 \bar{x}_1/\theta)$

$$f(\bar{x}_2) = (1/\Gamma n_2)\ (n_2/\theta)^{n_2} (\bar{x}_2)^{n_2 -1} \exp(-n_2 \bar{x}_2/\theta)$$

A Straight forward integration of (5.3.1.2) gives

$$R(\hat{\theta}_{DST_1}) \ = \ I_1 - \ I_2 - \left\{ I\left(\frac{x_2^2\,\phi}{2}, n_1 \right) - I\left(\frac{x_1^2\,\phi}{2}, n_1 \right) \right\} +$$

$$(I_3 + I_4) - (I_5 + I_6) - \left\{ I\left(\frac{x_1^2\,\phi}{2}, n_1 \right) - I\left(\frac{x_2^2\,\phi}{2}, n_1 \right) + 1 \right\}$$

_______(5.3.1.3)

Where $I(x;p) \ = \ (1/\Gamma p) \int_0^x \ e^{-x}\, x^{p-1}\, dx$ refers to the standard incomplete gamma function and

$$I_1 \ = \int_{\frac{x_1^2\phi}{2}}^{\frac{x_2^2\phi}{2}} \left[\left(2t^2/n_1\,\phi\,\chi^2 \right) - \left(2t/\ \chi^2 \right) + \phi \right]^p\ (1/\Gamma n_1)\ e^{-t}\ t^{n_1 - 1}\, dt$$

$$I_2 \ = \int_{\frac{x_1^2\phi}{2}}^{\frac{x_2^2\phi}{2}} p \ln \left[\left(2t^2/n_1\,\phi\,\chi^2 \right) - \left(2t/\ \chi^2 \right) + \phi \right] (1/\Gamma n_1)\ e^{-t}\ t^{n_1 - 1}\, dt$$

$$I_3 \ = \int_0^{\frac{x_1^2\phi}{2}} \int_0^{\infty} \frac{1}{(\Gamma n_1)(\Gamma n_2)(n_1 + n_2)^p} \cdot (t_1 + t_2)^p\ e^{-t_1}\, t_1^{\ n_1 - 1}\ \ e^{-t_2}\, t_2^{\ n_2 - 1}\, dt_1\, dt_2$$

$$I_4 \ = \int_{\frac{x_2^2\phi}{2}}^{\infty} \int_0^{\infty} \frac{1}{(\Gamma n_1)(\Gamma n_2)(n_1 + n_2)^p} \cdot (t_1 + t_2)^p\ e^{-t_1}\, t_1^{\ n_1 - 1}\ \ e^{-t_2}\, t_2^{\ n_2 - 1}\, dt_1\, dt_2$$

$$I_5 \ = \int_0^{\frac{x_1^2\phi}{2}} \int_0^{\infty} \frac{p}{(\Gamma n_1)(\Gamma n_2)(n_1 + n_2)^p} \cdot \ln (t_1 + t_2)\, e^{-t_1}\, t_1^{\ n_1 - 1}\ \ e^{-t_2}\, t_2^{\ n_2 - 1}\, dt_1\, dt_2$$

$$I_6 = \int_{\frac{x_2^2\phi}{2}}^{\infty}\int_0^{\infty} \frac{p}{(\Gamma n_1)(\Gamma n_2)(n_1+n_2)^p}\cdot \ln(t_1+t_2)\, e^{-t_1}\, t_1^{n_1-1}\; e^{-t_2}\, t_2^{n_2-1}\, dt_1\, dt_2$$

5.3.2 Risk of $\hat{\theta}_{DST2}$

Again, we obtain the risk of $\hat{\theta}_{DST_2}$ under $L_E(\hat{\theta},\theta)$ given by

$$R(\hat{\theta}_{DST_2}) = E[\hat{\theta}_{DST_2} \mid L_E(\hat{\theta},\theta)]$$

$$= E\left[\left(2n_1\bar{x}_1/\theta_0\,\chi^2\right)^2(\bar{x}_1-\theta_0)+\theta_0/\chi_1^2 < 2n_1\bar{x}_1/\theta_0 < \chi_2^2\right]\cdot p\left[\chi_1^2 < 2n_1\bar{x}_1/\theta_0 < \chi_2^2\right]$$
$$+E\left[\bar{x}_p \mid 2n_1\bar{x}_1/\theta_0 < \chi_1^2 \cup 2n_1\bar{x}_1/\theta_0 > \chi_2^2\right]\cdot p\left[2n_1\bar{x}_1/\theta_0 < \chi_1^2 \cup 2n_1\bar{x}_1/\theta_0 > \chi_2^2\right]$$

_______(5.3.2.1)

$$= \int_{\frac{\chi_1^2\theta_0}{2n_1}}^{\frac{\chi_2^2\theta_0}{2n_1}} \left[\left(\frac{2n_1\bar{x}_1}{\theta_0\,\chi^2}\right)^2(\bar{x}_1-\theta_0)+\theta_0\Big/\theta\right]^p f(\bar{x}_1)\, d\bar{x}_1$$

$$-\int_{\frac{\chi_1^2\theta_0}{2n_1}}^{\frac{\chi_2^2\theta_0}{2n_1}} p\ln\left[\left(\frac{2n_1\bar{x}_1}{\theta_0\,\chi^2}\right)^2(\bar{x}_1-\theta_0)+\theta_0\Big/\theta\right] f(\bar{x}_1)\, d\bar{x}_1$$

$$-\int_{\frac{\chi_1^2\theta_0}{2n_1}}^{\frac{\chi_2^2\theta_0}{2n_1}} f(\bar{x}_1)\, d\bar{x}_1 + \int_0^{\frac{\chi_1^2\theta_0}{2n_1}}\int_0^{\infty}\left[\left(\frac{\bar{x}_p}{\theta}\right)^p - p\ln\left(\frac{\bar{x}_p}{\theta}\right)-1\right] f(\bar{x}_1)\, f(\bar{x}_2)\, d\bar{x}_1\, d\bar{x}_2$$

$$+\int_{\frac{\chi_2^2\theta_0}{2n_1}}^{\infty}\int_0^{\infty}\left[\left(\frac{\bar{x}_p}{\theta}\right)^p - p\ln\left(\frac{\bar{x}_p}{\theta}\right)-1\right] f(\bar{x}_1)\, f(\bar{x}_2)\, d\bar{x}_1\, d\bar{x}_2$$

_______(5.3.2.2)

Where $f(\bar{x}_1) = (1/\Gamma n_1)\,(n_1/\theta)^{n_1}(\bar{x}_1)^{n_1-1}\exp(-n_1\bar{x}_1/\theta)$

$$f(\bar{x}_2) = (1/\Gamma n_2)\,(n_2/\theta)^{n_2}\,(\bar{x}_2)^{n_2-1}\exp(-n_2\bar{x}_2/\theta)$$

A straight forward integration of (5.3.2.2) gives:

$$R(\hat{\theta}_{DST_1}) \;=\; I_1 - I_2 - \left\{ I\left(\frac{x_2^2\,\phi}{2}, n_1\right) - I\left(\frac{x_1^2\,\phi}{2}, n_1\right)\right\} +$$

$$(I_3 + I_4) - (I_5 + I_6) - \left\{ I\left(\frac{x_1^2\,\phi}{2}, n_1\right) - I\left(\frac{x_2^2\,\phi}{2}, n_1\right) + 1\right\}$$

_______(5.3.2.3)

Where

$$I_1 = \int_{\frac{x_1^2\phi}{2}}^{\frac{x_2^2\phi}{2}} \left[\left(4t^3/n_1\phi^2\left(\chi^2\right)^2\right) - \left(4t^2/\phi\left(\chi^2\right)^2\right) + \phi\right]^p \left(1/\Gamma n_1\right) e^{-t}\, t^{n_1-1}\, dt$$

$$I_2 = \int_{\frac{x_1^2\phi}{2}}^{\frac{x_2^2\phi}{2}} p\ln\left[\left(4t^3/n_1\phi^2\left(\chi^2\right)^2\right) - \left(4t^2/\phi\left(\chi^2\right)^2\right) + \phi\right]\left(1/\Gamma n_1\right) e^{-t}\, t^{n_1-1}\, dt$$

$$I_3 = \int_0^{\frac{x_1^2\phi}{2}} \int_0^{\infty} \frac{1}{(\Gamma n_1)(\Gamma n_2)(n_1+n_2)^p} \cdot (t_1+t_2)^p\, e^{-t_1}\, t_1^{\,n_1-1}\; e^{-t_2}\, t_2^{\,n_2-1}\, dt_1\, dt_2$$

$$I_4 = \int_{\frac{x_2^2\phi}{2}}^{\infty} \int_0^{\infty} \frac{1}{(\Gamma n_1)(\Gamma n_2)(n_1+n_2)^p} \cdot (t_1+t_2)^p\, e^{-t_1}\, t_1^{\,n_1-1}\; e^{-t_2}\, t_2^{\,n_2-1}\, dt_1\, dt_2$$

$$I_5 = \int_0^{\frac{x_1^2\phi}{2}} \int_0^{\infty} \frac{p}{(\Gamma n_1)(\Gamma n_2)(n_1+n_2)^p} \cdot \ln(t_1+t_2)\, e^{-t_1}\, t_1^{\,n_1-1}\; e^{-t_2}\, t_2^{\,n_2-1}\, dt_1\, dt_2$$

$$I_6 = \int_{\frac{x_2^2\phi}{2}}^{\infty} \int_0^{\infty} \frac{p}{(\Gamma n_1)(\Gamma n_2)(n_1+n_2)^p} \cdot \ln(t_1+t_2)\, e^{-t_1}\, t_1^{\,n_1-1}\; e^{-t_2}\, t_2^{\,n_2-1}\, dt_1\, dt_2$$

5.4 **Relative Risks of** $\hat{\theta}_{DST_i}$

A natural way of comparing the risk of the proposed testimators, is to study its performance with respect to the best available estimator $\bar{x}_p$ in this case. For this purpose, we obtain the risk of $\bar{x}_p$ under $L_E(\hat{\theta},\theta)$ as:

$$R_E(\bar{x}_p) = E[\bar{x}_p \mid L_E(\hat{\theta},\theta)]$$

$$= \int_0^\infty \int_0^\infty \left[\left(\frac{\bar{x}_p}{\theta} \right)^p - p \ln\left(\frac{\bar{x}_p}{\theta} \right) - 1 \right] f(\bar{x}_1)\, f(\bar{x}_2)\, d\bar{x}_1\, d\bar{x}_2$$

______(5.4.1)

A straightforward integration of (5.4.1) gives

$$R_E(\bar{x}) = \left[\frac{\Gamma(n+P)}{\Gamma n\, (n^p)} - p\{\psi(n) - \ln(n)\} \right] - 1 \qquad \text{______(5.4.2)}$$

Where $\quad \psi(n) = (d/dn) \ln \Gamma n \quad$ refers to the Euler's psi function.

Now, we define the Relative Risk of $\hat{\theta}_{DST1}$ with respect to $\bar{x}_p$ under $L_E(\hat{\theta},\theta)$ as follows:

$$RR_1 = R_E(\bar{x}_p) \big/ R(\hat{\theta}_{DST_1}) \qquad \text{_______(5.4.3)}$$

Using (5.4.2) and (5.3.1.3) the expression for RR_1 given in (5.4.3) can be obtained;

Similarly, we define the Relative Risk of $\hat{\theta}_{DST_2}$ under $L_E(\hat{\theta},\theta)$ as follows

$$RR_2 = R_E(\bar{x}_p) \big/ R(\hat{\theta}_{DST_2}) \qquad \text{_______(5.4.4)}$$

The expression for RR_2 given in (5.4.4) which can be obtain by using equation (5.4.2) and (5.3.2.3).

