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GENERALIZED KONHAUSER MATRIX POLYNOMIAL AND ITS PROPERTIES

RESHMA SANJHIRA AND B. I. DAVE
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Abstract. We propose a generalized Konhauser matrix polynomial and obtain its properties such as the differential equation, inverse series relation and certain generating function relations involving Mittag-Leffler matrix function.

1. Introduction and Notations

Many of the Special Functions and most of their properties can be derived from the theory of Group representations [12]. Their matrix analogues often occur in Statistics, Number theory and in Lie Group theory [1, 5, 11]. In [6, 7, 10], are studied matrix differential equations and Frobenius method for the Laguerre, Hermite and Gegenbauer matrix polynomials. Interestingly in [10] is studied the quadrature matrix integration process with the help of matrix Laguerre polynomial. It is well known that the Konhauser polynomial

$$Z_m^\alpha(x;r) = \frac{\Gamma(rm+\alpha+1)}{\Gamma(m+1)} \sum_{n=0}^{m} (-1)^n \binom{m}{n} \frac{x^{rn}}{\Gamma(rn+\alpha+1)}, \quad (\Re(\alpha) > -1)$$

is the biorthogonal polynomial for the distribution function of the Laguerre polynomial [14]. This can also be viewed as a generalization of the Laguerre polynomial. In 2014, the above Konhauser polynomial $Z_m^\alpha(x;r)$ was further generalized by Prajapati, Ajudia and Agarwal in the form [13, Eq.(5), p.640]:

$$L_{[\frac{m}{q}]}^{(\alpha,\beta)}(z) = \frac{\Gamma(\alpha m+\beta+1)}{m!} \sum_{n=0}^{[\frac{m}{q}]} \frac{(-m)_{qn}}{\Gamma(\alpha n+\beta+1)} \frac{z^n}{n!}, \tag{1.1}$$

where $\alpha, \beta \in \mathbb{C}, m, q \in \mathbb{N}, \Re(\beta) > -1$ and $[\frac{m}{q}]$ denotes the integral part of $\frac{m}{q}$. Here, we define a matrix analogue of this polynomial and derive certain properties of it. In what follows, the following definitions and notations will be used. Throughout, we shall let A to be a matrix in $C^{p\times p}$ and $\sigma(A)$ to be the set of all eigenvalues of A. The matrix A is said to be positive stable matrix if $\Re(\lambda) > 0$ for all $\lambda \in \sigma(A)$. If $A_0, A_1, A_2, ...A_n$ are elements of $C^{p\times p}$ and $A_n \neq 0$ then

$$P_n(x) = A_n x^n + A_{n-1}x^{n-1} + A_{n-2}x^{n-2} + ... + A_1 x + A_0$$

is a matrix polynomial of degree n in x.

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The 2-norm of the matrix A, denoted by $\|A\|$, is defined by

$$\| A \| = \sup_{x\neq 0} \frac{\| Ax \|_2}{\| x \|_2} = max\{\sqrt{\lambda} : \lambda \in \sigma(A^*A)\},$$

where for a vector $x \in \mathbb{C}^p$, $\|x\|_2 = \left(x^T x\right)^{1/2}$ is Euclidean norm of x, and A^* denotes the transposed conjugate of A.

If $f(z)$ and $g(z)$ are holomorphic functions of a complex variable z which are defined on an open set Ω of the complex plane and if $\sigma(A) \subset \Omega$, then from the properties of the matrix functional calculus [3] it follows that

$$f(A)g(A) = g(A)f(A).$$

The reciprocal gamma function denoted by $\Gamma^{-1}(z) = [\Gamma(z)]^{-1} = \frac{1}{\Gamma(z)}$ is an entire function of complex variable z [4, p. 253] and thus for any matrix A in $C^{n\times n}$, the functional calculus [3] shows that $\Gamma^{-1}(A)$ is a well defined matrix function. If I denotes identity matrix of order p and $A + nI$ is invertible for every integer $n \geq 0$ then [8, Eq. (6) and (7), p.206]

$$(A)_n = \Gamma(A + nI)\Gamma^{-1}(A).$$

For positive stable matrices $C, D \in C^{p\times p}$, the Beta matrix function is denoted and defined by [8, Eq.(9), p.207] (also [9])

$$B(C, D) = \int_0^1 t^{C-I}(1 - t)^{D-I}dt. \tag{1.2}$$

Further, if $CD = DC$ and if $C + nI$, $D + nI$ and $C + D + nI$ are invertible for all nonnegative integers n then [8, Theorem 2, p. 209]

$$B(C, D) = \Gamma(C)\Gamma(D)\Gamma^{-1}(C + D). \tag{1.3}$$

For $A(k, n), B(k, n) \in C^{p\times p}$, $n, k \geq 0$ and $m \in \mathbb{N}$, there holds the double series identities (cf. [16, Eq.(1.7), p.606])

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]} B(k, n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k, n + mk) \tag{1.4}$$

and (cf. [2, Eq.(8), p.324])

$$\sum_{i=0}^{mn}\sum_{j=0}^{[i/m]} B(i, j) = \sum_{j=0}^{n}\sum_{i=0}^{mn-mj} B(i + mj, j). \tag{1.5}$$

For any matrix A in $C^{p\times p}$ and for $|x| < 1$, the following series expansion holds [8].

$$(1 - x)^{-A} = \sum_{n=0}^{\infty} \frac{(A)_n}{n!} x^n.$$

Also, we have the formula [16, Eq.(2.23), p.616]

$$(A)_{mk} = m^{mk} \prod_{i=1}^{m} \left(\frac{A + (i - 1)I}{m}\right)_k = \Delta(m; A). \tag{1.6}$$

In particular, for non negative integer n,

$$(-nI)_{mk} = (-1)^{mk} \frac{n!}{(n - mk)!} I = m^{mk} \prod_{i=1}^{m} \left(\frac{-n + i - 1}{m}\, I\right)_k. \tag{1.7}$$

We shall denote the zero matrix by O.

2. Generalized Konhauser matrix polynomial

We propose the extension of (1.1) as follows.

Definition 2.1. For the matrix A in $C^{p\times p}$

$$Z_{m^*}^{(A,\lambda)}(x^k;r) = \frac{\Gamma(A+rmI+I)}{m!}\sum_{n=0}^{\lfloor m/s\rfloor}(-mI)_{sn}\Gamma^{-1}(A+rnI+I)\frac{(\lambda x^k)^n}{n!}, \quad (2.1)$$

where $r,\lambda,\mu\in\mathbb{C}$, $k\in\mathbb{R}_{>0}$, $s\in\mathbb{N}$, $m\in\mathbb{N}\cup\{0\}$, $\Re(\lambda)>0$, $\Re(\mu)>-1$ for all eigen values $\mu\in\sigma(A)$ and the floor function $\lfloor u\rfloor = floor\ u$, represents the greatest integer $\leq u$.

It may be seen that when $r=k\in\mathbb{N}$ and $s=1$, this polynomial reduces to

$$Z_m^{(A,\lambda)}(x;k) = \Gamma(kmI+A+I)\sum_{n=0}^{m}\frac{(-1)^n(\lambda x)^{nk}}{(m-n)!n!}\ \Gamma^{-1}(knI+A+I)$$

studied by Varma, Çekim, and Taşdelen [18]. Further, if $k=1$ then this reduces to the Laguerre matrix polynomial [6]:

$$L_m^{(A,\lambda)}(x) = \sum_{n=0}^{m}\frac{(-1)^n}{n!(m-n)!}\ (A+I)_m\ [(A+I)_n]^{-1}\ (\lambda x)^n.$$

For the polynomial (2.1), we derive the differential equation and inverse series relation. Also, we show the relation of (2.1) with Mittag-Leffler matrix function which will be used in the generating function relations derived here. At last, the Euler(Beta) matrix transform is applied on this polynomial.

3. Differential Equations

If $\{A_i; i=1,2,\ldots,p\}$ and $\{B_j; j=1,2,\ldots,q\}$ are matrices in $C^{n\times n}$ and B_j+nI are invertible for all $n=0,1,2,\ldots$, then it is known that the generalized hypergeometric matrix function [16, Eq. (2.2), p. 608]:

$$\begin{aligned}&{}_pF_q(A_1,A_2,\ldots,A_p;B_1,B_2,\ldots,B_q;z)\\ &= \sum_{k=0}^{\infty}(A_1)_k(A_2)_k\cdots(A_p)_k\ [(B_1)_k]^{-1}[(B_2)_k]^{-1}\cdots[(B_q)_k]^{-1}\frac{z^k}{k!}\end{aligned} \quad (3.1)$$

satisfies the matrix differential equation [16, Eq. (2.10), p. 610]:

$$\left[\theta\prod_{j=1}^{q}(\theta I+B_j-I) - z\prod_{i=1}^{p}(\theta I+A_i)\right]\ {}_pF_q(z) = O, \quad (3.2)$$

where $\theta = zd/dz$ and O is the zero matrix of order n. Here, if we express the polynomial (2.1) in ${}_pF_q$ form then the equation (3.2) will readily yield the differential equation corresponding to the polynomial (2.1). In fact, assuming that the matrices occurring here commute with one another, we have, for $r,\ s\in\mathbb{N}$,

$$\begin{aligned}Z_{m^*}^{(A,\lambda)}(x^k;r) &= \frac{\Gamma(A+rmI+I)}{m!}\Gamma^{-1}(A+I)\sum_{n=0}^{\lfloor m/s\rfloor}\frac{(-mI)_{sn}(A+I)_{rn}^{-1}\ (\lambda x^k)^n}{n!}\\ &= \frac{\Gamma(A+rmI+I)}{m!}\Gamma^{-1}(A+I)\sum_{n=0}^{\lfloor m/s\rfloor}\left\{\prod_{i=1}^{s}\left(\frac{-m+i-1}{s}I\right)_n\right\}\end{aligned}$$

$$\times\left\{\prod_{j=1}^{r}\left(\frac{A+jI}{r}\right)_n^{-1}\right\}\frac{1}{n!}\left(\frac{\lambda x^k s^s}{r^r}\right)^n.$$

Hence, in (3.1), setting $p=s,\ q=r,\ A_i=(-m+i-1)I/s,\ B_j=(A+jI)/r,\ z=\lambda s^s x^k/r^r$, the equation immediately leads us to the differential equation for (2.1) of order $max.\{r+1,\ s\}$. This is stated in

Theorem 3.1. *If $r,s\in\mathbb{N}$ and the operator Θ is defined by $\Theta f(x)=\frac{x}{k}\frac{d}{dx}f(x)$ then $U=Z_{m^*}^{(A,\lambda)}(x^k;r)$ satisfies the equation*

$$\left[\left\{\Theta\prod_{j=1}^{r}\left(\Theta\ I+\frac{A+jI}{r}-I\right)\right\}\right.$$
$$\left.-\left(\frac{s^s}{r^r}\right)\ \lambda\ x^k\ \left\{\prod_{i=1}^{s}\left(\Theta\ I+\frac{-m+i-1}{s}I\right)\right\}\right]U=O.$$

4. Inverse series relations

For deriving the inverse series of the matrix polynomial (2.1), the following lemma will be used.

Lemma 4.1. *If $\{P_n\}$ and $\{Q_n\}$ are finite sequences of matrices in $C^{n\times n}$, then*

$$Q_n=\sum_{j=0}^{n}\frac{(-nI)_j}{j!}\ P_j\ \Leftrightarrow\ P_n=\sum_{j=0}^{n}\frac{(-nI)_j}{j!}\ Q_j.$$

Proof. Let us denote the right hand side of second series by T_n, then

$$\begin{aligned}T_n=\sum_{k=0}^{n}\frac{(-nI)_k}{k!}\ Q_k\ &=\ \sum_{k=0}^{n}\frac{(-1)^k n!}{k!\ (n-k)!}\ I\sum_{j=0}^{k}\frac{(-kI)_j}{j!}\ P_j\\&=\ \sum_{k=0}^{n}\frac{(-1)^k n!}{k!\ (n-k)!}I\ \sum_{j=0}^{k}\frac{(-1)^j\ k!}{j!\ (k-j)!}\ P_j\\&=\ \sum_{j=0}^{n}\binom{n}{j}\sum_{k=0}^{n-j}(-1)^k\ \binom{n-j}{k}\ P_j\\&=\ P_n+\sum_{j=0}^{n-1}\ \binom{n}{j}\ \sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k}\ P_j.\end{aligned}$$

Thus, $T_n=P_n$ and hence, first series implies the second series. Here we have used the simple fact that the inner sum vanishes being equal to $(1+a)^{n-j}P_j$ with $a=-1$. The converse part is similar hence its proof is omitted. □

Using this lemma, we now establish the inverse series relation in the next theorem.

Theorem 4.2. *For a matrix $A\in C^{p\times p},\ r,\lambda\in\mathbb{C},\ s\in\mathbb{N},\ m\in\mathbb{N}\cup\{0\}$,*

$$Z_{m^*}^{(A,\lambda)}(x^k;r)=\frac{\Gamma(A+rmI+I)}{m!}\sum_{j=0}^{\lfloor m/s\rfloor}(-mI)_{sj}\Gamma^{-1}(A+rjI+I)\frac{(\lambda x^k)^j}{j!}\qquad(4.1)$$

if and only if

$$\frac{(\lambda x^k)^m}{m!}I = \frac{\Gamma(A+rmI+I)}{(ms)!}\sum_{j=0}^{ms}(-msI)_j\Gamma^{-1}(A+rjI+I)Z_{j^*}^{(A,\lambda)}(x^k;r), \quad (4.2)$$

and for $m \neq sl,\ l \in \mathbb{N}$,

$$\sum_{j=0}^{m}(-mI)_j\ \Gamma^{-1}(A+rjI+I)\ Z_{j^*}^{(A,\lambda)}(x^k;r) = O. \quad (4.3)$$

Proof. We first show that the series (4.1) implies both (4.2) and (4.3). The proof of (4.1) implies (4.2) runs as follows. Denoting the right hand side of (4.2) by matrix Ξ_m, substituting the series expression for $Z_{j^*}^{(A,\lambda)}(x^k;r)$ from (4.1) and then using the double series relation (1.5), we get

$$\begin{aligned}
\Xi_m &= \frac{\Gamma(A+rmI+I)}{(ms)!}\sum_{j=0}^{ms}(-msI)_j\Gamma^{-1}(A+rjI+I)\ Z_{j^*}^{(A,\lambda)}(x^k;r)\\
&= \frac{\Gamma(A+rmI+I)}{(ms)!}\sum_{j=0}^{ms}\frac{(-msI)_j}{j!}\sum_{i=0}^{\lfloor j/s\rfloor}(-jI)_{si}\ \Gamma^{-1}(A+riI+I)\ \frac{(\lambda x^k)^i}{i!}\\
&= \sum_{j=0}^{ms}\sum_{i=0}^{\lfloor j/s\rfloor}\frac{\Gamma(A+rmI+I)\ (-1)^{j+si}\ \Gamma^{-1}(A+riI+I)}{(ms-j)!\ (j-si)!\ i!}(\lambda x^k)^i\\
&= \sum_{i=0}^{m}\sum_{j=0}^{ms-si}\frac{\Gamma(A+rmI+I)\ (-1)^{j}\ \Gamma^{-1}(A+riI+I)}{(ms-si-j)!\ j!\ i!}(\lambda x^k)^i\\
&= \frac{(\lambda x^k)^m}{m!}I + \sum_{i=0}^{m-1}\frac{\Gamma(A+rmI+I)\ \Gamma^{-1}(A+riI+I)}{(ms-si)!\ i!}(\lambda x^k)^i\\
&\quad \times \sum_{j=0}^{ms-si}(-1)^j\binom{ms-si}{j}.
\end{aligned}$$

Here the inner sum in the second term on the right hand side vanishes being equal to $(1+a)^{ms-si}$ with $a=-1$. Consequently, we arrive at $\Xi_m = \frac{(\lambda x^k)^m}{m!}I$. Next, to show further that (4.1) also implies (4.3), let us substitute the series expression for $Z_{j^*}^{(A,\lambda)}(x^k;r)$ from (4.1) to the left hand side of (4.3). Then in view of (1.5), we get

$$\begin{aligned}
&\sum_{j=0}^{m}(-mI)_j\Gamma^{-1}(A+rjI+I)\ Z_{j^*}^{(A,\lambda)}(x^k;r)\\
&= \sum_{j=0}^{m}\frac{(-1)^j m!}{(m-j)!}I\sum_{i=0}^{\lfloor j/s\rfloor}\frac{(-1)^{si}\ \Gamma^{-1}(A+riI+I)}{(j-si)!\ i!}(\lambda x^k)^i\\
&= \sum_{i=0}^{\lfloor m/s\rfloor}\frac{m!\ \Gamma^{-1}(A+ri+I)}{(m-si)!\ i!}(\lambda x^k)^i\sum_{j=0}^{m-si}(-1)^j\binom{m-si}{j} = O
\end{aligned}$$

if $m \neq sl,\ l \in \mathbb{N}$. This completes the proof of the first part. The proof of converse part which uses the technique due to Dave and Dalbhide [2], runs as follows. In order to show that the series (4.2) and the condition (4.3) together imply the series (4.1), we use Lemma 4.1 with

$$P_j = j!\ \Gamma^{-1}(A + rjI + I)\ Z^{(A,\lambda)}_{j^*}(x^k; r),$$

and consider one sided relation in the lemma, that is, the series on the left hand side implies the series on the right hand side. Then

$$Q_m = \sum_{j=0}^{m} (-mI)_j\ \Gamma^{-1}(A + rjI + I)\ Z^{(A,\lambda)}_{j^*}(x^k; r) \tag{4.4}$$

implies

$$Z^{(A,\lambda)}_{m^*}(x^k; r) = \frac{\Gamma(A + rmI + I)}{m!} \sum_{j=0}^{m} \frac{(-mI)_j}{j!}\ Q_j. \tag{4.5}$$

Since the condition (4.3) holds, $Q_m = O$ for $m \neq sl,\ l \in \mathbb{N}$, whereas

$$Q_{ms} = \sum_{j=0}^{ms} (-msI)_j\ \Gamma^{-1}(A + rjI + I)\ Z^{(A,\lambda)}_{j^*}(x^k; r).$$

Also the series (4.2) holds true, whence it follows that

$$\begin{aligned} Q_{ms} &= \sum_{j=0}^{ms} (-msI)_j\ \Gamma^{-1}(A + rjI + I)\ Z^{(A,\lambda)}_{j^*}(x^k; r) \\ &= \frac{(ms)!\ \Gamma^{-1}(A + rmI + I)}{m!} (\lambda x^k)^m. \end{aligned}$$

Consequently, the inverse pair (4.4) and (4.5) assume the form:

$$\begin{aligned} \frac{(\lambda x^k)^m}{m!} I &= \frac{\Gamma(A + rmI + I)}{(ms)!} \sum_{j=0}^{ms} (-msI)_j\ \Gamma^{-1}(A + rjI + I) \\ &\quad \times Z^{(A,\lambda)}_{j^*}(x^k; r) \end{aligned}$$

from which it follows that

$$\begin{aligned} Z^{(A,\lambda)}_{m^*}(x^k; r) &= \frac{\Gamma(A + rmI + I)}{m!} \sum_{j=0}^{\lfloor m/s \rfloor} \frac{(-mI)_{sj}}{(sj)!}\ Q_{sj} \\ &= \frac{\Gamma(A + rmI + I)}{m!} \sum_{j=0}^{\lfloor m/s \rfloor} \frac{(-mI)_{sj}\ \Gamma^{-1}(A + rjI + I)}{j!} (\lambda x^k)^j, \end{aligned}$$

subject to the condition (4.3). □

5. Mittag-Leffler Matrix Function

In 2007, Shukla and Prajapati [17] introduced a generalization of the Mittag-Leffler function in the form:

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \tag{5.1}$$

where $\alpha, \beta, \gamma \in \mathbb{C},\ \Re(\alpha, \beta, \gamma) > 0,\ q \in (0, 1) \cup \mathbb{N}$. Here we allow q to take value 0 in which case the series retains convergence behavior. Also, if α is allowed to assume value 0 then with $q = 0$ and $\beta = 1$, the reducibility of (5.1) to the exponential function e^z occurs. Thus, with $q \geq 0,\ \Re(\alpha) \geq 0,\ \Re(\beta, \gamma) > 0$ and $z \in \mathbb{C}$, (5.1) yields an instance

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)\ n!}. \tag{5.2}$$

We define here the matrix analogues of (5.1) and (5.2) as follows.