Now, it is observed that both RR_1 and RR_2 are a function of '∅' , 'n_1' , 'n_2' , 'α' and 'p'. To observe the behavior of the risk(s) of $\hat{\theta}_{DST_1}$ and $\hat{\theta}_{DST_2}$, we have taken several values of these viz α = 1%, 5%, 10%, (n_1, n_2) = (4,6), (4,8), (6,10), (4,12), ∅ = 0.2 (0.2) 1.6 and p = -3, -2, -1, 2, 3, 4 ; 'p' is the prime important factor and decides about the seriousness of over/under estimation in the real life situation. The recommendations regarding the applications of proposed testimators are provided as follows:

5.5 Recommendations for $\hat{\theta}_{DST_i}$

In this section we wish to compare the performance of $\hat{\theta}_{DST_1}$ and $\hat{\theta}_{DST_2}$ with respect to the best available (unbiased) estimator of $\bar{x}_p$.

(1) Taking $n_1 = 4$, $n_2 = 6$ and fixing α = 1% we have allowed the variation in 'p' which represents the degree of asymmetry. As the shrinkage factor depends on test statistics and hence on 'α'. It has been observed that the RR_1 values are higher than 1 (unity) in the whole range of ϕ, demonstrating that $\hat{\theta}_{DST_1}$ performs better than $\bar{x}_p$. For p = -3 (negative) and p = 2 (positive) its performance is 'best' however it performs better for the other values of 'p' also.

(2) It is also observed that $\hat{\theta}_{DST1}$ performs still better for $n_1 = 4$, $n_2 = 8$ ($n_2 = 2n_1$) i.e. perhaps second sample should be twice as much compared to the 1^{st} stage sample.

(3) For α = 5% and α = 10% a similar pattern of performance is observed however the magnitude of RR_1 is highest at α = 1%.

(4) For α = 10% and n_1 = 4, n_2 = 6, it observed in particular that RR_1 is highest for p = 2 (positive) and then followed by p = -3 (negative) a trend not observed earlier. However for other values of (n_1, n_2) considered here, p = -3 shows larger values of RR_1.

(5) In the next comparison stage, we have fixed p = -3 and have allowed variation in α values. Maximum gain in RR_1 is observed at $\phi = 1.0$. So, we have fixed ϕ = 1.0 again for the whole range and for all the combination(s) of (n_1, n_2). $\hat{\theta}_{DST1}$ fairs better than the usual estimator. It is also observed that there is a minor difference in the values of RR_1 for (4, 8) and (4, 12). So again second stage sample may be chosen in this light.

(6) It is observed that (6,10) sample combination does not give better control over risk as the values of RR_1 are smaller in magnitude compared to other RR_1 values.

(7) The data set considered here is n_1 = 4, n_2 = 10 and $\phi = 0.8$ (different from $\phi =$ 1.0 i.e. $\theta \neq \theta_0$) again for α = 1%. We have allowed the variation in the values of shape parameter 'p' and it has been observed that $\hat{\theta}_{DST1}$ dominates the usual unbiased estimator for all values of 'p' and the performance is at its best for p = -3.

(8) It has been observed that positive values of 'p' considered here, the maximum RR_1 values have been observed at p = 2, for different values of α^s. The highest values in terms of magnitude are observed at α = 1% for different n_1 and n_2 combination values.

The present investigation has also considered the square of S.F. and we have proposed another testimator viz $\hat{\theta}_{DST2}$. We have also studied the behaviour of Relative risk(s) of $\hat{\theta}_{DST2}$ with respect to $\bar{x}_p$ and have computed RR_2 values, to observe the behaviour of $\hat{\theta}_{DST2}$. For all the values of (n_1, n_2), ϕ, α and p considered for RR_1, we have computed RR_2 values for the same set of values. Following observations have been made.

1) It is observed that $\hat{\theta}_{DST2}$ performs better than the usual estimator $\bar{x}_p$. For all the values considered here. However the magnitude of RR_2 values are higher than RR_1 values, indicating a **better** control over risk by the proposed estimator $\hat{\theta}_{DST2}$.
2) Almost similar recommendations as above in case of $\hat{\theta}_{DST1}$ (1 - 8) follow here also. But definitely $\hat{\theta}_{DST2}$ has better performance compared to $\hat{\theta}_{DST1}$.

CONCLUSIONS:

The present chapter studies the risk properties of double stage shrinkage testimator(s) of the scale parameter (average life) of exponential life model using General Entropy Loss Function. Two choices of the shrinkage factor have been made making it dependent on the test statistics, hence the choice of 'α' plays an important role. We conclude that a lower value of level of significance i.e. α = 1% is suitable for almost all values of 'shape' parameter of the loss function but in particular when p = -3, at α = 1% its performance best for (n_1 = 4, n_2 = 8) and similar recommendation holds for p = 2 (positive).

The 'square' of S.F. gives better control over the relative risk as has been observed by Comparing the relative risk values. So, to conclude take α = 1% square of the shrinkage factor, p = -3 or p = 2 and (n_1 = 4, n_2 = 8).

Tables showing relative risk(s) of proposed testimator(s) with respect to the best available estimator.

Table : 5.5.1.1 **Relative Risk of** $\hat{\theta}_{DST_1}$ α = 1%, n_1 = 4, n_2 = 8

∅	p = -3	p = -2	p = -1	p = 2	p = 3	p = 4
0.20	1.324	0.684	0.321	0.873	0.836	0.94
0.40	2.225	1.741	1.218	1.147	1.53	1.187
0.60	4.729	3.217	2.278	2.453	2.241	1.664
0.80	6.102	4.69	3.54	3.62	3.33	2.444
1.00	9.883	6.898	5.369	5.147	5.883	5.062
1.20	4.447	4.8	3.533	3.998	3.576	3.01
1.40	2.284	3.01	2.017	2.712	2.016	1.833
1.60	1.614	1.688	1.786	1.089	1.032	0.916

Table : 5.5.1.2 **Relative Risk of** $\hat{\theta}_{DST_1}$ α = 1%, n_1 = 6, n_2 = 10

∅	p = -3	p = -2	p = -1	p = 2	p = 3	p = 4
0.20	0.953	0.431	0.153	0.159	0.483	0.831
0.40	1.224	0.445	0.209	0.282	0.643	0.897
0.60	2.002	1.096	1.388	1.904	1.618	1.783
0.80	5.362	4.12	3.789	4.027	3.218	2.866
1.00	9.259	8.591	6.399	7.714	6.367	5.165
1.20	5.908	4.197	4.218	5.025	3.737	3.165
1.40	2.954	2.985	2.58	3.073	1.633	1.421
1.60	1.722	1.329	1.766	1.513	0.823	0.698

Table : 5.5.1.3 **Relative Risk of** $\hat{\theta}_{DST_1}$ α = 5%, n_1 = 4, n_2 = 8

∅	p = -3	p = -2	p = -1	p = 2	p = 3	p = 4
0.20	1.103	0.564	0.461	0.598	0.641	0.86
0.40	1.237	0.76	0.987	0.819	0.753	1.689
0.60	2.103	1.771	1.639	1.313	1.468	2.096
0.80	4.103	3.631	3.238	2.819	2.694	2.437
1.00	6.684	6.075	5.758	5.097	4.771	4.413
1.20	4.04	3.068	2.587	3.111	2.581	1.988
1.40	2.15	2.019	1.627	2.098	1.627	1.077
1.60	1.04	0.957	0.804	1.203	0.787	0.653

Table : 5.5.1.4 **Relative Risk of** $\hat{\theta}_{DST_1}$ α = 5%, n_1 = 6, n_2 = 10

∅	p = -3	p = -2	p = -1	p = 2	p = 3	p = 4
0.20	0.702	0.52	0.147	0.177	0.248	0.845
0.40	0.806	0.834	0.258	0.886	0.548	1.053
0.60	1.702	1.44	1.249	1.202	1.023	1.241
0.80	2.891	2.484	2.841	2.514	2.702	2.311
1.00	5.888	5.073	4.069	4.667	3.992	3.893
1.20	3.214	2.334	2.23	2.871	2.667	2.378
1.40	1.066	0.417	1.077	1.911	1.754	1.716
1.60	0.025	0.16	0.678	0.363	0.429	0.927

Table : 5.5.1.5 **Relative Risk of** $\hat{\theta}_{DST_1}$ α = 10%, n_1 = 4, n_2 = 6

∅	p = -3	p = -2	p = -1	p = 2	p = 3	p = 4
0.20	0.685	0.716	0.338	0.784	0.236	0.669
0.40	1.404	0.938	0.535	0.809	1.092	1.585
0.60	1.949	1.789	1.331	1.409	1.37	2.07
0.80	2.554	2.537	2.486	3.054	2.418	2.382
1.00	3.934	3.443	3.3	4.881	3.789	3.148
1.20	2.282	2.631	2.737	2.097	2.087	2.07
1.40	1.136	1.306	1.206	1.598	1.34	1.286
1.60	0.075	0.168	0.591	0.784	0.672	0.94

Table : 5.5.2.1 **Relative Risk of** $\hat{\theta}_{DST_2}$ α = 1%, n_1 = 4, n_2 = 8

∅	p = -3	p = -2	p = -1	p = 2	p = 3	p = 4
0.20	1.154	0.626	0.311	0.739	0.683	0.939
0.40	2.047	1.697	1.213	0.815	1.399	1.084
0.60	3.742	1.212	2.277	1.715	2.221	1.785
0.80	4.696	2.893	4.553	2.974	3.468	1.866
1.00	6.42	7.955	5.443	4.653	4.094	2.279
1.20	4.318	5.692	3.556	2.173	2.259	2.267
1.40	3.548	2.514	2.147	1.516	1.416	1.441
1.60	1.491	1.63	1.882	0.868	0.768	0.763

5.6 Shrinkage Testimator for the Variance of a Normal Distribution

Let X be normally distributed with mean μ and variance σ^2, both unknown. It is assumed that the prior knowledge about σ^2 is available in the form of an initial estimate σ_0^2. We are interested in constructing an estimator of σ^2 using the sample observations and possibly the guess value σ_0^2. We define a double stage shrinkage testimator of σ^2 as follows:

1. Take a random sample x_{1i} $(i = 1,2,\_\_, n_1)$ of size n_1 from N(μ, σ^2) and compute $\bar{x}_1 = \frac{1}{n_1}\sum x_{1i}$, $s_1^2 = \frac{1}{n_1-1}\sum(x_{1i} - \bar{x}_1)^2$.
2. Test the hypothesis $H_0 : \sigma^2 = \sigma_0^2$ against the alternative $H_1 : \sigma^2 \neq \sigma_0^2$ at level α using the test statistic $\frac{\nu_1 s_1^2}{\sigma_0^2}$, which is distributed as χ^2 with $\nu_1 = (n_1 - 1)$ degrees of freedom.
3. If H_0 is accepted at α level of significance i.e. $x_1^2 < \frac{\nu_1 s_1^2}{\sigma_0^2} < x_2^2$,where x_1^2 and x_2^2 refer to lower and upper critical points of the unbiased portioning of the test statistic at a given level of significance α, take $k_1 s_1^2 + (1 - k_1)\sigma_0^2$ as the shrinkage estimator of σ^2 with shrinkage factor k_1 dependent on the test statistic.
4. If H_0 is rejected, take a second sample x_{2j} $(j = 1,2,\_\_, n_2)$ of size $n_2 = (n - n_1)$ compute $\bar{x}_2 = \frac{1}{n_2}\sum x_{2j}$, $s_2^2 = \frac{1}{n_2-1}\sum(x_{2j} - \bar{x}_2)^2$ and take $(\nu_1 s_1^2 + \nu_2 s_2^2)/(\nu_1 + \nu_2)$ where $\nu_2 = (n_2 - 1)$ as the estimator of σ^2.