Definition 5.1. For $A,B\in C^{p\times p}$, $\Re(\mu)>-1$ for all eigen values $\mu\in\sigma(A), r\in\mathbb{C}$ and $s\in\mathbb{N}$,

$$E^{B,sI}_{rI,A+I}(z)=\sum_{n=0}^{\infty}(B)_{sn}\Gamma^{-1}(A+rnI+I)\frac{z^n}{n!}. \tag{5.3}$$

Definition 5.2. For $A\in C^{p\times p}$, $r\in\mathbb{C}$, $\Re(\mu)>-1$ for all eigen values $\mu\in\sigma(A)$,

$$E_{rI,A+I}(z)=\sum_{n=0}^{\infty}\Gamma^{-1}(A+rnI+I)\frac{z^n}{n!}. \tag{5.4}$$

Putting $B=-mI$, where $m\in\mathbb{N}$ and $z=\lambda x^k$ in (5.3), and comparing it with the defined function (2.1), we obtain the relation:

$$E^{-mI,sI}_{rI,A+I}(\lambda x^k)=m!\ \Gamma^{-1}(A+rmI+I)Z^{(A,\lambda)}_{m^*}(x^k;r).$$

The functions (5.3) and (5.4) will be used in the generating function relations derived in the following section.

6. Generating Function relations

We derive the generating function relations for the matrix polynomial $Z^{(A,\lambda)}_{m^*}(x^k;r)$ in the form of Theorems 6.1, 6.3 and 6.5.

Theorem 6.1. *Let $r\in\mathbb{C}$, $s\in\mathbb{N}$ and A,B be the matrices in $C^{p\times p}$, $\Re(\mu)>-1$ for all eigenvalues $\mu\in\sigma(A)$, then for $|t|<1$,*

$$\sum_{m=0}^{\infty}(B)_m\ \Gamma^{-1}(A+rmI+I)\ Z^{(A,\lambda)}_{m^*}(x^k;r)\ t^m$$
$$=\ (1-t)^{-B}\ E^{B,sI}_{rI,A+I}\left(\lambda x^k(-t)^s(1-t)^{-sI}\right).$$

Proof. Observe that on substituting the series for $Z^{(A,\lambda)}_{m^*}(x^k;r)$ from (2.1) on the left hand side and using (1.4), we get

$$\sum_{m=0}^{\infty}(B)_m\ \Gamma^{-1}(A+rmI+I)\ Z^{(A,\lambda)}_{m^*}(x^k;r)\ t^m$$
$$=\sum_{m=0}^{\infty}(B)_m\Gamma^{-1}(A+rmI+I)\frac{\Gamma(A+rmI+I)}{m!}\sum_{n=0}^{\lfloor m/s\rfloor}\frac{m!(-1)^{sn}I\Gamma^{-1}(A+rnI+I)}{n!(m-sn)!}$$
$$\times(\lambda x^k)^n t^m$$
$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\lfloor m/s\rfloor}\frac{(-1)^{sn}(B)_m\Gamma^{-1}(A+rnI+I)}{n!\ (m-sn)!}\ (\lambda x^k)^n t^m$$
$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{sn}(B)_{m+sn}\Gamma^{-1}(A+rnI+I)}{n!\ m!}\ (\lambda x^k)^n t^{m+sn}$$
$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(B+snI)_m t^m}{m!}\frac{(-1)^{sn}(B)_{sn}\Gamma^{-1}(A+rnI+I)}{n!}\ (\lambda x^k)^n t^{sn}$$

$$= \sum_{n=0}^{\infty}(1-t)^{-B-snI}\,\frac{(-1)^{sn}(B)_{sn}\Gamma^{-1}(A+rnI+I)}{n!}\,(\lambda x^k)^n t^{sn}$$

$$= (1-t)^{-B}\sum_{n=0}^{\infty}\frac{(B)_{sn}\Gamma^{-1}(A+rnI+I)}{n!}\,(\lambda x^k(-t)^s(1-t)^{-sI})^n \qquad (6.1)$$

$$= (1-t)^{-B}\; E^{B,sI}_{rI,A+I}\left(\lambda x^k(-t)^s(1-t)^{-sI}\right).$$

This completes the proof. □

Corollary 6.2. *If $r \in \mathbb{N}$, then for $s \le r$ or $s = r+1$,*

$$\sum_{m=0}^{\infty}(B)_m\;(A+I)^{-1}_{rm}\;Z^{(A,\lambda)}_{m^*}(x^k;r)\;t^m \;=\; (1-t)^{-B}\;\times$$
$$ {}_sF_r\left(\frac{B}{s},\frac{B+I}{s},\ldots,\frac{B+(s-1)I}{s};\frac{A+I}{r},\frac{A+2I}{r},\ldots,\frac{A+rI}{r};\frac{s^s}{r^r}\lambda x^k R^s\right),$$

where $R = (-t)(1-t)^{-I}$.

Proof. For $r \in \mathbb{N}$, the infinite series on the right hand side in (6.1) assumes the form

$$(1-t)^{-B}\Gamma^{-1}(A+I)\sum_{n=0}^{\infty}(B)_{sn}(A+I)^{-1}_{rn}\frac{(\lambda x^k R^s)^n}{n!}.$$

In view of the formula (1.6) and the matrix function (3.1), this leads us to the corollary. □

If $(B)_m$ is dropped from the left hand side of this theorem, then it takes the following form.

Theorem 6.3. *In the usual notations and meaning, there holds the generating function relation:*

$$\sum_{m=0}^{\infty}\Gamma^{-1}(A+rmI+I)\;Z^{(A,\lambda)}_{m^*}(x^k;r)\;t^m = e^t\;E_{rI,A+I}\left(\lambda x^k(-t)^s\right).$$

Proof. The proof follows in a straight forward manner. In fact, by using the double series relation (1.4), we have

$$\sum_{m=0}^{\infty}\Gamma^{-1}(A+rmI+I)\;Z^{(A,\lambda)}_{m^*}(x^k;r)\;t^m$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\lfloor m/s\rfloor}\frac{(-1)^{sn}\Gamma^{-1}(A+rnI+I)}{n!\;(m-sn)!}(\lambda x^k)^n t^m$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{sn}\Gamma^{-1}(A+rnI+I)}{n!\;m!}(\lambda x^k)^n t^{m+sn}$$
$$= \sum_{m=0}^{\infty}\frac{t^m}{m!}\sum_{n=0}^{\infty}\frac{(-1)^{sn}\Gamma^{-1}(A+rnI+I)}{n!}(\lambda x^k)^n t^{sn}$$
$$= e^t\;E_{rI,A+I}\left(\lambda x^k(-t)^s\right).$$

□

Again, we have the following corollary. (cf. [16, Eq. (3.5), p. 619])

Corollary 6.4. *For $r \in \mathbb{N}$,*

$$\sum_{m=0}^{\infty} (A+I)_{rm}^{-1} \; Z_{m^*}^{(A,\lambda)}(x^k;r) \; t^m$$
$$= \; e^t \; {}_0F_r\left(--; \frac{A+I}{r}, \frac{A+2I}{r}, \ldots, \frac{A+rI}{r}; \frac{\lambda x^k(-t)^s}{r^r}\right).$$

The proof follows by proceeding as in corollary 6.2. Next, in the notations and meaning of Theorem 6.1, we have

Theorem 6.5. *Let a and b be complex constants which are not zero simultaneously, then there holds the generating function relation*

$$\sum_{n=0}^{\infty} Z_{n^*}^{(A,\lambda)}\left(\frac{x^k}{(a+bn)^s};r\right)(a+bn)^n \; \Gamma^{-1}(A+rnI+I) \; t^n$$
$$= \; e^{ax} \; (1-bte^{bx})^{-1} \; E_{rI,A+I}(\lambda x^k(-t)^s e^{bsx}).$$

Proof. Beginning with the left hand side, we have

$$\sum_{n=0}^{\infty} Z_{n^*}^{(A,\lambda)}\left(\frac{x^k}{(a+bn)^s};r\right)(a+bn)^n\Gamma^{-1}(A+rnI+I) \; t^n$$
$$= \; \sum_{n=0}^{\infty}\sum_{j=0}^{\lfloor n/s\rfloor} \frac{(-1)^{sj} \; \Gamma^{-1}(A+rjI+I)(\lambda x^k)^j}{(n-sj)! \; j!}(a+bn)^{n-sj}t^n$$
$$= \; \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} \frac{((-t)^s\lambda x^k)^j\Gamma^{-1}(A+rjI+I)}{j!}\frac{(a+bn+bsj)^n}{n!}t^n. \qquad (6.2)$$

We use here the Lagrange expansion formula [15, Eq. (18), p. 146]:

$$\frac{f(x)}{1-tg'(x)} = \sum_{n=0}^{\infty}\frac{t^n}{n!}\left[D^n f(x)(g(x))^n\right]_{x=0}, \quad (t = x/g(x))$$

by taking $f(x) = e^{(a+bsj)x}$ and $g(x) = e^{bx}$. Then we find that

$$\frac{e^{(a+bsj)x}}{1-bte^{bx}} = \sum_{n=0}^{\infty}(a+bsj+bn)^n \; \frac{t^n}{n!}.$$

Thus (6.2) simplifies to

$$\sum_{n=0}^{\infty} Z_{n^*}^{(A,\lambda)}\left(\frac{x^k}{(a+bn)^s};r\right)(a+bn)^n\Gamma^{-1}(A+rnI+I) \; t^n$$
$$= \; \sum_{j=0}^{\infty}\frac{\Gamma^{-1}(A+rjI+I)}{j!} \; ((-t)^s\lambda x^k)^j \; \frac{e^{(a+bsj)x}}{1-bte^{bx}}.$$

In view of (5.4), this yields the desired form. □

We again have the following corollary. (cf. [16, Eq. (3.14), p. 621])

Corollary 6.6. *For $r \in \mathbb{N}$, there holds the matrix generating function relation:*

$$\sum_{n=0}^{\infty} Z_{n^*}^{(A,\lambda)}\left(\frac{x^k}{(a+bn)^s};r\right)(a+bn)^n\ (A+I)_{rn}^{-1}\ t^n\ =\ e^{ax}\ (1-bte^{bx})^{-1}$$
$$\times\ {}_0F_r\left(--;\frac{A+I}{r},\frac{A+2I}{r},\ldots,\frac{A+rI}{r};\frac{\lambda x^k(-t)^s e^{bsx}}{r^r}\right).$$

7. Matrix Integral transform

Using the integral formula (7.1), we define Euler (Beta) matrix transform as follows.

Definition 7.1. For the matrices $P, Q \in C^{p\times p}$, a Beta matrix transform may be defined as

$$\mathfrak{B}\{f(x):P,Q\}=\int_0^1 x^{P-I}(1-x)^{Q-I}f(x)\ dx. \tag{7.1}$$

We apply this transform to the polynomial (2.1) in the following theorem.

Theorem 7.2. *If $A, P, Q \in C^{p\times p}$, P, Q are positive stable matrices, for $q = 0, 1, 2, \ldots,$ the matrices $P + qI$, Q are commutative, $P + qI, Q + qI$, $P + Q + qI$ are invertible and $k, r, s, m \in \mathbb{N}$, then*

$$\mathfrak{B}\left\{Z_{m^*}^{(A,\lambda)}(tx^k;r):P,Q\right\}=\frac{(A+I)_{rm}}{m!}\ \Gamma(Q)\Gamma^{-1}(P)\Gamma^{-1}(P+Q)$$
$$\times\ {}_{s+k}F_{r+k}\left[\begin{array}{lll}\Delta(s;-mI), & \Delta(k;P); & \frac{s^s}{r^r}t\\ \Delta(r;A+I), & \Delta(k;P+Q); & \end{array}\right],$$

where the notation $\Delta(j;C)$ carries the meaning as in (1.6).

Proof. From (7.1),

$$\mathfrak{B}\left\{Z_{m^*}^{(A,\lambda)}(tx^k;r):P,Q\right\}$$
$$=\int_0^1 x^{P-I}(1-x)^{Q-I}Z_{m^*}^{(A,\lambda)}(tx^k;r)dx$$
$$=\int_0^1 x^{P-I}(1-x)^{Q-I}\frac{\Gamma(rmI+A+I)}{m!}\sum_{n=0}^{\lfloor m/s\rfloor}\frac{(-m)_{sn}}{n!}\Gamma^{-1}(rnI+A+I)(tx^k)^n dx$$
$$=\frac{\Gamma(rmI+A+I)}{m!}\sum_{n=0}^{\lfloor m/s\rfloor}\frac{(-m)_{sn}}{n!}\Gamma^{-1}(rnI+A+I)t^n\int_0^1 x^{knI+P-I}(1-x)^{Q-I}dx$$
$$=\frac{\Gamma(rmI+A+I)}{m!}\sum_{n=0}^{\lfloor m/s\rfloor}\frac{(-m)_{sn}}{n!}\Gamma^{-1}(rnI+A+I)\ t^n\ \mathfrak{B}(knI+P,Q)$$
$$=\frac{\Gamma(rmI+A+I)}{m!}\sum_{n=0}^{\lfloor m/s\rfloor}\frac{(-m)_{sn}}{n!}\Gamma^{-1}(rnI+A+I)\ t^n\ \Gamma(knI+P)\ \Gamma(Q)$$

$$\times\Gamma^{-1}(knI+P+Q)$$
$$=\frac{(A+I)_{rm}}{m!}\sum_{n=0}^{\lfloor m/s\rfloor}(-m)_{sn}(A+I)_{rn}^{-1}(P)_{kn}(P+Q)_{kn}^{-1}\Gamma(Q)\Gamma(P)\Gamma^{-1}(P+Q)\frac{t^n}{n!}$$
$$=\frac{(A+I)_{rm}\ \Gamma(P)\Gamma(Q)\Gamma^{-1}(P+Q)}{m!}$$
$$\times\ {}_{s+k}F_{r+k}\left[\begin{array}{lll}\Delta(s;-mI), & \Delta(k;P); & \frac{s^s}{r^r}t\\ \Delta(r;A+I), & \Delta(k;P+Q); & \end{array}\right].$$

□

This theorem reduces to the Euler (Beta) transform given in [13, Theorem 9.4, p. 649] when the P, Q, A are scalars.

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Reshma Sanjhira
Dept. of Mathematical Sciences
P. D. Patel Institute of Applied Sciences
Faculty of Applied Sciences
Charotar University of Science and Technology
Changa-388 421, Dist: Anand, Gujarat, India;
& Research Scholar
Dept. of Mathematics, Faculty of Science
The Maharaja Sayajirao University of Baroda
Vadodara-390 002, Gujarat, India.
E-mail: *reshmashah.maths@charusat.ac.in*

B. I. Dave
Dept. of Mathematics, Faculty of Science
The Maharaja Sayajirao University of Baroda
Vadodara-390 002, Gujarat, India.
E-mail: *bidavemsu@yahoo.co.in*

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GENERALIZED MITTAG-LEFFLER MATRIX FUNCTION AND ASSOCIATED MATRIX POLYNOMIALS

RESHMA SANJHIRA, B. V. NATHWANI AND B. I. DAVE

Abstract. The Mittag-Leffler function has been found useful in solving certain problems in Science and Engineering. On the other hand, noticing the occurrence of certain matrix functions in Special functions' theory in general and in Statistics and Lie group theory in particular, we introduce here a matrix analogue of a recently generalized form of Mittag-Leffler function. This function yields the matrix analogues of the Saxena-Nishimoto's function, Bessel-Maitland function, Dotsenko function and the Elliptic Function. We obtain matrix differential equation and eigen matrix function property for the proposed matrix function. Also, a generalized Konhauser matrix polynomial is deduced and its inverse series relations and generating function are derived.

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1. Introduction

The function

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \tag{1.1}$$

where $z, \alpha \in \mathbb{C}$, $\Re(\alpha) > 0$, is due to Gosta Mittag-Leffler [17] which is well known as the Mittag-Leffler function. Wiman [26] generalized this in the form

$$E_{\alpha,\,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}. \tag{1.2}$$

This was further extended in different forms by T. R. Prabhakar [19], Shukla and Prajapati [25] and others. Recently, Nathwani and Dave [18] studied the

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generalized structure given by

$$E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r)=\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s}{\Gamma(\alpha n+\beta)\ [(\lambda)_{\mu n}]^r}\frac{z^n}{n!},\tag{1.3}$$

wherein the parameters α, β, γ, λ $\in\mathbb{C}$ with $\Re(\alpha,\beta,\gamma,\lambda)>0$, δ, $\mu>0$, $r\in\mathbb{N}\cup\{-1,0\}$ and $s\in\mathbb{N}\cup\{0\}$.
As indicated in [18], the function in (1.3) includes the Bessel-Maitland function: $J^{\mu}_{\nu}(z)$, Dotsenko function : ${}_2R_1(a,b;c,\omega;\nu;z)$, a particular form $(m=2)$ of extension of Mittag-Leffler function due to Saxena and Nishimoto given by $E_{\gamma,K}[(\alpha_j,\beta_j)_{1,2};z]$ and the Elliptic function : $K(k)=\frac{\pi}{2}\ {}_2F_1\left(1/2,1/2;1;k^2\right)$.
The Special matrix functions indeed appear in Statistics [7,13], Lie groups theory [16], and in a series of works during late nineties, on matrix analogues of Laguerre, Hermite and Legendre differential equations and the corresponding polynomial families [6,8,11,12,22].
Motivated by the matrix functions' study, we propose here a matrix analogue of (1.3) and derive the matrix differential equation and matrix eigen function. Also among certain specializations of this function, an extended Konhauser matrix polynomial and hence the Laguerre matrix polynomial are illustrated together with their inverse series relations and matrix generating function relation.

2. Preliminaries and some facts

In what follows, the following notations and definitions will be used in the work. We shall let a matrix A in $C^{p\times p}$ and $\sigma(A)$, the set of all eigenvalues of A. The matrix A is said to be positive stable matrix if $\Re(\lambda)>0$ for all $\lambda\in\sigma(A)$. If the matrices $A_0,A_1,A_2,....A_n$ are elements of $C^{p\times p}$ and $A_n\neq O$ then

$$P_n(x)=A_nx^n+A_{n-1}x^{n-1}+A_{n-2}x^{n-2}+...+A_1x+A_0\tag{2.1}$$

is a matrix polynomial of degree n in x.
The 2-norm of the matrix A denoted by $\|A\|$, is defined by

$$\parallel A\parallel=\sup_{x\neq 0}\frac{\parallel Ax\parallel_2}{\parallel x\parallel_2}=max\{\sqrt{\lambda}:\lambda\in\sigma(A^*A)\},\tag{2.2}$$

where for a vector $y\in\mathbb{C}^p$, $\|y\|_2=\left(y^Ty\right)^{1/2}$ is Euclidean norm of y, and A^* denotes the transposed conjugate of A.
If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z which are defined on an open set Ω of the complex plane, and if $\sigma(A)\subset\Omega$ then from the properties of the matrix functional calculus [3], it follows that

$$f(A)g(A)=g(A)f(A).\tag{2.3}$$

The reciprocal gamma function denoted by $\Gamma^{-1}(z) = (\Gamma(z))^{-1} = \frac{1}{\Gamma(z)}$ is an entire function of complex variable z [4, p. 253] and thus for any matrix A in $C^{p\times p}$, the functional calculus [3] shows that $\Gamma^{-1}(A)$ is a well defined matrix. If I denotes the identity matrix of order r and $A + nI$ is invertible for every integer $n \geq 0$ then [9]

$$A(A+I)\cdots(A+(n-1)I) = \Gamma(A+nI)\ \Gamma^{-1}(A).$$

The scalar factorial function

$$(a)_n = a(a+1)\cdots(a+n-1),$$

where $a \in \mathbb{C}, n \in \mathbb{N}, (a)_0 = 1$, has matrix analogue ([3], [10, Eq.(7), p.206]) :

$$(A)_n = A(A+I)\cdots(A+(n-1)I),$$

where $A \in C^{p\times p}, n \in \mathbb{N}, (A)_0 = I$.

Example 2.1. Let us evaluate $\Gamma(A)$, where

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}.$$

We have [1, Eq.(8), p. 64]

$$\Gamma(A) = \int_0^\infty e^{-t} t^{A-I} dt$$

where $t^P = e^{P\log t}$. We first find t^{A-I}, where

$$A - I = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Since for $n \geq 1$,

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}^n = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},$$

and

$$e^{B\log t} = \sum_{n=0}^{\infty} \frac{B^n (\log t)^n}{n!},$$

we find that

$$t^{A-I} = t^{\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ t-1 & t \end{bmatrix}.$$

Hence,

$$e^{-t} t^{A-I} = \begin{bmatrix} e^{-t} & 0 \\ e^{-t}(t-1) & te^{-t} \end{bmatrix}.$$

Consequently,

$$\Gamma\left(\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Alternatively, following [21], if $f(s)$ is a scalar function which is analytic in some region R of complex plane, then

$$f(s) = \sum_{k=0}^{\infty} \beta_k \; s^k.$$

Now if P be an $n \times n$ matrix with characteristic polynomial $\Delta(s)$ and eigenvalues λ_i then f(s) may be written as $f(s) = \Delta(s)Q(s) + R(s)$, where $R(s)$ is of degree $\leq n-1$. Now, from Cayley-Hamilton theorem,

$$f(\lambda_i) = R(\lambda_i) = \sum_{k=0}^{n-1} \alpha_k \lambda_i^k. \tag{2.4}$$

This yields the system of simultaneous equations in $\alpha_k' s$. Thus for matrix function $f(A)$, we have

$$f(A) = R(A) = \sum_{k=0}^{n-1} \alpha_k A^k.$$

The $\alpha_k' s$ are determined from (2.4). Thus, taking

$$P = A - I = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},$$

we find the eigen values to be $\lambda_1 = 0, \lambda_2 = 1$. The corresponding system of equations is $1 = \alpha_0 + 0; t = \alpha_0 + \alpha_1$. This provides us $\alpha_0 = 1, \alpha_1 = t - 1$. Now,

$$t^A = e^{P \log t} = \alpha_0 I + \alpha_1 P = \begin{bmatrix} 1 & 0 \\ t-1 & t \end{bmatrix},$$

Consequently,

$$\Gamma\left(\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\right) = \int_0^{\infty} e^{-t} \begin{bmatrix} 1 & 0 \\ t-1 & t \end{bmatrix} dt = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

For more details of matrix exponential, see [21].
The generalized hypergeometric matrix function is defined as follows [24, Eq. (2.2), p. 608].

Definition 2.2. If $\{A_i; i = 1, 2, \ldots, p\}$, and $\{B_j; j = 1, 2, \ldots, q\}$, are sequences of matrices in $C^{p \times p}$ such that $B_j + nI$ are invertible for all $n \geq 0$, then the generalized hypergeometric matrix function is defined as

$$\begin{aligned} & {}_rF_s\left(A_1, A_2, \ldots, A_r; B_1, B_2, \ldots, B_s; x\right) \\ &= \sum_{n=0}^{\infty} (A_1)_n (A_2)_n \ldots (A_r)_n \; [(B_1)_n]^{-1} [(B_2)_n]^{-1} \ldots [(B_s)_n]^{-1} \; \frac{x^n}{n!}. \end{aligned} \tag{2.5}$$

Here, the series converges for all x if $r \leq s$. If $r = s+1$, then the series converges for $|x| < 1$. If $r > s+1$, then the series diverges for all $x \neq 0$.

For any matrix A in $C^{p\times p}$ and for $|x| < 1$, the following series expansion holds [10].

$$(1-x)^{-A} = \sum_{n=0}^{\infty} \frac{(A)_n}{n!} x^n. \tag{2.6}$$

Also, we have the formula:

$$(A)_{mk} = m^{mk} \prod_{i=1}^{m} \left(\frac{A + (i-1)I}{m} \right)_k = \Delta(m; A). \tag{2.7}$$

In particular, for non negative integer n,

$$(-nI)_{mk} = (-1)^{mk} \frac{n!}{(n-mk)!} I = m^{mk} \prod_{i=1}^{m} \left(\frac{-n+i-1}{m} \; I \right)_k. \tag{2.8}$$

In these notations, we define here a generalized Mittag-Leffler matrix function as follows.

Definition 2.3.

$$E^{A,B,C}_{\alpha I,\delta I,\mu I}(\lambda z; s, r) = \sum_{n=0}^{\infty} [(A)_{\delta n}]^s \; \Gamma^{-1}(\alpha n I + B) \; [(C)_{\mu n}]^{-r} \; \frac{(\lambda z)^n}{n!}, \tag{2.9}$$

where A, B, C are positive stable matrices in $C^{p\times p}$, $\alpha, \lambda, z \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\delta, \mu > 0, r \in \{-1, 0\} \cup \mathbb{N}$ and $s \in \{0\} \cup \mathbb{N}$.

The matrix analogues of above stated functions (i)-(iv) are yielded by this function which are given below.

(i) Bessel-Maitland matrix function (cf. [5, Eq.(1.7.8), p.19]):

$$J^{\mu I}_{\nu I}(z) = \sum_{n=0}^{\infty} (-1)^n \; \Gamma^{-1}(\nu I + n\mu I + I) \; \frac{z^n}{n!},$$

(ii) Dotsenko matrix function (cf. [5, Eq.(1.8.9), p.24]):

$$\begin{aligned} {}_2R_1(aI, bI; cI, \omega I; \nu; z) \quad = \quad & \sum_{n=0}^{\infty} \Gamma(aI + nI)\Gamma^{-1}(aI) \; \Gamma(bI + n\frac{\omega}{\nu} I) \\ & \times \Gamma^{-1}(bI) \; \Gamma(cI)\Gamma^{-1}(cI + n\frac{\omega}{\nu} I) \frac{z^n}{n!}, \end{aligned}$$

(iii) Saxena and Nishimoto's matrix function (cf. [23]):

$$E_{\gamma I,K}[(\alpha_j I, B_j)_{1,2}; z] = \sum_{n=0}^{\infty} (\gamma I)_{Kn} \Gamma^{-1}(\alpha_1 nI + B_1) \Gamma^{-1}(\alpha_2 nI + B_2) \frac{z^n}{n!},$$

where $z, \gamma, \alpha_j, \beta_j \in \mathbb{C}, \Re(\alpha_1 + \alpha_2) > \Re(K) - 1, \Re(K) > 0$, and

(iv) the Elliptic matrix function (cf. [15, Eq.(1), p.211]):

$$K(I; k) = \frac{\pi}{2}\, {}_2F_1\left(\begin{array}{ccc} \frac{1}{2}I, & \frac{1}{2}I; & k^2 \\ I; & & \end{array} \right).$$

3. Differential Equation

Let us take

$$\frac{\delta^{s\delta}}{\alpha^{\alpha}\ \mu^{r\mu}} = u,\ \frac{d}{dz} = D,\ zD = \theta,\ \prod_{m=0}^{\delta-1} \left[\left(\theta I + \frac{A + mI}{\delta} \right) \right]^s = \Delta_m^{(\delta, A; s)},$$

$$\prod_{j=0}^{\alpha-1} \left(\theta I + \frac{B + jI}{\alpha} - I \right)^m = \Upsilon_j^{(\alpha, B; m)}. \tag{3.1}$$

In these notations, we derive the differential equation satisfied by (2.9).

Theorem 3.1. *Let* $\alpha, \mu, \delta \in \mathbb{N}$ *then* $y = E_{\alpha I, \delta I, \mu I}^{A,B,C}(\lambda z; s, r)$ *satisfies the equation*

$$\left[\Upsilon_k^{(\mu, C; r)}\ \Upsilon_j^{(\alpha, B; 1)}\ \theta - \frac{\delta^{s\delta} \lambda z}{\alpha^{\alpha}\ \mu^{r\mu}}\ \Delta_m^{(\delta, A; s)} \right] y = 0. \tag{3.2}$$

Proof. We first assume that the matrices occurring here are commutative with one another, then we have

$$\begin{aligned} y &= \sum_{n=0}^{\infty} [(A)_{\delta n}]^s\ \Gamma^{-1}(\alpha nI + B)\ [(C)_{\mu n}]^{-r}\ \frac{(\lambda z)^n}{n!} \\ &= \sum_{n=0}^{\infty} \Gamma^{-1}(B)\, [(A)_{\delta n}]^s\ [(B)_{\alpha n}]^{-1}\ [(C)_{\mu n}]^{-r}\ \frac{(\lambda z)^n}{n!} \\ &= \sum_{n=0}^{\infty} \delta^{s\delta n} \Gamma^{-1}(B) \left[\left(\frac{A}{\delta} \right)_n \right]^s \left[\left(\frac{A + I}{\delta} \right)_n \right]^s \cdots \left[\left(\frac{A + (\delta - 1)I}{\delta} \right)_n \right]^s \\ &\quad \times \alpha^{-\alpha n} \left[\left(\frac{B}{\alpha} \right)_n \right]^{-1} \left[\left(\frac{B + I}{\alpha} \right)_n \right]^{-1} \cdots \left[\left(\frac{B + (\alpha - 1)I}{\alpha} \right)_n \right]^{-1} \\ &\quad \times \mu^{-r\mu n} \left[\left(\frac{C}{\mu} \right)_n \right]^{-r} \left[\left(\frac{C + I}{\mu} \right)_n \right]^{-r} \cdots \left[\left(\frac{C + (\mu - 1)I}{\mu} \right)_n \right]^{-r} \\ &\quad \times \frac{(\lambda z)^n}{n!}. \end{aligned}$$

Thus,

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} \Gamma^{-1}(B) \frac{\delta^{s\delta n}}{\alpha^{\alpha n}\ \mu^{r\mu n}} \left\{ \prod_{m=0}^{\delta-1} \left[\left(\frac{A+mI}{\delta} \right)_n \right]^s \right\} \\
&\quad \times \left\{ \prod_{j=0}^{\alpha-1} \left(\frac{B+jI}{\alpha} \right)_n^{-1} \right\} \left\{ \prod_{k=0}^{\mu-1} \left[\left(\frac{C+kI}{\mu} \right)_n^{-1} \right]^r \right\} \frac{(\lambda z)^n}{n!}. \qquad (3.3)
\end{aligned}
$$

Now take

$$
\prod_{m=0}^{\delta-1} \left[\left(\frac{A+mI}{\delta} \right)_n \right]^s = P_n,\ \prod_{j=0}^{\alpha-1} \left(\frac{B+jI}{\alpha} \right)_n = Q_n, \prod_{k=0}^{\mu-1} \left[\left(\frac{C+kI}{\mu} \right)_n^{-1} \right]^r = R_n^{-1},
$$

then the function (2.9) takes the form

$$
y = \Gamma^{-1}(B) \sum_{n=0}^{\infty} u^n\ P_n\ Q_n^{-1}\ R_n^{-1} \frac{(\lambda z)^n}{n!}.
$$

Now,

$$
\begin{aligned}
\theta\ y &= \Gamma^{-1}(B) \sum_{n=0}^{\infty} u^n\ P_n\ Q_n^{-1}\ R_n^{-1} \frac{1}{n!}\ \theta\ (\lambda z)^n \\
&= \Gamma^{-1}(B) \sum_{n=1}^{\infty} u^n\ P_n\ Q_n^{-1}\ R_n^{-1} \frac{1}{(n-1)!}\ (\lambda z)^n.
\end{aligned}
$$

Further, for the matrices which commute with one another, we have

$$
\begin{aligned}
\Upsilon_j^{(\alpha,B;1)}\ \theta I\ y &= \Gamma^{-1}(B) \sum_{n=1}^{\infty} u^n\ \frac{P_n\ Q_n^{-1}\ R_n^{-1}}{(n-1)!} \prod_{j=0}^{\alpha-1} \left(\theta I + \frac{B+jI}{\alpha} - I \right) \\
&\quad \times (\lambda z)^n \\
&= \Gamma^{-1}(B) \sum_{n=1}^{\infty} u^n\ \frac{P_n\ Q_n^{-1}\ R_n^{-1}}{(n-1)!} \prod_{j=0}^{\alpha-1} \left(nI + \frac{B+jI}{\alpha} - I \right) \\
&\quad \times (\lambda z)^n \\
&= \Gamma^{-1}(B) \sum_{n=1}^{\infty} u^n \frac{P_n\ Q_{n-1}^{-1}\ R_n^{-1}}{(n-1)!}\ (\lambda z)^n.
\end{aligned}
$$

Finally,

$$\begin{aligned}
&\Upsilon_k^{(\mu,C;r)}\ \Upsilon_j^{(\alpha,B;1)}\ \theta I\ y\\
=\ & \Gamma^{-1}(B)\sum_{n=1}^{\infty} u^n\ \frac{P_n\ Q_{n-1}^{-1}\ R_n^{-1}}{(n-1)!}\ \prod_{k=0}^{\mu-1}\left[\left(\theta I+\frac{C+kI}{\mu}-I\right)\right]^r\ (\lambda z)^n\\
=\ & \Gamma^{-1}(B)\sum_{n=1}^{\infty} u^n\ \frac{P_n\ Q_{n-1}^{-1}\ R_n^{-1}}{(n-1)!}\ \prod_{k=0}^{\mu-1}\left[\left(nI+\frac{C+kI}{\mu}-I\right)\right]^r\ (\lambda z)^n\\
=\ & \Gamma^{-1}(B)\sum_{n=1}^{\infty} u^n\ \frac{P_n\ Q_{n-1}^{-1}\ R_{n-1}^{-1}}{(n-1)!}\ (\lambda z)^n.
\end{aligned}$$

Thus,

$$\Upsilon_k^{(\mu,C;r)}\ \Upsilon_j^{(\alpha,B;1)}\ \theta\ y = \Gamma^{-1}(B)\sum_{n=0}^{\infty} u^{n+1}P_{n+1}\ Q_n^{-1}\ R_n^{-1} \times\frac{(\lambda z)^{n+1}}{n!}. \tag{3.4}$$

On the other hand,

$$\begin{aligned}
\Delta_m^{(\delta,A;s)}\ y &= \Gamma^{-1}(B)\sum_{n=1}^{\infty} u^n\ \frac{P_n\ Q_n^{-1}\ R_n^{-1}}{n!}\ \prod_{m=0}^{\delta-1}\left[\left(\theta I+\frac{A+mI}{\delta}\right)\right]^s \times(\lambda z)^n\\
&= \Gamma^{-1}(B)\sum_{n=0}^{\infty} u^n\ \frac{P_n\ Q_n^{-1}\ R_n^{-1}}{n!}\ \prod_{m=0}^{\delta-1}\left[\left(nI+\frac{A+mI}{\delta}\right)\right]^s \times(\lambda z)^n\\
&= \Gamma^{-1}(B)\sum_{n=0}^{\infty} u^n\ P_{n+1}\ Q_n^{-1}\ R_n^{-1}\ \frac{(\lambda z)^n}{n!},
\end{aligned}$$

that is,

$$u(\lambda z)\Delta_m^{(\delta,A;s)}\ y = \Gamma^{-1}(B)\sum_{n=0}^{\infty} u^{n+1}P_{n+1}\ Q_n^{-1}\ R_n^{-1}\ \frac{(\lambda z)^{n+1}}{n!}. \tag{3.5}$$

On comparing (3.4) and (3.5), we get (3.2). □

3.1. **Eigen function property.** In deriving the eigen function property for the function (2.9), we will require the operators:

$$\Theta_j^{(\delta,A;m)} = \prod_{m=0}^{\delta-1}\left[\left(-\theta I+\frac{A+mI}{\delta}-I\right)\right]^{-s}, \tag{3.6}$$

and

$$\Omega_{\Theta;\Upsilon} = (\lambda u)^{-1}\ D\ \Theta_m^{(\delta,A;-s)}\ \Upsilon_k^{(\mu,C;r)}\ \Upsilon_j^{(\alpha,B;1)}. \tag{3.7}$$

Theorem 3.2. *Let $\alpha,\mu,\delta\in\mathbb{N}$ then $E^{A,B,C}_{\alpha I,\delta I,\mu I}(\lambda z;s,r)$ is an eigen function with respect to the operator $\Omega_{\Theta;\Upsilon}$. That is,*

$$\Omega_{\Theta;\Upsilon}\left(E^{A,B,C}_{\alpha I,\delta I,\mu I}(\zeta\lambda z;s,r)\right) = \zeta\ E^{A,B,C}_{\alpha I,\delta I,\mu I}(\lambda z;s,r). \tag{3.8}$$

Proof. We first note that

$$w = E^{A,B,C}_{\alpha I,\delta I,\mu I}(\zeta\lambda z;s,r) = \Gamma^{-1}(B)\sum_{n=0}^{\infty}(\zeta u)^n\ P_n\ \ Q_n^{-1}\ R_n^{-1}\frac{(\lambda z)^n}{n!}.$$

Now in view of (3.1),

$$\begin{aligned}
\Upsilon_j^{(\alpha,B;1)}\ w \quad &= \quad \Gamma^{-1}(B)\sum_{n=1}^{\infty}(\zeta u)^n\frac{P_n\ Q_n^{-1}\ R_n^{-1}}{n!}\prod_{j=0}^{\alpha-1}\left(\theta I+\frac{B+jI}{\alpha}-I\right)\\
&\quad\ \times(\lambda z)^n\\
&= \quad \Gamma^{-1}(B)\sum_{n=1}^{\infty}(\zeta u)^n\ \frac{P_n\ Q_n^{-1}\ R_n^{-1}}{n!}\prod_{j=0}^{\alpha-1}\left(nI+\frac{B+jI}{\alpha}-I\right)\\
&\quad\ \times(\lambda z)^n\\
&= \quad \Gamma^{-1}(B)\sum_{n=1}^{\infty}(\zeta u)^n\ P_n\ Q_{n-1}^{-1}\ R_n^{-1}\frac{(\lambda z)^n}{n!}.
\end{aligned}$$

Next

$$\begin{aligned}
&\Upsilon_k^{(\mu,C;r)}\ \Upsilon_j^{(\alpha,B;1)}\ w\\
=\ &\Gamma^{-1}(B)\sum_{n=1}^{\infty}(\zeta u)^n\ \frac{P_n\ Q_{n-1}^{-1}\ R_n^{-1}}{n!}\prod_{k=0}^{\mu-1}\left[\left(\theta I+\frac{C+kI}{\mu}-I\right)\right]^r(\lambda z)^n\\
=\ &\Gamma^{-1}(B)\sum_{n=1}^{\infty}(\zeta u)^n\ \frac{P_n\ Q_{n-1}^{-1}\ R_n^{-1}}{n!}\prod_{k=0}^{\mu-1}\left[\left(nI+\frac{C+kI}{\mu}-I\right)\right]^r(\lambda z)^n\\
=\ &\Gamma^{-1}(B)\sum_{n=1}^{\infty}(\zeta u)^n\ P_n\ Q_{n-1}^{-1}\ R_{n-1}^{-1}\ \frac{(\lambda z)^n}{n!}.
\end{aligned}$$

Further, from (3.6),

$$
\begin{aligned}
&\Theta_m^{(\delta,A;-s)}\ \Upsilon_k^{(\mu,C;r)}\ \Upsilon_j^{(\alpha,B;1)}\ w \\
= \ & \Gamma^{-1}(B)\sum_{n=0}^{\infty}(\zeta u)^n\ \frac{P_n\ Q_{n-1}^{-1}\ R_{n-1}^{-1}}{n!}\ \Theta_m^{(\delta,A;-s)}(\lambda z)^n \\
= \ & \Gamma^{-1}(B)\sum_{n=1}^{\infty}(\zeta u)^n\ \frac{P_n\ Q_{n-1}^{-1}\ R_{n-1}^{-1}}{n!}\prod_{m=0}^{\delta-1}\left[\left(-\theta I+\frac{A+mI}{\delta}-I\right)\right]^{-s} \\
& \times(\lambda z)^n \\
= \ & \Gamma^{-1}(B)\sum_{n=0}^{\infty}(\zeta u)^n\frac{P_n\ Q_{n-1}^{-1}\ R_{n-1}^{-1}}{n!}\prod_{m=0}^{\delta-1}\left[\left(nI+\frac{A+mI}{\delta}-I\right)\right]^{-s} \\
& \times(\lambda z)^n \\
= \ & \Gamma^{-1}(B)\sum_{n=0}^{\infty}(\zeta u)^n\ P_{n-1}\ Q_{n-1}^{-1}\ R_{n-1}^{-1}\ \frac{(\lambda z)^n}{n!}.
\end{aligned}
$$

Finally, using (3.7) we get

$$
\begin{aligned}
&\Omega_{\Theta;\Upsilon}\left(E_{\alpha I,\delta I,\mu I}^{A,B,C}(\zeta\lambda z;s,r)\right) \\
= \ & (\lambda u)^{-1}\ D\ \Theta_m^{(\delta,A;-s)}\ \Upsilon_k^{(\mu,C;r)}\ \Upsilon_j^{(\alpha,B;1)}\ w \\
= \ & \Gamma^{-1}(B)\sum_{n=0}^{\infty}\zeta^n(\lambda u)^{n-1}\ P_{n-1}\ Q_{n-1}^{-1}\ R_{n-1}^{-1}\frac{1}{n!}Dz^n \\
= \ & \Gamma^{-1}(B)\sum_{n=1}^{\infty}\zeta^n(\lambda u)^{n-1}\ P_{n-1}\ Q_{n-1}^{-1}\ R_{n-1}^{-1}\ \frac{z^{n-1}}{(n-1)!} \\
= \ & \Gamma^{-1}(B)\sum_{n=0}^{\infty}\zeta^{n+1}u^n\ P_n\ \ Q_n^{-1}\ R_n^{-1}\ \frac{(\lambda z)^n}{n!} \\
= \ & \Gamma^{-1}(B)\zeta\ \sum_{n=0}^{\infty}\zeta^n u^n\ P_n\ Q_n^{-1}\ R_n^{-1}\ \frac{(\lambda z)^n}{n!} \\
= \ & \Gamma^{-1}(B)\ \zeta\ E_{\alpha I,\delta I,\mu I}^{A,B,C}(\zeta\lambda z;s,r).
\end{aligned}
$$

□

The properties corresponding to the special cases (i)-(iv) listed in section 2 may be deduced by suitably specializing the parameters involved in the above derived properties.