To summarize, we define the double- stage shrinkage testimators $\hat{\sigma}_{DST1}^2$ and $\hat{\sigma}_{DST2}^2$ of σ^2 as follows:

$$\hat{\sigma}^2_{DST\,1} = \begin{cases} k\, s_1^2 + (1-k)\sigma_0^2\ , & if\ \ H_0\ \ is\, accepted \\ s_p^2 = \dfrac{(\nu_1 s_1^2 + \nu_2 s_2^2)}{(\nu_1 + \nu_2)}, & if\ H_0\ \ is\ \ rejected \end{cases}$$

Estimators of this type with k arbitrary and lying between 0 and 1 have been proposed by Katti (1962), Shah(1964), Arnold and Al-Bayyati (1970), Waikar and Katti (1971), Pandey (1979) and k being dependent on the test statistics by Waikar, Schuurman and Raghunandan (1984), Pandey, Srivastava and Malik (1988).

$$\hat{\sigma}^2_{DST\,2} = \begin{cases} k_1\, s_1^2 + (1-k_1)\sigma_0^2\ , & if\ \ H_0\ \ is\, accepted \\ s_p^2 = \dfrac{(\nu_1 s_1^2 + \nu_2 s_2^2)}{(\nu_1 + \nu_2)}, & if\ H_0\ \ is\ \ rejected \end{cases}$$

Where k_1 being dependent on test statistic and is given by $k_1 = \frac{\nu_1 s_1^2}{{\sigma_0}^2\, \chi^2}$

We have studied the risk properties of these testimators under GELF defined in section 4.1.1.

5.7 Risk of Testimators

In this section we derive the risk of proposed testimators which are defined in the previous section.

5.7.1 Risk of $\hat{\sigma}^2_{DST1}$

The risk of $\hat{\sigma}^2{}_{DST\,1}$ under $L_E\left(\hat{\sigma}^2, \sigma^2\right)$ is defined by

$$R(\hat{\sigma}^2{}_{DST_1}) = E[\,\hat{\sigma}^2{}_{DST_1} \big|\, L_E\left(\hat{\sigma}^2, \sigma^2\right)]$$

$$= E\left[ks_1^{\,2} + (1-k)\sigma_0^2 \middle/ \chi_1^2 < \frac{\upsilon_1 s_1^{\,2}}{\sigma_0^2} < \chi_2^2 \right] \cdot p\left[\chi_1^2 < \frac{\upsilon_1 s_1^{\,2}}{\sigma_0^2} < \chi_2^2 \right]$$

$$+ E\left[s_p^{\,2} \;\middle|\; \frac{\upsilon_1 s_1^{\,2}}{\sigma_0^2} < \chi_1^2 \;\cup\; \frac{\upsilon_1 s_1^{\,2}}{\sigma_0^2} > \chi_2^2 \right] \cdot p\left[\frac{\upsilon_1 s_1^{\,2}}{\sigma_0^2} < \chi_1^2 \cup \frac{\upsilon_1 s_1^{\,2}}{\sigma_0^2} > \chi_2^2 \right]$$

______(5.7.1.1)

$$= \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} \left[\frac{k\left(s_1^{\,2} - \sigma_0^2\right) + \sigma_0^2}{\sigma^2} \right]^p f(s_1^{\,2})\, ds_1^{\,2} - \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} p \;\ln\left[\frac{k\left(s_1^{\,2} - \sigma_0^2\right) + \sigma_0^2}{\sigma^2} \right] f(s_1^{\,2})\, ds_1^{\,2}$$

$$- \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} f(s_1^{\,2})\, ds_1^{\,2} + \int_0^{\frac{\chi_1^2\sigma_0^2}{\upsilon}} \int_0^{\infty} \left[\left(\frac{s_p^{\,2}}{\sigma^2} \right)^p - p\;\ln\left(\frac{s_p^{\,2}}{\sigma^2} \right) - 1 \right] f(s_1^{\,2})\, f(s_2^{\,2})\, ds_1^{\,2} ds_2^{\,2}$$

$$+ \int_{\frac{\chi_2^2\sigma_0^2}{\upsilon}}^{\infty} \int_0^{\infty} \left[\left(\frac{s_p^{\,2}}{\sigma^2} \right)^p - p\;\ln\left(\frac{s_p^{\,2}}{\sigma^2} \right) - 1 \right] f(s_1^{\,2})\, f(s_2^{\,2})\, ds_1^{\,2} ds_2^{\,2}$$

______(5.7.1.2)

Where $f(s_1^{\,2}) = \dfrac{1}{2^{\upsilon_1/2}\,\Gamma\left(\upsilon_1/2\right)} \left(s_1^{\,2}\right)^{\frac{\upsilon_1}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon_1 s_1^{\,2}}{\sigma^2}\right)}\, ds_1^{\,2}$

$$f(s_2^{\,2}) = \frac{1}{2^{\upsilon_2/2}\,\Gamma\left(\upsilon_2/2\right)} \left(s_2^{\,2}\right)^{\frac{\upsilon_2}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon_2 s_2^{\,2}}{\sigma^2}\right)}\, ds_2^{\,2}$$

Straight forward integration of (5.7.1.2) gives

$$R\left(\hat{\sigma}^2_{DST1}\right) = \left(\frac{\sigma^2}{\nu_1}\right)^{\nu_1/2}\left(\frac{\sigma^2}{\nu_2}\right)^{\nu_1/2}\left[\begin{array}{c} I_1 - I_2 - \left\{I\left(\chi_2^2\lambda,\frac{\nu_1}{2}\right) - I\left(\chi_1^2\lambda,\frac{\nu_1}{2}\right)\right\} \\ -\left[I\left(\chi_1^2\lambda,\frac{\nu_1}{2}\right) - I\left(\chi_2^2\lambda,\frac{\nu_1}{2}\right)+1\right] \\ +\frac{\Gamma\left(\frac{\nu_1}{2}+p\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\left(\frac{\nu_1}{2}\right)^p}\left[I\left(\chi_1^2\lambda,\frac{\nu_1}{2}+p\right) - I\left(\chi_2^2\lambda,\frac{\nu_1}{2}+p\right)+1\right] \\ - I_3 - I_4 \end{array}\right]$$

_____(5.7.1.3)

Where $I(x;p) = (1/\Gamma p)\int_0^x e^{-x}x^{p-1}\,dx$ refers to the standard incomplete gamma function, $\lambda = \frac{\sigma_0^2}{\sigma^2}$, and

$$I_1 = \frac{1}{2^{\frac{\nu_1}{2}}\,\Gamma\left(\frac{\nu_1}{2}\right)}\int_{x_1^2\lambda}^{x_2^2\lambda}\left(k\left(\frac{t_1}{\nu_1}-\lambda\right)+\lambda\right)^p e^{-\left(\frac{1}{2}\right)t_1}\, t_1^{\frac{\nu_1}{2}-1}\,dt_1$$

$$I_2 = \frac{p}{2^{\frac{\nu_1}{2}}\,\Gamma\left(\frac{\nu_1}{2}\right)}\int_{x_1^2\lambda}^{x_2^2\lambda} ln\left(k\left(\frac{t_1}{\nu_1}-\lambda\right)+\lambda\right) e^{-\left(\frac{1}{2}\right)t_1}\, t_1^{\frac{\nu_1}{2}-1}\,dt_1$$

$$I_3 = \frac{p}{2^{\nu_1/2}\,\Gamma\left(\frac{\nu_1}{2}\right)}\int_0^{x_1^2\lambda} ln\left(\frac{t_1}{\nu_1}\right) e^{-\left(\frac{1}{2}\right)t_1}\, t^{\frac{\nu_1}{2}-1}\,dt_1$$

$$I_4 = \frac{p}{2^{\nu_1/2}\,\Gamma\left(\frac{\nu_1}{2}\right)}\int_{x_2^2\lambda}^{\infty} ln\left(\frac{t_1}{\nu_1}\right) e^{-\left(\frac{1}{2}\right)t_1}\, t^{\frac{\nu_1}{2}-1}\,dt_1$$

5.7.2 <u>Risk of</u> $\hat{\boldsymbol{\sigma}}^2_{DST2}$

Again, we obtain the risk of $\hat{\sigma}^2{}_{DST_2}$ under $L_E(\hat{\theta},\theta)$ with respect to s_p^2, given by

$$R(\hat{\sigma}^2{}_{DST_2}) = E\left[\hat{\sigma}^2{}_{DST_2}\middle| L_E(\hat{\theta},\theta)\right]$$

$$= E\left[k_1 s_1{}^2 + (1-k_1)\sigma_0^2 \middle/ \chi_1^2 < \frac{\upsilon_1 s_1{}^2}{\sigma_0^2} < \chi_2^2\right]\cdot p\left[\chi_1^2 < \frac{\upsilon_1 s_1{}^2}{\sigma_0^2} < \chi_2^2\right]$$

$$+ E\left[s_p{}^2 \middle| \frac{\upsilon_1 s_1{}^2}{\sigma_0^2} < \chi_1^2 \cup \frac{\upsilon_1 s_1{}^2}{\sigma_0^2} > \chi_2^2\right]\cdot p\left[\frac{\upsilon_1 s_1{}^2}{\sigma_0^2} < \chi_1^2 \cup \frac{\upsilon_1 s_1{}^2}{\sigma_0^2} > \chi_2^2\right]$$

_____ (5.7.2.1)

$$= \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} \left[\frac{\frac{\upsilon_1 s_1^{\,2}}{\sigma_0^2\,\chi^2}\left(s_1^{\,2}-\sigma_0^2\right)+\sigma_0^2}{\sigma^2} \right]^p f(s_1^{\,2})\, ds_1^{\,2} - \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} p\;\ln\left[\frac{\frac{\upsilon_1 s_1^{\,2}}{\sigma_0^2\,\chi^2}\left(s_1^{\,2}-\sigma_0^2\right)+\sigma_0^2}{\sigma^2} \right] f(s_1^{\,2})\, ds_1^{\,2}$$

$$- \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon}} f(s_1^{\,2})\, ds_1^{\,2} + \int_0^{\frac{\chi_1^2\sigma_0^2}{\upsilon}} \int_0^{\infty} \left[\left(\frac{s_p^{\,2}}{\sigma^2}\right)^p - p\;\ln\left(\frac{s_p^{\,2}}{\sigma^2}\right) - 1 \right] f(s_1^{\,2})\, f(s_2^{\,2})\, ds_1^{\,2} ds_2^{\,2}$$

$$+ \int_{\frac{\chi_2^2\sigma_0^2}{\upsilon}}^{\infty} \int_0^{\infty} \left[\left(\frac{s_p^{\,2}}{\sigma^2}\right)^p - p\;\ln\left(\frac{s_p^{\,2}}{\sigma^2}\right) - 1 \right] f(s_1^{\,2})\; f(s_2^{\,2})\; ds_1^{\,2}\, ds_2^{\,2}$$

_______(5.7.2.2)

Where $f(s_1^{\,2}) = \frac{1}{2^{\upsilon_1/2}\,\Gamma\left(\upsilon_1/2\right)} \left(s_1^{\,2}\right)^{\frac{\upsilon_1}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon_1 s_1^{\,2}}{\sigma^2}\right)} ds_1^{\,2}$

and $f(s_2^{\,2}) = \frac{1}{2^{\upsilon_2/2}\,\Gamma\left(\upsilon_2/2\right)} \left(s_2^{\,2}\right)^{\frac{\upsilon_2}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon_2 s_2^{\,2}}{\sigma^2}\right)} ds_2^{\,2}$