4. Extended Konhauser matrix polynomial

A generalized Konhauser polynomial due to Nathwani and Dave is given by [18, Eq. (22), p. 71]

$$B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r)=\frac{\Gamma(\alpha n+\beta+1)}{(n!)^s}\sum_{j=0}^{[n/m]}\frac{[(-n)_{mj}]^s\ x^{kj}}{\Gamma(\alpha j+\beta+1)[(\lambda)_{\mu j}]^r\ j!}.$$

We propose its matrix analogue as follows.

Definition 4.1. For the matrix A and B in $C^{p\times p}$,

$$\begin{aligned}Z_{m^*}^{(A,B,\mu,r)}(\lambda x^k;s,p) &= \frac{\Gamma(A+rmI+I)}{(m!)^s}\sum_{n=0}^{\lfloor m/\delta\rfloor}(-m)_{\delta n}^s\\ &\quad\times[(B)_{\mu n}^p]^{-1}\ \Gamma^{-1}(A+rnI+I)\frac{(\lambda x^k)^n}{n!},\end{aligned} \tag{4.1}$$

where $r\in\mathbb{C}$; $m\in\mathbb{N}\cup\{0\}, s,p,k\in\mathbb{R}_{>0}$, $\Re(\mu)>-1$, for all eigen values $\mu\in\sigma(A)$, λ is a complex number with $\Re(\lambda)>0$ and the floor function $\lfloor u\rfloor=$ *floor* u, represents the greatest integer $\leq u$.

The Laguerre matrix polynomial [14, Eq.(10), p. 3]:

$$L_m^{(A,\lambda)}(x) = \sum_{n=0}^{m}\frac{(-1)^n\lambda^n}{n!(m-n)!}(A+I)_m\ [(A+I)_n]^{-1}x^n \tag{4.2}$$

occurs as a special case of this polynomial when $\mu=0, k=\delta=r=s=1$. We now derive its inverse series and generating function relations in the following sections.

5. Inverse series relations

In deriving the inverse series of the matrix polynomial (4.1), the following lemma will be used.

Lemma 5.1. *If $\{G_n\}$ and $\{H_n\}$ are finite sequences of matrices in $C^{p\times p}$, then*

$$H_n=\sum_{j=0}^{n}\frac{(-nI)_j}{j!}\ G_j\ \Leftrightarrow\ G_n=\sum_{j=0}^{n}\frac{(-nI)_j}{j!}\ H_j.$$

Proof. Denoting by matrix T_n the right hand side of second series, we have

$$\begin{aligned} T_n &= \sum_{k=0}^{n} \frac{(-nI)_k}{k!}\ H_k \\ &= \sum_{k=0}^{n} \frac{(-1)^k n!\ I}{k!\ (n-k)!} \sum_{j=0}^{k} \frac{(-kI)_j}{j!}\ G_j \\ &= \sum_{k=0}^{n} \frac{(-1)^k n!\ I}{k!\ (n-k)!} \sum_{j=0}^{k} \frac{(-1)^j I\ k!}{j!\ (k-j)!}\ G_j \\ &= \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{n-j} (-1)^k \binom{n-j}{k}\ G_j \\ &= G_n + \sum_{j=0}^{n-1} \binom{n}{j} \sum_{k=0}^{n-j} (-1)^k \binom{n-j}{k}\ G_j. \end{aligned}$$

Thus, $T_n = G_n$ and hence, first series implies the second series. The converse follows along the same line hence omitted. □

Using this lemma, we now establish the inverse series relation for $s = 1$ in

Theorem 5.2. *For $A, B \in C^{p\times p}$ and $\delta = 2, 3, \ldots,$*

$$\begin{aligned} Z_{m^*}^{(A,B,\mu,r)}(\lambda x^k; 1, p) &= \frac{\Gamma(A + rmI + I)}{m!} \sum_{n=0}^{\lfloor m/\delta \rfloor} (-mI)_{\delta n} \\ &\quad \times [(B)_{\mu n}^{p}]^{-1}\ \Gamma^{-1}(A + rnI + I) \frac{(\lambda x^k)^n}{n!}, \end{aligned} \tag{5.1}$$

if and only if

$$\begin{aligned} \frac{(\lambda x^k)^m}{m!} I &= \frac{\Gamma(A + rmI + I)}{(m\delta)!} (B)_{\mu m}^{p} \sum_{j=0}^{m\delta} (-m\delta I)_j \\ &\quad \times \Gamma^{-1}(A + rjI + I)\ Z_{j^*}^{(A,B,\mu,r)}(\lambda x^k; 1, p), \end{aligned} \tag{5.2}$$

and for $m \neq \delta l,\ l \in \mathbb{N}$,

$$\sum_{j=0}^{m} (-mI)_j\ \Gamma^{-1}(A + rjI + I)\ Z_{j^*}^{(A,B,\mu,r)}(\lambda x^k; 1, p) = 0. \tag{5.3}$$

Proof. We first show that the series (5.1) implies both (5.2) and (5.3). The proof of (5.1) implies (5.2) runs as follows.

Denoting the right hand side of (5.2) by some matrix Ξ_m, and then substituting for $Z_{j^*}^{(A,B,\mu,r)}(\lambda x^k;1,p)$ from (5.1), we get

$$\begin{aligned}
\Xi_m &= \frac{\Gamma(A+rmI+I)}{(m\delta)!}(B)^p_{\mu m}\sum_{j=0}^{m\delta}(-m\delta I)_j\Gamma^{-1}(A+rjI+I)\\
&\quad\times Z_{j^*}^{(A,B,\mu,r)}(\lambda x^k;1,p)\\
&= \frac{\Gamma(A+rmI+I)}{(m\delta)!}(B)^p_{\mu m}\sum_{j=0}^{m\delta}(-m\delta I)_j\sum_{i=0}^{\lfloor j/\delta\rfloor}(-jI)_{\delta i}\ (B)^{-p}_{\mu i}\\
&\quad\times\Gamma^{-1}(A+riI+I)\frac{(\lambda x^k)^i}{i!}\\
&= \sum_{j=0}^{m\delta}\sum_{i=0}^{\lfloor j/\delta\rfloor}\frac{(-1)^{j+\delta i}\Gamma(A+rmI+I)(B)^p_{\mu m}(B)^{-p}_{\mu i}}{(m\delta-j)!\ (j-\delta i)!\ i!}\Gamma^{-1}(A+riI+I)\\
&\quad\times(\lambda x^k)^i\\
&= \sum_{i=0}^{m}\sum_{j=0}^{m\delta-\delta i}\frac{(-1)^j\Gamma(A+rmI+I)\ (B)^p_{\mu m}(B)^{-p}_{\mu i}}{(m\delta-\delta i-j)!\ j!\ i!}\Gamma^{-1}(A+riI+I)\\
&\quad\times(\lambda x^k)^i\\
&= \frac{(\lambda x^k)^m}{m!}I+\sum_{i=0}^{m-1}\frac{\Gamma(A+rmI+I)\ (B)^p_{\mu m}(B)^{-p}_{\mu i}}{(m\delta-\delta i)!\ i!}\Gamma^{-1}(A+riI+I)\\
&\quad\times(\lambda x^k)^i\sum_{j=0}^{m\delta-\delta i}(-1)^j\binom{m\delta-\delta i}{j}.
\end{aligned}$$

Here the inner sum in the second term on the right hand side vanishes, consequently, we arrive at $\Xi_m=\frac{(\lambda x^k)^m}{m!}I$.

In order to show further that (5.1) also implies (5.3), let us substitute for $Z_{j^*}^{(A,B,\mu,r)}(\lambda x^k;1,p)$ from (5.1) to the left hand side of (5.3), we then get

$$\begin{aligned}
&\sum_{j=0}^{m}(-mI)_j\Gamma^{-1}(A+rjI+I)\ Z_{j^*}^{(A,B,\mu,r)}(\lambda x^k;1,p)\\
&= \sum_{j=0}^{m}\frac{(-1)^j m!}{(m-j)!}I\sum_{i=0}^{\lfloor j/\delta\rfloor}\frac{(-1)^{\delta i}\ (B)^{-p}_{\mu i}}{(j-\delta i)!\ i!}\Gamma^{-1}(A+riI+I)(\lambda x^k)^i\\
&= \sum_{i=0}^{\lfloor m/\delta\rfloor}\frac{m!\ (B)^{-p}_{\mu i}\Gamma^{-1}(A+ri+I)}{(m-\delta i)!\ i!}(\lambda x^k)^i\sum_{j=0}^{m-\delta i}(-1)^j\binom{m-\delta i}{j}\\
&= 0
\end{aligned}$$

if $m\neq\delta l,\ l\in\mathbb{N}$. Thus completing the first part. The proof of converse part which uses the technique due to Dave and Dalbhide [2], runs as follows. In order to show that the series (5.2) and the condition (5.3) together imply the series (5.1), we use Lemma 4.1 with

$$G_j = j!\ \Gamma^{-1}(A + rjI + I)\ Z_{j^*}^{(A,B,\mu,r)}(\lambda x^k; 1, p),$$

and consider one sided relation in the lemma that is, the series on the left hand side implies the series on the right hand side. Then

$$H_m = \sum_{j=0}^{m}(-mI)_j\ \Gamma^{-1}(A + rjI + I)\ Z_{j^*}^{(A,B,\mu,r)}(\lambda x^k; 1, p) \tag{5.4}$$

$$\Rightarrow$$

$$Z_{j^*}^{(A,B,\mu,r)}(\lambda x^k; 1, p) = \frac{\Gamma(A + rmI + I)}{m!} \sum_{j=0}^{m} \frac{(-mI)_j}{j!}\ H_j. \tag{5.5}$$

Since the condition (5.3) holds, $H_m = 0$ for $m \neq \delta l,\ l \in \mathbb{N}$, whereas

$$H_{m\delta} = \sum_{j=0}^{m\delta}(-m\delta I)_j\ \Gamma^{-1}(A + rjI + I)\ Z_{j^*}^{(A,B,\mu,r)}(\lambda x^k; 1, p).$$

But since the series (5.2) holds true,

$$H_{m\delta} = \frac{(m\delta)!\ \Gamma^{-1}(A + rmI + I)}{m!}(\lambda x^k)^m.$$

Consequently, the inverse pair (5.4) and (5.5) assume the form:

$$\begin{aligned}\frac{(\lambda x^k)^m}{m!} I &= \frac{\Gamma(A + rmI + I)}{(m\delta)!} \sum_{j=0}^{m\delta}(-m\delta I)_j\ \Gamma^{-1}(A + rjI + I) \\ &\quad \times Z_{j^*}^{(A,B,\mu,r)}(\lambda x^k; 1, p)\end{aligned}$$

$$\Rightarrow$$

$$\begin{aligned}Z_{j^*}^{(A,B,\mu,r)}(\lambda x^k; 1, p) &= \frac{\Gamma(A + rmI + I)}{m!} \sum_{j=0}^{\lfloor m/\delta \rfloor} \frac{(-mI)_{\delta j}}{(\delta j)!}\ H_{\delta j} \\ &= \frac{\Gamma(A + rmI + I)}{m!} \sum_{j=0}^{\lfloor m/\delta \rfloor} \frac{(-mI)_{\delta j}}{j!} \\ &\quad \times \Gamma^{-1}(A + rjI + I)\ (\lambda x^k)^j,\end{aligned}$$

subject to the condition (5.3). □

If $\delta = 1$, then Theorem 5.2 takes the following form.

Corollary 5.3. *In the usual notations and meaning,*

$$\begin{aligned} Z_m^{(A,B,\mu,r)}(\lambda x^k;1,p) &= \frac{\Gamma(A+rmI+I)}{m!}\sum_{n=0}^{m}(-mI)_n \\ &\quad \times[(B)_{\mu n}^p]^{-1}\ \Gamma^{-1}(A+rnI+I)\frac{(\lambda x^k)^n}{n!}, \end{aligned} \tag{5.6}$$

if and only if

$$\begin{aligned} (\lambda x^k)^m I &= \Gamma(A+rmI+I)(B)_{\mu m}^p\sum_{j=0}^{m}(-mI)_j \\ &\quad \times\Gamma^{-1}(A+rjI+I)\ Z_j^{(A,B,\mu,r)}(\lambda x^k;1,p). \end{aligned} \tag{5.7}$$

This is an evident consequence of Lemma 5.1 with

$$H_m = \Gamma^{-1}(A+rmI+I)m!Z_{m^*}^{(A,B,\mu,r)}(\lambda x^k;1,p)$$

and

$$G_m = \Gamma^{-1}(A+rmI+I)[(B)_{\mu m}^p]^{-1}(\lambda x^k)^m.$$

The Konhauser matrix polynomial and its inverse [24, Eq.(3.2) and (3.29), p.618, 626] follow from from the Corollary 5.3 when $\mu = 0$. Further, if $k = 1$, then we obtain the inverse pair for the Laguerre matrix polynomial (4.2) [14, Eq. (26), p. 5].

6. GENERATING FUNCTION

We obtain here a matrix generating function relation involving the L-exponential function

$$e_k(x) = \sum_{n=0}^{\infty}\frac{x^n}{(n!)^{k+1}}$$

of order k, due to Ricci and Tavkhelidze [20].

Theorem 6.1. *In the usual notations and meaning, there holds the generating function relation:*

$$\sum_{m=0}^{\infty}(A+I)_{rm}^{-1}\ Z_{m^*}^{(A,B,\mu,r)}(\lambda x^k;s,p)\ t^{ms} = e_{s-1}(t^s)$$

$$\times\ {}_0F_{r+\mu p}\left(-;\left(\frac{B}{\mu}\right)^p,\ldots,\left(\frac{B+(\mu-1)I}{\mu}\right)^p,\frac{A+I}{r},\ldots,\frac{A+rI}{r};Y\right),$$

where $Y = (-t)^{\delta s}\ \lambda x^k/r^r\ \mu^{\mu p}$.

Proof. Beginning with the left hand side, we have

$$\begin{aligned}
&\sum_{m=0}^{\infty}(A+I)_{rm}^{-1}\ \ Z_{m^*}^{(A,B,\mu,r)}(\lambda x^k;s,r)t^{ms}\\
=\ &\sum_{m=0}^{\infty}\sum_{n=0}^{\lfloor m/\delta\rfloor}\frac{(-1)^{s\delta n}(A+I)_{rn}^{-1}}{n!\ [(m-\delta n)!]^s}(B)_{\mu n}^{-p}(\lambda x^k)^n t^{ms}\\
=\ &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{s\delta n}(A+I)_{rn}^{-1}}{n!\ (m!)^s}(B)_{\mu n}^{-p}(\lambda x^k)^n t^{ms+\delta sn}\\
=\ &\sum_{m=0}^{\infty}\frac{t^{ms}}{(m!)^s}\sum_{n=0}^{\infty}\frac{(-1)^{s\delta n}(A+I)_{rn}^{-1}}{n!}(B)_{\mu n}^{-p}(\lambda x^k)^n t^{\delta sn},
\end{aligned}$$

which is the right hand side expression. □

When $\delta = 1 = r = s = k$ and $\mu = 0$ then this reduces to the generating function for $L_n^{(A,\lambda)}(x)$ stated in (4.2).

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RESHMA SANJHIRA,
Assistant Professor, Department of Mathematical Sciences, P. D. Patel Institute of Applied Sciences, Faculty of Applied Sciences, Charotar University of Science and Technology, Changa-388 421, Dist: Anand, India
Research Scholar, Department of Mathematics, Faculty of Science, The Maharaja Sayajirao University of Baroda, Vadodara-390 002, India
e-mail address: *reshmashah.maths@charusat.ac.in*

B. V. NATHWANI,
Assistant Professor, Ramrao Adik Institute of Technology, Sector 7, Nerul, Navi Mumbai-400706, India
e-mail address: *bharti.nathwani@yahoo.com*

B. I. DAVE,
Professor, Department of Mathematics, Faculty of Science, The Maharaja Sayajirao University of Baroda, Vadodara-390002, India
e-mail address: *bidavemsu@yahoo.co.in*

e-mail for correspondence: *bidavemsu@yahoo.co.in*

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A GENERAL INVERSE MATRIX SERIES RELATION AND ASSOCIATED POLYNOMIALS – II

Reshma Sanjhira

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ABSTRACT. We propose a matrix analogue of a general inverse series relation with an objective to introduce the generalized Humbert matrix polynomial, Wilson matrix polynomial, and the Rach matrix polynomial together with their inverse series representations. The matrix polynomials of Kiney, Pincherle, Gegenbauer, Hahn, Meixner-Pollaczek etc. occur as the special cases. It is also shown that the general inverse matrix pair provides the extension to several inverse pairs due to John Riordan [*An Introduction to Combinatorial Identities*, Wiley, 1968].

1. Introduction

Let

$$a_n = \sum_{k=0}^{n} \alpha_{n,k}\, b_k, \qquad b_n = \sum_{k=0}^{n} \beta_{n,k}\, a_k$$

be the given series. They are said to form a pair of inverse series relation if one of the series when substituted into the other, yields the expression involving the Kronecker delta:

$$\delta_{nk} = \begin{cases} 0, & \text{if } k \neq n \\ 1, & \text{if } k = n. \end{cases}$$

To illustrate this, consider the pair of the series

$$f(n) = \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!}\, g(k), \qquad g(n) = \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!}\, f(k).$$

Here, if the second series is substituted into the first series then the inner sum simplifies to the form

$$\sum_{k=j}^{n} \frac{(-1)^{k+j}}{(n-k)!\,(k-j)!} = \delta_{nj},$$

which proves the one sided inverse series relation. The other part follows similarly.
It appears from the literature that the inverse series relations were systematically studied by H. W. Gould (see [5,12–14] and subsequently by L. Carlitz (see [3–5]).

For last few decades, the theory of hypergeometric function and its various generalization were provided matrix extension from varied point of view due to their applications in Physics, Statistics,

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Group representation theory, Engineering, Probability theory, Control theory, Lie theory, Medical imaging etc. (see [2, 6, 8, 10, 11, 17, 21]).

In this paper, our objective is to obtain a matrix analogue of Gould's general inverse series relation [14] by means of a general matrix inversion pair(GMIP) and thereby deduce the generalized Humbert matrix polynomial together with its inverse series. The matrix analogues of the particular polynomials belonging to this generalized matrix polynomial follow readily along with their inverse series relations. Besides this, we also deduce matrix inverse pairs by means of GMIP which extend the Riordan's inverse pairs belonging to the Gould classes, Simpler Legendre classes and Legendre-Chebyshev classes (see [23: Ch. 2]).

We first list out in the following, the notations and basic formulas for the matrix case.

1.1. Preliminaries

Throughout this paper, the notation $\mathbf{C}^{p\times p}$ stands for the linear space of all square matrices with in general complex entries.

Let $A_0, A_1, A_2, \ldots, A_n$ be the matrices in $\mathbf{C}^{p\times p}$ and A_n is a non-zero matrix. Then a matrix polynomial of degree n in x is given by [18]

$$P(x) = A_n x^n + A_{n-1}x^{n-1} + A_{n-2}x^{n-2} + \cdots + A_1 x + A_0. \tag{1.1}$$

The 2-norm of a matrix A denoted by $\|A\|_2$ is defined by

$$\| A \| = \sup_{x=0} \frac{\| Ax \|_2}{\| x \|_2} = \max\{\sqrt{\lambda} : \lambda \in \sigma(A^*A)\}, \tag{1.2}$$

where for a vector $u \in \mathbf{C}^p, \|u\|_2 = \sqrt{(u^T u)}$ is the Euclidean norm of u, and A^* denotes the transposed conjugate of A. The spectrum $\sigma(A)$ denotes the set of all eigenvalues of A. For a complex number α, the notation $\Re(\alpha)$ indicates the real part of α.

Definition 1. A square matrix A is said to be positive stable matrix in $\mathbf{C}^{p\times p}$ if A satisfies the condition:

$$\Re(\mu) > 0 \qquad \text{for all } \mu \in \sigma(A), \tag{1.3}$$

where $\sigma(A)$ is the set of all eigenvalues of A and μ is in general, a complex number.

For a positive stable matrix A, the gamma matrix function exists as an Euler integral (see [24] and [25: Ex 2.1, p. 164]). Also, if $f(z)$ and $g(z)$ are defined on an open set G of the complex z-plane and they are analytic on G and if $G \supset \sigma(A)$, then from [9], we have

$$f(A)g(A) = g(A)f(A). \tag{1.4}$$

Moreover, if B in $\mathbf{C}^{p\times p}$ is a matrix for which $\sigma(B) \subset G$, and $AB = BA$, then

$$f(A)g(B) = g(B)f(A). \tag{1.5}$$

The reciprocal gamma function $\frac{1}{\Gamma(z)} = (\Gamma(z))^{-1} = \Gamma^{-1}(z)$ is an entire function of complex variable z (see [16: p. 253]) and thus for any matrix A in $\mathbf{C}^{p\times p}$, the functional calculus [9] shows that $\Gamma^{-1}(A)$ is a well defined matrix.

If I denotes the identity matrix of order p and $A + nI$ is invertible for every integer $n \geq 0$ then ([7, 19])

$$\Gamma^{-1}(A) = A(A + I)\cdots(A + (n-1)I)\ \Gamma^{-1}(A + nI). \tag{1.6}$$

From this, the functional equation of the gamma matrix function

$$A\,\Gamma(A) = \Gamma(A + I) \tag{1.7}$$

follows readily for $n = 1$. For a matrix A in $\mathbf{C}^{p\times p}$, the Pochhammer matrix symbol is defined by [19]

$$(A)_n = \begin{cases} I, & \text{if } n = 0 \\ A(A+I)\cdots(A+(n-1)I), & \text{if } n \geq 1. \end{cases}$$

Also, if $I - A - nI$ is invertible for all $n \geq 0$, then

$$(A)_{n-k} = (-1)^k \; n! \; (A)_n \; (I - A - nI)_k^{-1}. \tag{1.8}$$

If $A - nI$ is invertible for all $n \geq 1$, then in view of the product

$$(-A+I)_n(A-I)^{-1}(A-2I)^{-1}\cdots(A-nI)^{-1} = (-1)^n \; I,$$

we define

$$(A)_{-n} = (A-I)^{-1}(A-2I)^{-1}\cdots(A-nI)^{-1} = (-1)^n(-A+I)_n^{-1}.$$

Hence,

$$\Gamma(A-nI)\Gamma^{-1}(A) = (A)_{-n} = (-1)^n(-A+I)_n^{-1}. \tag{1.9}$$

For $A(k,n) \in \mathbf{C}^{p\times p}$, $n, k \geq 0$ and $m \in \mathbf{N}$, there hold the double series identities [22]:

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} A(k,j) = \sum_{j=0}^{\infty}\sum_{k=0}^{j} A(k,j-k), \tag{1.10}$$

and

$$\sum_{k=0}^{N}\sum_{j=0}^{N-k} A(k,j) = \sum_{j=0}^{N}\sum_{k=0}^{j} A(k,j-k). \tag{1.11}$$

We shall denote the zero matrix by O.

We define the action of two operators, namely the shift operator E and the forward difference operator Δ on a matrix polynomial function (1.1) as follows.

Definition 2. Let $P(x)$ be a matrix polynomial of degree n in x of the form (1.1), then (see [15; p. 175])

$$E(P(x)) := P(x+h) \tag{1.12}$$

and

$$\Delta P(x) := P(x+h) - P(x), \tag{1.13}$$

where $h > 0$.

Also, for $r \in \mathbf{N}$, $E^r P(x) = E(E^{r-1})P(x)$, and $\Delta^r P(x) = \Delta(\Delta^{r-1})P(x)$. For $r = 0$, $E^0 := \mathbf{I}$, $\Delta^0 := \mathbf{I}$.

We now illustrate that how a particular matrix polynomial and its inverse matrix series are deduced from a general inversion pair. For that we consider the generalized Konhauser matrix polynomial and its inverse series (see [25; Cor. 5.3] and [27; Eq. (3.2) and (3.29), pp. 618, 626]), with $\mu = 0$) given by

$$Z_n^{(A,B,\mu,r)}(\lambda x^s; 1, l) = \frac{\Gamma(A+rnI+I)}{n!}\sum_{k=0}^{n}\frac{(-nI)_k}{k!}\;[(B)_{\mu k}^{l}]^{-1}\;\Gamma^{-1}(A+rkI+I)\;(\lambda x^s)^k,$$

$$(\lambda x^s)^n \; I = \Gamma(A+rnI+I)\;(B)_{\mu n}^{l}\;\sum_{k=0}^{n}(-nI)_k\;\Gamma^{-1}(A+rkI+I)\;Z_k^{(A,B,\mu,r)}(\lambda x^s; 1, l),$$

where $l, s, \mu \in \mathbb{R}_{>0}, r \in \mathbb{C}$, and a general matrix inversion pair:

$$\mathbf{u}(n) = \sum_{k=0}^{n}\overline{\alpha}_{n,k}\;\mathbf{v}(k), \qquad \mathbf{v}(n) = \sum_{k=0}^{n}\overline{\beta}_{n,k}\;\mathbf{u}(k). \tag{1.14}$$

Then the substitutions $\mathbf{v}(j) = (\lambda x^s)^j[(B)^l_{\mu j}]^{-1}\Gamma^{-1}(A + rjI + I)/j!$, $\overline{\alpha}_{n,j} = (-1)^j/(n-j)! = \overline{\beta}_{n,j}$ and $\mathbf{u}(j) = \Gamma^{-1}(A + rjI + I)Z_j^{(A,B,\mu,r)}(\lambda x^s; 1, l)$ provide the above generalized Konhauser matrix polynomial and its inverse. Further, if $s = 1$ and $r = 1$, then this matrix polynomial and its inverse get reduced to the Laguerre matrix polynomial and its inverse [18: pp. 3,5].

2. Auxiliary results

The following is the matrix analogue of an n^{th} difference of a matrix polynomial of degree less than n.

Lemma 2.1. *If $P(x)$ is a matrix polynomial of degree less than n then*

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}P(a+hk) = O, \tag{2.1}$$

where $n \geq 1$ and a, h are constants.

Proof. Since the $\deg[P(x)] < n$, in view of the Definition 2, we have

$$\Delta^n P(x) = O.$$

Now ([15: p. 177]),

$$\begin{aligned} O = \Delta^n P(x) &= (E - I)^n P(x) \\ &= \left[E^n - \binom{n}{1}E^{n-1} + \binom{n}{2}E^{n-2} + \cdots + (-1)^n I\right]P(x) \\ &= E^n P(x) - \binom{n}{1}E^{n-1}P(x) + \cdots + (-1)^n P(x) \\ &= P(x+nh) - \binom{n}{1}P(x+(n-1)h) + \cdots + (-1)^n P(x) \\ &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\ P(x+kh), \end{aligned}$$

which leads to the lemma for $x = a$. □

In the similar notations of (1.14), we prove the following lemma which will be useful in the proof of the GMIP in Section 3.

Lemma 2.2. *For a positive stable matrix $A \in C^{p\times p}$,*

$$\mathbf{f}_j = \sum_{k=0}^{j}(-1)^k\binom{j}{k}\Gamma(A - (nr - mrk + k - j)I)\ \mathbf{g}_k \tag{2.2}$$

$$\Longleftrightarrow$$

$$\mathbf{g}_j = \sum_{k=0}^{j}(-1)^k\binom{j}{k}\ \Gamma^{-1}(A + (1 - nr + mrj - j + k)I) \times (A - (nr - mrk)I)\ \mathbf{f}_k. \tag{2.3}$$

Proof. We prove that the series in (2.3) $\Rightarrow$ (2.2). For that we assume that the series in (2.3) holds true. If $\mathbf{F}_j$ stands for the right hand side of (2.2) then on substituting the series for $\mathbf{g}_k$ from (2.3), we get

$$\mathbf{F}_j = \sum_{k=0}^{j}(-1)^k\binom{j}{k}\Gamma(A-(nr-mrk+k-j)I)\sum_{i=0}^{k}(-1)^i\binom{k}{i} \\ \times\Gamma^{-1}(A+(1-nr+mrk-k+i)I)(A-nrI+mriI)\mathbf{f}_i.$$

Using the double series relation (1.11), this becomes

$$\begin{aligned}\mathbf{F}_j &= \sum_{i=0}^{j}\sum_{k=0}^{j-i}(-1)^k\binom{j}{k+i}\binom{k+i}{i}\Gamma(A-(nr-mr(k+i)+(k+i)-j)I)\\ &\quad\times\ \Gamma^{-1}(A+(1-nr+mr(k+i)-k)I)\ (A-nrI+mriI)\ \mathbf{f}_i\\ &= \sum_{i=0}^{j}\binom{j}{i}\sum_{k=0}^{j-i}(-1)^k\binom{j-i}{k}\ \Gamma(A-nrI+mr(k+i)I-(k+i)I+jI)\\ &\quad\times\Gamma^{-1}(A+(1-nr+(k+i)(mr-1)+i)I)\ (A-nrI+mriI)\ \mathbf{f}_i\\ &= \sum_{i=0}^{j-1}\binom{j}{i}\sum_{k=0}^{j-i}(-1)^k\binom{j-i}{k}\Gamma(A-(nr-mr(k+i)+(k+i)-j)I)\\ &\quad\times\Gamma^{-1}(A+(1-nr+mrk+mri-k)I)\ (A-nrI+mriI)\ \mathbf{f}_i+\mathbf{f}_j.\end{aligned} \tag{2.4}$$

Here the product of the matrix gamma function and the inverse matrix gamma function simplifies to a matrix polynomial in k, that is,

$$\Gamma(A-(nr-mr(k+i)+(k+i)-j)I) \\ \times\Gamma^{-1}(A+(1-nr+mrk+mri-k)I)=\sum_{s=0}^{j-i-1}\overline{\alpha}_s k^s,$$

where the coefficient matrices $\overline{\alpha}_s=\overline{\alpha}_s(A,m,n,r,i)$. For illustration, let us take $j=5; i=2$ and denote $A-nrI-mriI$ by B and $(mr-1)I$ by N then we have

$$\begin{aligned}\Gamma(B+kN+3I)\Gamma^{-1}(B+kN+I) &= (B+kNI+2I)(B+kNI+I)\\ &= k^2N^2+k(2NB+3N)+(B^2+3B+2I)=\sum_{r=0}^{2}\overline{s}_r k^r,\end{aligned}$$

say, where $\overline{s}_2=N^2\neq O$. Hence from (2.4), we have

$$\mathbf{F}_j=\mathbf{f}_j+\sum_{i=0}^{j-1}\binom{j}{i}\left[\sum_{k=0}^{j-i}(-1)^k\binom{j-i}{k}\sum_{s=0}^{j-i-1}\overline{\alpha}_s\ k^s\right](A-(nr-mri)I)\ \mathbf{f}_i.$$

Here, the two inner series on the right hand side are $(j-i)^{th}$ difference of the polynomial of degree $j-i-1$, hence in view of Lemma 2.1, $\mathbf{F}_j=\mathbf{f}_j$.

We see that the diagonal elements $\Gamma^{-1}(A-nrI+mrNI-NI+jI)$ of the block matrix corresponding to the series (2.2) and those given by $\Gamma(A-nrI+mrNI-NI+jI)$ of the block matrix corresponding to the series (2.3) are non singular matrices for every matrix $A\neq nrI-mrNI+NI-jI$, $j=0,1,2,\ldots$, hence the inverse of each block matrix is unique. Since (2.3) $\Rightarrow$ (2.2), it follows that (2.2) $\Leftrightarrow$ (2.3). □

3. Main result

Consider the series

$$\mathbf{u}(a)=\sum_{k=0}^{M}\overline{\mu}_{n,k}\mathbf{v}(a+bk),\qquad \mathbf{v}(a)=\sum_{k=0}^{M}\overline{\sigma}_{n,k}\mathbf{u}(a+bk),$$

where M is a non negative integer or infinity depending upon whether b is a negative integer or a positive integer. In particular, let a be a non negative integer n. If b is a negative integer $-m$, $m\in N$, then $M=[n/m]$ and if b is a positive integer then $M=\infty$.

As a main result, we derive the inverse matrix series relations in

Theorem 3.1. *Let $A-sI$, $s\in\{0\}\cup\mathbf{N}$, be a positive stable matrix in $\mathbf{C}^{p\times p}$ then*

$$\mathbf{U}(n)=\sum_{k=0}^{M}\frac{\eta^k\Gamma^{-1}(A+(1-nr-brk-k)I)}{k!}\mathbf{V}(n+bk) \tag{3.1}$$

if and only if

$$\mathbf{V}(n)=\sum_{k=0}^{M}(-\eta)^k\ \Gamma(A-(nr-k)I)\ \frac{A-(nr+brk)I}{k!}\mathbf{U}(n+bk). \tag{3.2}$$

P r o o f. Throughout the proof, we let n to be a non negative integer. We first choose $b=-m, m\in\mathbf{N}$, then $M=[n/m]$.

Now, if $\mathbf{U}$ denotes the right hand member of the series (3.1), then on substituting the series from (3.2) for $\mathbf{V}(n-mk)$, we get

$$\begin{aligned}\mathbf{U}=&\sum_{mk=0}^{n}\frac{\eta^k\ \Gamma^{-1}(A+(1-nr+mrk-k)I)}{k!}\sum_{mj=0}^{n-mk}\frac{(-\eta)^j\Gamma(A-(nr-mrk-j)I)}{j!}\\&\times(A-(nr-mrk-mrj)I)\mathbf{U}(n-mk-mj).\end{aligned}$$

From the double series relation (1.11), we further have

$$\begin{aligned}\mathbf{U}=&\sum_{mj=0}^{n}\frac{(-\eta)^j}{j!}\sum_{k=0}^{j}(-1)^k\binom{j}{k}\ \Gamma^{-1}(A-(nr-mrk+k-1)I)\\&\times\ \Gamma(A-(nr-mrk+k-j)I)(A-(nr-mrj)I)\mathbf{U}(n-mj).\end{aligned} \tag{3.3}$$

Now, for $r,s=0,1,2,\dots$, the matrices $A-rI$ and $\Gamma(A-sI)$ are commutative, where the matrices $A-sI$ are positive stable, hence

$$\begin{aligned}&\Gamma^{-1}(A-(nr-mrk+k-1)I)\Gamma(A-(nr-mrk+k-j)I)\\&=\prod_{i=1}^{j-1}(A-(nr-mrk+k+i)I)=\sum_{s=0}^{j-1}\overline{\zeta}_s\ k^s,\end{aligned}$$

where $\overline{\zeta}_0=\prod_{i=1}^{j-1}(A-nrI-iI)$, $\overline{\zeta}_{j-1}=(mr-1)^{j-1}I$ and for $0<s<j-1$,

$$\overline{\zeta}_s=\sum_{\substack{u_1,u_2,\dots u_s=1\\u_1\neq u_2\neq\cdots\neq u_s}}^{j-1}\left\{\prod_{i=1}^{s}(A-nrI+u_iI)\right\}.$$

Therefore from (3.3), we get

$$\mathbf{U} = \mathbf{U}(n) + \sum_{mj=1}^{n} \frac{(-\eta)^j}{j!} \left[\sum_{k=0}^{j} (-1)^k \binom{j}{k} \sum_{s=0}^{j-1} \overline{\zeta}_s \, k^s \right] (A - (nr - mrj)I)\mathbf{U}(n - mj).$$

In view of Lemma 2.1, the second term on the right hand side will be a null matrix for all $j \geq 1$; consequently $\mathbf{U} = \mathbf{U}(n)$. With this, the proof of the first part is completed.

In order to prove the converse part, let us take

$$\mathbf{V} = \sum_{mk=0}^{n} \frac{(-\eta)^k}{k!} \Gamma(A - (nr - k)I)(A - (nr - mrk)I)\mathbf{U}(n - mk),$$

then substituting the series (3.1) for $\mathbf{U}(n - mk)$ and using the double sum (1.11), we arrive at

$$\begin{aligned} \mathbf{V} = & \sum_{mj=0}^{n} \frac{\eta^j}{j!} \sum_{k=0}^{j} (-1)^k \binom{j}{k} \Gamma^{-1}(A + (1 - nr + mrj - j + k)I) \\ & \times \; \Gamma(A - (nr - k)I)(A - (nr - mrk)I)\mathbf{V}(n - mj). \end{aligned} \tag{3.4}$$

We now show that the inner series in (3.4) is equal to $\delta_{jo}I$. Here, denoting the inner series in (3.4) by $\mathbf{g}_j$, and replacing $\Gamma(A - (nr - k)I)$ by $\mathbf{f}_k$ then in view of (1.4) and (1.5), we get

$$\begin{aligned} \mathbf{g}_j = & \sum_{k=0}^{j} (-1)^k \binom{j}{k} \Gamma^{-1}(A + (1 - nr + mrj - j + k)I) \\ & \times (A - (nr - mrk)I)\mathbf{f}_k. \end{aligned} \tag{3.5}$$

The inverse series of this occurs from Lemma 2.2, in the form:

$$\mathbf{f}_j = \sum_{k=0}^{j} (-1)^k \binom{j}{k} \Gamma(A - (nr - mrk + k - j)I) \; \mathbf{g}_k. \tag{3.6}$$

If

$$\mathbf{g}_s = \binom{0}{s} I$$

is set in the inverse series (3.6), then we obtain $\mathbf{f}_j = \Gamma(A - (nr - j)I)$ (as before), and using the same substitution in (3.5), there occurs the matrix series orthogonality relation:

$$\begin{aligned} \binom{0}{j} I = \delta_{jo} I = & \sum_{k=0}^{j} (-1)^k \binom{j}{k} \Gamma^{-1}(A + (1 - nr + mrj - j + k)I) \\ & \times \; \Gamma(A - (nr - k)I)(A - (nr - mrk)I). \end{aligned}$$

Thus (3.4) becomes

$$\mathbf{V} = \mathbf{V}(n) + \sum_{mj=1}^{n} \frac{\eta^j}{j!} \; \mathbf{V}(n - mj)\delta_{jo},$$

giving $\mathbf{V} = \mathbf{V}(n)$. This completes the proof of the converse part and hence the proof of the theorem when $M = [n/m]$.

The proof for the case $M = \infty$ runs almost parallel to the above proof. We assume that the sequences $\{\mathbf{U}(n)\}$ and $\{\mathbf{V}(n)\}$ are such that $\|\mathbf{U}(n)\| < \infty$ and $\|\mathbf{V}(n)\| < \infty$. In order to prove the first part, we denote the right hand side of (3.1) by $\mathbf{R}$ and substitute the series for $\mathbf{V}(n-mk)$ from (3.2), to get

$$\mathbf{R} = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{\eta^{k+j}\Gamma^{-1}(A+(1-nr-brk-k)I)}{k!\ j!} \times\ \Gamma(A-(nr+brk-j)I)\ (A-(nr+brk+brj)I)\mathbf{U}(n+bk+bj).$$

This, with the help of (1.10) takes the form

$$\begin{aligned}\mathbf{R} &= \sum_{j=0}^{\infty}(-1)^j\eta^j\sum_{k=0}^{j}(-1)^k\binom{j}{k}\ \Gamma^{-1}(A+(1-nr-brk-k)I)\\ &\quad\times\Gamma(A-(nr+brk-j+k)I)\ \frac{(A-(nr+brj)I)}{j!}\mathbf{U}(n+bj)\\ &= \mathbf{U}(n)+\sum_{j=1}^{\infty}(-1)^j\eta^j\sum_{k=0}^{j}(-1)^k\binom{j}{k}\Gamma^{-1}(A+(1-nr-brk-k)I)\\ &\quad\times\Gamma(A-(nr+brk-j+k)I)\ \frac{(A-(nr+brj)I)}{j!}\mathbf{U}(n+bj).\end{aligned} \tag{3.7}$$

Since the inner series in this last expression is resembling with the inner series occurring in (3.3), it follows that the expression (3.7) yields the relation $\mathbf{R}=\mathbf{U}(n)$. Conversely, let us put

$$\sum_{k=0}^{\infty}\frac{(-\eta)^k}{k!}\Gamma(A-(nr-k)I)\ (A-(nr+brk)I)\mathbf{U}(n+bk)=\mathbf{S}.$$

Then on making use of the series (3.1) and (1.10) in turn, we find that

$$\begin{aligned}\mathbf{S} &= \sum_{j=0}^{\infty}\frac{(-\eta)^j}{j!}\sum_{k=0}^{j}(-1)^k\binom{j}{k}\ \Gamma^{-1}(A+(1-nr-brj-j+k)I)\\ &\quad\times\Gamma(A-(nr-k)I)(A-(nr+brk)I)\mathbf{V}(n+bj).\end{aligned} \tag{3.8}$$

Again, that the inner series in (3.7) is of the similar form as that of (3.4). Thus following the same arguments, we find the following orthogonal relation implied by the inner series in (3.7).

$$\begin{aligned}\binom{0}{j}I &= \sum_{k=0}^{j}(-1)^k\binom{j}{k}\Gamma^{-1}(A+(1-nr-brj-j+k)I)\Gamma(A-(nr-k)I)\\ &\quad\times(A-(nr+brk)I).\end{aligned} \tag{3.9}$$

With this orthogonal relation, the expression (3.8) gives

$$\mathbf{S}=\mathbf{V}(n); \tag{3.10}$$

this completes the proof of the second part and hence the proof of the theorem. □