Straight forward integration of (5.7.2.2) gives

$$R\left(\hat{\sigma}^2_{DST2}\right) = \left(\frac{\sigma^2}{\nu_1}\right)^{\nu_1/2} \left(\frac{\sigma^2}{\nu_2}\right)^{\nu_1/2} \left[\begin{array}{c} \mathrm{I}_1 - \mathrm{I}_2 - \left\{ \mathrm{I}\left(\chi_2^2\lambda, \frac{\nu_1}{2}\right) - \mathrm{I}\left(\chi_1^2\lambda, \frac{\nu_1}{2}\right) \right\} \\ - \left[\mathrm{I}\left(\chi_1^2\lambda, \frac{\nu_1}{2}\right) - \mathrm{I}\left(\chi_2^2\lambda, \frac{\nu_1}{2}\right) + 1 \right] \\ + \frac{\Gamma\left(\frac{\nu_1}{2}+\mathrm{p}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\left(\frac{\nu_1}{2}\right)^{\mathrm{p}}} \left[\mathrm{I}\left(\chi_1^2\lambda, \frac{\nu_1}{2}+\mathrm{p}\right) - \mathrm{I}\left(\chi_2^2\lambda, \frac{\nu_1}{2}+\mathrm{p}\right) + 1 \right] \\ - \mathrm{I}_3 - \mathrm{I}_4 \end{array} \right]$$

_____(5.7.2.3)

Where $I(x;p) = (1/\Gamma p) \int_0^x e^{-x} x^{p-1} dx$ refers to the standard incomplete gamma function, $\lambda = \frac{\sigma_0^2}{\sigma^2}$, and

$$I_1 = \frac{1}{2^{\nu_1/2} \; \Gamma\left(\frac{\nu_1}{2}\right)} \int_{x_1^2\lambda}^{x_2^2\lambda} \left(\frac{{t_1}^2}{\nu_1 \lambda \chi^2} - \frac{t_1}{\chi^2} + \lambda\right)^p e^{-\left(\frac{1}{2}\right)t_1} \, {t_1}^{\frac{\nu_1}{2}-1} \, dt_1$$

$$I_2 = \frac{p}{2^{\nu_1/2} \; \Gamma\left(\frac{\nu_1}{2}\right)} \int_{x_1^2\lambda}^{x_2^2\lambda} ln\left(\frac{{t_1}^2}{\nu_1 \lambda \chi^2} - \frac{t_1}{\chi^2} + \lambda\right) e^{-\left(\frac{1}{2}\right)t_1} \, {t_1}^{\frac{\nu_1}{2}-1} \, dt_1$$

$$I_3 = \frac{p}{2^{\nu_1/2} \; \Gamma\left(\frac{\nu_1}{2}\right)} \int_0^{x_1^2\lambda} ln\left(\frac{t_1}{\nu_1}\right) e^{-\left(\frac{1}{2}\right)t_1} \, {t_1}^{\frac{\nu_1}{2}-1} \, dt_1$$

$$I_4 = \frac{p}{2^{\nu_1/2} \; \Gamma\left(\frac{\nu_1}{2}\right)} \int_{x_2^2\lambda}^{\infty} ln\left(\frac{t_1}{\nu_1}\right) e^{-\left(\frac{1}{2}\right)t_1} \, {t_1}^{\frac{\nu_1}{2}-1} \, dt_1$$

5.8 Relative Risk of $\hat{\sigma}^2_{DSTi}$

A natural way of comparing the risk of the proposed testimators, is to study its performance with respect to the best available estimator s_p^2 in this case. For this purpose, we obtain the risk of s_p^2 under $L_E(\hat{\sigma}^2, \sigma^2)$ as:

$$R_E({s_p}^2) = E[{s_p}^2 \mid L(\hat{\sigma}^2, \sigma^2)]$$

$$= \int_0^\infty \int_0^\infty \left[\left({s_p}^2/\sigma^2\right)^p - p \ln\left({s_p}^2/\sigma^2\right) - 1\right] f({s_1}^2) \, f({s_2}^2) \, d{s_1}^2 d{s_2}^2$$

______(5.8.1)

Where $f({s_1}^2) = \frac{1}{2^{\upsilon_1/2} \, \Gamma\left(\upsilon_1/2\right)} \left({s_1}^2\right)^{\frac{\upsilon_1}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon_1 {s_1}^2}{\sigma^2}\right)} d{s_1}^2$

and $$f(s_2^{\,2}) = \frac{1}{2^{\upsilon_2/2}\,\Gamma\left(\upsilon_2/2\right)} \left(s_2^{\,2}\right)^{\frac{\upsilon_2}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon_2 s_2^{\,2}}{\sigma^2}\right)} ds_2^{\,2}$$

A straightforward integration of (5.8.1) gives

$$R_E(s_p^{\,2}) = \left[\frac{\Gamma\left(\frac{\nu_1}{2}+P\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\left(\frac{\nu_1}{2}\right)^p} - p\left\{\psi\left(\frac{\nu_1}{2}\right) - \ln\left(\frac{\nu_1}{2}\right)\right\}\right] - 1 \qquad \text{\_\_\_\_\_\_(5.8.2)}$$

Where $\psi(n) = (d/dn)\ln\Gamma n$ refers to the Euler's psi function.

Now, we define the Relative Risk of $\hat{\sigma}^2{}_{DST_i}$, $i = 1,2$ with respect to s^2 under $L(\hat{\sigma}^2, \sigma^2)$ as follows:

$$RR_1 = \frac{R_E(\mathrm{s_p}^{\,2})}{R(\hat{\sigma}^2{}_{DST1})} \qquad \text{\_\_\_\_\_\_\_(5.8.3)}$$

Using (5.8.2) and (5.7.1.3) the expression for RR_1 given in (5.8.3) can be obtained; it is observed that RR_1 is a function of 'λ', '(ν_1, ν_2)', 'α', 'k' and 'p'. In order to study the risk behaviour of $\hat{\sigma}^2_{DST1}$ we have considered the following values of these quantities. k = 0.2 (0.2) 1.0, λ = 0.2 (0.2) 2.0, p = -3, -2.5, -2.0, -1.5, -1.0, 1.0 and 1.5, α = 1% and 0.1%, (ν_1, ν_2) = (5,5), (5,8), (5,10), (5,12).

Finally, we define the Relative Risk of $\hat{\sigma}^2{}_{DST_2}$ by

$$RR_2 = \frac{R_E(\mathrm{s_p}^{\,2})}{R(\hat{\sigma}^2{}_{DST2})} \qquad \text{\_\_\_\_\_\_\_(5.8.4)}$$

The expression for RR_2 is given by (5.8.4) can be obtained by using (5.8.2) and (5.7.2.3). Again we observed that RR_2 is a function of 'λ', '(ν_1, ν_2)', 'α' and 'p'. We have considered same values of these as in case of RR_1 not 'k'. i.e. λ = 0.2

(0.2) 2.0, p = -3, -2.5, -2.0, -1.5, -1.0, 1.0 and 1.5, α = 1% and 0.1% , (ν_1, ν_2) = (5,5), (5,8), (5,10), (5,12).

5.9 Recommendations for $\hat{\sigma}^2_{DSTi}$

In this section we wish to compare the performance of $\hat{\sigma}^2_{DST_1}$ and $\hat{\sigma}^2_{DST_2}$ with respect to the best available (unbiased) estimator of σ^2.

5.9.1 Recommendations for $\hat{\sigma}^2_{DST1}$

There will be several tables of RR_1 , some of these tables are assembled at the end of the chapter. Recommendations for the use of $\hat{\sigma}^2_{DST1}$ are as follows:

1. For (ν_1, ν_2) = (5, 5), α = 1% the following table provides the effective ranges of 'λ' for different choice of 'k' (shrinkage factor) values. Various degrees of asymmetries are also presented.

k	λ	p
0.2	$0.6 \le \lambda \le 2.0$	p = -3 to -1.5
0.4	$0.6 \le \lambda \le 2.0$	p = -3 to -1.5
0.6	$0.8 \le \lambda \le 1.6$	p = -1
0.8	$0.8 \le \lambda \le 1.4$	p = 1 & 1.5

   From the above table it is observed that the range of 'λ' decreases as 'k' increases and it remains true for extreme negative and positive values of 'p'.

2. As (ν_1, ν_2) change i.e. (5,8), (5,10) the values of RR_1 also change in their magnitude but still higher than unity. A high value of ν_2 is not recommended. In this case, also for $0.2 \le k \le 0.8$ the effective range of 'λ' varies slightly as in the above table as for p = -3 it is $0.6 \le \lambda \le 2.0$ for k = 0.2 where as for k = 0.8 it becomes $1.0 \le \lambda \le 2.0$ for p = 1.5 .

3. Next, we have considered α = 0.1% as it is reported by several authors that shrinkage testimators perform better for smaller level of significance. RR_1 values obtained for this choice of 'α' are better than those obtained for earlier value of 'α' as their magnitude is higher. A higher value of relative risk indicates better performance of the proposed estimator.
4. The effective ranges of 'λ' are more or less the same obtained previously i.e. for p = -3 it is $0.6 \leq \lambda \leq 2.0$ and for p = +1 it is $0.8 \leq \lambda \leq 1.6$ However as mentioned above the numerical values are larger.
5. As (ν_1, ν_2) change to (5,8) and (5,10) the RR_1 values are better in the range of 0.6 to 2.0 for 'λ' when p is upto -1.75, however 'λ' range becomes 0.8 to 1.8 for p = -1.5 and -1.0. This range reduces further to 0.8 to 1.6 for both the positive values of 'p'.

5.9.2 <u>Recommendation for</u> $\hat{\sigma}^2_{DST2}$

There will be several tables of RR_2 some of these are assembled at the end of the chapter. The recommendations are as follows:

1. For α = 1% and all the negative values of 'p' i.e. -3 upto -1.5 $\hat{\sigma}^2_{DST2}$ performs better than s_p^2 for fairly large range of λ i.e. $0.6 \leq \lambda \leq 2.0$. However for p = -1 this shrinks and it becomes $0.8 \leq \lambda \leq 1.6$. For p = +1 and 1.5 the values of

   RR_2 are better i.e. greater than unity for a range of 'λ' i.e. $0.8 \leq \lambda \leq 1.6$ for p = +1 however for p = 1.5 it becomes $0.8 \leq \lambda \leq 1.4$ it reduces very slightly. So, $\hat{\sigma}^2_{DST2}$ can be considered for various degrees of positive / negative asymmetry. This behaviour is observed for (ν_1, ν_2) = (5,5).
2. As we have considered another data sets for (ν_1, ν_2) it is observed that as ν_2 increases i.e. (5, 8), (5,10) etc. still $\hat{\sigma}^2_{DST2}$ behaves nicely for different

positive / negative values of ‘p’. But it is observed that the performance is better for larger negative values of ‘p’ as compared to positive values of ‘p’. Further, it is noted that the magnitude of RR_2 values decrease as ν_2 increases. However it does not change the effective ranges of ‘λ’ i.e. again for p = -3 it is $0.6 \leq \lambda \leq 2.0$ which reduces by 0.2 units as ‘p’ changed from -3 to -1 but even for p = -1, it is $0.6 \leq \lambda \leq 1.6$. For much higher values of ν_2 i.e. ν_2 = 12 and more the performance is not very good.

3. Next we reduce ‘α’ further to α = 0.1% then still better values of RR_2 are obtained in the sense that they are higher in magnitude as compared to those obtained for α = 1%.
4. The effective ranges of ‘λ’ are more or less same as obtained previously i.e. for p = -3 it is $0.6 \leq \lambda \leq 2.0$ and for p = +1 it becomes $0.8 \leq \lambda \leq 1.6$. Again, it performs better for both positive/ negative degrees of asymmetry for almost all the data set considered here. But the magnitude of RR_2 values are higher uniformly than those obtained at α = 1%.
5. It is recommended that use large negative value of ‘p’, smaller level of significance and a small sample (ν_1, ν_2).