4. Particular cases

In Theorem 3.1, putting $r = 1, b = -m, m \in \mathbf{N}$, $\mathbf{V}(n-mk) = (-mx)^{n-mk}c^{A-(n-mk)I}/(n-mk)!$ and replacing η by ηc^{-I}, an explicit representation of the generalized Humbert polynomials $P_n^A(m,x,\eta,c)\Gamma^{-1}(A+I) = \mathbf{U}(n)$ occurs in the form:

$$P_n^A(m,x,\eta,c) = \sum_{k=0}^{[n/m]} \eta^k \frac{c^{A-(n-(m-1)k)I}}{(n-mk)!\,k!}\Gamma^{-1}(A+(1-n+mk-k)I)\Gamma(A+I)(-mx)^{n-mk} \quad (4.1)$$

and its inverse series in the form:

$$\begin{aligned}\frac{(-mx)^n}{n!}I = &\sum_{k=0}^{[n/m]} (-\eta)^k\; c^{nI-kI-A}\frac{(A-nI+mkI)(A-nI+kI)^{-1}}{k!}\\ &\times \Gamma(A+(1+k-n)I)P_{n-mk}^A(m,x,\eta,c).\end{aligned} \quad (4.2)$$

The formula (1.9) provides the alternative form of the polynomial (4.1), given by [28: Eq. (37), p. 3626]

$$P_n^A(m,x,\eta,c) = \sum_{k=0}^{[n/m]} (-\eta)^k\; c^{-(A+(n-mk-k)I)}\frac{(-A)_{n-mk+k}}{(n-mk)!\,k!}(mx)^{n-mk}. \quad (4.3)$$

The inverse series

$$\frac{(mx)^n}{n!}I = \sum_{k=0}^{[n/m]} \eta^k\frac{(A-nI+mkI)(A-nI+kI)^{-1}c^{nI-kI-A}}{k!}(-A)_{n-k}^{-1}P_{n-mk}^A(m,x,\eta,c) \quad (4.4)$$

follows similarly. With $c = \eta = 1$, (4.3) and (4.4) yield the pair of Humbert matrix polynomial: $\Pi_{n,m}^A(x)$ and its inverse in the form [28: Eq. (40), p. 3626]:

$$\begin{aligned}&\Pi_{n,m}^A(x) = \sum_{k=0}^{[n/m]} (-1)^k\frac{(-A)_{n-mk+k}}{(n-mk)!\,k!}\,(mx)^{n-mk}\\ &\Longleftrightarrow\\ &\frac{(mx)^n}{n!}I = \sum_{k=0}^{[n/m]} \frac{(A-nI+mkI)(A-nI+kI)^{-1}}{k!}(-A)_{n-k}^{-1}\Pi_{n-mk,m}^A(x).\end{aligned} \quad (4.5)$$

In fact, this polynomial constitutes the class $\{\Pi_{n,m}^A(x); n = 0,1,2,\ldots\}$ of polynomials which includes several well known polynomials as well as *not so well known* polynomials. The following are amongst them.
If $A = \frac{1}{m}I$, then (4.5) reduces to the pair of Kinney matrix polynomial and its inverse series relation as follows.

$$\begin{aligned}&K_n^A(m,x) = \sum_{k=0}^{[n/m]} (-1)^k\frac{((-1/m)I)_{n-mk+k}}{(n-mk)!\,k!}\,(mx)^{n-mk}\\ &\Longleftrightarrow\\ &\frac{(-mx)^n}{n!}I = \sum_{k=0}^{[n/m]} \frac{\left(-\frac{1}{m}I-(n-mk)I\right)\,\left(-\frac{1}{m}I-(n-k)I\right)^{-1}}{k!}((-1/m)I)_{n-k}^{-1}\;K_{n-mk}^A(m,x).\end{aligned}$$

For $m=3$, and $A=-\frac{1}{2}I$ the pair (4.5) yields the pair of Pincherle matrix polynomial and its inverse series relation in the form [22: p. 212] :

$$\mathcal{P}_n(x)=\sum_{k=0}^{[n/3]}(-1)^k\,\frac{((1/2)I)_{n-2k}}{(n-3k)!\;k!}\,(3x)^{n-3k}$$
$$\Longleftrightarrow$$
$$\frac{(3x)^n}{n!}I=\sum_{k=0}^{[n/3]}\frac{((1/2)I+(n-3k)I)\;\;((1/2)I+(n-k)I)^{-1}}{k!}((1/2)I)^{-1}_{n-k}\;\mathcal{P}_{n-3k}(x).$$

The Gegenbauer matrix polynomial and its inverse are the special cases $m=2$ of (4.5) which occur in the form [26: Eq. (15), p. 104]:

$$C_n^A(x)=\sum_{k=0}^{[n/2]}(-1)^k\frac{(-A)_{n-k}}{(n-2k)!\;k!}\,(2x)^{n-2k}$$
$$\Longleftrightarrow$$
$$\frac{(2x)^n}{n!}I=\sum_{k=0}^{[n/2]}\frac{(A+(n-2k)I)(A+(n-k)I)^{-1}}{k!}(-A)^{-1}_{n-k}\;C^A_{n-2k}(x).$$

5. Alternative forms of theorem – I

In this section, several alternative forms of Theorem 3.1 are deduced which will be used in the next section for deducing the matrix versions of the Wilson polynomials, Racah polynomials and Riordan's inverse pairs. We begin with Theorem 3.1 and substitute $\eta=1$ and replace $\mathbf{V}(n)$ by $\Gamma(A-nrI+I)\mathbf{V}(n)$ to get

$$\begin{aligned}\mathbf{U}(n)&=\sum_{k=0}^{M}\frac{\Gamma(A-(nr+brk-1)I)}{k!}\Gamma^{-1}(A+(1-nr-brk-k)I)\mathbf{V}(n+bk)\\ &\Longleftrightarrow \\ \mathbf{V}(n)&=\sum_{k=0}^{M}(-1)^k\frac{(A-(nr+brk)I)}{k!}\Gamma(A-(nr-k)I)\Gamma^{-1}(A-(nr-1)I)\mathbf{U}(n+bk).\end{aligned}\tag{5.1}$$

This pair enables us to obtain some more alternative pairs. In fact, on multiplying both the relation in (5.1) by $A-nrI$ and putting $(A-nrI)\mathbf{U}(n)=\mathbf{U}_1(n)$ and $(A-nrI)\mathbf{V}(n)=\mathbf{V}_1(n)$, we obtain the pair:

$$\begin{aligned}\mathbf{U}_1(n)&=\sum_{k=0}^{M}\frac{\Gamma(A-(nr+brk)I)}{k!}\;\Gamma^{-1}(A-(nr+brk+k-1)I)\\ &\quad\times(A-nrI)\;\mathbf{V}_1(n+bk)\\ &\Longleftrightarrow\\ \mathbf{V}_1(n)&=\sum_{k=0}^{M}(-1)^k\frac{\Gamma(A-(nr-k)I)}{k!}\Gamma^{-1}(A-nrI)\mathbf{U}_1(n+bk)\end{aligned}\tag{5.2}$$

Further on replacing A by $A+I$ and r by $-r$, (5.2) gets transformed to

$$
\begin{aligned}
U_1(n) &= \sum_{k=0}^{M} \frac{\Gamma(A+(nr+brk+1)I)}{k!}\Gamma^{-1}(A+(nr+brk-k+2)I) \\
&\quad \times (A+nrI+I)V_1(n+bk) \\
\Longleftrightarrow & \\
V_1(n) &= \sum_{k=0}^{M}(-1)^k \frac{\Gamma(A+(nr+k+1)I)}{k!}\Gamma^{-1}(A+nrI+I)U_1(n+bk).
\end{aligned}
\tag{5.3}
$$

Using the formula (1.9) in (5.1), we obtain the pair:

$$
\begin{aligned}
\mathbf{U}_2(n) &= \sum_{k=0}^{M} \frac{(-1)^k}{k!}(-A+nrI+brkI)_k \mathbf{V}_2(n+bk) \\
\Longleftrightarrow & \\
\mathbf{V}_2(n) &= \sum_{k=0}^{M} \frac{(A-(nr+brk)I)}{k!}\ (A-nrI+kI)^{-1}(-A+nrI)^{-1}_{-k} U_2(n+bk).
\end{aligned}
\tag{5.4}
$$

Also, applying the formula (1.9) in (5.2) and then replacing $-A$ by A, we get

$$
\begin{aligned}
\mathbf{U}_2(n) &= \sum_{k=0}^{M} \frac{(-1)^k}{k!}(A+nrI)(A+nrI+brkI+kI)^{-1} \\
&\quad \times (A+(nr+brk+1)I)_k\ \mathbf{V}_2(n+bk) \\
\Longleftrightarrow & \\
\mathbf{V}_2(n) &= \sum_{k=0}^{M} \frac{(A+nrI+I)^{-1}_{-k}}{k!}\mathbf{U}_2(n+bk).
\end{aligned}
\tag{5.5}
$$

6. Application

6.1. Wilson matrix polynomials and Racah matrix polynomials

In Theorem 3.1 putting $b=-1$, $\eta=1$, $r=2$ and reversing the series and assuming that $A-NI$ is invertible for all $N \geq 0$, we obtain

$$
\begin{aligned}
\mathbf{U}(n) &= \sum_{k=0}^{n} \frac{\Gamma^{-1}(A-nI-kI+I)}{(n-k)!}\ \mathbf{V}(k) \\
\Longleftrightarrow & \\
\mathbf{V}(n) &= \sum_{k=0}^{n} \frac{(-1)^{n-k}(A-2kI)(A-nI-kI)^{-1}\Gamma(A-nI-kI+I)}{(n-k)!}\ \mathbf{U}(k).
\end{aligned}
$$

Replacing $\mathbf{U}(n)$ by $\mathbf{U}(n)\Gamma^{-1}(A+I)$, we find

$$\mathbf{U}(n)=\sum_{k=0}^{n}\frac{\Gamma^{-1}(A-nI-kI+I)\Gamma(A+I)}{(n-k)!}\mathbf{V}(k)$$
$$\Longleftrightarrow$$
$$\mathbf{V}(n)=\sum_{k=0}^{n}\frac{(-1)^{n-k}(A-2kI)(A-nI-kI)^{-1}\Gamma(A-(n+k-1)I)\Gamma^{-1}(A+I)}{(n-k)!}\mathbf{U}(k).$$

In view of formula (1.9), this pair may be written in the form:

$$\mathbf{U}(n)=\sum_{k=0}^{n}\frac{(-A)_{n+k}}{(n-k)!}\mathbf{V}(k)$$
$$\Longleftrightarrow$$
$$\mathbf{V}(n)=\sum_{k=0}^{n}\frac{(-1)^{n-k}(A-2kI)(A-nI-kI)^{-1}(-A)_{n+k}^{-1}}{(n-k)!}\mathbf{U}(k).$$

If the second series is re-written in slightly different form, it becomes

$$\mathbf{U}(n)=\sum_{k=0}^{n}\frac{(-A)_{n+k}}{(n-k)!}\mathbf{V}(k)$$
$$\Longleftrightarrow$$
$$\mathbf{V}(n)=\sum_{k=0}^{n}\frac{(-1)^{n-k}(-A+2kI)(-A)_{n+k+1}^{-1}}{(n-k)!}\ \mathbf{U}(k).$$

Finally, using the formulas $(A)_{m+n}=(A)_m(A+mI)_n$and $(-1)^k(n!)I/(n-k)!=(-nI)_k$, we get the pair:

$$\begin{aligned}\mathbf{U}(n)&=\frac{(-A)_n}{n!}\sum_{k=0}^{n}(-1)^k(-nI)_k(-A+nI)_k\ \mathbf{V}(k)\\ &\Longleftrightarrow\\ \mathbf{V}(n)&=\frac{(-A)_n^{-1}}{n!}\sum_{k=0}^{n}(-1)^k(-nI)_k(2kI-A)(-A+nI)_{k+1}^{-1}\ \mathbf{U}(k).\end{aligned}\tag{6.1}$$

This inverse pair readily provides us the matrix polynomials of Wilson as well as those of Racah. In fact, the Wilson matrix polynomials together with the inverse series relation are obtainable from the pair (6.1) if A is replaced by $A+B+C+D-I$ and

$$\mathbf{V}(n)=\frac{(-1)^n(A+\mathrm{i}xI)_n(A-\mathrm{i}xI)_n}{n!}\ (A+B)_n^{-1}(A+C)_n^{-1}(A+D)_n^{-1}.$$

With this choice, (6.1) yields the Wilson matrix polynomials:

$$\begin{aligned}&P_{n,l,s}(x^2)(A+B)_n^{-1}(A+C)_n^{-1}(A+D)_n^{-1}\\ &=\sum_{k=0}^{n}\frac{(-nI)_{sk}}{k!}(A+B+C+D+nI+I)_{lk}\\ &\quad\times(A+\mathrm{i}xI)_k(A-\mathrm{i}xI)_k(A+B)_k^{-1}(A+C)_k^{-1}(A+D)_k^{-1}.\end{aligned}$$

The inverse series occurs from the second series of (6.1) in the form:

$$\frac{(A+\mathrm{i}xI)_n(A-\mathrm{i}xI)_n}{n!}(A+B)_n^{-1}(A+C)_n^{-1}(A+D)_n^{-1}$$
$$=\sum_{k=0}^{n}(-1)^{n-k}\frac{(-nI)_k}{n!}(A+B+C+D+2kI-I)$$
$$\times (A+B+C+D+kI-I)_{n+1}^{-1}(A+B)_k^{-1}(A+C)_k^{-1}(A+D)_k^{-1}P_{k,l,s}(x^2).$$

Similarly, replacing $-A$ by $A+B+I$ and putting

$$\mathbf{V}(n)=\frac{(-xI)_n(xI+D+E+I)_n}{n!}(A+I)_n^{-1}(B+E+I)_n^{-1}(D+I)_n^{-1}$$

in (6.1), we obtain the following pair of inverse series relation of the Racah matrix polynomials.

$$R_n(x(xI+D+E+I);A,B,D,E)$$
$$=\sum_{k=0}^{n}\frac{(-nI)_k}{k!}(A+B+nI+I)_k(-x)_k(xI+D+E+I)_k(A+I)_k^{-1}$$
$$\times (B+E+I)_k^{-1}(D+I)_k^{-1};$$

$$\frac{(-xI)_n(xI+D+E+I)_n}{k!}[(A+I)_n]^{-1}(B+E+I)_n^{-1}[(D+I)_n]^{-1}$$
$$=\sum_{k=0}^{n}(-1)^{n-k}\frac{(-nI)_k}{n!}(A+B+2kI+I)\,(A+B+kI+I)_{n+1}^{-1}$$
$$\times R_k(x(xI+D+E+I);A,B,D,E).$$

Since the scalar polynomials encompass several polynomials namely the polynomilas of Hahn, dual Hahn, continuous Hahn, continuous dual Hahn, Meixner-Pollaczek, Meixner, Krawtchouk, Jacobi etc. (see [1: p. 46] for complete reducibility chart), their extended matrix polynomials' versions would follow directly from these two matrix polynomials together with their inverse series relations.

6.2. Riordan's matrix pairs

John Riordan [23: Ch. 2] studied and classified a number of inverse series in to several classes. In this section, the Gould classes, simpler Legendre classes and the Legendre-Chebyshev classes are extended to the matrix series forms with the help of the alternating inverse pairs of Section 5.

7. Conclusion

It is seen that the main result, that is Theorem 3.1, is found useful in inverting the matrix polynomials and also, some of the Riordan's inverse pairs. It is noteworthy that there exist many more inverse series relations in [23: Ch. 2] which can be extended further in matrix forms. Besides this, the inverse pairs derived by H. W. Gould [12] and those by L. Carlitz ([3–5]) can also be extended to matrix forms for further study. Coming to the generalized functions, there is studied the Mittag-Leffler matrix function in [25]. By means of this function, a generalized Konhauser matrix polynomial was defined and some properties were studied. Since there are many generalized functions exist in the literature, there is vast scope of developing the matrix theory by extending such functions to the matrix forms.

TABLE 1. Gould matrix classes $F(n) = \sum C_{n,k}G(k)$; $G(n) = \sum(-1)^{n-k}D_{n,k}F(k)$, $(h_{r,s} = qr - s,\ B = A + I)$

Inv. pair $(a = n)$	b	r	A	$C_{n,k}$	$D_{n,k}$
(5.1)	-1	$1-q$	A	$\frac{\Gamma(B+h_{k,k}I)}{(n-k)!}\Gamma^{-1}(B+h_{k,n}I)$	$\frac{(A+(h_{k,k})I)}{(n-k)!}\Gamma^{-1}(B+h_{n,n}I)$ $\times\Gamma(A+(h_{n,k})I)$
(5.3)	-1	$q-1$	A	$\frac{(B+h_{n,n}I)}{(n-k)!}\Gamma(B+h_{k,k}I)$ $\times\Gamma^{-1}(B+(h_{k,n}+1)I)$	$\frac{\Gamma(B+h_{n,k}I)}{(n-k)!}\Gamma^{-1}(B+h_{n,n}I)$
(5.5)	1	$q-1$	A	$\frac{(A+(h_{n,n})I)}{(k-n)!}\Gamma(A+(h_{k,n})I)$ $\times\Gamma^{-1}(B+h_{k,k}I)$	$\frac{\Gamma(B+h_{n,n}I)}{(k-n)!}\Gamma^{-1}(B+h_{n,k}I)$
(5.4)	1	$q-1$	$-A$ $-I$	$\frac{\Gamma(B+h_{k,n}I)}{(k-n)!}\Gamma^{-1}(B+h_{k,k}I)$	$\frac{(B+h_{k,k}I)}{(k-n)!}\Gamma(B+h_{n,n}I)$ $\times\Gamma^{-1}(B+(h_{n,k}+1)I)$

TABLE 2. A. Simpler Legendre matrix classes-I $F(n) = \sum C_{n,k}\ G(k)$; $G(n) = \sum(-1)^{n-k}\ D_{n,k}\,F(k)\ (B = A + I)$

Inv. pair $a = n$	b	r	A	$C_{n,k}$	$D_{n,k}$
(5.4)	-1	2	$-A$ $-I$	$\frac{\Gamma(B+nI+kI)}{(n-k)!}\Gamma^{-1}(B+2kI)$	$\frac{(B+2kI)}{(n-k)!}\Gamma(B+2nI)$ $\times(B+nI+kI)^{-1}\Gamma^{-1}(B+nI+kI)$
(5.5)	-1	2	A	$\frac{(A+2nI)}{(n-k)!}\Gamma(A+nI+kI)$ $\times\Gamma^{-1}(B+2kI)$	$\frac{\Gamma(B+2nI)}{(n-k)!}\Gamma^{-1}(B+nI+kI)$
(5.3)	1	2	A	$\frac{(B+2nI)}{(k-n)!}\Gamma(B+2kI)$ $\times\Gamma^{-1}(A+nI+kI+2I)$	$\frac{\Gamma(B+nI+kI)}{(k-n)!}\Gamma^{-1}(B+2nI)$
(5.1)	1	-2	A	$\frac{\Gamma(B+2kI)}{(k-n)!}\Gamma^{-1}(B+nI+kI)$	$\frac{(A+2kI)}{(k-n)!}\Gamma(A+nI+kI)$ $\times\Gamma(A+nI+kI)\Gamma^{-1}(B+2nI)$

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TABLE 2. B. Simpler Legendre matrix classes-II $F(n) = \sum C_{n,k} G(n-2k)$; $G(n) = \sum (-1)^k D_{n,k} F(n-2k) \quad (B = A + I)$

Inv. pair $(a = n)$	b	r	A	$C_{n,k}$	$D_{n,k}$
(5.5)	-2	2	A	$\frac{(A+2nI)}{k!}\Gamma(A+2nI-3kI)$ $\times\Gamma^{-1}(B+2nI-4kI)$	$\frac{\Gamma(B+2nI)}{k!}\Gamma^{-1}(B+2nI-kI)$
(5.4)	-2	2	$-A$	$\frac{\Gamma(B+2nI-3kI)}{k!}\Gamma^{-1}(B+2nI-4kI)$ $-I$	$\frac{(B+2nI-4kI)}{k!}(B+2nI-kI)^{-1}$ $\times\Gamma(B+2nI)\,\Gamma^{-1}(B+2nI-kI)$

TABLE 3. Legendre-Chebyshev matrix classes $F(n) = \sum C_{n,k} G(k)$; $G(n) = \sum (-1)^{n-k} D_{n,k} F(k)$, $U = A + cnI$, $V = A + ckI$; $U + I = C$, $V + I = D$

Inv. pair $(a = n)$	b	A	r	$C_{n,k}$	$D_{n,k}$
(5.5)	-1	A	c	$\frac{U\Gamma(V+nI-kI)}{(n-k)!}\Gamma^{-1}(D)$	$\frac{\Gamma(C)}{(n-k)!}\Gamma^{-1}(U-nI+kI+I)$
(5.5)	1	A	c	$\frac{U\Gamma(V+kI-nI)}{(k-n)!}\Gamma^{-1}(D)$	$\frac{\Gamma(C)}{(k-n)!}\Gamma^{-1}(C-kI+nI)$
(5.1)	-1	A	$-c$	$\frac{\Gamma(D)}{(n-k)!}\Gamma^{-1}(D+kI-nI)$	$\frac{V\,\Gamma(U+nI-kI)}{(n-k)!}\Gamma^{-1}(C)$
(5.1)	1	A	$-c$	$\frac{D}{(k-n)!}\Gamma^{-1}(D-kI+nI)$	$\frac{V\Gamma(U+kI-nI)}{(k-n)!}\Gamma^{-1}(C)$
(5.4)	-1	-A	c	$\frac{\Gamma(D+nI-kI)}{(n-k)!}\Gamma^{-1}(D)$	$\frac{D\,\Gamma(C)}{(n-k)!}\Gamma^{-1}(C-nI+kI+I)$
(5.4)	1	$-A$	c	$\frac{\Gamma(D+kI-nI)}{(k-n)!}\Gamma^{-1}(D)$	$\frac{D\,\Gamma(C)}{(k-n)!}\Gamma^{-1}(C+nI-kI+I)$
(5.3)	-1	A	c	$\frac{C\,\Gamma(D)}{(n-k)!}\Gamma^{-1}(D-nI+kI+I)$	$\frac{\Gamma(C+nI-kI)}{(n-k)!}\Gamma^{-1}(C)$
(5.3)	1	A	c	$\frac{C\,\Gamma(V+I)}{(k-n)!}\Gamma^{-1}(D+nI-kI+I)$	$\frac{\Gamma(C-nI+kI)}{(k-n)!}\Gamma^{-1}(C)$

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Department of Mathematical Sciences
P. D. Patel Institute of Sciences
Charotar University of Science and Technology
Changa-388 421, Dist: Anand
INDIA

Research Scholar at Department of Mathematics,
The Maharaja Sayajirao University of Baroda,
Vadodara-390 002
INDIA

E-mail: reshmasanjhira1508@gmail.com

Gmail

reshmma sanjhira <reshmasanjhira1508@gmail.com>

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On the matrix function ${}_pR_q(A, B; z)$ and its fractional calculus properties

Ravi Dwivedi * and Reshma Sanjhira †

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Abstract

The main objective of the present paper is to introduce and study the function ${}_pR_q(A, B; z)$ with matrix parameters and investigate the convergence of this matrix function. The contiguous matrix function relations, differential formulas and the integral representation for the matrix function ${}_pR_q(A, B; z)$ are derived. Certain properties of the matrix function ${}_pR_q(A, B; z)$ have also been studied from fractional calculus point of view. Finally, we emphasize on the special cases namely the generalized matrix M-series, the Mittag-Leffler matrix function and its generalizations and some matrix polynomials.