CONCLUSIONS:

We have propose two double stage shrinkage testimator(s) for the variance of a Normal distribution viz. $\hat{\sigma}^2_{DST1}$ and $\hat{\sigma}^2_{DST2}$. It is observed that both the testimators dominate the usual unbiased estimator of σ^2 for various sample sizes, degrees of asymmetries, levels of significance and a wide range of ‘λ’. It is found that the use of GEL is beneficial for those situations where underestimation is more harmful than overestimation or vice- versa. In particular for p = -3 and p = 1.0, α = 0.1% and (ν_1, ν_2) = (5,5) both the testimator(s) perform at their best. However for other values also the performance is satisfactory. So, it is recommended take smaller sample sizes, smaller level of significance for both positive and negative values of degrees of asymmetry. In particular $\hat{\sigma}^2_{DST2}$ may be preferred as it removes the arbitraryness in the choice of shrinkage factor. So, it can be mentioned that shrinkage testimators perform better under GELF.

Tables showing relative risk(s) of proposed testimator(s) with respect to the best available estimator.

Table : 5.9.1.1 **Relative Risk of** $\hat{\sigma}^2{}_{DST_1}$ α = 0.1% ,(ν_1, ν_2) = (5,5), p = -3

λ	k = 0.2	k = 0.4	k = 0.6	k = 0.8
0.20	0.129	0.179	0.192	0.149
0.40	0.598	0.669	0.564	0.337
0.60	2.371	1.989	1.272	0.596
0.80	4.35	4.172	2.479	0.936
1.00	7.854	6.174	5.214	1.365
1.20	5.518	4.561	4.974	2.173
1.40	4.134	4.009	3.873	2.43
1.60	3.264	3.496	3.612	1.966
1.80	2.854	2.903	2.845	1.426
2.00	2.03	1.868	1.981	1.237

Table : 5.9.1.2 **Relative Risk of** $\hat{\sigma}^2_{DST_1}$ α = 1% ,(ν_1, ν_2) = (5,5), p = -3

λ	k = 0.2	k = 0.4	k = 0.6	k = 0.8
0.20	0.149	0.19	0.196	0.153
0.40	0.596	0.646	0.551	0.346
0.60	2.024	1.756	1.198	0.611
0.80	5.209	3.658	2.122	0.94
1.00	6.541	4.929	3	1.301
1.20	4.996	4.615	3.434	1.65
1.40	3.588	3.75	3.372	1.93
1.60	2.712	2.998	3.049	2.104
1.80	2.176	2.464	2.69	2.176
2.00	1.823	2.085	2.367	2.167

Table : 5.9.2.1 **Relative Risk of** $\hat{\sigma}^2_{DST_2}$ α = 1% , (ν_1, ν_2) = (5,5)

λ	p = -3	p = -2.5	p = --2	p = -1.5	p = -1.0	P = 1	P = 1.5
0.20	0.265	0.263	0.263	0.246	0.182	0.172	0.282
0.40	0.874	0.71	0.594	0.481	0.326	0.279	0.426
0.60	2.409	1.917	1.528	1.199	0.839	0.647	0.828
0.80	4.943	4.708	4.041	3.324	2.612	1.629	1.6
1.00	5.67	6.903	7.338	7.522	5.091	4.169	2.647
1.20	4.464	5.371	5.886	6.642	6.524	5.442	2.541
1.40	3.3	3.597	3.627	3.772	3.394	2.829	1.547
1.60	2.529	2.548	2.399	2.327	3.175	1.43	0.899
1.80	2.041	1.942	1.745	1.617	1.972	0.844	0.566
2.00	1.715	1.563	1.364	1.226	1.401	0.561	0.387

Table : 5.9.2.2 **Relative Risk of** $\hat{\sigma}^2_{DST_2}$ α = 0.1% , (ν_1, ν_2) = (5,5)

λ	p = -3	p = -2.5	p = --2	p = -1.5	p = -1.0	P = 1	P = 1.5
0.20	0.235	0.234	0.236	0.226	0.176	0.174	0.278
0.40	0.858	0.697	0.6	0.52	0.409	0.379	0.507
0.60	2.857	2.144	1.711	1.415	1.146	0.9	0.977
0.80	3.38	5.283	3.631	4.47	3.602	2.143	1.984
1.00	6.92	6.46	6.591	6.884	6.081	4.099	3.357
1.20	5.342	5.642	5.793	4.171	6.213	3.131	2.613
1.40	4.746	4.677	4.464	3.557	3.007	1.588	1.373
1.60	3.506	3.577	2.72	2.12	1.749	0.887	0.768
1.80	2.351	2.554	1.902	1.458	1.179	0.562	0.482
2.00	1.661	1.982	1.453	1.099	0.876	0.392	0.33

Chapter – 6

SHRINKAGE TESTIMATION IN WEIBULL DISTRIBUTION

6.1 Introduction

Weibull distribution is a continuous distribution. It is named after Swedish physicist Walodi Weibull (1939). He used this distribution to model data from problems dealing with yield strength of Bofor's Steel, fibre strength of Indian Cotton etc. In the context of life testing and reliability estimation this model fits well for the situations with changing failure rates i.e. when the failure rates increasing or decreasing. The Weibull distribution interpolates between Exponential distribution when $\beta = 1$ and a Rayleish distribution when $\beta = 2$. As 'β' converges to infinity the Weibull distribution converges to Dirac Delta distribution. This distribution is very widely used in Survival Analysis, Reliability and Engineering and Industrial Engineering. The Weibull distribution is alos useful in describing wear out, fatigue failure, vaccum tube failures, ball bearing failures etc.

In weather forecasting to describe wind speed distributions it is extensively used as the shape parameter of this distribution matches with natural distribution. Also, in general insurance to model the size of re-insurance claims, this model is appropriate.

In hydrology the Weibull distribution is applied to extreme events such as annual maximum one-day rainfalls and river discharges.

THE MODEL:

When we assume that some power (say) p^{th} of the failure time is distributed Exponentially, we get the Weibull distribution whose pdf is given by

$$f(x;\,\theta,\beta) = \begin{cases} \left(\frac{\beta}{\theta}\right)\left(\frac{x}{\theta}\right)^{\beta-1}\exp\left(-\left(\frac{x}{\theta}\right)^{\beta}\right), & x > 0,\ \theta,\beta > 0 \\ 0 & , o.w. \end{cases}$$

_____(6.1.1)

The parameters θ and β are called the life and shape parameter respectively. This distribution is also useful in describing the wear out or fatigue failures. Cohen (1965), Harter and Moore (1965) have derived maximum likelihood estimators in Weibull distribution based on Complete and Censored samples. Bain and Antle (1967) have given estimators which are practical and based on Monte - Carlo methods. Mann and Fertig (1975) have derived simplified efficient point and interval estimator for Weibull distribution parameters, which are either maximum likelihood or best linear invariant.

Singh and Bhatkulikar (1978) have studied shrunken estimators in Weibull distribution, Pandey and Singh (1984) considered estimating the shape parameter of this distribution by shrinkage towards an interval.

Due to considerable handling of manufactured items or past information as in the case of wind speed data and in many other situations in life testing and reliability estimation where one may have an initial estimate of the shape parameter β in the form of either a guess β_0 or an interval (β_1, β_2) $(\beta_1 < \beta_2)$ in which β lies. We have proposed shrinkage testimators for the shape parameter both using point and interval guess. We have studied the risk properties of the proposed estimator using an asymmetric loss function.

In section 6.2 two shrinkage testimators for the shape parameter using point guess have been proposed and their risks are derived in section 6.3. In section 6.4 we have derived the relative risks of the proposed testimators with respect to $\hat{\beta}$ the best available estimator in the absence of any other information. Section 6.5 is devoted to the recommendations and conclusions of $\hat{\beta}_{ST_1}$ and $\hat{\beta}_{ST_2}$.

In section 6.6 deals with the shrinkage testimation of 'β' by shrinkage towards an interval. Section 6.7 is devoted to the derivation of the risk of the proposed testimator and its relative risk is derived in section 6.8. Section 6.9 gives the recommendations and conclusions of the proposed testimator.

6.2 SHRINKAGE TESTIMATOR USING POINT GUESS

Let x have the distribution

$$f(x;\ \theta,\beta) = \left(\frac{\beta}{\theta}\right)\left(\frac{x}{\theta}\right)^{\beta-1}\exp\left(-\left(\frac{x}{\theta}\right)^{\beta}\right), \qquad x > 0,\ \ \theta,\beta\ > 0$$

Suppose that a guess of β (say) β_0 is given and a random sample of size 'n' ($x_1, x_2, \ldots\ldots, x_n$) is available from this distribution and we are interested in constructing an estimator of β using the sample information and hopefully the guess value β_0. Let $x_1 \leq x_2 \ldots\ldots.. \leq x_r$ denote the r smallest ordered observations in a sample of size n from the Weibull distribution.

Then, the shrinkage Testimator of β (say) $\hat{\beta}_{ST_1}$ can be proposed as follows:

1. First test with a sample of size 'n' the null hypothesis $H_0 : \beta = \beta_0$ against the alternative $H_1 : \beta \neq \beta_0$ where β_0 is the point guess value of β.
2. If H_0 is accepted at α % level of significance i.e. $\chi_1^2 < \frac{2\,\mathrm{Tr}}{\beta_0} < \chi_2^2$, where $Tr = -\sum\left(x_{(i)} - x_{(r)}\right)$; N = $n\ k_{r,n}$ and χ_1^2 , χ_2^2 refer to critical points of

unbiased portioning of χ^2 distribution with 2N degrees of freedom, N = $n\,k_{r,n}$. Then use the conventional shrinkage estimator $\hat{\beta}_s = k\hat{\beta} + (1-k)\beta_0$ with shrinkage factor 'k' otherwise ignore β_0 and use an unbiased estimator $\hat{\beta}$ of β.

Estimators of this type with 'k' arbitrary and lying between '0' and '1' have been proposed by Singh and Bhatkulikar (1978) and have calculated relative efficiencies with respect to $\hat{\beta}$. In the present study we have studied the risk properties of shrinkage testimators of β using an asymmetric loss function.

We have proposed another testimator $\hat{\beta}_{ST_2}$ which removes the arbitrariness in the choice of a shrinkage factor for a given 'α'. Also, in all such studies we have considered a two sided alternative. i.e. $H_1 : \beta \neq \beta_0$ which appears more appropriate for shrinkage problems. It is to be noted that a UMP test of $H_0 : \beta = \beta_0$ against $H_1 : \beta \neq \beta_0$ does not exist. However, a UMPU test of an equivalent hypothesis H_0 : b = 1 against H_1 : b ≠ 1 exist and is to reject H_0 whenever 2Tr $< \chi_1^2$ and 2Tr $> \chi_2^2$. So, we define

$$\hat{\beta}_{ST_1} = \begin{cases} k\left(\frac{N-1}{Tr} - 1\right) + 1\,, & if\ H_0\ is\ accepted \\ \frac{N-1}{Tr}\,, & otherwise \end{cases} \qquad \text{\_\_\_\_(6.2.1)}$$

Where $0 \leq k \leq 1$. Again, we take $k = \frac{2\,Tr}{\chi^2}$ and define another shrinkage testimator $\hat{\beta}_{ST_2}$ of β as follows:

$$\hat{\beta}_{ST_2} = \begin{cases} \frac{2\,Tr}{\chi^2}\left(\frac{N-1}{Tr} - 1\right) + 1\,, & if\ H_0\ is\ accepted \\ \frac{N-1}{Tr}\,, & otherwise \end{cases} \qquad \text{\_\_\_\_(6.2.2)}$$

6.3 Risk of Testimators

In this section we derive the risk of these two testimators which are defined in the previous section.

6.3.1 Risk of $\hat{\boldsymbol{\beta}}_{ST_1}$

The risk of $\hat{\beta}_{ST_1}$ under L(Δ) is defined by

$R(\hat{\beta}_{ST_1}) = E[\hat{\beta}_{ST_1} | L(\Delta)]$

$$= E\left[\left(k\left(\frac{N-1}{Tr}-1\right)+1\right)/\frac{\chi_1^2}{2} < Tr < \frac{\chi_2^2}{2}\right] \cdot P\left[\frac{\chi_1^2}{2} < Tr < \frac{\chi_2^2}{2}\right]$$

$$+ E\left[\left(\frac{N-1}{Tr}\right)/Tr < \frac{\chi_1^2}{2} \cup Tr > \frac{\chi_2^2}{2}\right] \cdot P\left[Tr < \frac{\chi_1^2}{2} \cup Tr > \frac{\chi_2^2}{2}\right]$$

____(6.3.1.1)

$$= e^{-a} \int_{\frac{\chi_1^2}{2}}^{\frac{\chi_2^2}{2}} e^{a\left(\frac{k\left(\frac{N-1}{Tr}-1\right)+1}{\beta}\right)} f(Tr)dTr - a \int_{\frac{\chi_1^2}{2}}^{\frac{\chi_2^2}{2}} \left[\frac{k\left(\frac{N-1}{Tr}-1\right)+1}{\beta}-1\right] f(Tr)dTr$$

$$- \int_{\frac{\chi_1^2}{2}}^{\frac{\chi_2^2}{2}} f(Tr)dTr + e^{-a} \int_0^{\frac{\chi_1^2}{2}} e^{\left(\frac{N-1}{Tr}\right)} f(Tr)dTr + e^{-a} \int_{\frac{\chi_2^2}{2}}^{\infty} e^{\left(\frac{N-1}{Tr}\right)} f(Tr)dTr$$

$$-a \int_0^{\frac{\chi_1^2}{2}} \left(\frac{N-1}{Tr}-1\right) f(Tr)dTr - a \int_{\frac{\chi_2^2}{2}}^{\infty} \left(\frac{N-1}{Tr}-1\right) f(Tr)dTr - \int_0^{\frac{\chi_1^2}{2}} f(Tr)dTr - \int_{\frac{\chi_2^2}{2}}^{\infty} f(Tr)dTr$$

______(6.3.1.2)

$$f\,(Tr) = \frac{1}{b^N\,\Gamma(N)}\,e^{\frac{-Tr}{b}}\,(Tr)^{N-1}\,dTr \quad ;\, Tr > 0,\; b > 0$$

Straight forward integration of (6.3.1.2) gives

$$R(\hat{\beta}_{ST_1}) = \begin{bmatrix} I_1 - \left\{ I\left(\frac{\chi_2^2}{2b}, N-1\right) - I\left(\frac{\chi_1^2}{2b}, N-1\right) \right\} ak \\ + \left\{ I\left(\frac{\chi_2^2}{2b}, N\right) - I\left(\frac{\chi_1^2}{2b}, N\right) \right\} a(kb - b + 1) \\ - \left\{ I\left(\frac{\chi_2^2}{2b}, N\right) - I\left(\frac{\chi_1^2}{2b}, N\right) \right\} + \ I_2 + I_3 - \\ a\left\{ I\left(\frac{\chi_1^2}{2b}, N-1\right) - I\left(\frac{\chi_2^2}{2b}, N-1\right) + 1 \right\} \\ + a\left\{ I\left(\frac{\chi_1^2}{2b}, N\right) - I\left(\frac{\chi_2^2}{2b}, N\right) + 1 \right\} \\ - \left\{ I\left(\frac{\chi_1^2}{2b}, N\right) - I\left(\frac{\chi_2^2}{2b}, N\right) + 1 \right\} \end{bmatrix}$$

_____(6.3.1.3)

Where $I(x;p) \;=\; (1/\Gamma p)\int_0^x e^{-x}\,x^{p-1}\,dx$ refers to the standard incomplete gamma function and b = (1/β) and

$$I_1 \;=\; \frac{e^{a(b-1)}}{\Gamma(N)}\,e^{-abk}\int_{\frac{x_1^2}{2b}}^{\frac{x_2^2}{2b}} e^{\left[\frac{ak\,(N-1)}{t}\right]}\;e^{-t}\;(t)^{\,N-1}\;dt$$

$$I_2 \;=\; \frac{e^{-a}}{\Gamma(N)}\int_0^{\frac{x_1^2}{2b}} e^{\left[\frac{a\,(N-1)}{t}\right]}\;e^{-t}\;(t)^{\,N-1}\;dt$$

$$I_3 = \frac{e^{-a}}{\Gamma(N)} \int_{\frac{x_2^2}{2b}}^{\infty} e^{\left[\frac{a(N-1)}{t}\right]} e^{-t} (t)^{N-1} dt$$

6.3.2 Risk of $\widehat{\boldsymbol{\beta}}_{ST_2}$

The risk of $\hat{\beta}_{ST_2}$ under L(Δ) is defined by

$$R(\hat{\beta}_{ST_2}) = E[\hat{\beta}_{ST_2} | L(\Delta)]$$

$$= E\left[\left(\tfrac{2Tr}{\chi^2}\left(\tfrac{N-1}{Tr} - 1\right) + 1\right) / \tfrac{\chi_1^2}{2} < Tr < \tfrac{\chi_2^2}{2}\right] \cdot P\left[\tfrac{\chi_1^2}{2} < Tr < \tfrac{\chi_2^2}{2}\right]$$

$$+ E\left[\left(\tfrac{N-1}{Tr}\right) / Tr < \tfrac{\chi_1^2}{2} \cup Tr > \tfrac{\chi_2^2}{2}\right] \cdot P\left[Tr < \tfrac{\chi_1^2}{2} \cup Tr > \tfrac{\chi_2^2}{2}\right]$$

____(6.3.2.1)

$$= e^{-a} \int_{\frac{\chi_1^2}{2}}^{\frac{\chi_2^2}{2}} e^{a\left(\frac{\frac{2Tr}{\chi^2}\left(\frac{N-1}{Tr}-1\right)+1}{\beta}\right)} f(Tr)dTr - a \int_{\frac{\chi_1^2}{2}}^{\frac{\chi_2^2}{2}} \left[\frac{\frac{2Tr}{\chi^2}\left(\frac{N-1}{Tr}-1\right)+1}{\beta} - 1\right] f(Tr)dTr$$

$$- \int_{\frac{\chi_1^2}{2}}^{\frac{\chi_2^2}{2}} f(Tr)dTr + e^{-a} \int_{0}^{\frac{\chi_1^2}{2}} e^{\left(\frac{N-1}{Tr}\right)} f(Tr)dTr + e^{-a} \int_{\frac{\chi_2^2}{2}}^{\infty} e^{\left(\frac{N-1}{Tr}\right)} f(Tr)dTr$$

$$- a \int_{0}^{\frac{\chi_1^2}{2}} \left(\frac{N-1}{Tr} - 1\right) f(Tr)dTr - a \int_{\frac{\chi_2^2}{2}}^{\infty} \left(\frac{N-1}{Tr} - 1\right) f(Tr)dTr - \int_{0}^{\frac{\chi_1^2}{2}} f(Tr)dTr - \int_{\frac{\chi_2^2}{2}}^{\infty} f(Tr)dTr$$

______(6.3.2.2)

$$f(Tr) = \frac{1}{b^N \Gamma(N)} e^{\frac{-Tr}{b}} (Tr)^{N-1} dTr \quad ; Tr > 0,\ b > 0$$

Straight forward integration of (6.3.2.2) gives

$$R\left(\hat{\beta}_{ST_2}\right) = \left[\begin{array}{c} \dfrac{e^{a(b-1)}}{\left[1+\dfrac{2ab^2}{\chi^2}\right]^N} e^{\left(\frac{2ab\,(N-1)}{\chi^2}\right)} \left\{I\left(\dfrac{\chi_2^2}{2b}, N\right) - I\left(\dfrac{\chi_1^2}{2b}, N\right)\right\} \\ -\left\{I\left(\dfrac{\chi_2^2}{2b}, N\right) - I\left(\dfrac{\chi_1^2}{2b}, N\right)\right\} a\left(\dfrac{2b(N-1)}{\chi^2} + b - 1\right) \\ +\left\{I\left(\dfrac{\chi_2^2}{2b}, N+1\right) - I\left(\dfrac{\chi_1^2}{2b}, N+1\right)\right\}\left(\dfrac{2ab^2N}{\chi^2}\right) \\ -\left\{I\left(\dfrac{\chi_2^2}{2b}, N\right) - I\left(\dfrac{\chi_1^2}{2b}, N\right)\right\} + I_1 + I_2 \\ -a\left\{I\left(\dfrac{\chi_1^2}{2b}, N-1\right) - I\left(\dfrac{\chi_2^2}{2b}, N-1\right) + 1\right\} \\ + a\left\{I\left(\dfrac{\chi_1^2}{2b}, N\right) - I\left(\dfrac{\chi_2^2}{2b}, N\right) + 1\right\} \\ -\left\{I\left(\dfrac{\chi_1^2}{2b}, N\right) - I\left(\dfrac{\chi_2^2}{2b}, N\right) + 1\right\} \end{array}\right]$$

_____(6.3.2.3)

Where $I(x;p) = (1/\Gamma p)\int_0^x e^{-x}\, x^{p-1}\, dx$ refers to the standard incomplete gamma function and b = (1/β) ; N = $n\, k_{r,n}$ and

$$I_1 = \frac{e^{-a}}{\Gamma(N)} \int_0^{\frac{x_1^2}{2b}} e^{\left[\frac{a(N-1)}{t}\right]} e^{-t}\, (t)^{N-1}\, dt$$

$$I_2 = \frac{e^{-a}}{\Gamma(N)} \int_{\frac{x_2^2}{2b}}^{\infty} e^{\left[\frac{a(N-1)}{t}\right]} e^{-t}\, (t)^{N-1}\, dt$$

6.4 Relative Risk of $\hat{\boldsymbol{\beta}}_{ST_i}$

A natural way of comparing the risk of the proposed testimators, is to study its performance with respect to the best available estimator $\hat{\beta}$ in this case. For this purpose, we obtain the risk of $\hat{\beta}$ under $L_E(\hat{\beta},\beta)$ as:

$$R_E(\hat{\beta}) = E[\hat{\beta}|\, L\,(\hat{\beta},\beta)]$$

$$= e^{-a}\int_0^{\infty} e^{a\left(\frac{\hat{\beta}}{\beta}\right)} f(Tr)dTr - a\int_0^{\infty}\left(\frac{\hat{\beta}}{\beta}-1\right)f(Tr)dTr - \int_0^{\infty} f(Tr)dTr$$

____(6.4.1)

A straight forward integration of (6.4.1) gives

$$R_E(\hat{\beta}) = \frac{e^{-a}}{\Gamma(N)}\int_0^{\infty} e^{a\left(\frac{N-1}{t}\right)} e^{-t}\, t^{(N-1)}dt\ -1 \qquad \text{_____(6.4.2)}$$

Now, we define the Relative Risk of $\hat{\beta}_{ST_i}$; $i = 1{,}2$ with respect to $\hat{\beta}$ under L(Δ) as follows

$$RR_1 = \frac{R_E(\hat{\beta})}{R(\hat{\beta}_{ST_1})}$$

_____(6.4.3)

Using (6.4.2) and (6.3.1.3) the expression for RR_1 is given by (6.4.3). It is observed that RR_1 is a function of N, α, k, b and 'a'.