Keywords: Hypergeometric function, Mittag-Leffer function, Matrix functional calculus.

AMS Subject Classification (2010): 15A15, 33E12, 33C65.

1 Introduction

Special matrix functions play an important role in mathematics and physics. In particular, special matrix functions appear in the study of statistics [7], probability theory [28] and Lie theory [13, 16], to name a few. The theory of special matrix functions has been initiated by Jódar and Cortés who studied matrix analogues of gamma, beta and Gauss hypergeometric functions [17, 18]. Dwivedi and Sahai generalize the study of one variable special matrix functions to n-variables [10]-[12]. Some of the extended work of Appell matrix functions have been given in [4]. Certain polynomials in one or more variables have been introduced and studied from matrix point of view, see [1]-[3],[6, 8, 24, 26]. Recently, the generalized Mittag-Leffler matrix function have been introduced and studied in [23].

It appears from the literature that the function ${}_pR_q(\alpha, \beta; z)$ were systematically studied in [9]. In this article, we introduce a new class of matrix function, namely ${}_pR_q(A, B; z)$ and discuss its regions of convergence. We also give contiguous matrix function relations, integral representations and differential formulas satisfied by the matrix function ${}_pR_q(A, B; z)$.

*Department of Mathematics and Astronomy Lucknow University Lucknow 226007, India. E-mail: dwivedir999@gmail.com

†Department of Mathematics, Faculty of Science, The Maharaja Sayajirao University of Baroda, Vadodara-390002, India: Department of Mathematical Sciences, P. D. Patel Institute of Applied Sciences, Faculty of Applied Sciences, Charotar University of Science and Technology, Changa-388 421, India E-mail: reshmasanjhira1508@gmail.com (Corresponding author)

The matrix analogues of generalized M-series ${}_pM_q^{\alpha,\beta}$ $(\gamma_1, \ldots, \gamma_p, \delta_1, \ldots, \delta_q; z)$, Mittag-Leffler functions and its generalizations have been presented as special cases of the matrix function ${}_pR_q(A, B; z)$. The paper is organized as follows:

In Section 2, we list the basic definitions and results from special matrix functions that are needed in the sequel. In Section 3, we introduce the matrix function ${}_pR_q(A, B; z)$ and prove a theorem on its absolute convergence. In Section 4, we give contiguous matrix function relations and differential formulas satisfied by ${}_pR_q(A, B; z)$. In Section 5, an integral representation of the matrix function ${}_pR_q(A, B; z)$ motivated by the integral of beta matrix function has been given. In Section 6, the fractional order integral and differential transforms of the matrix function ${}_pR_q(A, B; z)$ have been determined. Finally, in Section 7, we present the Gauss hypergeometric matrix function and its generalization, the matrix M-series, the Mittag-Leffler matrix function and its generalizations and some matrix polynomials as special cases of ${}_pR_q(A, B; z)$.

2 Preliminaries

Let the spectrum of a matrix A in $\mathbb{C}^{r\times r}$, denoted by $\sigma(A)$, be the set of all eigenvalues of A. Then, for a matrix $A \in \mathbb{C}^{r\times r}$ such that A is positive stable, that is, $\beta(A) = \min\{\, \Re(z) \mid z \in \sigma(A)\,\} > 0$, the gamma matrix function is defined by [17]

$$\Gamma(A) = \int_0^\infty e^{-t}\, t^{A-I}\, dt$$

and the reciprocal gamma matrix function is defined as [17]

$$\Gamma^{-1}(A) = A(A+I)\ldots(A+(n-1)I)\Gamma^{-1}(A+nI),\;\; n \geq 1. \tag{2.1}$$

The Pochhammer symbol for $A \in \mathbb{C}^{r\times r}$ is given by [18]

$$(A)_n = \begin{cases} I, & \text{if } n = 0, \\ A(A+I)\ldots(A+(n-1)I), & \text{if } n \geq 1. \end{cases} \tag{2.2}$$

This gives

$$(A)_n = \Gamma^{-1}(A)\;\Gamma(A+nI), \qquad n \geq 1. \tag{2.3}$$

If $A \in \mathbb{C}^{r\times r}$ is a positive stable matrix and $n \geq 1$ is an integer, then the gamma matrix function can also be defined in the form of a limit as [17]

$$\Gamma(A) = \lim_{n\to\infty} (n-1)!\,(A)_n^{-1}\, n^A. \tag{2.4}$$

If A and B are positive stable matrices in $\mathbb{C}^{r\times r}$, then the beta matrix function is defined as [17]

$$\mathfrak{B}(A, B) = \int_0^1 t^{A-I}\,(1-t)^{B-I} dt. \tag{2.5}$$

Furthermore, if A, B and $A+B$ are positive stable matrices in $\mathbb{C}^{r\times r}$ such that $AB = BA$, then the beta matrix function is defined as [17]

$$\mathfrak{B}(A, B) = \Gamma(A)\,\Gamma(B)\,\Gamma^{-1}(A+B). \tag{2.6}$$

Using the Schur decomposition of A, it follows that [15, 29]

$$\|e^{tA}\| \leq e^{t\alpha(A)} \sum_{k=0}^{r-1} \frac{(\|A\| r^{1/2} t)^k}{k!}, \quad t \geq 0. \tag{2.7}$$

We shall use the notation $\Gamma\begin{pmatrix} A_1, \dots, A_p \\ B_1, \dots, B_q \end{pmatrix}$ for $\Gamma(A_1) \cdots \Gamma(A_p)\, \Gamma^{-1}(B_1) \cdots \Gamma^{-1}(B_q)$.

3 The matrix function ${}_pR_q(A, B; z)$

Jódar and Cortés [18] defined the Gauss hypergeometric function with matrix parameters denoted by ${}_2F_1(A, B; C; z)$, where A, B, C are matrices in $\mathbb{C}^{r\times r}$, and determined its region of convergence and integral representation. A natural generalization of the Gauss hypergeometric matrix function is obtained in [10] by introducing an arbitrary number of matrices as parameters in the numerator and denominator and referring to this generalization as the generalized hypergeometric matrix function, ${}_pF_q(A_1, \dots, A_p; B_1, \dots, B_q; z)$. We now give an extension of the generalized hypergeometric matrix function. Let A, B, C_i and D_j, $1 \leq i \leq p$, $1 \leq j \leq q$, be matrices in $\mathbb{C}^{r\times r}$ such that $D_j + kI$ are invertible for all integers $k \geq 0$. Then, we define the matrix function ${}_pR_q(A, B; z)$ as

$$\begin{aligned} {}_pR_q(A, B; z) &= {}_pR_q\left(\begin{matrix} C_1, \dots, C_p \\ D_1, \dots, D_q \end{matrix} \mid A, B; z\right) \\ &= \sum_{n\geq 0} \Gamma^{-1}(nA + B)\, (C_1)_n \dots (C_p)_n\, (D_1)_n^{-1} \dots (D_q)_n^{-1} \frac{z^n}{n!}. \end{aligned} \tag{3.1}$$

In the following theorem, we find the regions in which the matrix function ${}_pR_q(A, B; z)$ either converges or diverges.

Theorem 3.1. *Let $A, B, C_1, \dots, C_p, D_1, \dots, D_q$ be positive stable matrices in $\mathbb{C}^{r\times r}$. Then the matrix function ${}_pR_q(A, B; z)$ defined in (3.1) converges or diverges in one of the following regions:*

1. *If $p \leq q + 1$, the matrix function converges absolutely for all finite z.*
2. *If $p = q + 2$, function converges for $|z| < 1$ and diverges for $|z| > 1$.*
3. *If $p = q + 2$ and $|z| = 1$, the function converges absolutely for $\beta(D_1) + \dots + \beta(D_q) > \alpha(C_1) + \dots + \alpha(C_p)$.*
4. *If $p > q + 2$, the function diverges for all $z \neq 0$.*

Proof. Let $U_n(z)$ denote the general term of the series (3.1). Then, we have

$$\begin{aligned} \|U_n(z)\| &\leq \|\Gamma^{-1}(nA + B)\| \prod_{i=1}^{p} \|(C_i)_n\| \prod_{j=1}^{q} \|(D_j)_n^{-1}\| \frac{|z|^n}{n!} \\ &\leq \|\Gamma^{-1}(nA + B)\| \prod_{i=1}^{p} \left\| \frac{(C_i)_n n^{C_i} n^{-C_i} (n-1)!}{(n-1)!} \right\| \end{aligned}$$

$$\times \prod_{j=1}^{q} \left\| \frac{(D_j)_n^{-1} n^{D_j} n^{-D_j} (n-1)!}{(n-1)!} \right\| \frac{|z|^n}{n!}. \tag{3.2}$$

The limit definition of gamma matrix function (2.4) and Schur decomposition (2.7) yield

$$\|U_n(z)\| \leq N\ S\ ((n-1)!)^{p-q-2}\ n^{\sum_{i=1}^{p} \alpha(C_i) - \sum_{j=1}^{q} \beta(D_j) - 1}\ |z|^n, \tag{3.3}$$

where $N = \|\Gamma^{-1}(C_1)\| \cdots \|\Gamma^{-1}(C_p)\| \|\Gamma(D_1)\| \cdots \|\Gamma(D_q)\|$ and

$$S = \left(\sum_{k=0}^{r-1} \frac{(\max\{\|C_1\|, \ldots, \|C_p\|, \|D_1\|, \ldots, \|D_q\|\}\, r^{\frac{1}{2}}\ \ln n)^k}{k!} \right)^{p+q}. \tag{3.4}$$

Thus, it can be easily calculated from (3.3) and comparison theorem of numerical series that the matrix series (3.1) converges or diverges in one of the region listed in Theorem 3.1. □

4 Contiguous matrix function relations

In this section, we shall obtain contiguous matrix function relations and differential formulas satisfied by the matrix function ${}_pR_q(A, B; z)$. The following abbreviated notations will be used throughout the subsequent sections:

$$\begin{aligned}
R &= {}_pR_q(A, B; z) = {}_pR_q\left(\begin{matrix} C_1, \ldots, C_p \\ D_1, \ldots, D_q \end{matrix} \mid A, B; z \right), \\
R(C_i+) &= {}_pR_q\left(\begin{matrix} C_1, \ldots, C_{i-1}, C_i + I, C_{i+1}, \ldots, C_p \\ D_1, \ldots, D_q \end{matrix} \mid A, B; z \right), \\
R(C_i-) &= {}_pR_q\left(\begin{matrix} C_1, \ldots, C_{i-1}, C_i - I, C_{i+1}, \ldots, C_p \\ D_1, \ldots, D_q \end{matrix} \mid A, B; z \right), \\
R(D_j-) &= {}_pR_q\left(\begin{matrix} C_1, \ldots, C_p \\ D_1, \ldots, D_{j-1}, D_j - I, D_{j+1}, \ldots, D_q \end{matrix} \mid A, B; z \right), \\
{}_pR_q(A, B+I; z) &= {}_pR_q\left(\begin{matrix} C_1, \ldots, C_p \\ D_1, \ldots, D_q \end{matrix} \mid A, B+I; z \right), \\
{}_pR_q(A, B-I; z) &= {}_pR_q\left(\begin{matrix} C_1, \ldots, C_p \\ D_1, \ldots, D_q \end{matrix} \mid A, B-I; z \right).
\end{aligned} \tag{4.1}$$

Following Desai and Shukla [9], we can find $(p+q-1)$ contiguous matrix function relations of bilateral type that connect either R, $R(C_1+)$ and $R(C_i+)$, $1 \leq i \leq p$ or R, $R(C_1+)$ and $R(D_j-)$, $1 \leq j \leq q$. Let $C_i, 1 \leq i \leq p$ be positive stable matrices in $\mathbb{C}^{r\times r}$ such that $C_iC_k = C_kC_i, 1 \leq k \leq p, k < i$, $C_iA = AC_i$ and $C_iB = BC_i$. Then, we have

$$R(C_i+) = \sum_{n\geq 0} C_i^{-1}(C_i + nI)\Gamma^{-1}(nA + B)\,(C_1)_n \cdots (C_p)_n\,(D_1)_n^{-1} \cdots (D_q)_n^{-1}\,\frac{z^n}{n!}. \tag{4.2}$$

If $\theta = z\frac{d}{dz}$ is a differential operator, then we get

$$(\theta + C_i)R = \sum_{n\geq 1}(C_i + nI)\Gamma^{-1}(nA + B)\,(C_1)_n \cdots (C_p)_n\,(D_1)_n^{-1}\cdots(D_q)_n^{-1}\,\frac{z^n}{n!}. \tag{4.3}$$

Equations (4.2) and (4.3) together yield

$$(\theta + C_i)R = C_i\,R(C_i+), \quad i = 1, \dots, p. \tag{4.4}$$

In particular, for $i = 1$, we write

$$(\theta + C_1)R = C_1\,R(C_1+). \tag{4.5}$$

Similarly for matrices $D_j \in \mathbb{C}^{r\times r}, 1 \leq j \leq q$ such that $D_jD_k = D_kD_j, 1 \leq k \leq q, k > j$, we obtain a set of q equations, given by

$$\theta\,R + R\,(D_j - I) = R(D_j-)(D_j - I). \tag{4.6}$$

Now, eliminating θ from (4.4) and (4.6) gives rise to $(p+q-1)$ contiguous matrix function relations of bilateral type

$$C_i\,R - R\,(D_j - I) = C_i\,R(C_i+) - R(D_j-)(D_j - I),\ 1 \leq i \leq p, 1 \leq j \leq q. \tag{4.7}$$

Equations (4.4) and (4.5) produce $(p-1)$ contiguous matrix function relations

$$(C_1 - C_i)R = C_1R(C_1+) - C_i\,R(C_i+), \quad i = 2, \dots, p. \tag{4.8}$$

Furthermore, Equations (4.5) and (4.6) leads to q contiguous matrix function relations

$$C_1\,R - R\,(D_j - I) = C_1\,R(C_1+) - R(D_j-)(D_j - I),\ 1 \leq j \leq q. \tag{4.9}$$

The set of matrix function relations given in (4.8) and (4.9) are simple contiguous matrix function relations.

Next, we give matrix differential formulas satisfied by the matrix function ${}_pR_q(A, B; z)$.

4.1 Matrix differential formulas

Theorem 4.1. *Let A, B, $C_1, \dots, C_p$, $D_1, \dots, D_q \in \mathbb{C}^{r\times r}$ such that each $D_j+kI, 1 \leq j \leq q$ is invertible for all integers $k \geq 0$. Then the matrix function ${}_pR_q(A, B; z)$ satisfies the matrix differential formulas*

$$\begin{aligned}\left(\frac{d}{dz}\right)^r {}_pR_q(A, B; z) &= (C_1)_r \cdots (C_p)_r\ {}_pR_q\left(\begin{array}{c} C_1 + rI, \dots, C_p + rI \\ D_1 + rI, \dots, D_q + rI \end{array} \mid A, rA + B; z\right) \\ &\quad \times (D_1)_r^{-1}\cdots(D_q)_r^{-1},\ C_lC_m = C_mC_l,\ C_lA = AC_l,\ C_lB = BC_l, \\ &\quad D_iD_j = D_jD_i, 1 \leq l, m \leq p,\ 1 \leq i, j \leq q;\end{aligned} \tag{4.10}$$

$$\left(\frac{d}{dz}\right)^r ({}_pR_q(A, B; z)z^{D_j-I}) = {}_pR_q\left(\begin{array}{c} C_1, \dots, C_p \\ D_1, \dots, D_{j-1}, D_j - rI, D_{j+1}, \dots, D_q \end{array} \mid A, B; z\right)$$

$$\times (-1)^r z^{D_j-(r+1)I}(I-D_j)_r,\ D_iD_j = D_jD_i; \qquad (4.11)$$

$$\left(z^2\frac{d}{dz}\right)^r (z^{C_i-(r-1)I}\,{}_pR_q(A,B;z))$$
$$= (C_i)_r\, z^{C_i+rI}\,{}_pR_q\left(\begin{array}{c} C_1,\dots,C_{i-1},C_i+rI,C_{i+1},\dots,C_p \\ D_1,\dots,D_q \end{array} \mid A,B;z\right), C_iC_j = C_jC_i$$
$$C_iA = AC_i,\ C_iB = BC_i,\ 1 \le i,j \le p. \qquad (4.12)$$

Proof. Differentiating the Equation (3.1) with respect to z, we get

$$\begin{aligned}\frac{d}{dz}\,{}_pR_q(A,B;z) &= \sum_{n\ge1}\Gamma^{-1}(nA+B)\,(C_1)_n\dots(C_p)_n\,(D_1)_n^{-1}\dots(D_q)_n^{-1}\,\frac{z^{n-1}}{(n-1)!}\\ &= \sum_{n\ge0}\Gamma^{-1}(nA+A+B)\,(C_1)_{n+1}\dots(C_p)_{n+1}\,(D_1)_{n+1}^{-1}\dots(D_q)_{n+1}^{-1}\,\frac{z^n}{n!}\\ &= (C_1)_1\cdots(C_p)_1\,{}_pR_q\left(\begin{array}{c} C_1+I,\dots,C_p+I \\ D_1+I,\dots,D_q+I\end{array} \mid A,A+B;z\right)\\ &\quad\times (D_1)_1^{-1}\cdots(D_q)_1^{-1}. \qquad (4.13)\end{aligned}$$

Proceeding similarly r-times, we get the required relation (4.10). Using the commutativity of matrices considered in the hypothesis and the way (4.10) is proved, we are able to prove (4.11) and (4.12). □

Theorem 4.2. *Let A, B, $C_1,\dots,C_p$, $D_1,\dots,D_q \in \mathbb{C}^{r\times r}$ such that each $D_j+kI, 1\le j\le q$ is invertible for all integers $k\ge 0$ and A, $B-I$ are positive stable. Then the matrix function ${}_pR_q(A,B;z)$ defined in (3.1) satisfies the matrix differential formula*

$$zA\frac{d}{dz}\,{}_pR_q(A,B;z) = {}_pR_q(A,B-I;z) - (B-I)\,{}_pR_q(A,B;z), \quad AB = BA. \qquad (4.14)$$

Proof. Using the definition of matrix function ${}_pR_q(A,B;z)$ and $z\frac{d}{dz}z^n = nz^n$ in the left hand side of (4.14), we get

$$\begin{aligned}zA\frac{d}{dz}\,{}_pR_q(A,B;z) &= \sum_{n\ge0} nA\Gamma^{-1}(nA+B)\,(C_1)_n\dots(C_p)_n\,(D_1)_n^{-1}\dots(D_q)_n^{-1}\,\frac{z^n}{n!}\\ &= \sum_{n\ge0}\Gamma^{-1}(nA+B-I)\,(C_1)_n\dots(C_p)_n\,(D_1)_n^{-1}\dots(D_q)_n^{-1}\,\frac{z^n}{n!}\\ &\quad - (B-I)\sum_{n\ge0}\Gamma^{-1}(nA+B)\,(C_1)_n\dots(C_p)_n\,(D_1)_n^{-1}\dots(D_q)_n^{-1}\\ &\quad\times\frac{z^n}{n!}, \quad AB = BA\\ &= {}_pR_q(A,B-I;z) - (B-I)\,{}_pR_q(A,B;z). \qquad (4.15)\end{aligned}$$

This completes the proof of (4.14). □

5 Integral representation

We now find an integral representation of the matrix function ${}_pR_q(A,B;z)$ using the integral of the beta matrix function.