Again, we define the Relative Risk of $\hat{\beta}_{ST_2}$ by

$$RR_2 = \frac{R_E(\hat{\beta})}{R(\hat{\beta}_{ST_2})}$$

______(6.4.4)

The expression for RR_2 is given by (6.4.4) which can be obtained by using (6.4.2) and (6.3.2.3). Again we observed that RR_2 is a function of N, α, k, b and 'a'.

We have calculated N from the values of $k_{r,n}$ given in Bain (1972). For n = 10, 20, 30 the values of N are respectively.

r/n	0.2	0.3	0.4	0.5	0.6
N	1.0540	2.1720	3.3690	4.6670	6.0980
N	3.1666	5.4420	7.8880	10.5540	13.5120
N	5.2770	8.7120	12.4110	16.4460	20.9370

6.5 Recommendations for $\hat{\boldsymbol{\beta}}_{ST_i}$

In this section we wish to compare the performance of $\hat{\beta}_{ST_1}$ and $\hat{\beta}_{ST_2}$ with respect to the best available (unbiased) estimator of $\hat{\beta}$.

6.5.1 Recommendations for $\hat{\boldsymbol{\beta}}_{ST_1}$

The risk of $\hat{\beta}_{ST_1}$ with respect to $\hat{\beta}$ the best available estimator of β is a function of 'k', a, b, 'α', N (i.e. n and $k_{r,n}$). For different values of r/n these values are given by Bain (1972). Now we have taken k = 0.2 (0.2) 0.8, a = -0.5, -0.75, -1.0, -1.25, -1.50 and b = 0.2 (0.2) 1.8, N is given in table and α = 1% and 5%.

For a = -1.5, it performs better for the whole range of 'k' and 'b' considered here. The values of RR_1 are better for almost all the values of N obtained for different censoring fractions however as N increases i.e. 'r/n' increases there. We observe that RR_1 values are still better. Implying that one can take larger censoring factions. However for other values of 'a' up to – 0.5 $\hat{\beta}_{ST_1}$ performs better than $\hat{\beta}$ but the magnitude of RR_1 values decrease slightly (but greater than unity).

When we change 'α' to 5% the similar kind of pattern of RR_1 values is observed as observed for α =1% . But the magnitude of relative risk values lowers down but still greater than unity for the whole range of shrinkage factor (i.e. $0.2 \leq k \leq 0.8$) and the whole range of 'b' i.e. b = 0.2 (0.2) 1.8. Again changing the censoring fraction to some higher values, it is observed that the magnitude of RR_1 increases with larger values of 'N' i.e. again it may be suggested that a larger censoring faction can be considered.

We have also tried some positive values of 'a' the degree of asymmetry but it is observed that RR_1 values are not good in such situations indicating that the asymmetric loss function is more useful for those situations where under-estimation is more serious.

6.5.2 Recommendations for $\hat{\boldsymbol{\beta}}_{ST_2}$

It is observed that RR_2 is a function of 'a', 'b', 'α' and N (i.e. n and $k_{r,n}$) we have considered a= -0.5, -0.75, -1.0, -1.25, -1.5 b= 0.2 (0.2) 1.8, N is again tabulated for different values of $^{r}/_{n}$ in table and α = 1% and α = 5%. There will be several tables for RR_2 values for the above values considered. We have assembled some of the tables at the end of the chapter. However our recommendations based on all the computations are as follows.

1. $\hat{\beta}_{ST_2}$ performs better than $\hat{\beta}$ for almost all the values considered as above. However for a = - 1.5 and α = 1% its performance is at its best for the whole range of 'b'.

2. When the value of α = 5% still the performance of $\hat{\beta}_{ST_2}$ is good though there is a slight decrease to in the values of RR_2.

3. As N increases (i.e. r/n increases) implying that a higher censoring fraction is admissible the RR_2 values increase indicating a better control over the risk of $\hat{\beta}_{ST_2}$. This is in contrast to the behaviour of MSE of $\hat{\beta}$ under the 'SELF'. As the recommendations with MSE criterion is to use small censoring fraction.

4. As the degree of asymmetry becomes positive the RR_2 values are not good in the sense that they are less in magnitude even lesser than unity. So, it is suggested that only negative degree(s) of asymmetries be considered.

5. RR_2 values are higher in magnitude compared to RR_1.

CONCLUSIONS:

We have proposed two shrinkage testimators $\hat{\beta}_{ST_1}$ and $\hat{\beta}_{ST_2}$ for the shape parameter β . It is suggested to use $\hat{\beta}_{ST_1}$ as it (k = k) performs better than the shrinkage factor dependent on test statistics i.e. $\hat{\beta}_{ST_2}$. In particular smaller level of significance i.e. α =1% coupled with proper censoring fraction is recommended for various degrees of asymmetry. The best performance is achieved at a = -1.5.

Tables showing relative risk(s) of proposed testimator(s) with respect to the best available estimator.

Table : 6.5.1.1 **Relative Risk of $\hat{\boldsymbol{\beta}}_{ST_1}$** α = 1% , n = 10, k = 0.2

b	a = -0.5	a = -0.75	a = -1	a = -1.25	a = -1.5
0.2	1.016	1.048	1.076	1.101	1.126
0.4	1.099	1.176	1.253	1.333	1.416
0.6	1.174	1.297	1.427	1.569	1.724
0.8	1.233	1.395	1.573	1.771	1.997
1.0	1.272	1.459	1.666	1.902	2.175
1.2	1.287	1.48	1.695	1.94	2.223
1.4	1.278	1.461	1.663	1.891	2.151
1.6	1.249	1.411	1.586	1.783	2.004
1.8	1.205	1.339	1.484	1.646	1.828

Table : 6.5.1.2 **Relative Risk of $\hat{\boldsymbol{\beta}}_{ST_1}$** α = 1% , n = 10, k = 0.4

b	a = -0.5	a = -0.75	a = -1	a = -1.25	a = -1.5
0.2	1.009	1.037	1.06	1.081	1.101
0.4	1.073	1.135	1.195	1.254	1.315
0.6	1.134	1.231	1.329	1.433	1.543
0.8	1.187	1.316	1.453	1.602	1.766
1.0	1.229	1.385	1.554	1.743	1.956
1.2	1.257	1.431	1.623	1.838	2.085
1.4	1.271	1.452	1.652	1.878	2.136
1.6	1.271	1.449	1.644	1.864	2.112
1.8	1.258	1.425	1.606	1.807	2.032

Table : 6.5.1.3 **Relative Risk of $\hat{\boldsymbol{\beta}}_{ST_1}$** α = 5% , n = 10, k = 0.2

b	a = -0.5	a = -0.75	a = -1	a = -1.25	a = -1.5
0.2	1.026	1.062	1.094	1.124	1.153
0.4	1.095	1.173	1.25	1.329	1.414
0.6	1.158	1.276	1.402	1.538	1.689
0.8	1.205	1.355	1.52	1.705	1.916
1.0	1.23	1.397	1.583	1.793	2.033
1.2	1.232	1.4	1.584	1.791	2.025
1.4	1.214	1.368	1.535	1.719	1.923
1.6	1.181	1.313	1.455	1.608	1.776
1.8	1.138	1.246	1.361	1.485	1.621

Table : 6.5.1.4 **Relative Risk of** $\hat{\beta}_{ST_2}$ α = 5% , n = 20

b	a = -0.5	a = -0.75	a = -1	a = -1.25	a = -1.5
0.2	0. 633	0.594	0.558	0.526	0.497
0.4	0.859	0.834	0.808	0.783	0.76
0.6	1.438	1.474	1.489	1.495	1.496
0.8	2.497	2.744	2.935	3.094	3.234
1.0	3.729	4.3	4.782	5.212	5.611
1.2	3.597	3.91	4.12	4.279	4.411
1.4	2.445	2.456	2.444	2.433	2.429
1.6	1.574	1.525	1.487	1.464	1.453
1.8	1.078	1.033	1.005	0.991	0.986

Table : 6.5.2.1 **Relative Risk of** $\hat{\beta}_{ST_2}$ α = 1% , n = 10

b	a = -0.5	a = -0.75	a = -1	a = -1.25	a = -1.5
0.2	1.02	1.053	1.082	1.11	1.137
0.4	1.108	1.192	1.276	1.363	1.455
0.6	1.181	1.31	1.446	1.593	1.757
0.8	1.23	1.39	1.563	1.756	1.974
1.0	1.25	1.422	1.609	1.818	2.053
1.2	1.235	1.4	1.576	1.768	1.977
1.4	1.188	1.329	1.472	1.617	1.758
1.6	1.113	1.218	1.31	1.378	1.389
1.8	1.018	1.077	1.099	1.033	0.71

Table : 6.5.2.2 **Relative Risk of** $\hat{\beta}_{ST_2}$ α = 5% , n = 10

b	a = -0.5	a = -0.75	a = -1	a = -1.25	a = -1.5
0.2	1.034	1.074	1.109	1.143	1.177
0.4	1.106	1.188	1.27	1.356	1.446
0.6	1.159	1.275	1.397	1.529	1.674
0.8	1.18	1.312	1.454	1.609	1.778
1.0	1.158	1.286	1.42	1.56	1.706
1.2	1.095	1.198	1.295	1.38	1.443
1.4	1	1.061	1.094	1.074	0.948
1.6	0.883	0.887	0.817	0.58	0.515
1.8	0.755	0.679	0.422	0.44	0.416

6.6 Shrinkage Testimator of β by Shrinkage towards an Interval

Suppose 'n' items are put to life test and the experiment is continued until 'r' failures are observed. Let these failure times be $x_1, x_2 ...., x_r$ and suppose further that they follow Weibull distribution. Then following Thompson (1968,b) a shrinkage of β can be defined as

$$\hat{\beta}_{ST_1} = k\left(\frac{h-2}{t}\right) + (1-k)\left(\frac{\beta_1+\beta_2}{2}\right)$$

which was proposed by Pandey and Singh (1984). The properties of this estimator were studied by minimizing the Mean Square Error.

We propose another shrunken estimator of β as follows:

$$\hat{\beta}_{ST_3} = \begin{cases} \beta_1 & , \ if \ \ t > \dfrac{h-2}{\beta_1} \\ k\left(\dfrac{h-2}{t}\right) + (1-k)\left(\dfrac{\beta_1+\beta_2}{2}\right) & , if \ \ \dfrac{h-2}{\beta_2} \le t \le \dfrac{h-2}{\beta_1} \\ \beta_2 & , \ if \ \ t < \dfrac{h-2}{\beta_2} \end{cases}$$

Where (β_1, β_2) $(\beta_2 < \beta_1)$ is the guess interval in which β is supposed to lie. So, here we have proposed a shrinkage testimator for the shape parameter of Weibull distribution by shrinking the guess towards an interval. The shrinkage factor 'k' lies between '0' and '1'. We have derived the risk of $\hat{\beta}_{ST_3}$ under an asymmetric loss function. L(Δ) which is defined and discussed earlier.

As Weibull distribution finds applications in many real life problems. This estimation procedure could find its place whenever the guess value of β lies in some interval. One these could be estimation of pollutants in water (say) the arsenic contents in the water lies in some interval $(\beta_1,\ \beta_2)$ and then utilizing this guess interval a better estimate of β can be proposed along the lines discussed in the chapter. The use of asymmetric loss will facilitate a proper control over the 'risk' by choosing the degrees of asymmetry appropriately.