Theorem 5.1. *Let A, B, $C_1, \ldots, C_p$, $D_1, \ldots, D_q$ be matrices in $\mathbb{C}^{r\times r}$ such that $C_pD_j = D_jC_p, 1 \leq j \leq q$ and $C_p, D_q, D_q - C_p$ are positive stable. Then, for $|z| < 1$, the matrix function ${}_pR_q(A,B;z)$ defined in (3.1) can be presented in integral form as*

$$\begin{aligned}{}_pR_q(A,B;z) = &\int_0^1 {}_{p-1}R_{q-1}\left(\begin{array}{c} C_1,\ldots,C_{p-1} \\ D_1,\ldots,D_{q-1}\end{array} \mid A,B;t\,z\right) t^{C_p-I}(1-t)^{D_q-C_p-I}dt \\ &\times \Gamma\left(\begin{array}{c} D_q \\ C_p, D_q-C_p\end{array}\right).\end{aligned} \tag{5.1}$$

Proof. Since $C_p, D_q, D_q - C_p$ are positive stable and $C_pD_q = D_qC_p$, we have [18]

$$(C_p)_n(D_q)_n^{-1} = \left(\int_0^1 t^{C_p+(n-1)I}(1-t)^{D_q-C_p-I}dt\right)\Gamma\left(\begin{array}{c} D_q \\ C_p, D_q-C_p\end{array}\right). \tag{5.2}$$

Using (5.2) in (3.1), we get

$$\begin{aligned}{}_pR_q(A,B;z) = &\sum_{n\geq 0}\int_0^1 \Gamma^{-1}(nA+B)\,(C_1)_n\cdots(C_{p-1})_n\,(D_1)_n^{-1}\cdots(D_{q-1})_n^{-1} \\ &\times \frac{z^n}{n!}\,t^{C_p+(n-1)I}\,(1-t)^{D_q-C_p-I}dt\;\Gamma\left(\begin{array}{c} D_q \\ C_p, D_q-C_p\end{array}\right).\end{aligned} \tag{5.3}$$

To interchange the integral and summation, consider the product of matrix functions

$$\begin{aligned}S_n(z,t) = &\Gamma^{-1}(nA+B)\,(C_1)_n\cdots(C_{p-1})_n\,(D_1)_n^{-1}\cdots(D_{q-1})_n^{-1}\,\frac{z^n}{n!}\,t^{C_p+(n-1)I} \\ &\times (1-t)^{D_q-C_p-I}\;\Gamma\left(\begin{array}{c} D_q \\ C_p, D_q-C_p\end{array}\right).\end{aligned} \tag{5.4}$$

For $0 < t < 1$ and $n \geq 0$, we get

$$\begin{aligned}&\|S_n(z,t)\| \\ &\leq \left\|\Gamma\left(\begin{array}{c} D_q \\ C_p, D_q-C_p\end{array}\right)\right\| \left\|\Gamma^{-1}(nA+B)\,(C_1)_n\cdots(C_{p-1})_n\,(D_1)_n^{-1}\cdots(D_{q-1})_n^{-1}\,\frac{z^n}{n!}\right\| \\ &\quad\times \|t^{C_p-I}\|\|(1-t)^{D_q-C_p-I}\|.\end{aligned} \tag{5.5}$$

The Schur decomposition (2.7) yields

$$\begin{aligned}\|t^{C_p-I}\|\;\|(1-t)^{D_q-C_p-I}\| \leq\; &t^{\alpha(C_p)-1}(1-t)^{\alpha(D_q-C_p)-1}\left(\sum_{k=0}^{r-1}\frac{(\|C_p-I\|\;r^{1/2}\;\ln t)^k}{k!}\right) \\ &\times\left(\sum_{k=0}^{r-1}\frac{(\|D_q-C_p-I\|\;r^{1/2}\;\ln(1-t))^k}{k!}\right).\end{aligned} \tag{5.6}$$

Since $0 < t < 1$, we have

$$\|t^{C_p-I}\| \, \|(1-t)^{D_q-C_p-I}\| \leq \mathcal{A}\, t^{\alpha(C_p)-1}(1-t)^{\alpha(D_q-C_p)-1}, \tag{5.7}$$

where

$$\mathcal{A} = \left(\sum_{k=0}^{r-1} \frac{(\max\{\|C_p - I\|, \|D_q - C_p - I\|\}\, r^{1/2})^k}{k!}\right)^2. \tag{5.8}$$

The matrix series $\Gamma^{-1}(nA+B)\,(C_1)_n \cdots (C_{p-1})_n\,(D_1)_n^{-1} \cdots (D_{q-1})_n^{-1} \frac{z^n}{n!}$ converges absolutely for $p \leq q+2$ and $|z| < 1$; suppose it converges to S'. Thus, we get

$$\sum_{n \geq 0} \|S_n(z,t)\| \leq f(t) = N S' \mathcal{A}\, t^{\alpha(C_p)-1}\,(1-t)^{\alpha(D_q-C_p)-1}. \tag{5.9}$$

Since $\alpha(C_p), \alpha(D_q - C_p) > 0$, the function $f(t)$ is integrable and by the dominated convergence theorem [14], the summation and the integral can be interchanged in (5.3). Using $C_pD_j = D_jC_p,\ 1 \leq j \leq q$, we get (5.1). □

6 Fractional calculus of the matrix function ${}_pR_q(A, B; z)$

Let $x > 0$ and $\mu \in \mathbb{C}$ such that $\Re(\mu) > 0$. Then the Riemann-Liouville type fractional order integral and derivatives of order μ are given by [19, 27]

$$(\mathbf{I}_a^\mu f)(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} f(t)dt \tag{6.1}$$

and

$$\mathbf{D}_a^\mu f(x) = (\mathbf{I}_a^{n-\mu}\mathbf{D}^n f(x)), \quad \mathbf{D} = \frac{d}{dx}. \tag{6.2}$$

Bakhet and his co-workers, [5], studied the fractional order integrals and derivatives of Wright hypergeometric and incomplete Wright hypergeometric matrix functions using the operators (6.1) and (6.2). To obtain such they used the following lemma:

Lemma 6.1. *Let A be a positive stable matrix in $\mathbb{C}^{r\times r}$ and $\mu \in \mathbb{C}$ such that $\Re(\mu) > 0$. Then the fractional integral operator (6.1) yields*

$$\mathbf{I}^\mu(x^{A-I}) = \Gamma(A)\Gamma^{-1}(A+\mu I)x^{A+(\mu-1)I}. \tag{6.3}$$

In the next two theorems, we find the fractional order integral and derivative of matrix function ${}_pR_q(A, B; z)$.

Theorem 6.1. *Let $A,\ B,\ C_1,\ \ldots, C_p,\ D_1, \ldots, D_q$ be matrices in $\mathbb{C}^{r\times r}$ and $\mu \in \mathbb{C}$ such that $D_iD_j = D_jD_i, 1 \leq i, j \leq q$ and $\Re(\mu) > 0$. Then the fractional integral of the matrix function ${}_pR_q(A, B; z)$ is given by*

$$\begin{aligned}
&\mathbf{I}^\mu[{}_pR_q(A, B; z)z^{D_j-I}] \\
&= {}_pR_q\left(\begin{array}{c} C_1, \ldots, C_p \\ D_1, \ldots, D_{j-1}, D_j + \mu I, D_{j+1}, \ldots, D_q \end{array} \mid A, B; z\right) z^{D_j+(\mu-1)I} \\
&\quad \times \Gamma(D_j)\Gamma^{-1}(D_j+\mu I).
\end{aligned} \tag{6.4}$$

Proof. From Equation (6.1), we have

$$\begin{aligned}
&\mathbf{I}^{\mu}[{}_pR_q(A,B;z)z^{D_j-I}] \\
&= \frac{1}{\Gamma(\mu)}\int_0^z (z-t)^{\mu-1}{}_pR_q(A,B;t)t^{D_j-I}dt \\
&= \frac{1}{\Gamma(\mu)}\sum_{n\geq 0}(C_1)_n\dots(C_p)_n\left(\int_0^z (z-t)^{\mu-1}t^{D_j+(n-1)I}dt\right)(D_1)_n^{-1}\dots(D_q)_n^{-1}\frac{1}{n!} \\
&= \sum_{n\geq 0}(C_1)_n\dots(C_p)_n\left(\mathbf{I}^{\mu}\, z^{D_j+(n-1)I}\right)(D_1)_n^{-1}\dots(D_q)_n^{-1}\frac{1}{n!}.
\end{aligned} \tag{6.5}$$

Using the Lemma 6.1, we get

$$\begin{aligned}
\mathbf{I}^{\mu}[{}_pR_q(A,B;z)z^{D_j-I}] &= \frac{1}{\Gamma(\mu)}\sum_{n\geq 0}(C_1)_n\dots(C_p)_n\Gamma(D_j+nI)\Gamma^{-1}(D_j+nI+\mu I) \\
&\quad\times z^{D_j+(n+\mu-1)I}(D_1)_n^{-1}\dots(D_q)_n^{-1}\frac{1}{n!} \\
&= {}_pR_q\left(\begin{matrix} C_1,\dots,C_p \\ D_1,\dots,D_{j-1},D_j+\mu I,D_{j+1},\dots,D_q\end{matrix}\mid A,B;z\right) \\
&\quad\times z^{D_j+(\mu-1)I}\,\Gamma(D_j)\Gamma^{-1}(D_j+\mu I).
\end{aligned} \tag{6.6}$$

This completes the proof. □

Theorem 6.2. *Let A, B, C_1, $\dots,C_p$, $D_1,\dots,D_q$ be matrices in $\mathbb{C}^{r\times r}$ and $\mu\in\mathbb{C}$ such that $D_iD_j=D_jD_i, 1\leq i,j\leq q$ and $\Re(\mu)>0$. Then the fractional integral of the matrix function ${}_pR_q(A,B;z)$ is given by*

$$\begin{aligned}
&\mathbf{D}^{\mu}[{}_pR_q(A,B;z)z^{D_j-I}] \\
&= {}_pR_q\left(\begin{matrix} C_1,\dots,C_p \\ D_1,\dots,D_{j-1},D_j-\mu I,D_{j+1},\dots,D_q\end{matrix}\mid A,B;z\right)z^{D_j-(\mu-1)I} \\
&\quad\times\Gamma(D_j)\Gamma^{-1}(D_j-\mu I).
\end{aligned} \tag{6.7}$$

Proof. The fractional derivative operator (6.2) and Theorem 6.1 together yield

$$\begin{aligned}
&\mathbf{D}^{\mu}[{}_pR_q(A,B;z)z^{D_j-I}] \\
&= \left(\frac{d}{dz}\right)^r{}_pR_q\left(\begin{matrix} C_1,\dots,C_p \\ D_1,\dots,D_{j-1},D_j+(r-\mu)I,D_{j+1},\dots,D_q\end{matrix}\mid A,B;z\right)z^{D_j+(r-\mu-1)I} \\
&\quad\times\Gamma(D_j)\Gamma^{-1}(D_j+(r-\mu)I).
\end{aligned} \tag{6.8}$$

Now, proceeding exactly in the same manner as in Theorem 4.1, we get (6.7). □

7 Special Cases

The matrix function ${}_pR_q(A,B;z)$ reduces to several special matrix functions. These matrix functions are considered as matrix generalizations of respective classical matrix functions such as the generalized hypergeometric matrix function, the Gauss hypergeometric matrix function, the confluent hypergeometric matrix function, the matrix M-series, the Wright matrix function and the Mittag-Leffler matrix function and its generalizations. We also discuss some matrix polynomials as particular cases.

We Start with the special case, $A = B = I$ and $C_p = I$. The matrix function ${}_pR_q(A, B; z)$ reduces to

$$ {}_pR_q\left(\begin{array}{c} C_1, \dots, C_{p-1}, I \\ D_1, \dots, D_q \end{array} \mid I, I; z \right) = \sum_{n \geq 0} (C_1)_n \dots (C_{p-1})_n \, (D_1)_n^{-1} \dots (D_q)_n^{-1} \, \frac{z^n}{n!} $$
$$ = {}_{p-1}F_q(C_1, \dots, C_{p-1}, D_1, \dots, D_q; z), \tag{7.1} $$

which is known as generalized hypergeometric matrix function with $p-1$ matrix parameters in the numerator and q in the denominator [10]. For $C_1 = A_1, C_2 = B_1, C_3 = I, D_1 = C$ and $A = B = I$, the matrix function ${}_pR_q(A, B; z)$ reduces to the Gauss hypergeometric matrix function ${}_2F_1(A_1, B_1; C; z)$. Similarly, for $C_1 = A_1, C_2 = I, D_1 = C$ and $A = B = I$, ${}_pR_q(A, B; z)$ reduces to the confluent hypergeometric matrix function ${}_1F_1(A_1; C; z)$.

For $C_p = I$, the matrix function ${}_pR_q(A, B; z)$ leads to the matrix analogue of the generalized M-series [25].

$$ {}_pR_q\left(\begin{array}{c} C_1, \dots, C_{p-1}, I \\ D_1, \dots, D_q \end{array} \mid A, B; z \right) = \sum_{n \geq 0} \Gamma^{-1}(nA + B)(C_1)_n \dots (C_{p-1})_n $$
$$ \times (D_1)_n^{-1} \dots (D_q)_n^{-1} \, z^n $$
$$ = {}_{p-1}M_q^{(A,B)}(C_1, \dots, C_{p-1}, D_1, \dots, D_q; z). \tag{7.2} $$

With $p = 1, q = 0$, $C_1 = I$ and $B = I$, the matrix function ${}_pR_q(A, B; z)$ reduces to

$$ {}_1R_0\left(\begin{array}{c} I \\ - \end{array} \mid A, I; z \right) = \sum_{n \geq 0} \Gamma^{-1}(nA + I) z^n = E_A(z), \tag{7.3} $$

for $p = 1, q = 0$ and $C_1 = I$, the matrix function ${}_pR_q(A, B; z)$ gives

$$ {}_1R_0\left(\begin{array}{c} I \\ - \end{array} \mid A, B; z \right) = \sum_{n \geq 0} \Gamma^{-1}(nA + B) \, z^n = E_{A,B}(z), \tag{7.4} $$

with one matrix parameter, $C_1 = C$, ${}_pR_q(A, B; z)$ becomes

$$ {}_1R_0\left(\begin{array}{c} C \\ - \end{array} \mid A, B; z \right) = \sum_{n \geq 0} \Gamma^{-1}(nA + B) \, (C)_n \, \frac{z^n}{n!} = E_{A,B}^{C}(z) \tag{7.5} $$

and for two numerator matrix parameter, $C_1 = C$, $C_2 = I$ and one denominator matrix parameter $D_1 = D$, ${}_pR_q(A, B; z)$ reduces to

$$ {}_2R_1\left(\begin{array}{c} C, I \\ D \end{array} \mid A, B; z \right) = \sum_{n \geq 0} \Gamma^{-1}(nA + B) \, (C)_n \, (D)_n^{-1} z^n = E_{A,B}^{C,D}(z). \tag{7.6} $$

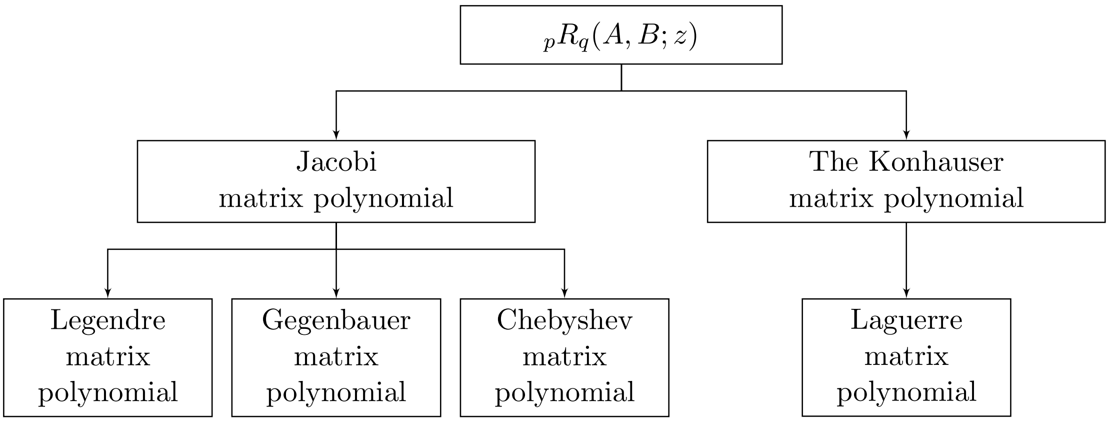
We define the matrix functions obtained in (7.3)-(7.6) as the matrix analogue of the classical Mittag-Leffler function [20], Wiman's function [30], the generalized Mittag-Leffler function in three parameters [21] and the generalized Mittag-Leffler function in four parameters [22], respectively.

For $p = q = 0$, with replacement of B by $B + I$ and z by $-z$, the matrix function ${}_pR_q(A, B; z)$ turns into the generalized Bessel-Maitland matrix function [23]

$$ {}_0R_0\left(\begin{array}{c} - \\ - \end{array} \mid A, B+I; -z\right) = \sum_{n\geq 0} \frac{\Gamma^{-1}(nA+B+I)\,(-z)^n}{n!} = J_A^B(z). \tag{7.7} $$

The matrix polynomials such as the Jacobi matrix polynomial, the generalized Konhauser polynomial, the Laguerre matrix polynomial, the Legendre matrix polynomial, the Chebyshev matrix polynomial and the Gegenbauer matrix polynomial can be presented as the particular cases of the matrix function ${}_pR_q(A, B; z)$.The matrix polynomial dependency chart is given below:

Figure 1: Special cases



More explicitly, we say that the Jacobi matrix polynomial can be written in term of the matrix function ${}_pR_q(A, B; z)$, for $p = 2$, $q = 1$, $C_1 = A + C + (k+1)I$, $C_2 = -kI$, $D_1 = C + I$, $A = 0$, $B = C + I$ and $z = \frac{1+x}{2}$, as

$$ P_k^{(A,C)}(x) = \frac{(-1)^k}{k!}\,{}_2R_1\left(\begin{array}{c} A+C+(k+1)I, -kI \\ C+I \end{array} \mid 0, C+I; \frac{1+x}{2}\right) \times \Gamma(C+(k+1)I). \tag{7.8} $$

For $p = 2$, $q = 1$, $C_1 = (k+1)I$, $C_2 = -kI$, $D_1 = D$, $A = 0$ and $z = \frac{1-x}{2}$, the matrix function ${}_pR_q(A, B; z)$ reduced into the Legendre matrix polynomial

$$ P_k(x, D) = {}_2R_1\left(\begin{array}{c} (k+1)I, -kI \\ D \end{array} \mid 0, B; \frac{1-x}{2}\right). \tag{7.9} $$

Similarly, the Gegenbauer matrix polynomial in term of the matrix function ${}_pR_q(A, B; z)$ can be expressed as

$$ C_k^D(x) = \frac{(2D)_k}{k!}\,{}_2R_1\left(\begin{array}{c} 2D+kI, -kI \\ D+\frac{1}{2}I \end{array} \mid 0, B; \frac{1-x}{2}\right). \tag{7.10} $$

The Konhauser matrix polynomial in term of the matrix

$$Z_m^C(x,k) = \frac{\Gamma(C+(km+1)I)}{\Gamma(m+1)} {}_1R_0\left(\begin{array}{c} -mI \\ - \end{array} \mid kI, C+I; x^k\right). \tag{7.11}$$

The Laguerre matrix polynomial can be obtain by taking $k = 1$ in Equation (7.11).

Note that the properties of these matrix functions and polynomials can be deduced from the corresponding properties of the matrix function ${}_pR_q(A, B; z)$.

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