6.7 Risk of Testimators

The risk of $\hat{\beta}_{ST_3}$ under $L_E(\hat{\beta},\beta)$ is defined by

$$R(\hat{\beta}_{ST_3}) = E[\,\hat{\beta}_{ST_3} \mid L_E(\hat{\beta},\beta)]$$

$$\begin{aligned}
&= E\left[\beta_1 \middle/ t > \frac{h-2}{\beta_1}\right]\cdot p\left[t > \frac{h-2}{\beta_1}\right] \\
&\quad + E\left[k\left(\frac{h-2}{t}\right) + (1-k)\left(\frac{\beta_1+\beta_2}{2}\right) \middle/ \frac{h-2}{\beta_2} \le t \le \frac{h-2}{\beta_1}\right]\cdot p\left[\frac{h-2}{\beta_2} \le t \le \frac{h-2}{\beta_1}\right] \\
&\quad + E\left[\beta_2 \middle/ t < \frac{h-2}{\beta_2}\right]\cdot p\left[t < \frac{h-2}{\beta_2}\right]
\end{aligned}$$

______(6.7.1)

$$\begin{aligned}
= \; & e^{-a}\int_{\frac{h-2}{\chi_1^2}}^{\infty} e^{a\left(\frac{\beta_1}{\beta}\right)} f(t)dt - a\int_{\frac{h-2}{\chi_1^2}}^{\infty}\left[\frac{\beta_1}{\beta}-1\right] f(t)dt - \int_{\frac{h-2}{\chi_1^2}}^{\infty} f(t)dt \\
& e^{-a}\int_{\frac{h-2}{\chi_1^2}}^{\frac{h-2}{\chi_2^2}} e^{a\left(\frac{k\left(\frac{h-2}{t}\right)+(1-k)\left(\frac{\beta_1+\beta_2}{2}\right)}{\beta}\right)} f(t)dt - a\int_{\frac{h-2}{\chi_1^2}}^{\frac{h-2}{\chi_2^2}}\left[\frac{k\left(\frac{h-2}{t}\right)+(1-k)\left(\frac{\beta_1+\beta_2}{2}\right)}{\beta}-1\right] f(t)dt \\
& -\int_{\frac{h-2}{\chi_1^2}}^{\frac{h-2}{\chi_2^2}} f(t)dt + e^{-a}\int_{0}^{\frac{h-2}{\chi_2^2}} e^{a\left(\frac{\beta_2}{\beta}\right)} f(t)dt - a\int_{0}^{\frac{h-2}{\chi_2^2}}\left[\frac{\beta_2}{\beta}-1\right] f(t)dt - \int_{0}^{\frac{h-2}{\chi_2^2}} f(t)dt
\end{aligned}$$

______(6.7.2)

Where $f(t) = \frac{\beta^{\left(\frac{h}{2}\right)} t^{\left(\frac{h}{2}-1\right)}}{2^{\left(\frac{h}{2}\right)} \Gamma\left(\frac{h}{2}\right)} e^{\left(-\frac{\beta t}{2}\right)} dt \ , \ t > 0$

Straight forward integration of (6.7.2) gives

$$R\left(\hat{\beta}_{ST_3}\right) = \left[\begin{array}{c} \left\{1 - I\left(\frac{h-2}{\chi_1^2}, \frac{h}{2}\right)\right\}\left[e^{a\left(\frac{\beta_1}{\beta}-1\right)} - a\left(\frac{\beta_1}{\beta} - 1\right) - 1\right] \\ + \left\{I\left(\frac{h-2}{\chi_2^2}, \frac{h}{2}\right)\right\}\left[e^{a\left(\frac{\beta_2}{\beta}-1\right)} - a\left(\frac{\beta_2}{\beta} - 1\right) - 1\right] \\ + \left\{I\left(\frac{h-2}{\chi_1^2}, \frac{h}{2}\right) - I\left(\frac{h-2}{\chi_2^2}, \frac{h}{2}\right)\right\} \\ \left[\begin{array}{c} e^{a\left(k\phi + (1-k)\left(\frac{\beta_1+\beta_2}{2}\right)-1\right)} - \\ a\left(k\phi + (1-k)\left(\frac{\beta_1+\beta_2}{2}\right) - 1\right) - 1 \end{array} \right] \end{array} \right]$$

_____(6.7.3)

Where $I(x;p) = (1/\Gamma p) \int_0^x e^{-x} x^{p-1} dx$ refers to the standard incomplete gamma function and $\phi = \frac{\hat{\beta}}{\beta}$; $h = \frac{2}{Var\left(\hat{b}_s/_b\right)}$; $\hat{\beta} = \frac{h-2}{t}$; $\hat{b}_s = \frac{\sum \left(x_{(i)} - x_{(r)}\right)}{n \ k_{r,n}}$

6.8 Relative Risk of $\hat{\beta}_{ST_3}$

A natural way to compare the performance of $\hat{\beta}_{ST_3}$ is to compare its performance with respect to the unbiased estimator $\hat{\beta}$. For this we obtain the risk of $\hat{\beta}$ under L(Δ) and the risk of $\hat{\beta}_{ST_3}$ has already been obtained in the previous section. Now the risk of $\hat{\beta}$ is defined as

$$R_E(\hat{\beta}) = E\left[\hat{\beta} | \ L\left(\hat{\beta}, \beta\right)\right]$$

$$= e^{-a} \int_0^\infty e^{a\left(\frac{\hat{\beta}}{\beta}\right)} f(Tr)dTr - a\int_0^\infty \left(\frac{\hat{\beta}}{\beta} - 1\right) f(Tr)dTr - \int_0^\infty f(Tr)dTr$$

____(6.8.1)

A straight forward integration of (6.8.1) gives

$$R_E(\hat{\beta}) = \frac{e^{-a}}{\Gamma(N)} \int_0^\infty e^{a\left(\frac{N-1}{t}\right)} e^{-t} t^{(N-1)} dt - 1$$ _____(6.8.2)

Now, we define the Relative Risk of $\hat{\beta}_{ST_3}$ with respect to $\hat{\beta}$ under L(Δ) as follows

$$RR_3 = \frac{R_E(\hat{\beta})}{R(\hat{\beta}_{ST_3})}$$ _____(6.8.3)

The expression for RR_3 is given by (6.8.3) which can be obtained by using (6.8.2) and (6.7.3). It is observed that RR_3 expression is a function of 'k', β_1, β_2, $\emptyset$, a, n and α. In order to study its behaviour numerically we have taken k = 0.2 (0.2) 1.0, β_1, β_2, $\emptyset$ are taken as $0.2 \leq \beta_1 \leq 1.4$, $0.4 \leq \beta_2 \leq 1.6$ and $0.1 \leq \emptyset \leq 1.3$, a = ± 3 to ± 1 and α = 1% , 5%. There will be several tables of RR_3 for the above values considered. Some of these tables have been assembled at the end of the chapter. However our recommendations based on all these tables are summarized as follows.

6.9 Recommendations for $\hat{\beta}_{ST_3}$

1. $\hat{\beta}_{ST_3}$ dominates $\hat{\beta}$ for all the positive and negative values of 'a' i.e., it behaves well in both over / under estimation situations. However its performance is best for a = -3 and a = +1. These values are observed for the

almost the whole range of ‘k’ i.e. $0.2 \leq k \leq 0.8$ and $0.1 \leq \emptyset \leq 1.3$. The magnitude of RR_3 values are higher for n = 5.

2. As the level of significance is changed to 5% still $\hat{\beta}_{ST_3}$ performs better but now the RR_3 values are slightly lower than those obtained for α = 1%. Suggesting a lower level of significance should be preferred.
3. $\hat{\beta}_{ST_3}$ performs better than $\hat{\beta}$ for $0.3 \leq \frac{\beta_1+\beta_2}{2} \leq 1.5$ for all the values of ‘a’ considered here. However the range in this case is slightly increased as compared to the performance of $\hat{\beta}_{ST_3}$ under Minimum Mean Square Error (MMSE) criterion. (Ref: Pandey and Singh (1984)) where they have reported the range from (0.6 to 1.5). So the use of an asymmetric loss function improves the effective range where the proposed testimator performs better than the usual one.
4. The performance of $\hat{\beta}_{ST_3}$ is not good for higher values of ‘n’ and higher level of significance.

CONCLUSION:

A shrinkage testimator for the shape parameter of Weibull distribution is proposed and its risk properties are studied under an asymmetric loss function. It has been observed that the proposed testimator dominates the usual unbiased estimator for fairly large range of departures of parameter(s). The proposed testimator is useful for larger negative values of degrees of asymmetry in particular a = -3 and different positive values of ‘a’ in particular a = +1. A lower sample size and smaller level of significance report the best performance of $\hat{\beta}_{ST_3}$.

Tables showing relative risk(s) of proposed testimator(s) with respect to the best available estimator.

Table : 6.9.1 **Relative Risk of $\hat{\boldsymbol{\beta}}_{ST_3}$** α = 1% , n = 5, k = 0.2

φ	a = -1	a = -2	a = -3	a = 1	a = 2	a = 3
0.1	1.333	1.403	1.493	1.243	1.216	1.196
0.3	1.438	1.507	1.593	1.341	1.308	1.283
0.5	1.66	1.733	1.819	1.549	1.507	1.473
0.7	2.436	2.532	2.639	2.275	2.207	2.147
0.9	3.923	3.858	3.913	3.388	2.772	2.249
1.1	2.245	2.253	2.261	2.228	2.219	2.21
1.3	1.566	1.583	1.599	1.527	1.506	1.482

Table : 6.9.2 **Relative Risk of $\hat{\boldsymbol{\beta}}_{ST_3}$** α = 1% , n = 7, k = 0.2

φ	a = -1	a = -2	a = -3	a = 1	a = 2	a = 3
0.1	1.315	1.381	1.467	1.23	1.204	1.186
0.3	1.413	1.478	1.559	1.321	1.29	1.267
0.5	1.62	1.688	1.77	1.515	1.476	1.443
0.7	2.331	2.42	2.521	2.18	2.117	2.061
0.9	3.363	3.549	3.747	3.022	2.867	2.72
1.1	2.258	2.266	2.274	2.241	2.233	2.224
1.3	1.58	1.597	1.612	1.543	1.522	1.499

Table : 6.9.3 **Relative Risk of $\hat{\boldsymbol{\beta}}_{ST_3}$** α = 5% , n = 5, k = 0.2

ϕ	**a = -1**	**a = -2**	**a = -3**	**a = 1**	**a = 2**	**a = 3**
0.1	1.319	1.386	1.472	1.233	1.206	1.188
0.3	1.418	1.484	1.566	1.325	1.294	1.27
0.5	1.628	1.698	1.78	1.522	1.482	1.449
0.7	2.352	2.443	2.545	2.199	2.136	2.079
0.9	2.986	2.799	3.164	2.868	2.499	2.548
1.1	2.255	2.263	2.271	2.238	2.23	2.221
1.3	1.577	1.594	1.609	1.539	1.518	1.496

Table : 6.9.4 **Relative Risk of $\hat{\boldsymbol{\beta}}_{ST_3}$** α = 5% , n = 7, k = 0.2

ϕ	**a = -1**	**a = -2**	**a = -3**	**a = 1**	**a = 2**	**a = 3**
0.1	1.315	1.381	1.467	1.23	1.204	1.185
0.3	1.413	1.478	1.559	1.321	1.29	1.266
0.5	1.62	1.688	1.769	1.515	1.476	1.443
0.7	2.33	2.419	2.52	2.179	2.116	2.06
0.9	2.701	2.851	3.01	2.428	2.303	2.185
1.1	2.258	2.266	2.274	3.242	2.233	2.224
1.3	1.58	1.597	1.612	2.543	1.522	1.5

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